AD-A192 104 NONLINEAR WAVE PROPAGATION(U) CLARKSON UNIV POTSDAM NY DEPT OF MATHEMATICS AND COMPUTER SCIENCE M J ABLOWITZ 23 NOV 87 AFOSR-TR-88-0065 AFOSR-84-0005 1/2

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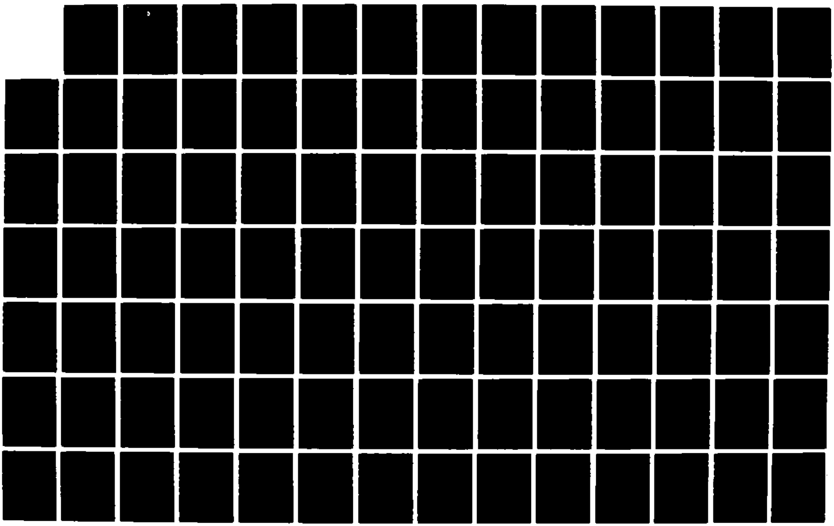

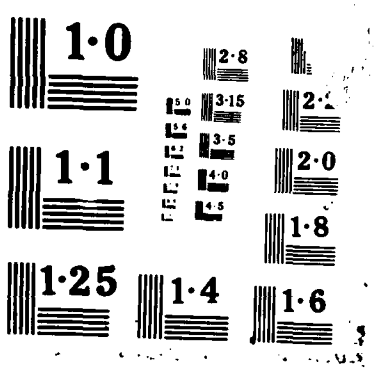
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REPORT DOCUMENTATION PAGE

AD-A192 104

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	1b. RESTRICTIVE MARKINGS
	3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S)	5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TR- 88-0065
6a. NAME OF PERFORMING ORGANIZATION Clarkson University	6b. OFFICE SYMBOL (If applicable)
7a. NAME OF MONITORING ORGANIZATION AFOSR	
6c. ADDRESS (City, State and ZIP Code) Department of Mathematics and Computer Science Potsdam, NY 13676	7b. ADDRESS (City, State and ZIP Code) Bldg 410 Bolling AFB DC
8a. NAME OF FUNDING/SPONSORING ORGANIZATION Air Force Office of Scientific Research /NM	8b. OFFICE SYMBOL (If applicable)
9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER AFOSR 84-0005	
8c. ADDRESS (City, State and ZIP Code) Bolling AFB Washington, DC 20332	10. SOURCE OF FUNDING NOS.

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
61102F	2304	A4	

11. TITLE (Include Security Classification)
Nonlinear Wave Propagation

12. PERSONAL AUTHOR(S)
Mark J. Ablowitz

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT
Final ~~Technical~~ Report	FROM 11/1/86 TO 10/31/87	11/23/87	5 text + reprints

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB. GR.	

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

This year has been an active and productive period for the group at Clarkson involved with nonlinear wave propagation. We have continued to make progress in the study of nonlinear evolution equations, their properties and their solutions for both one plus one and multidimensional nonlinear evolution equations. We are continuing our studies of Painleve equations and nonlinear partial difference equations which can be used as numerical approximations to various soliton equations.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT	21. ABSTRACT SECURITY CLASSIFICATION
UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS ☐	

22a. NAME OF RESPONSIBLE INDIVIDUAL	22b. TELEPHONE NUMBER (Include Area Code)	22c. OFFICE SYMBOL
Arje Nachman ~~Mark J. Ablowitz~~	202-767-4940 ~~315-268-2375~~	

DD FORM 1473, 83 APR EDITION OF 1 JAN 73 IS OBSOLETE.

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SECURITY CLASSIFICATION OF THIS PAGE

AFOSR-TR- 88-0065

ANNUAL TECHNICAL REPORT

NONLINEAR WAVE PROPAGATION

AFOSR GRANT AFOSR-84-0005

by

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and

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88 2 26 037

November 1, 1986 - October 31, 1987

This year has been an active and productive period for the group at Clarkson involved with nonlinear wave propagaton. We have continued to make progress in the study of nonlinear evolution equations, their properties and their solutions for both one plus one and multidimensional nonlinear evolution equations. We are continuing our studies of Painlevé equations and nonlinear partial difference equations which can be used as numerical approximations to various soliton equations.

We have recently considered a singular integral version of the sine-Gordon equation:

$$Hu_t = \sin u \tag{1}$$

where

$$Hu(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(\xi)}{\xi - x}d\xi$$

is the Hilbert transform of u. An interesting feature of (1) is the fact that all solutions arise from bound states of an associated isospectral problem. This is in contrast to say the KdV equation where only the soliton sector arises from bound states.

Areas of study include:

- Exact solutions of nonlinear equations of physical significance.
- Inverse scattering, DBAR method.
- Solutions to nonlinear singular integro-differential equations.
- Applications of solitons to nonlinear optics, fluid dynamics, theoretical physics etc.

Recent publications of M.J. Ablowitz supported by this research grant include the following:

Multidimensional Nonlinear Evolution Equations and Inverse Scattering, M.J. Ablowitz and A.I. Nachman, Physica 18D, p. 223-241, 1986.

On the solution of the generalized wave and generalized sine-Gordon equations, M.J. Ablowitz, R. Beals and K. Tenenblat, Stud. Appl. Math., 74, pp. 177-203 1986.

Solutions of Multidimensional Extensions of the Anti-Self Dual Yang-Mills Equations, M.J. Ablowitz, D.J. Costa and K. Tenenblat, Stud. Appl. Math. 77:37-46 1987.

An Example of a Problem Arising in a Finite Difference Context: Direct and Inverse Problem for the Discrete Analogue of the Equation O. Ragnisco, P.M. Santini, S. Chitlaru-Briggs and M.J. Ablowitz, J. Math. Phys. 28, 777 1987.

Note on Solutions to a Class of Nonlinear Singular Integro-Differential equations, M.J. Ablowitz, A.S. Fokas and M.D. Kruskal, Phys. Lett. A. Vol. 120, 5 pp. 215-218 1987.

A Method of Solution for Painlevé Equations: Painleve IV, V, A.S. Fokas, U. Mugan and M.J. Ablowitz, INS#73 preprint 1987.

Exactly Solvable Multidimensional Nonlinear Equations and Inverse Scattering, M.J. Ablowitz, Proceedings of Nonlinear Evolution Equations, Solitons and the IST, Oberwolfach, Germany 1986, Ed. by M.J. Ablowitz, M.D. Kruskal and B. Fuchssteiner, World Scientific Publ. Co.

