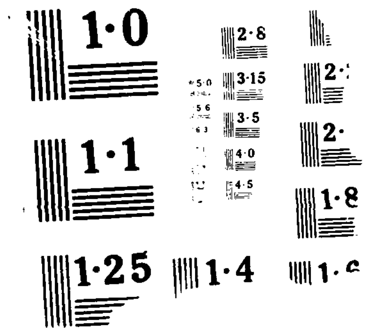
AD-A193 172 DISTORTION ESTIMATES FOR NEGATIVE SCHWARZIAN MAPS(U) 1/1
CORNELL UNIV ITHACA NY MATHEMATICAL SCIENCES INST
J GUCKENHEIMER ET AL 29 FEB 88 ARO-23306.157-MA
UNCLASSIFIED DAAG29-85-C-0018 F/G 12/1 NL


1·0
2·8
3·15
3·5
1·1
4·0
4·5
1·8
1·25
1·4

UNCLASSIFIED MASTER COPY - FOR REPRODUCTION PURPOSES

SECURITY CLASSIFICATION OF THIS PAGE

ORT DOCUMENTATION PAGE

2

AD-A193 172

APR 1 4 1988

1b. RESTRICTIVE MARKINGS

2b. DECLASSIFICATION / DOWNGRADING SCHEDULE

3. DISTRIBUTION/AVAILABILITY OF REPORT
Approved for public release; distribution unlimited.

4. PERFORMING ORGANIZATION REPORT NUMBER(S)

5. MONITORING ORGANIZATION REPORT NUMBER(S)
ARO 23306.157-MA

6a. NAME OF PERFORMING ORGANIZATION
Mathematical Science Institute

6b. OFFICE SYMBOL (If applicable)
MSI

7a. NAME OF MONITORING ORGANIZATION
U. S. Army Research Office

6c. ADDRESS (City, State, and ZIP Code)
294 Caldwell Hall - Cornell University
Ithaca, NY 14853

7b. ADDRESS (City, State, and ZIP Code)
P. O. Box 12211
Research Triangle Park, NC 27709-2211

8a. NAME OF FUNDING/SPONSORING ORGANIZATION
U. S. Army Research Office

8b. OFFICE SYMBOL (If applicable)

9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
DAAG29-85-C-0018

8c. ADDRESS (City, State, and ZIP Code)
P. O. Box 12211
Research Triangle Park, NC 27709-2211

10. SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO	PROJECT NO.	TASK NO.	WORK UNIT ACCESSION NO.

11. TITLE (Include Security Classification)
DISTORTION ESTIMATES FOR NEGATIVE SCHWARZIAN MAPS

12. PERSONAL AUTHOR(S)
JOHN GUCKENHEIMER AND STEWART JOHNSON

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
Technical Report	FROM ______ TO ______	2/29/88	12

16. SUPPLEMENTARY NOTATION The view, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.

17. COSATI CODES

FIELD	GROUP	SUB-GROUP

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
The Schwarzian derivative was defined H.A. Schwarz in connection with the study of conformal maps of the complex plane.

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

One dimensional maps with a negative schwarzian derivative are shown to have area preserving properties: the distortion dis(f) = f"/(f')2 varies inversely proportional to the vertical distance from a critical point for maps f with negative Schwarzian; maps consisting of monotone branches mapping across an interval either have a sigma-finite absolutely continuous ergodic measure or a universal attractor at theends of the interval.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT
☐ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS

21. ABSTRACT SECURITY CLASSIFICATION
Unclassified

22a. NAME OF RESPONSIBLE INDIVIDUAL

22b. TELEPHONE (Include Area Code)

22c. OFFICE SYMBOL

DD FORM 1473, 84 MAR 83 APR edition may be used until exhausted. All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE
UNCLASSIFIED

ARO 23306.157-M

Distortion Estimates for Negative Schwarzian Maps

*John Guckenheimer**
*Stewart Johnson***

Abstract: One dimensional maps with a negative Schwarzian derivative are shown to have area preserving properties: the distortion $dis(f) = \left|\frac{f''}{(f')^2}\right|$ *varies inversely proportional to the vertical distance from a critical point for maps* f *with negative Schwarzian: maps consisting of monotone branches mapping across an interval either have a sigma-finite absolutely continuous ergodic measure or a universal attractor at the ends of the interval.*

I. The Schwarzian Derivative

The Schwarzian derivative was defined H.A. Schwarz in connection with the study of conformal maps of the complex plane. The derivative has found an interesting application in the study of one dimensional maps where the assumption of a negative Schwarzian has been used to establish topological conjugacy between unimodal maps with identical kneading sequences [3]. This and other one dimensional applications arise from inherent measure preserving properties of maps with a negative Schwarzian derivative. We attempt in this paper to make these properties more explicit.

Definition The Schwarzian derivative $\mathcal{S} : C^3(\Re) \to C^0(\Re)$ is defined as

$$(\mathcal{S}f)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

The following composition rule holds:

$$(\mathcal{S}\, f \circ g)(x) = (\mathcal{S}g)(x) \ + \ (g')^2(\mathcal{S}f)(g(x))$$

Thus if f has a negative Schwarzian, so will all the iterates $f^n = f \circ f \circ \cdots \circ f$. Lemmas 1A & 1B and corollary 1 are standard results.

Lemma 1A For an interval I and $f \in C^3(I)$ with $f' > 0$

$$(\mathcal{S}f) < 0 \iff (f')^{-\frac{1}{2}} \text{ is convex}$$
$$(\mathcal{S}f) = 0 \iff (f')^{-\frac{1}{2}} \text{ is linear}$$
$$(\mathcal{S}f) > 0 \iff (f')^{-\frac{1}{2}} \text{ is concave}$$

* *Research supported by NSF, AFOSR and the ARO through the Mathematical Sciences Institute at Cornell University*

** *Research supported by the ARO through the Mathematical Sciences Institute at Cornell University*

proof $2(f')^{\frac{1}{2}}\frac{d^2}{dx^2}(f')^{-\frac{1}{2}} = -(\mathcal{S}f)$

end of proof

Similarly;

Lemma 1B For an interval I and $f \in C^3(I)$ with $f' < 0$

$$
\begin{aligned}
(\mathcal{S}f) < 0 &\iff (f')^{-\frac{1}{2}} \textit{is concave} \\
(\mathcal{S}f) = 0 &\iff (f')^{-\frac{1}{2}} \textit{is linear} \\
(\mathcal{S}f) > 0 &\iff (f')^{-\frac{1}{2}} \textit{is convex}
\end{aligned}
$$

proof $2(f')^{\frac{1}{2}}\frac{d^2}{dx^2}(f')^{-\frac{1}{2}} = -(\mathcal{S}f)$

end of proof

A function f is said to be fractional linear if it can be written in the form $f(x) = \frac{ax+b}{cx+d}$ with $ad - bc \neq 0$; that is, if f is a hyperbola.

Corollary 1 For $f \in C^3(I)$, $\mathcal{S}(f) \equiv 0$ iff f is fractional linear.

proof f is fractional linear iff $\frac{d}{dx}(f')^{-\frac{1}{2}} \equiv 0$

end of proof

The following simple geometric observation illustrates that estimates on the distortion and slope of functions with a negative Schwarzian derivative can be made by comparison with fractional linear transformations.

Lemma 2 For an interval I and $f, h \in C^3(I)$ with $\mathcal{S}(f) < 0$, $\mathcal{S}(h) \equiv 0$ and any $a \in I$, if

$$
\begin{aligned}
f(a) &= h(a) \\
f'(a) &= h'(a) > 0 \\
f''(a) &= h''(a)
\end{aligned}
$$

then

$$
\begin{aligned}
f(x) < h(x) \ \textit{for}\ x > a \\
f(x) > h(x) \ \textit{for}\ x < a
\end{aligned}
$$

proof $(h')^{-\frac{1}{2}}$ is linear, $(f')^{-\frac{1}{2}}$ is convex, and they are equal with identical derivative at $x = a$, see figure 1. Hence $(g')^{-\frac{1}{2}} > (h')^{-\frac{1}{2}} > 0$ implying $0 < f' < h'$ for $x \in I$.

end of proof

Symmetric properties hold for decreasing functions by reflecting the x axis. Figure 2 represents the possibilities.

Definition A hyperbola h will be said to match f at a point $x = a$ if it satisfies the hypothesis of lemma 2 at the point $x = a$.

A corollary to lemma 2 is Singer's observation [9] that an increasing function of negative Schwarzian cannot go from concave to convex without having a singularity.

Corollary 2 If $f \in C^3(I)$ and $\mathcal{S}(f) < o$ then $|f'|$ cannot have a positive minimum.

proof In order for $|f'|$ to have a positive minimum it would have to have an inflection point. If $f' > 0$ it can be matched at the inflection point by a line of positive slope. By lemma 2, f would have to be below the line to the right and above the line to the left, hence no minimum for $|f'|$ is attained. The argument for $f' < 0$ is similar.

end of proof

The following corollary implies, among other things, that a monotone function of negative Schwarzian intersects a hyperbola at most three times.

Corollary 3 For $g, f \in C^3(I)$, f monotone increasing in I with $\mathcal{S}(g) \equiv o$ and $\mathcal{S}(f) < o$ there can be at most one point $x = b$ such that $f(b) = g(b)$ and $f'(b) > g'(b)$.

proof If there were two such points there would have to be a point $x = a$ between them such that $f(a) = g(a)$ and $f'(a) < g'(a)$. If a hyperbola h matches the function f at the point $x = a$ then h must intersect g in three places by lemma 2. But two hyperbolae can intersect at most twice.

end of proof

The following corollary is a consequence of lemma 1.