Topics Associated with Nonlinear Evolution Equations and Inverse Scatteing in Multidimensions, M.J. Ablowitz, Ed. by M. Lakshmanan, Proceedings of "Solitons", Winter School, Tiruchirapalli, India, January, 1987, INS#76 preprint.

Publications (continued)

Numerical Simulation of the Modified Korteweg-deVries Equation, Thiab R. Taha and M.J. Ablowitz, INS#77 preprint, February 1987.

Hodograph Transformations on Linearizable Partial Differential Equations, P.A. Clarkson, A.S. Fokas and M.J. Ablowitz, INS#78 preprint, April 1987.

Davey-Stuartson I-A Quantum 2+1 Dimensional Itegrable System, C.L. Schultz, M.J. Ablowitz and D. Bar Yaacov, INS#82 preprint May, 1987.

Solutions of Multidimensional Extensions of the Anti-Self-Dual Yang-Mills Equation

By Mark J. Ablowitz, David G. Costa, and Keti Tenenblat**

Motivated by recent work on the generalized wave and Sine-Gordon equatons, various multidimensional extensions of the classical self-dual Yang-Mills equation are developed. A method to obtain a broad class of solutions is given.

The advent of the inverse scattering transform (IST) has allowed mathematicians and physicists to linearize and solve certain classes of nonlinear partial differential equations. A review of much of this work can be found in texts on the subject (see for example [1]). One such equation of physical interest is the sine-Gordon equation (SGE). The SGE arises naturally in the study of surfaces of constant negative curvature in differential geometry. Classical work by Bäcklund [2] and Bianchi [3] developed special solutions as well as transformations between solutions. The IST encompasses the classical approach in a natural way and allows one to find a far broader class of solutions to the SGE.

Natural geometric generalizations of the classical results were obtained in [4,5], in which a multidimensional version of the sine-Gordon equation, called the generalized sine-Gordon equation (GSGE), and related transformations were found. Similar results were obtained for nonlinear generalizations of the wave equation (GWE) [6]. In [7] the associated linear equation and the IST for the GWE and GSGE were developed. It was found in [7] that the linear problems for the GWE and GSGE are given by systems of ordinary differential equations which can be transformed to a nearly standard form. The solutions of the

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*Partially supported by CNPq, Brasil.

Published by Elsevier Science Publishing Co., Inc. 0022-2526/87/$3.50

generalized equations are obtained via factorization of a certain Riemann-Hilbert boundary value problem.

Motivated by this, one can look for solutions to multidimensional extensions of other well-known nonlinear systems. In this paper we consider extensions [see Equations (12)–(14)] of the anti-self-dual Yang-Mills equation given by Pohlmeyer [8]. The point of view we take is to develop multidimensional analogues of the associated linear problem. Solutions of Equations (13), (14) below are associated to local frames on vector bundles over $\mathbf{C}^n$. The solutions are obtained via the so-called $\bar{\partial}$ method (which itself generalizes the notion of a Riemann-Hilbert factorization problem). Recently there has been considerable development of the $\bar{\partial}$ approach, and here we mention the reviews in [9–13].

A version of the anti-self-dual Yang-Mills equation is given by

$$\frac{\partial}{\partial \bar{x}_1}\left(\Omega^{-1}\frac{\partial \Omega}{x_1}\right) + \frac{\partial}{\partial \bar{x}_2}\left(\Omega^{-1}\frac{\partial \Omega}{\partial x_2}\right) = 0, \tag{1}$$

where Ω is a positive matrix valued function of $(x_1, x_2) \in \mathbf{C}^2$; see [8]. We obtain extensions of this equation for a matrix valued function $\mathfrak{U}(x)$, $x = (x_1, \ldots, x_n) \in \mathbf{C}_n$, as follows:

Consider linear problems of the form

$$D_z^j m(x,z) = A_j(x) m(x,z), \qquad 1 \le j \le n, \tag{2}$$

$x = (x_1, \ldots, x_n) \in \mathbf{C}^n$, $z \in \mathbf{C}$, where $D_z^j = \mathscr{L}_1^j + z\mathscr{L}_2^j$ are commuting derivations acting on m, and $\mathscr{L}_1^j, \mathscr{L}_2^j$ are first-order differential operators in the variables $x_j, \bar{x}_j$. Commutativity implies

$$D_z^i A_j - D_z^j A_i + [A_i, A_j] = 0. \tag{3}$$

As examples choose three distinct sets of derivations:

$$D_z^j = \frac{\partial}{\partial x_j} + z S_j \frac{\partial}{\partial \bar{x}_{j+1}}, \tag{4}$$

where we denote $x_{n+1} = x_1$ and $S_j = (-1)^j$;

$$D_z^j = \frac{\partial}{\partial x_j} + z \sum_{k \ne j} S_{jk} \frac{\partial}{\partial \bar{x}_k}, \tag{5}$$

where

$$S_{jk} = \begin{cases} 1 & \text{if } j > k, \\ -1 & \text{if } j < k; \end{cases}$$

and

$$D_z^j = \frac{\partial}{\partial x_j} + z r_j \frac{\partial}{\partial \bar{x}_{j-r_j(n/2)}}, \tag{6}$$

where n is an even integer and

$$r_j = \begin{cases} -1 & \text{if} \quad j \le n/2, \\ \ \ 1 & \text{if} \quad j > n/2. \end{cases}$$

Applying D_z^j as given above into (3) in each case, we have

$$\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} + [A_i, A_j] = 0 \tag{7}$$

and respectively

$$S_j \frac{\partial A_i}{\partial \bar{x}_{j+1}} - S_i \frac{\partial A_j}{\partial \bar{x}_{i+1}} = 0, \tag{8}$$

$$\sum_{l \neq j} S_{jl} \frac{\partial A_i}{\partial \bar{x}_l} - \sum_{k \neq i} S_{ik} \frac{\partial A_j}{\partial \bar{x}_k} = 0, \tag{9}$$

$$r_j \frac{\partial A_i}{\partial \bar{x}_{j-r_j(n/2)}} - r_i \frac{\partial A_j}{\partial \bar{x}_{i-r_i(n/2)}} = 0. \tag{10}$$

If we introduce Ω by

$$A_j = \Omega^{-1} \frac{\partial \Omega}{\partial x_j}, \tag{11}$$

we obtain respectively the equations

$$S_j \frac{\partial}{\partial \bar{x}_{j+1}} \left(\Omega^{-1} \frac{\partial \Omega}{\partial x_i} \right) - S_i \frac{\partial}{\partial \bar{x}_{i+1}} \left(\Omega^{-1} \frac{\partial \Omega}{\partial x_j} \right) = 0, \tag{12}$$

$$\sum_{l \neq j} S_{jl} \frac{\partial}{\partial \bar{x}_l} \left(\Omega^{-1} \frac{\partial \Omega}{\partial x_i} \right) - \sum_{k \neq i} S_{ik} \frac{\partial}{\partial \bar{x}_k} \left(\Omega^{-1} \frac{\partial \Omega}{\partial x_j} \right) = 0, \tag{13}$$

$$r_j \frac{\partial}{\partial \bar{x}_{j-r_j(n/2)}} \left(\Omega^{-1} \frac{\partial \Omega}{\partial x_i} \right) - r_i \frac{\partial}{\partial \bar{x}_{i-r_i(n/2)}} \left(\Omega^{-1} \frac{\partial \Omega}{\partial x_j} \right) = 0. \tag{14}$$

Whenever $n = 2$, each of these equations reduces to (1).