Corollary 4 For $f, h \in C^3(I)$, $a, b \in I$ with $\mathcal{S}(f) < o$ and $\mathcal{S}(h) \equiv o$, and $f', h' > 0$, if $f'(a) = h'(a)$ and $f'(b) = h'(b)$ then $f'(x) > h'(x)$ for $x \in (a, b)$.

proof $(h')^{-\frac{1}{2}}$ is linear, $(f')^{-\frac{1}{2}}$ is convex and they are equal at a and b, hence $(f')^{-\frac{1}{2}} < (h')^{-\frac{1}{2}}$ for $x \in (0, 1)$.

end of proof

II. A Terse Proof of the Folklore Theorem

Definition For an interval I and $f \in C^2(I)$ the distortion $dis : C^2(I) \mapsto C^0(I)$ is defined by

$$dis(f)(x) = \frac{f''(x)}{(f'(x))^2}$$

For $f, g \in C^2(I)$,the following composition rule holds for distortion:

$$dis(f \circ g)(x) = dis(f)(g(x)) + \frac{1}{f'(g(x))} \cdot dis(g)(x)$$

If $y = f(x)$ then $dis(f)(x)$ is the rate of change of $log(f'(x))$ with respect to y: $dis(y)(x) = \frac{d}{dy} log(\frac{dy}{dx})$. Thus estimates on $dis(f)$ can be used to control the change in f' over a given range. Furthermore, if $dis(f)$ is bounded and $|f'| > L > 1$, then by the composition rule $dis(f^n)$ is uniformly bounded for all n. The utility of these ideas is demonstrated in the following classic theorem first stated and proved by Adler [1]. For further exposition of these ideas and an expanded treatment of the theorem please refer to [2] and [4].

Folklore Theorem For an interval I, suppose $f \in C^2(I)$, $f : I \mapsto I$ and I can be written as a disjoint union of intervals.$I = \bigcup_{J \in \mathcal{J}} J$, with $f : J \xrightarrow{1-1\ onto} I$ for each interval $J \in \mathcal{J}$. If $\exists D < \infty, L > 1$ such that $|f'| > L$ and $dis(f) < D$ then there exists a finite ergodic invariant measure for f which is absolutley continuous with respect to Lebesgue measure.

proof Let $\mathcal{J}_n$ be the collection of intervals over which f^n is monotone, hence $\forall J \in \mathcal{J}_n$, $f^n : J \xrightarrow{1-1\ onto} I$. Let ℓ represent Lebesgue measure.

By the composition rule for distortion it follows that $dis(f^n) < \frac{DL}{L-1}$. For any $x, y \in J \in \mathcal{J}_n$,

$$|\ln\left|\frac{(f^n)'(x)}{(f^n)'(y)}\right|| = \int_x^y |\frac{d}{dt}(\ln|f^{n\prime}(t)|)dt| = \int_x^y |\frac{f^{n\prime\prime}(u)}{f^{n\prime}(u)}du|$$
$$< \frac{DL}{L-1}\int_x^y |f^{n\prime}(u)du| < \ell(I) \cdot \frac{DL}{L-1}$$

Thus for any measurable $E \subset I$,

$$\ell(f^{-n}(E)) = \sum_{J \in \mathcal{J}_n} \ell(f^{-n}(E) \cap J) < \sum_{J \in \mathcal{J}_n} e^{\frac{DL}{L-1}} \cdot \frac{\ell(E)}{\ell(I)} \cdot \ell(J) = e^{\frac{DL}{L-1}} \cdot \frac{\ell(E)}{\ell(I)}$$

Similarly,

$$\ell(f^{-n}(E)) > e^{-\frac{DL}{L-1}} \cdot \frac{\ell(E)}{\ell(I)}$$

Therefore any weak limit μ of $\frac{1}{n}\sum_{i=1}^{n} \ell \circ f^{-i}$ is non-vanishing, invariant with respect to f, and absolutely continuous with respect to Lebesgue measure. If $\ell(E) > 0$, then since $\mathcal{J}$ generates, $\forall \epsilon > 0, \exists n, \exists J \in \mathcal{J}_n$ such that $\frac{\ell(J-E)}{\ell(J)} < \epsilon$ and hence $\ell(f^n(E)) > 1 - \epsilon e^{\frac{DL}{L-1}}$. It follows that f is ergodic.

end of proof

Adler's proof that f, μ is weakly Bernoulli on $\mathcal{J}_n$ is given in the appendix.

III. Estimates on Distortion

The distortion $dis(f)$ is invariant under changes of scale in the domain and is multiplied by the inverse of a scaling factor in the range. In analyzing the distortion and slope of negative Schwarzian maps, it then suffices to consider a generic monotone branch mapping a unit interval onto itself.

Let $\mathcal{F}$ be the collection of maps f which are C^3, monotone, have negative Schwarzian and map the unit interval $I = [0,1]$ onto itself with $f(0) = 0$ and $f(1) = 1$. Likewise, let $\mathcal{G}$ be the collection of hyperbolae g which are monotone and map the unit interval I into itself with $g(0) = 0$ and $g(1) = 1$.