In order to obtain solutions for these equations we use a general result, Proposition 4. Our approach is similar to the one used in [12] for Equation (1).

Let $\mathscr{X}$ denote the space of $N \times N$ matrix-valued functions $m(x, z) \in L^{\infty}_{\text{loc}}$, $x \in U$, U an open domain of $\mathbf{C}^n$, $z \in \mathbf{C}$, such that m is a locally bounded function of x with values in L^{∞}_z. Let $\mathscr{V}$ denote the space of $N \times N$ matrix-valued functions $v(x, z) \in L^{\infty}_{\text{loc}}$, $x \in U$, such that v is a locally bounded function of x with values in $L^1_z \cap L^{\infty}_z$. We introduce the following notation:

$$L^p_{(1),z} = \{f(z): z \in \mathbf{C};\, f(z), zf(z) \in L^p\}, \qquad 1 \le p \le \infty.$$

We will denote by $\mathscr{X}_{(1)}$ [respectively $\mathscr{V}_{(1)}$] the space of functions $m(x, z) \in \mathscr{X}$ [$v(x, z) \in \mathscr{V}$] such that $\partial m/\partial x_j$ and $\partial m/\partial \bar{x}_j$ [$\partial v/\partial x_j$ and $\partial v/\partial \bar{x}_j$] are locally bounded functions of x with values in $L^{\infty}_{(1),z}$ [$L^1_{(1),z} \cap L^{\infty}_{(1),z}$]. We observe that the domain U of the variable x can eventually be all of $\mathbf{C}^n$. Introduce the operator

$$Cf(z) = \frac{1}{2\pi i}\int_{\mathbf{C}} \frac{f(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}.$$

Lemma 1. *Let $f(z) \in L^1 \cap L^{\infty}$. Then $Cf \in L^{\infty}$ and*

$$|Cf|_{L^{\infty}} \le |f|_{L^{\infty}} + \frac{1}{2\pi}|f|_{L^1} \le |f|_{L^1 \cap L^{\infty}}.$$

Proof: Write $Cf(z)$ as a sum of integrals over the ball $|\zeta - z| \le 1$ and its complement. The estimate follows readily.

Lemma 2. *Let $f(\eta, \zeta) \in L^1_{\text{loc}}$, $\eta \in \mathbf{C}^k$, $\zeta \in \mathbf{C}$, such that $\partial f/\partial \eta_j$ exists in the distribution sense. If f and $\partial f/\partial \eta_j \in L^1_{\zeta}$, then we have*

$$\frac{\partial}{\partial \eta_j}\int_{\mathbf{C}} f(\eta, \zeta)\, d\zeta \wedge d\bar{\zeta} = \int_{\mathbf{C}} \frac{\partial f(\eta, \zeta)}{\partial \eta_j}\, d\zeta \wedge d\bar{\zeta}$$

in the distribution sense.

Proof: The proof follows from the definition of the weak derivative $\partial f/\partial \eta_j$ by using convenient test functions $\psi(\eta), \theta(\zeta)$ and applying Fubini's theorem. □

From here on, we assume $V(x, z) \in \mathscr{V}_{(1)}$ fixed, and we introduce the operator

$$Tm = mV.$$

Lemma 3. *Let $V(x, z) \in \mathscr{V}_{(1)}$. Then $CT: \mathscr{X} \to \mathscr{X}$ and $CT: \mathscr{X}_{(1)} \to \mathscr{X}_{(1)}$ are well defined.*

Proof: We show that $T: \mathscr{X}_{(1)} \to \mathscr{V}_{(1)}$ and $C: \mathscr{V}_{(1)} \to \mathscr{X}_{(1)}$ are well defined. It will be clear that the proof also shows that $T: \mathscr{X} \to \mathscr{V}$ and $C: \mathscr{V} \to \mathscr{X}$ are well defined.

(i) For $m \in \mathscr{X}_{(1)}$ we have that $Tm = mV \in L^\infty_{\text{loc}}$, since V and $m \in L^\infty_{\text{loc}}$. And, since m and V (or $\partial m/\partial x_j, \partial m/\partial \bar{x}_j$ and $\partial V/\partial x_j, \partial V/\partial \bar{x}_j$, respectively) are locally bounded functions in x with values in L^∞_z and $L^r_z \cap L^\infty_z$ (or $L^\infty_{(1),z}$ and $L^r_{(1),z} \cap L^\infty_{(1),z}$), it follows that Tm is a locally bounded function in x with values in $L^r_z \cap L^\infty_z$.

Moreover, for $m \in \mathscr{X}_{(1)}$, there exist the distributional derivatives $\partial(Tm)/\partial x_j$, $\partial(Tm)\partial \bar{x}_j$, locally bounded in x with values in $L^r_{(1),z} \cap L^\infty_{(1),z}$, given by

$$\frac{\partial(Tm)}{\partial x_j} = \frac{\partial m}{\partial x_j} V + m\frac{\partial V}{\partial x_j}.$$

Similarly for $\partial(Tm)/\partial \bar{x}_j$. Therefore, $Tm \in \mathscr{V}_{(1)}$ and $T: \mathscr{X}_{(1)} \to \mathscr{V}_{(1)}$ is well defined.

(ii) Given $v \in \mathscr{V}_{(1)}$, Lemma 1 implies for a.a. x

$$|Cv(x,z)|_{L^\infty_z} \le |v(x,z)|_{L^r_z \cap L^\infty_z}.$$

Since v is locally bounded in x, it follows that Cv is locally bounded in x with values in L^∞_z. From Lemma 2, we have the existence of $\partial(Cv)/\partial x_j$, $\partial(Cv)/\partial \bar{x}_j$, and

$$\frac{\partial(Cv)}{\partial x_j} = C\left(\frac{\partial v}{\partial x_j}\right). \tag{15}$$

Now, Lemma 1 yields

$$\left|C\left(\frac{\partial v}{\partial x_j}\right)\right|_{L^\infty_z} \le \left|\frac{\partial v}{\partial x_j}\right|_{L^r_z \cap L^\infty_z} \tag{16}$$

and

$$\begin{aligned}\left|zC\left(\frac{\partial v}{\partial x_j}\right)\right|_{L^\infty_z} &\le \frac{1}{2\pi}\left|\frac{\partial v}{\partial x_j}\right|_{L^r_z} + \left|C\left(z\frac{\partial v}{\partial x_j}\right)\right|_{L^\infty_z} \\ &\le \left|\left(\frac{\partial v}{\partial x_j}\right)\right|_{L^r_z \cap L^\infty_z} + \left|z\frac{\partial v}{\partial x_j}\right|_{L^r_z \cap L^\infty_z}.\end{aligned} \tag{17}$$

Similar estimates hold for $\partial Cv/\partial \bar{x}_j$. Since $\partial v/\partial x_j, \partial v/\partial \bar{x}_j$ are locally bounded in x with values in $L^r_{(1),z} \cap L^\infty_{(1),z}$, it follows from (15), (16), (17) that $\partial(Cv)/\partial x_j$ and $\partial(Cv)/\partial \bar{x}_j$ are locally bounded in x with values in $L^\infty_{(1),z}$. Therefore, $Cv \in \mathscr{X}_{(1)}$ and $C: \mathscr{V}_{(1)} \to \mathscr{X}_{(1)}$ is well defined. □

Let $D_z = \mathscr{L}_1 + z\mathscr{L}_2$ be a derivation acting on functions m, where $\mathscr{L}_1$ and $\mathscr{L}_2$ are first order differential operators in the variables $x_j, \bar{x}_j$ with constant coefficients. We note that if $m \in \mathscr{X}_{(1)}$ then $D_z m$ is a locally bounded function in x with values in L^∞_z. Moreover, for $v \in \mathscr{V}_{(1)}$, $D_z v$ is a locally bounded function in x with values in $L^r_z \cap L^\infty_z$. Therefore, $[D_z, T]: \mathscr{X}_{(1)} \to \mathscr{V}$ is well defined. Similarly $[D_z, C]: \mathscr{V}_{(1)} \to \mathscr{X}$ is well defined.

PROPOSITION 4. *Let* $V(x,z) \in \mathscr{V}_{(1)}$, $x \in \Omega \subset \mathbf{C}^n$, $z \in \mathbf{C}$, *such that*

(i) $D_z V = 0$,
(ii) $I - CT: \mathscr{X} \to \mathscr{X}$ *is* 1-1.

If $m(x,z) \in \mathscr{X}_{(1)}$ *satisfies* $(I - CT)m = \mathbf{1}$, *then* m *is a solution of the equation*

$$D_z m = Q(x)m,$$

where

$$Q(x) = -\frac{1}{2\pi i}\int_{\mathbf{C}} \mathscr{L}_2(Tm(x,\zeta))\, d\zeta \wedge d\bar{\zeta}.$$

Proof: Since D_z is a derivation, it follows that

$$D_z(\mathbf{1}) = 0. \tag{18}$$

From now on, in view of the above remark, we shall use the fact that the commutators $[D_z, T]$ and $[D_z, C]$ are well defined on $\mathscr{X}_{(1)}$ and $\mathscr{V}_{(1)}$ respectively.

Now (i) clearly implies that

$$[D_z, T]m = 0 \tag{19}$$

for all $m \in \mathscr{X}$. On the other hand, using Lemma 2, we obtain that, for any $v \in \mathscr{V}_{(1)}$,

$$D_z(Cv) = C(D_z v) - \frac{1}{2\pi i}\int_{\mathbf{C}} (\mathscr{L}_2 v)\, d\zeta \wedge d\bar{\zeta};$$

hence

$$[D_z, C]v = -\frac{1}{2\pi i}\int_{\mathbf{C}} \mathscr{L}_2 v\, d\zeta \wedge d\bar{\zeta}. \tag{20}$$

Applying D_z to the equation $(I - CT)m = \mathbf{1}$ and using (18) it follows that

$$D_z m - (D_z C)Tm = 0;$$

hence

$$\begin{aligned} D_z M &= [D_z, C]Tm + CD_z Tm \\ &= Q(x) + CTD_z m, \end{aligned}$$

where we have used (20) with $v = Tm$, $m \in \mathscr{X}_{(1)}$, and (19) in the last equality. Therefore,

$$(I - CT)D_z m = Q(x). \tag{21}$$

Now, we have

$$Q(x) = Q(x)\mathbf{1} = Q(x)(I-CT)m = (I-CT)Q(x)m,$$

which together with (21) and (ii) yields $D_z m = Q(x)m$. □

For future use, we suppose that the given $V(x,z)\in\mathscr{V}_{(1)}$ is such that

$$\sup_{x\in U}|V(x,\cdot)|_{L_z^1\cap L_z^\infty} = \delta < 1. \tag{22}$$

Lemma 5. *If* (22) *holds, then*

(i) $I-CT\colon \mathscr{X}\to\mathscr{X}$ *is* 1-1 *and onto,*
(ii) $m = (I-CT)^{-1}\mathbf{1}\in\mathscr{X}_{(1)}$,
(iii) $m(x,z)$ *is of class* C^k *in* x *provided* $V(x,z)$ *is of class* C^k *in* x.

Proof: (i): For each fixed $x\in U$ the linear operator $m(x,\cdot)\to CTm(x,\cdot)$ is bounded from L_z^∞ to L_z^∞, with norm less than or equal to $\delta<1$. Therefore $I-CT$ is 1-1 and onto with an inverse given by the Neumann series

$$(I-CT)^{-1} = \sum_{l=0}^{\infty}(CT)^l. \tag{23}$$

(ii): We must show that there exist $\partial m/\partial x_j, \partial m/\partial \bar{x}_j$ as locally bounded functions of x with values in $L^\infty_{(1),z}$. In view of (23), it suffices to prove that the partial sums

$$\sum_{l=0}^{q}\frac{\partial}{\partial x_j}(CT)^l\mathbf{1} \tag{24}$$

form a Cauchy sequence. For that we use the following straightforward estimates, which hold for each fixed $x\in U$ and integer $l\ge 0$:

$$|Cg|_{L_z^\infty} \le |g|_{L_z^1\cap L_z^\infty}, \qquad g\in\mathscr{V}, \tag{25}$$

$$|zCg|_{L_z^\infty} \le |g|_{L_z^1\cap L_z^\infty} + |zg|_{L_z^1\cap L_z^\infty}, \qquad g\in\mathscr{V}_{(1)}, \tag{26}$$

$$\left|(CT)^l f\right|_{L_z^\infty} \le |V|^l_{L_z^1\cap L_z^\infty}|f|_{L_z^\infty} = \delta^l|f|_{L_z^\infty}, \qquad f\in\mathscr{X}, \tag{27}$$

$$\left|(TC)^l g\right|_{L_z^1\cap L_z^\infty} \le \delta^l|g|_{L_z^1\cap L_z^\infty}, \qquad g\in\mathscr{V}, \tag{28}$$

$$\left|z(TC)^l g\right|_{L_z^1\cap L_z^\infty} \le \delta^l\left(l|g|_{L_z^1\cap L_z^\infty} + |zg|_{L_z^1\cap L_z^\infty}\right), \qquad g\in\mathscr{V}_{(1)}. \tag{29}$$