We examine how the distortion and slope of these maps depends upon vertical displacement from the ends. For a fixed $\epsilon > 0$, define the middle ϵ-portion of f to be that part extending from $(f^{-1}(\epsilon), \epsilon)$ to $(f^{-1}(1-\epsilon), 1-\epsilon)$. If f is matched at any point of this middle portion by a hyperbola h then by lemma 2, h will extend out the top and bottom of the unit square. By a horizontal dilation and translation this hyperbola can be brought into the class $\mathcal{G}$. Thus the minimal slope and maximal distortion that $f \in \mathcal{F}$ could have in the middle ϵ-portion will be attained by a hyperbolae in $\mathcal{G}$.

It is a straightforward to calculate the minimal slope and maximal distortion over the middle ϵ-portion of branches in $\mathcal{G}$. These are therefore bounds for the class $\mathcal{F}$ as well.

More precisely, we have the following lemmas.

Lemma 3 For $g \in \mathcal{G}$ and $\frac{1}{2} > \epsilon > 0$, if $\epsilon < g(x) < 1-\epsilon$ then $dis(g)(x) < \frac{2}{\epsilon}$ and $g'(x) > 4\epsilon(1-\epsilon)$

proof The class $\mathcal{G}$ can be parametrized by location of the vertical asymptote: $\mathcal{G} = \{g_k(x) = \frac{x(k-1)}{x-k}\}_{k<0,k>1}$. It is readily calculated that

$$g_k'(g_k^{-1}(y)) = \frac{(y-k)^2}{k(k-1)} \qquad \text{and} \qquad dis(g_k)(g_k^{-1}(y)) = \frac{1}{y-k}$$

Minimizing and maximizing these functions over $\epsilon < y < 1-\epsilon$, $k < 0$, $k > 1$ yields the lemma.

Lemma 4 For $f \in \mathcal{F}$ and $\frac{1}{2} > \epsilon > 0$, if $\epsilon < f(x) < 1-\epsilon$ then $f'(x) > 4\epsilon(1-\epsilon)$

proof Let g be the hyperbola through the points $(0,0)$, $(x, f(x))$ and $(1,1)$. Then the vertical asymptote of g lies outside the interval $[0,1]$ making $g \in \mathcal{G}$. By corollary 3, $f'(x) > g'(x)$, hence $f'(x) > 4\epsilon(1-\epsilon)$

end of proof

Lemma 5 For $f \in \mathcal{F}$ and $\frac{1}{2} > \epsilon > 0$, if $\epsilon < f(a) < 1-\epsilon$ then $dis(f)(a) < \frac{2}{\epsilon}$.

proof Let $b = f(a)$, and let h be the hyperbola that matches f at the point (a, b). By lemma 2, $0 < h^{-1}(0) < h^{-1}(1) < 1$. Then $\tilde{h}(x) = h(h^{-1}(0) + x(h^{-1}(1) - h^{-1}(0))) \in \mathcal{G}$. By the composition rule for distortion,

$$dis(f)(a) = dis(h)(h^{-1}(b)) = \frac{(dis(\tilde{h})(\tilde{h}^{-1}(b))}{h^{-1}(1) - h^{-1}(0)} < \frac{2/\epsilon}{h^{-1}(1) - h^{-1}(0)} < \frac{2}{\epsilon}$$

end of proof

IV. An Illustrative Theorem

The following theorem addresses the same type of maps as the Folklore theorem: transformations on an interval consisting of monotone branches mapping across the interval. The assumptions of bounded distortion and expansion are replaced by the assumption of a negative Schwarzian. The conclusion is that either there is a sigma-finite absolutely continuous ergodic invariant measure, or there is a universal attractor consisting of one or both endpoints of the interval.

The simplest form of attraction at the endpoints is if one or both are attracting fixed points, or together they form an attracting periodic orbit. If there are infinitely many branches near an endpoint a more complex form of attraction is possible, in which a majority of points near an endpoint are mapped closer to the endpoint. In this case, any neighborhood of the endpoint is mapped across the interval, but most points near the endpoint will tend to drift closer to the endpoint as they are iterated.

If there are a finite number of branches near the endpoints, then each endpoint is either a fixed point, a preimage of a fixed point, or part of a periodic orbit of period two and one need only check stability at these points to determine the dynamics of the map.

Theorem Let I be an interval, $f \in C^3(I)$, $f : I \mapsto I$ and assume I can be written as a disjoint union of intervals, $I = \bigcup_{J \in \mathcal{J}} J$, with $f : J \xrightarrow{1-1\ onto} I$ for each $J \in \mathcal{J}$, and $\ell(x : |f'(x)| = 1) = 0$. If $\mathcal{S}(f) < o$ then either (a) $\forall \epsilon > 0, l - a.e.\ x \in I, \exists N, \forall n > N,\ f^n(x) \in (0, \epsilon) \bigcup (1 - \epsilon, 1)$ or (b) there exists a sigma-finite ergodic invariant measure for f which is absolutley continuous with respect to Lebesgue measure.

Working with this type of map is simpler with the assumption that the intervals I and $J \in \mathcal{J}$ are closed. This means that many of the endpoints of intervals $J \in \mathcal{J}$ will have two images. Since we are concerned with positive Lebesgue measure phenomenon this will not lead to any difficulties. The following definitions and lemmas will ease the proof of the theorem.