Observing the identities

$$\sum_{l=p}^{q}\frac{\partial}{\partial x_j}(CT)^l\mathbf{1} = \sum_{\mu=1}^{p}(CT)^{\mu-1}C\left\{\sum_{l=p}^{q}\left[(CT)^{l-\mu}\mathbf{1}\right]\frac{\partial V}{\partial x_j}\right\}, \tag{30}$$

$$z\sum_{l=p}^{q}\frac{\partial}{\partial x_j}(CT)^l\mathbf{1} = \sum_{\mu=1}^{p} zC(TC)^{\mu-1}\left\{\sum_{l=p}^{q}\left[(CT)^{l-\mu}\mathbf{1}\right]\frac{\partial V}{\partial x_j}\right\}, \tag{31}$$

we use (25) and (27) repeatedly to estimate (30) as

$$\left|\sum_{l=p}^{q}\frac{\partial}{\partial x_j}(CT)^l\mathbf{1}\right|_{L_z^\infty} \le \sum_{\mu=1}^{p}\delta^{\mu-1}\left|\frac{\partial V}{\partial x_j}\right|_{L_z^1\cap L_z^\infty}\frac{\delta^{-\mu}}{1-\delta}$$

$$= \frac{p\delta^{p-1}}{1-\delta}\left|\frac{\partial V}{\partial x_j}\right|_{L_z^1\cap L_z^\infty}, \tag{32}$$

and, similarly, we use (26)–(29) to estimate (31) as

$$\left|z\sum_{l=p}^{q}\frac{\partial}{\partial x_j}(CT)^l\mathbf{1}\right|_{L_z^\infty} - \sum_{\mu=1}^{p}\left\{\mu\frac{\delta^{p-1}}{1-\delta}\left|\frac{\partial V}{\partial x_j}\right|_{L_z^1\cap L_z^\infty} + \frac{\delta^{p-1}}{1-\delta}\left|z\frac{\partial V}{\partial x_j}\right|_{L_z^1\cap L_z^\infty}\right\}$$

$$= \frac{p(p+1)\delta^{p-1}}{2(1-\delta)}\left|\frac{\partial V}{\partial x_j}\right|_{L_z^1\cap L_z^\infty} + \frac{p\delta^{p-1}}{1-\delta}\left|z\frac{\partial V}{\partial x_j}\right|_{L_z^1\cap L_z^\infty}.$$

These last two estimates show that the partial sums (24) form a Cauchy (hence convergent) sequence of locally bounded functions of x with values in $L^\infty_{(1),z}$. Therefore, there exist $\partial m/\partial x_j$ in $L^\infty_{\mathrm{loc}}(U, L^\infty_{(1),z})$. Similarly, there exist $\partial m/\partial \bar{x}_j$. The proof of (ii) is complete.

(iii): If $V(x,z)$ is of class C^1 in x, then, for fixed $z\in\mathbf{C}$ and an arbitrary compact set $K\subset U$, we obtain from (32) that

$$\sup_{x\in K}\left|\sum_{l=p}^{q}\frac{\partial}{\partial x_j}(CT)^l\mathbf{1}(x,z)\right| \le \frac{p\delta^{p-1}}{1-\delta}\sup_{x\in K}\left|\frac{\partial V}{\partial x_j}(x,z)\right|,$$

which goes to zero as $q\ge p\to 0$. This implies that $m(x,z)$ is also of class C' in x. □

Now we will use Proposition 4 with the derivations D_z^j given by (4)–(6) for Equations (12)–(14) respectively. In each case we must choose $V(x,z)\in\mathscr{V}_{(1)}$ in such a way that the hypotheses (i) and (ii) are satisfied.

We consider the change of variables

$$u_j = zx_j + S_{j+1}\bar{x}_{j+1} \qquad \text{for } D_z^j \text{ given by (4)}, \tag{33}$$

$$u_j = z\sum_{k\neq j}S_{jk}x_k + \bar{x}_j \qquad \text{for } D_z^j \text{ given by (5)}, \tag{34}$$

$$u_j = zr_jx_{j-r_j(n/2)} + \bar{x}_j \qquad \text{for } D_z^j \text{ given by (6)}. \tag{35}$$

Then using (4)–(6) we have that $D_z^jV = 0$, for each j, whenever V is a holomorphic function on u_j. Therefore, we consider $V(u_1,\ldots,u_n,z)$ holomorphic on the variables u_j given respectively by (33)–(35). In fact, we shall take $V(u_1,\ldots,u_n,z)$ as a polynomial in the u_j's:

$$V = \sum_{|\alpha|\le p} C_\alpha(z)u^\alpha, \tag{36}$$

where multiindex notation is being used.

Now, we take $U \subset \mathbf{C}^n$ to be a bounded domain and consider $K = \bar{U}$. Then, we choose the $C_\alpha(z)$'s, $|\alpha| \le p$, in such a way that the linear operator $m(x,\cdot) \mapsto CTm(x,\cdot)$ is bounded from L_z^∞ to L_z^∞ with norm $\le \delta < 1$ for all $x \in K$. The corresponding V in (36) satisfies the hypothesis of Proposition 4 in view of Lemma 5. Therefore, for such a V fixed, we obtain $m(x,z) = (I - CT)^{-1}\mathbf{1}$, which satisfies the equation

$$D_z^j m(x,z) = A_j(x)m(x,z)$$

with

$$A_j(x) = -\frac{1}{2\pi}\int_{\mathbf{C}} \mathscr{L}_{\bar{z}}^j(Tm(x,\zeta))\,d\zeta \wedge d\bar{\zeta},$$

where $D_z^j = \mathscr{L}_1^j + z\mathscr{L}_2^j$ is given respectively by (4)–(6). It follows that A_j satisfy (7) and respectively (8)–(10). Therefore Ω given by (11) satisfies (12)–(14) respectively.

The matrices $m(x,z)$ can be interpreted as local frames on vector bundles over $\mathbf{C}^n$ for Equations (13) and (14). These bundles when compactified may be viewed as fibre bundles of a complex projective space $P_{(\mathbf{C})}^{n+1}$ over S^{2n}. The coordinates u_j defined in (34) and (35) arise from the following fibration. Considering $(u_1,\ldots,u_n,z,1)$ as coordinates in $P_{(\mathbf{C})}^{(n+1)}$, we take

$$x^t = -\bar{z}B(I - |z|^2B^2)^{-1}u^t + (I - |z|^2B^2)^{-1}\bar{u}^t,$$

where $x = (x_1,\ldots,x_n)$, $u = (u_1,\ldots,u_n)$, and

$$B = \begin{cases} \begin{pmatrix} 0 & -1 & \cdots & -1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ 1 & \cdots & 1 & 0 \end{pmatrix} & \text{for (34)}, \\[2ex] \left(\begin{array}{c|c} 0 & -I_{n/2} \\ \hline I_{n/2} & 0 \end{array}\right) & \text{for (35)}. \end{cases}$$

Conversely, the fibre above x is given by $(u_1, \ldots, u_n, z, 1)$, where

$$u^t = zBx^t + \bar{x}^t,$$

as in (34) and (35) respectively. We observe that since $B + B^t = 0$, it follows that

$$I - |z|^2 B^2 = (I - |z|B)(I + |z|B)$$

is invertible.