If $f : I \mapsto I$ and $n(x)$ is a positive integer valued function on I then $f^{n(x)}(x)$ is called a stopping time map with stopping rule $n(x)$.

Definition For $f : I \mapsto I$ and $E \subset I$ the first return map f_E mapping a subset of E to E is defined as $f_E(x) = f^{n(x)}(x)$ where $n(x) = min\{n : f^n(x) \in E\}$. This transformation is well defined if $n(x)$ is finite for ℓ-a.e. point $x \in E$

The following is well known. The reader is refered to the work of Rohklin [7] for discusion and proof.

Lemma 6 If $f : I \mapsto I$ and $E \subset I$ is such that f_E has a finite invariant ergodic measure. then f has a sigma-finite invariant measure. Either both are absolutely continuous or both are singular with respect to Lebesgue measure.

Definition For $f : I \mapsto I$ and $\mathcal{P}.\mathcal{Q}$ partitions of I, f is said to be Markov from $\mathcal{P}$ to $\mathcal{Q}$ if $\mathcal{P}$ refines $\mathcal{Q}$ and f maps each element of $\mathcal{P}$ monotonely onto some union of elements in $\mathcal{Q}$. If f is Markov from $\mathcal{P}$ to $\mathcal{P}$ it is said to be Markov on $\mathcal{P}$

The proof of the following lemma is left to the reader.

Lemma 7 Let $f : I \mapsto I$ be Markov from $\mathcal{P}$ to $\mathcal{Q}$ and let $E \subset I$ be a union of elements in $\mathcal{Q}$. If the first return map f_E is well defined then it is Markov from $E \cap \mathcal{P}$ to $E \cap \mathcal{Q}$.

The following lemma is subsumed in the proofs of Pianigiani and Yorke [6]. An independent proof is given here for clarity.

Lemma 8 If $\mathcal{F}$ is a collection of disjoint intervals in I with $\ell(I - \bigcup_{J \in \mathcal{F}} J) > 0$ and $f : \bigcup_{J \in \mathcal{F}} J \mapsto I$ such that $\forall J \in \mathcal{F}$, $f : J \xrightarrow{1-1\ onto} I$ and for some $D < \infty$, $dis(f^n) < D$ wherever f^n is defined, then for ℓ-a.e. $x \in I$, $\exists n \geq 0$ such that $f^n(x) \in I - \bigcup_{J \in \mathcal{F}} J$

proof Let $\gamma = I - \bigcup_{J \in \mathcal{F}} J$. For x and y in the domain of a single monotone branch of f, $|\frac{f'(x)}{f'(y)}| < e^D$ (refer to the proof of the Folklore theorem). Therefore $\ell(f^{-1}(\gamma)) > e^{-D}\ell(\gamma) \cdot \ell(\bigcup_{J \in \mathcal{F}} J)$. Now $f^2 = f \circ f$ satisfies the same hypothesis as f with a new union of disjoint intervals $\bigcup_{J' \in \mathcal{F}'} J' = \bigcup_{J \in \mathcal{F}} J - f^{-1}(\gamma)$ and $\gamma' = I - \bigcup_{J' \in \mathcal{F}'} J' = \gamma \bigcup f^{-1}(\gamma)$ with $\ell(\gamma') > (1 + e^{-D}\ell(\bigcup_{J \in \mathcal{F}} J)) \cdot \ell(\gamma)$. The lemma follows by induction: $\ell(f^{-n}(\gamma)) < (1 - e^{-D}(1 - \ell(\gamma)))^n \ell(\gamma)$

end of proof

The idea to proving the theorem is straightforward: a dichotomy is established between attraction at the endpoints and the existence of a stopping time map that satisfies the hypothesis of the Folklore theorem. Rohklin's theorem gives a sigma-finite invariant measure for the original transformation from the invariant measure of the stopping time map. The details of the proof involve accounting for all possible behavior at the endpoints.

proof of theorem If the endpoints of I form a period two orbit, then replace the map f with the map f^2. Then either there are an infinite number of branches at an endpoint, or the endpoint is a repelling fixed point, an attracting fixed point, or a preimage of the other endpoint which is fixed. Let p be an endpoint that is not a preimage of the other endpoint. The monotone sequence $a_k \to p$ and integers u_k are defined as follows.

If p is a repelling fixed point, let the interval $[p, r]$ be the domain of the branch of f at p. Then there is a unique monotone sequence $a_k \to p$ in $[p, r]$ such that $a_0 = r$ and $f(a_k) = a_{k-1}$ for $k = 1, 2, \ldots$. Let $u_k = k$ for $k = 0, 1, \ldots$.

If p is an attracting fixed point, let the interval $[p, r]$ be the domain of attraction. Then r is a repelling fixed point. Let a_0 be any point in the interior of $[p, r]$, and let $a_k = f^k(a_0)$. Then $a_k \to p$ monotonely. Let $u_k = 1$ for $k = 0, 1, \ldots$.

If there are an infinite number of branches near p then let a_k be any monotone sequence approaching p such that each a_k lies at the endpoint of the domain of a monotone branch of f. Let $u_k = 1$ for $k = 0, 1, \ldots$.