References

1. M. J. Ablowitz and H. Segur, Solitons and the inverse scattering transform, *SIAM J. Appl. Math.* 4 (1981).
2. A. V. Bäcklund, *Concerning Surfaces with Constant Negative Curvature* (transl. by E. M. Coddington), New Era Printing Co., Lancaster, Pa., 1905.
3. L. Bianchi, *Lezionei di Geometria Differenziale* (Nicola Zanichelli, Ed.), Bologna, 1927.
4. K. Tenenblat and C. L. Terng, Bäcklund's theorem for n-dimensional submanifolds of R^{2n-1}, *Ann. of Math.* 111:477–490 (1980).
5. C. L. Terng, A higher dimension generalization of the sine-Gordon equation and its soliton theory, *Ann. of Math.* 111:491–510 (1980).
6. K. Tenenblat, Bäcklund's theorem for submanifolds of space forms and a generalized wave equation, *Bol. Soc. Brasil. Math.* 16:67–92 (1985).
7. M. J. Ablowitz, R. Beals, and K. Tenenblat, On the solution of the generalized wave and generalized sine-Gordon equations, *Stud. Appl. Math.* 74:177–203 (1986).
8. K. Pohlmeyer, On the Lagrangian theory of anti-self-dual fields in four-dimensional Euclidean space, *Comm. Math. Phys.* 72:37–47 (1980).
9. A. S. Fokas and M. J. Ablowitz, The inverse scattering transform for multidimensional (2 + 1) problems, in *Nonlinear Phenomena*, Lecture Notes in Physics 189, Springer, 1983.
10. M. J. Ablowitz and A. S. Fokas, Comments on the inverse scattering transform and related nonlinear evolution equations, in *Nonlinear Phenomena*, Lecture Notes in Physics 189, Springer, 1983.
11. M. J. Ablowitz and A. Nachman, A multidimensional inverse scattering method, *Phys. D* 18D:293 (1986).
12. R. Beals and R. R. Coifman, Multidimensional inverse scattering and nonlinear PDE, *Proc. Sympos. Pure Math.* 43:45–70 (1985).
13. R. Beals and R. R. Coifman, The D-bar approach to inverse scattering and nonlinear evolutions, *Phys. D* 18D:242 (1986).

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(Received November 20, 1986)

On the Solution of the Generalized Wave and Generalized Sine-Gordon Equations

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The generalized wave equation and generalized sine-Gordon equations are known to be natural multidimensional differential geometric generalizations of the classical two-dimensional versions. In this paper we associate a system of linear differential equations with these equations and show how the direct and inverse problems can be solved for appropriately decaying data on suitable lines. An initial–boundary–value problem is solved for these equations.

1. Introduction

In 1967 Gardner, Greene, Kruskal, and Miura [1] discovered that the Cauchy problem, with suitably decaying initial data on the line, associated with the Korteweg-deVries (KdV) equation could be solved by making use of ideas from the theory of scattering and inverse scattering. Subsequently a number of nonlinear equations of physical interest have been solved by variants of this method, often referred to as the inverse-scattering transform (I.S.T.). Accounts of these techniques, associated algebraic structure, and amenable nonlinear equations can be found in texts on this subject (see for example [2]).

An equation which fits into this framework is the sine-Gordon equation:

$$u_{tt} - u_{xx} - \kappa \sin u = 0. \tag{1.1}$$

The sine-Gordon equation is of interest to physicists and mathematicians. It was first solved by I.S.T. in [3]. In physics it arises in the study of Josephson junctions, particle physics, stability of fluid motions, etc. In mathematics it has arisen classically in the study of differential geometry. In this paper we shall describe a method which enables us to carry out the I.S.T. for certain nonlinear

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Published by Elsevier Science Publishing Co., Inc. 0022-2526/86/$3.50

n-dimensional generalizations of the sine-Gordon and wave equations ($\kappa = 0$) which arise in the study of differential geometry.

Originally, the sine-Gordon equation was derived in the study of surfaces of constant negative curvature contained in Euclidean space $\mathbf{R}^3$. There is an intimate connection between such surfaces and solutions of the equation. Indeed, in 1875 Bäcklund [4] considered the following. Let M and $\overline{M}$ be surfaces in $\mathbf{R}^3$ and $l: M \to \overline{M}$ be a diffeomorphism such that for any point p in M and corresponding point $\bar{p} = l(p)$ one has the following:

(a) the line determined by p and $\bar{p}$ is tangent to M and $\overline{M}$ at p and $\bar{p}$ respectively;

(b) the distance $d(p, \bar{p}) = r > 0$ is a constant independent of p;

(c) the angle between the normal vectors $N(p)$ and $\overline{N}(p)$ to the surfaces is a constant θ independent of p.

Bäcklund proved that under these conditions the surfaces M and $\overline{M}$ have constant Gaussian curvature $\kappa = \bar{\kappa} = -(\sin^2\theta)/r^2$ which can be normalized to be -1. Moreover he showed that given any surface $M \subset \mathbf{R}^3$ with curvature $\kappa = -1$ there exists a two-parameter family of surfaces $\overline{M}$ with curvature $\bar{\kappa} = -1$ related to M by diffeomorphisms which satisfy (a)–(c).

The analytic interpretation of these results originated in what is now called a Bäcklund transformation, which provides new solutions to the sine-Gordon equation from a given one. Later Bianchi [5] obtained a permutability theorem for surfaces which provides superposition formulae for the sine-Gordon equation.

Motivated in part by the work of [6], the natural geometric generalizations of these results were obtained in [7,8] by considering hyperbolic (constant sectional curvature equal to -1] n-dimensional submanifolds M^n of the Euclidean space $\mathbf{R}^{2n-1}$. The geometric results for hyperbolic manifolds M^n contained in $\mathbf{R}^{2n-1}$ were extended [9] to manifolds M^n of constant sectional curvature $\kappa < 1$ ($\kappa < -1$) contained in the unit spheres S^{2n-1} (hyperbolic space H^{2n-1}). In particular, the zero-curvature submanifolds of the unit sphere correspond to solutions of a generalized wave equation (GWE) which is a homogeneous version of the generalized sine-Gordon equation (GSGE) associated with embeddings in Euclidean space.

The higher-dimensional version of Bäcklund's results takes the following form:

$$dX + XA^tX = A - XB, \tag{1.2}$$

where

$$dX = \sum_{j=1}^{n} \frac{\partial X}{\partial x_j} dx_j,$$

$$A_{ij} = \beta_i(z) a_{ij} dx_j,$$

$$B_{ij} = \frac{1}{a_{1i}} \frac{\partial a_{1j}}{\partial x_i} dx_j - \frac{1}{a_{1j}} \frac{\partial a_{1i}}{\partial x_j} dx_i, \qquad 1 \le i, j \le n, \tag{1.3}$$

and $a = \{a_{ij}\} \in \mathbf{R}^{n\times n}$. Equations (1.2)–(1.3) reduce to the Bäcklund transformation for the generalized sine-Gordon equation (GSGE) when

$$\beta_i(z) = \left(z^2 + (2\delta_{i1} - 1)\right)/2z, \tag{1.4}$$

and for the generalized wave equation (GWE) when

$$\beta_i(z) = -(1 - z^2)/2z \equiv \lambda(z). \tag{1.5}$$

The compatibility condition required for the existence of solutions to these Bäcklund transformations results in a system of second-order partial differential equations for an orthogonal $n\times n$ matrix $a = \{a_{ij}\}$ in (1.2) which is a function of n independent variables $a = a(x_1, x_2, \ldots, x_n)$. The equation has the form

$$\frac{\partial}{\partial x_i}\left(\frac{1}{a_{1i}}\frac{\partial a_{1j}}{\partial x_i}\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{a_{1j}}\frac{\partial a_{1i}}{\partial x_j}\right)$$

$$+ \sum_{k\neq i,j} \frac{1}{a_{1k}^2}\frac{\partial a_{1i}}{\partial x_k}\frac{\partial a_{1j}}{\partial x_k} = \varepsilon a_{1i}a_{1j}, \qquad i \neq j,$$

$$\frac{\partial}{\partial x_k}\left(\frac{1}{a_{1j}}\frac{\partial a_{1i}}{\partial x_j}\right) = \frac{1}{a_{1k}a_{1j}}\frac{\partial a_{1i}}{\partial x_k}\frac{\partial a_{1k}}{\partial x_j}, \qquad i, j, k \text{ distinct},$$

$$\frac{\partial a_{jk}}{\partial x_i} = \frac{a_{ji}}{a_{1i}}\frac{\partial a_{1k}}{\partial x_i}, \qquad i \neq k, \tag{1.6}$$

where $\varepsilon = 1$ for the GSGE and $\varepsilon = 0$ for the GWE.