Let q be the other endpoint. If q does not map to p then define the monotone sequence $b_k \to q$ and the integers v_k using the same criteria and definitions as above for p, a_k, and u_k. If $f(q) = p$ let $[r, q]$ be the domain of the branch of f at q, let $b_k \to q$ be the monotone sequence in $[r, q]$ such that $f(b_k) = a_k$ for $k = 0, 1, \ldots$, and let $v_k = u_k$ for $k = 0, 1, \ldots$.

Let $\mathcal{P}$ be the partition consisting of the intervals of monotonicity of f and let $\mathcal{Q}_k$ be the partition $\{(a_k, b_k), (a_i, a_{i+1}), (b_i, b_{i+1}), i = k, k+1, \ldots\}$. Let $w_k = max\{u_k, v_k\}$. Then by the above construction, f^{w_k} is markov from $\mathcal{P} \vee \mathcal{Q}_k$ to $\mathcal{Q}_k$ for any $k \geq 0$. For any $k \geq 0$ it then follows from lemma 7 that $f_{(a_k, b_k)}$ consists of monotone branches mapping onto the interval (a_k, b_k).

If for some $k \geq 0$ there is a set $\gamma \subset [a_k, b_k]$ with $\ell(\gamma) > 0$ such that $f^n(\gamma) \subset [p, a_k] \bigcup [b_k, q]$ for all n then by lemma 8, for ℓ-a.e. point $x \in [a_k, b_k]$ there exists N such that $f^n(x) \in [p, a_k] \bigcup [b_k, q]$ for all $n > N$.

If such a γ exists for all $k \geq 0$ then for any $\epsilon > 0$ and ℓ-a.e $x \in I$ there exists N such that $f^n(x) \in [0, \epsilon] \bigcup [1 - \epsilon, 1]$ for all $n > N$.

Otherwise, there will exist a k such that $f_{[a_k, b_k]}$ is defined for ℓ-a.e $x \in [a_k, b_k]$. For such a k, let $\tilde{f} = f_{(a_k, b_k)}$ so that $\tilde{f} : [a_k, b_k] \mapsto [a_k, b_k]$. For any $n \geq 1$, every branch of $\tilde{f}^n$ is a middle portion of some branch of f^n for some n. With $\delta = min\{p - a_k, q - b_k\}$ it follows that $dis(\tilde{f}^n) < \frac{2}{\delta}$ for all $n \geq 1$.

The proof that $\tilde{f}$ possesses a finite invariant measure which is absolutely continuous with respect to Lebesgue measure proceeds as in the Folklore theorem: $|\ln|\frac{(\tilde{f}^n)'(x)}{(\tilde{f}^n)'(x)}||$ is bounded by $\frac{1}{\delta}$ and hence any weak limit μ of $\frac{1}{n}\sum_{i=1}^{n} \ell \circ \tilde{f}^{-i}$ is invariant for $\tilde{f}$ and absolutely continuous with respect to Lebesgue measure.

To prove ergodicity it is necessary to show that the partition $\mathcal{F}$ of $[a_k, b_k]$ formed by intervals of monotonicity of $\tilde{f}$ is a generating partition for $\tilde{f}$. It follows from the above that $\tilde{f}$ has no attracting fixed points or orbits (otherwise μ would have a singularity). Suppose $\mathcal{F}$ does not generate and let J be the maximal open interval in $\bigvee_{i=0}^{-\infty} \tilde{f}^i(\mathcal{F})$. If $\tilde{f}^n(J) = \tilde{f}^m(J)$ for some $n \neq m$ then $\tilde{f}^n(J)$ would have to contain an attracting periodic orbit. If $\tilde{f}^n(J) \cap \tilde{f}^m(J) \neq \emptyset$ but $\tilde{f}^n(J) \neq \tilde{f}^m(J)$ then J was not maximal. Then $\tilde{f}^n(J)$ must all be disjoint, and therefore increasingly small. This would force μ to have a singularity. Hence no such J exists and $\mathcal{F}$ generates.

The proof of ergodicity again proceeds the same as in the Folklore Theorem. If $\ell(E) > 0$ then for all $\epsilon > 0$, there is some n and some $J \in \bigvee_{i=0}^{-n} f^i(\mathcal{F})$ such that $\frac{\ell(J-E)}{\ell(J)} < \epsilon$. Then $\ell(f^n(E)) > 1 - \epsilon \cdot e^{\frac{2}{\delta}}$. Hence μ is ergodic.

Then f_k^n, and hence f, possesses a sigma-finite invariant ergodic measure which is absolutely continuous with respect to Lebesgue measure.

<u>**end of proof**</u>

<u>Appendix</u>

Adler's proof that f with invariant measure μ given by the Folklore theorem is weakly Bernoulli is given. For measurable sets A and B, the measure of A relative to B is $\ell(A|B) = \frac{1}{\ell(B)}\ell(A \cap B)$.

The span of a set of partitions $\bigvee_{i=0}^{n-1} \mathcal{J}_i$ is the partition formed by all intersections $J_0 \cap J_1 \cap \ldots \cap J_{n-1}$ with $J_i \in \mathcal{J}_i$. For a partition $\mathcal{J}$, let $\mathcal{J}_n = \bigvee_{i=0}^{n-1} f^{-i}(\mathcal{J})$. For example, if $\mathcal{J}$ is the intervals of monotonicity of f then $\mathcal{J}_n$ will be the intervals of monotonicity of f^n.

A transformation f is said to be Bernoulli if there is a partition $\mathcal{J}$ such that $\forall n$, $\forall A, B \in \mathcal{J}_n$, $\mu(f^{-n}(A) \cap B) = \mu(A)\mu(B)$. Thus as points are iterated under f their first n locations in the partition $\mathcal{J}$ is independent of their next n locations. The isomorphism theorem due to Ornstien [5] states the two Bernoulli transformations with the same entropy are isomorphic.

In general it is difficult to exhibit such a partition for a given Bernoulli transformation. It is easier to demonstrate the following property called weak Bernoullicity which implies Bernoullicity (see [8]).

The transformation f is said to be weakly Bernoulli on the partition $\mathcal{J}$ if

$$\sum_{A,B \in \mathcal{J}_n} |\mu(f^{-(m+n)}(A) \cap B) - \mu(A)\mu(B)| \to 0 \quad \text{as} \quad m \to \infty \quad \text{uniformly in} \quad n$$

This implies that as points are iterated under f a block of n locations in the partition $\mathcal{J}$ becomes increasingly independent of a second block of n locations as the points are iterated longer between blocks. To prove weak Bernoullicity it suffices to show that

$$\frac{\mu(f^{-(m+n)}(A) \cap B)}{\mu(A)\mu(B)}\mu(f^{-(m+n)}(A) \cap B) \to 1 \qquad \text{as} \qquad m \to \infty$$

uniformly over all n and all $A, B \in \mathcal{J}_n$.

We let $\mathcal{J}_n$ be the partition of intervals over which f^n is monotone.

Lemma A $\forall$ measurable sets A, $\forall m, n > 0$, $\forall J_1, J_2 \in \mathcal{J}_{m+n}$ such that $f^m(J_1) = f^m(J_2)$, $\exists c$ such that

$$|\ell(f^{-(m+n)}(A)|J_1) - \ell(f^{-(m+n)}(A)|J_2)| < cL^{-n}\ell(A)$$

proof $\ell(f^m(J_i)) < L^{-n}$. Thus $\forall x, y \in J_i$,

$$|\ln\left|\frac{(f^m)'(x)}{(f^m)'(y)}\right|| < \ell(f^m(J_i)) \cdot \frac{DL}{L-1} < \frac{DL^{-n+1}}{L-1}$$

Hence,

$$1 - c'L^{-n} < \left|\frac{(f^m)'(x)}{(f^m)'(y)}\right| < 1 + c'L^{-n}$$

for some c'. Since $f^m : f^{-(m+n)}(A) \cap J_i \xrightarrow{1-1\ onto} f^{-n}(A) \cap f^m(J_i)$ and $f^m(J_1) = f^m(J_2)$ it follows that

$$1 - c'L^{-n} < \frac{\ell(f^{-(m+n)}(A)|J_1)}{\ell(f^{-(m+n)}(A)|J_2)} < 1 + c'L^{-n}$$

Combining this with $\ell(f^{-(m+n)}(A)|J_i) < e^{\frac{DL}{L-1}}\frac{\ell(A)}{\ell(I)}$ yields the lemma for $c = \frac{c'}{\ell(I)}e^{\frac{DL}{L-1}}$

end of proof

The following is a property of averages.

Lemma B For $a_i, b_i > 0$, $\sum a_i = \sum b_i = 1$, and any numbers A_i, B_i,

$$\sum A_i a_i - \sum B_i b_i \leq (Sup\{A_i\} - Inf\{A_i\})(1 - Inf\{\tfrac{b_i}{a_i}\}) + Sup\{A_i - B_i\}$$

proof $\sum_{a_i \geq b_i} a_i - b_i = \sum_{b_i > a_i} b_i - a_i$ hence

$$\begin{aligned}
\sum A_i a_i - \sum B_i b_i &= \sum A_i a_i - \sum A_i b_i + \sum A_i b_i - \sum B_i b_i \\
&= \sum A_i(a_i - b_i) + \sum (A_i - B_i) b_i \\
&\leq \sum_{a_i > b_i} Sup\{A_i\}(a_i - b_i) - \sum_{b_i > a_i} Inf\{A_i\}(b_i - a_i) + Sup\{A_i - B_i\} \\
&= (Sup\{A_i\} - Inf\{A_i\}) \sum_{a_i > b_i}(a_i - b_i) + Sup\{A_i - B_i\} \\
&\leq (Sup\{A_i\} - Inf\{A_i\})(1 - Inf\{\tfrac{b_i}{a_i}\}) + Sup\{A_i - B_i\}
\end{aligned}$$