We observe that when $n = 2$ and $\varepsilon = 1$ (GSGE), the orthogonal matrix $a = \{a_{ij}\}$ given by

$$a = \begin{pmatrix} \cos\frac{1}{2}u & \sin\frac{1}{2}u \\ -\sin\frac{1}{2}u & \cos\frac{1}{2}u \end{pmatrix} \tag{1.7}$$

for the function $u = u(x, t)$ reduces the GSGE to the classical sine-Gordon equation (1.1). We note also that if the parameter z in (1.2) is given by $z = \tan\frac{1}{2}\theta$, then θ is the constant in Bäcklund's statement (c) above. On the other hand when $n = 2$ and $\varepsilon = 0$, then with (1.7) the GWE reduces to the wave equation (1.1) with $\kappa = 0$. When $n \geq 3$ the generalization of the wave equation discussed here is nonlinear. A Bäcklund transformation and a superposition formula for the GWE were obtained in [9].

The Bäcklund transformations (1.2) described above are in fact matrix Riccati equations. Linearizations of such a system can be performed in a straightforward manner (see for example [10]). Introducing the transformation

$$X = UV^{-1}, \tag{1.8}$$

where U, V are $n \times n$ matrix functions of $x_1, \ldots, x_n$, the following linear system is deduced:

$$\begin{pmatrix} dU \\ dV \end{pmatrix} = \begin{pmatrix} 0 & A \\ A' & B \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} \tag{1.9}$$

with the components of A, B given in (1.3). Compatibility ensures that the orthogonal matrix $a = \{a_{ij}\}$ satisfies the GSGE with (1.4) and GWE with (1.5). Alternatively, if we call

$$\begin{pmatrix} U \\ V \end{pmatrix} = \psi,$$

the following linear system of $2n$ o.d.e.'s are obtained:

$$\frac{\partial \psi}{\partial x_j} = \lambda \tilde{A}_j \psi + C_j \psi, \tag{1.10}$$

where $\tilde{A}_j, C_j$ are $2n \times 2n$ matrices with the block structure

$$\tilde{A}_j = \begin{pmatrix} 0 & \tilde{a}_j \\ \tilde{a}_j' & 0 \end{pmatrix}, \qquad C_j = \begin{pmatrix} 0 & 0 \\ 0 & \gamma_j \end{pmatrix}. \tag{1.11}$$

Here $\tilde{a}_j, \tilde{\gamma}_j$ are $n \times n$ matrices having the following structure:

$$\begin{aligned} \tilde{a}_j &= \left(\frac{\delta}{\lambda} - 1\right) e_1 a_j + a_j, \\ a_j &= a e_j \end{aligned} \tag{1.12}$$

where $e_j = \{e_j\}_{ik}$ is the unit matrix

$$\{e_j\}_{ik} = \begin{cases} 1 & i = k = j, \\ 0 & \text{otherwise}, \end{cases} \tag{1.13}$$

and in component form γ_j takes the form

$$(\gamma_j)_{kl} = (1 - \delta_{kj}) \frac{1}{a_{1k}} \frac{\partial a_{1j}}{\partial x_k} \delta_{lj} - (1 - \delta_{lj}) \frac{1}{a_{1l}} \frac{\partial a_{1j}}{\partial x_l} \delta_{kj}. \tag{1.14}$$

In (1.12) a is the orthogonal matrix $\mathbf{R}^n \to SO(n)$ associated with the GWE when $\delta = \lambda$ and with the GSGE when $\delta = \frac{1}{2}(z + 1/z)$, $\lambda = \frac{1}{2}(z - 1/z)$, and γ_j is the matrix (1.14): $\mathbf{R}_n \to M_n(\mathbf{R})$, $\gamma_j + \gamma_j' = 0$. *Although γ_j is determined by a, it will be convenient to treat $(a, \gamma_1, \cdots \gamma_n)$ as the data. Then both* (1.6) *and* (1.14) *arise as the compatibility conditions for the scattering problem* (1.10).

Since we shall separately examine the two cases GSW and GSGE, we write down the explicit scattering problems which are compatible with each of these equations.

For the GWE the scattering problem takes the form $[\psi = \psi(x, \lambda)]$

$$\frac{\partial \psi}{\partial x_j} = \lambda A_j \psi + C_j \psi \tag{1.15}$$

with

$$A_j = \begin{pmatrix} 0 & a_j \\ a_j' & 0 \end{pmatrix}, \tag{1.16}$$

and e_j is given in (1.14) and C_j given by (1.11), (1.14).

For the GSGE the scattering problem for $\psi = \psi(x, z)$ is

$$\frac{\partial \psi}{\partial x_j} = \delta(z)\begin{pmatrix} 0 & e_1 a_j \\ a_j' e_1 & 0 \end{pmatrix}\psi$$

$$+ \lambda(z)\begin{pmatrix} 0 & (I - e_1)a_j \\ a_j'(I - e_1) & 0 \end{pmatrix}\psi + C_j\psi, \tag{1.17a}$$

$\delta(z)$, $\lambda(z)$, C_j given above, or equivalently

$$\frac{\partial \psi}{\partial x_j} = \frac{z}{2}A_j\psi + \frac{z}{2}B_j\psi + C_j\psi, \tag{1.17b}$$

where

$$B_j = \begin{pmatrix} 0 & ua_j \\ a_j' u & 0 \end{pmatrix}, \qquad u = \operatorname{diag}(+1, -1, \ldots, -1).$$

In this paper we show how the direct and inverse scattering problems associated with the GWE (1.15) and the GSGE (1.17) can be solved for matrix potentials tending to the identity sufficiently fast in certain "generic" directions (to be discussed later). It is along such directions (lines) that suitable initial values for the entries of a (x) and the matrices $\gamma_j(x)$ can be specified. In Sections 2–4 the analysis for the GWE is given, and in Sections 5–8 the analogous problems are discussed for the GSGE.

Finally, we remark that solving the n-dimensional GWE and GSGE reduces to the study of the scattering and inverse scattering associated with a coupled system of n one-dimensional o.d.e.'s. This is in marked contrast to other attempts to isolate solvable (local) multidimensional nonlinear evolution equations which are the compatibility condition of two Lax-type operators

$$L\psi = \lambda\psi, \tag{1.18}$$

$$\psi_t = M\psi, \tag{1.19}$$

where L is a partial differential operator with the variable t entering only parametrically. Although nonlinear evolution equations in three independent variables can be associated with suitable Lax pairs (e.g. the Kadomtsev-Petviashvili, Davey-Stewartson, and three-wave interaction equations—see for example the review [11]), little progress has been made in more than three independent variables. In this context one has to overcome a serious constraint inherent in the scattering theory for higher-dimensional partial differential operators in order to be able to find associated solvable nonlinear equations: namely, the scattering data generally satisfy a nonlinear equation (see [12–14]). The analysis discussed herein completely avoids such problems, since the linear system is simply a compatible set of n linear one-dimensional scattering problems. On the other hand, these results demonstrate that the initial-value problem is posed with given data along lines and not on $(n-1)$-dimensional manifolds.