end of proof

Definition For a measurable set A, let

$$D_n(A) = Sup_{J_1, J_2 \in \mathcal{J}_n}\{\ell(A|J_1) - \ell(A|J_2)\}$$

Lemma C $\exists \epsilon, c > 0$ such that $\forall k, m, n > 0, m > k$,

$$D_n(f^{-(n+m)}(A)) < (1-\epsilon) D_{n+k}(f^{-(n+m)}(A)) + cL^{-n}\ell(A)$$

proof For $J_1, J_2 \in \mathcal{J}_n$, let $\{J_1^i\}$ be such that $J_1^i \in \mathcal{J}_{n+k}$, $J_1^i \subset J_1$, let $\{J_2^i\}$ be such that $J_2^i \in \mathcal{J}_{n+k}$, $J_2^i \subset J_2$, and let the indices be such that $f^m(J_1^i) = f^m(J_2^i)$. Then

$$D_n(f^{-(n+m)}(A)) = Sup_{J_1, J_2 \in \mathcal{J}_n} \Big\{ \sum_i \ell(f^{-(n+m)}(A)|J_1^i) \cdot \ell(J_1^i|J_1) - \sum_i \ell(f^{-(n+m)}(A)|J_2^i) \cdot \ell(J_2^i|J_2) \Big\}$$

If $\epsilon = e^{-\frac{DL}{L-1}}$ then $\frac{\ell(J_1^i|J_1)}{\ell(J_2^i|J_2)} < 1-\epsilon$, and it follows from lemmas A and B that

$$\begin{aligned} & D_n(f^{-(n+m)}(A)) \\ \le\ & Sup_{J_1, J_2 \in \mathcal{J}_n} \Big\{ \big(Sup_i\, \ell(f^{-(n+m)}(A)|J_1^i) - Inf_i\, \ell(f^{-(n+m)}(A)|J_1^i) \big) \cdot (1-\epsilon) \\ & + Sup_i \big\{ \ell(f^{-(n+m)}((A)|J_1^i) - \ell(f^{-(n+m)}((A)|J_2^i) \big\} \\ <\ & (1-\epsilon) D_{n+k}(f^{-(n+m)}(A)) + cL^{-m}\ell(A) \end{aligned}$$

end of proof

To prove Bernoullicity, it follows from lemma C that $\frac{1}{\ell(A)} D_n(f^{-(n+m)}(A)) \to 0$ as $m \to \infty$ uniformly over all n, all measurable sets A. Therefore

$$\frac{\ell(f^{-(n+m)}(A)|J)}{\ell(f^{-(n+m)}(A))} \to 0 \qquad \text{as} \qquad m \to \infty$$

uniformly over all n, all $J \in \mathcal{J}_n$ and all measurable sets A. If $\{J'\}$ is the set of interval components of $f^{-k}(J)$ then

$$\frac{\ell(f^{-(n+m+k)}(A)|f^{-k}(J))}{\ell(f^{-(n+m+k)}(A))} = \sum_{J'} \ell(J') \frac{\ell(f^{-(n+m+k)}(A)|J')}{\ell(f^{-(n+m+k)}(A))} \to 0 \quad \text{as} \quad m \to \infty$$

uniformly over all n, all $J \in \mathcal{J}_n$ and all measurable sets A. Taking the limit as $k \to \infty$ yields $\frac{1}{\mu(A)}\mu(f^{-(n+m)}(A)|J) \to 1$ as $m \to \infty$ uniformly over all n, all $J \in \mathcal{J}_n$ and all measurable A. Thus f, μ is weakly Bernoulli on $\mathcal{J}_1$

References

[1] Adler.R.L. (1975); Continued Fractions and Bernoulli Trials, *Ergodic Theory; Moser.J., Phillips.E., Varadhan.S., (ed.s) Lecture Notes N.Y. Courant Inst. Math. Sci.*

[2] Bowen.R.(1979); Invariant Measures for Markov Maps of the Interval. *Comm. Math. Phys.* **69**, 1-17

[3] Guckenheimer.J. (1979); Sensitive Dependence on Initial Conditions for One-Dimemsional Maps, *Comm. Math. Phys.* **70**, 133-160.

[4] Johnson.S. (1986); Continuous Measures in One Dimension, *Preprint*

[5] Ornstein.D.S. (1970); Bernoulli Shifts With the Same Entropy are Isomorphic. *Adv. in Math* **4**, 337-352.

[6] Pianigiani.G. & Yorke.J.A. (1979); Expanding Maps on Sets Which Are Almost Invariant: Decay and Chaos. *Trans. A.M.S.* **252**, 351-366.

[7] Rokhlin.V.A. (1961); Exact Endomorphisms of Lebesque Spaces. *Izv. Akad. Nauk. Mat. SSSR* **25**, 499-530; (1964)*A.M.S. Transl. (2)* **39**, 1- 36.

[8] Sheilds.P. (1973); The Theory of Bernoulli Shifts.

Chicago Lectures in Mathematics, Univ. of Chicago Press.

[9] Singer.D. (1978); Stable Orbits and Bifurcations of Maps of the Interval. *SIAM J. App. Math.* **35**, 260-267.

END
DATE
FILMED
DTIC
7-88