2. The forward problem for the GWE

We consider here the spectral problem (1.15), assuming the associated compatibility conditions, i.e. the GWE. The strategy is to transform (1.15) to a standard form and to associate to it a Riemann-Hilbert factorization problem as in [15]. The transformation uses the $2n \times 2n$ orthogonal matrices

$$U_1 = \begin{pmatrix} a & 0 \\ 0 & I \end{pmatrix}, \quad U_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} I & -I \\ I & I \end{pmatrix}, \quad U = U_1 U_2. \tag{2.1}$$

If ψ is a fundamental matrix solution of (1.15), then the function

$$\tilde{\psi}(x,\lambda) = U(x)^{-1}\psi(x,\lambda) \tag{2.2}$$

satisfies

$$\frac{\partial\tilde{\psi}}{\partial x_j} = \lambda J_j \tilde{\psi} + Q_j \tilde{\psi}, \tag{2.3}$$

where

$$J_j = U^{-1} A_j U = U_2^{-1}\begin{pmatrix} 0 & e_j \\ e_j & 0 \end{pmatrix} U_2 = \begin{pmatrix} e_j & 0 \\ 0 & -e_j \end{pmatrix} \tag{2.4}$$

and

$$Q_j = U^{-1} C_j U - U^{-1}\frac{\partial}{\partial x_j} U = U_2^{-1}\begin{pmatrix} \alpha_j & 0 \\ 0 & \gamma_j \end{pmatrix} U_2 \tag{2.5}$$

where

$$\alpha_j = -a' \frac{\partial a}{\partial x_j}. \tag{2.6}$$

Conversely, (2.2)–(2.6) imply that ψ is a solution of (1.15). We look for a solution $\tilde{\psi}$ in the form

$$\tilde{\psi}(x,\lambda) = m(x,\lambda)e^{\lambda x \cdot J}, \qquad x \cdot J = \sum_{j=1}^{n} x_j J_j. \tag{2.7}$$

Then (2.3) is equivalent to

$$\frac{\partial m}{\partial x_j} = \lambda\left[J_j, m\right] + Q_j m. \tag{2.8}$$

These equations imply that $\det m$ is constant. We look for m such that

$$m(\cdot,\lambda) \text{ is bounded}, \qquad \det m(x,\lambda) \equiv 1. \tag{2.9}$$

Proposition 2.1. *Suppose that for some $\lambda \in \mathbf{C}$, m_1 and m_2 are two solutions of* (2.8), (2.9). *Then there is a matrix $W(\lambda) \in \mathrm{SL}(2n,\mathbf{C})$ such that*

$$m_2(x,\lambda) = m_1(x,\lambda)e^{\lambda x \cdot J}W(\lambda)e^{-\lambda x \cdot J} \tag{2.10}$$

Moreover, if $\lambda \notin i\mathbf{R}$ then W is diagonal.

Proof: One checks that

$$\frac{\partial}{\partial x_j}\left[e^{-\lambda x \cdot J}m_1(x,\lambda)^{-1}m_2(x,\lambda)e^{\lambda x \cdot J}\right] = 0, \tag{2.11}$$

so the matrix in brackets, $W(\lambda)$, is independent of x. Now (2.9) implies $\exp(\lambda x \cdot J)W(\lambda)\exp(-\lambda x \cdot J)$ is bounded with respect to x, which is only possible if $\lambda \in \mathbf{R}$ or $W(\lambda)$ is diagonal.

We study the problem (2.8), (2.9) by restricting to lines in $\mathbf{R}^n$. Let w be a unit vector in $\mathbf{R}$, and y a vector orthogonal to w. Along the line

$$L(w,y) = \{y + sw : s \in \mathbf{R}\} \tag{2.12}$$

we consider the restriction of m:

$$\tilde{m}(s,\lambda) \equiv m(s,\lambda; w; y) \equiv m(y + sw, \lambda). \tag{2.13}$$

Then (2.8) gives

$$\frac{\partial \tilde{m}}{\partial s} = \lambda[J_w, \tilde{m}] + Q\tilde{m};$$

$$J_w \equiv w \cdot J \equiv \sum w_j J_j,$$

$$Q(s) \equiv Q(s, w, y) \equiv \sum w_j Q_j(y + sw). \tag{2.14}$$

DEFINITION 2.1. The data $\{\alpha_j, \gamma_j\}$ are ***small in the direction*** w if the operator norm of the associated matrix function Q satisfies

$$\int_{-\infty}^{\infty} \|Q(s, w, y)\| ds \leq k < 1 \tag{2.15}$$

for some constant k and all y orthogonal to w.

DEFINITION 2.2. The data $\{\alpha_j, \gamma_i\}$ are *asymptotically flat* in the direction w if each derivative of each entry of the matrices α_j, γ_j is rapidly decreasing at infinity on each line $L(w, y)$, uniformly with respect to y. Thus, for each such matrix entry f, each integer $N > 0$, and each multiindex β,

$$\left|\left(\frac{-\partial}{\partial x}\right)^{\beta} f(y + sw)\right| \leq C(1 + |s|)^{-N} \tag{2.16}$$

for every $y \perp w$ and $s \in \mathbf{R}$.

DEFINITION 2.3. The direction w is *oblique* if the $2n$ numbers $\{\pm w_j\}$ are distinct.

THEOREM 2.2. Suppose the data $\{\alpha_j, \gamma_j\}$ are small and asymptotically flat in some oblique direction w. Then for each $\lambda \in \mathbf{C} \setminus i\mathbf{R}$ there is a unique $m(\cdot, \lambda)$ which solves the problem (2.8) and (2.9) and satisfies the asymptotic condition

$$\lim_{s \to -\infty} m(y + sw, \lambda) = I, \qquad \text{all } y \perp w. \tag{2.17}$$

Moreover m is bounded, $m(s, \cdot)$ is holomorphic on $\mathbf{C} \setminus i\mathbf{R}$, and the limits

$$m_{\pm}(x, \lambda) = \lim_{\epsilon \to 0+} m(x, \lambda \pm \epsilon) \tag{2.18}$$

exist and are smooth functions on $\mathbf{R}^n \times i\mathbf{R}$. Also

$$\lim_{\lambda_1 \to \infty} m(x, \lambda) = I, \tag{2.19}$$

uniformly with respect to x.

Before discussing the proof of this theorem, let us consider the implications. For $\lambda \in i\mathbf{R}$ the limits $m_{\pm}$ give two solutions of (2.8), (2.9). Therefore Proposition 2.1 implies the following.

COROLLARY 2.3. *There is a matrix-valued function* $V: i\mathbf{R} \to SL(2n, \mathbf{C})$ *such that*

$$m_{+}(x, \lambda) = m_{-}(x, \lambda) e^{\lambda x \cdot J} V(\lambda) e^{-\lambda x \cdot J} \tag{2.20}$$

for all $x \in \mathbf{R}^n$, $\lambda \in i\mathbf{R}$.

DEFINITION 2.4. The function V is the *scattering data* associated to (a, γ_j) and the direction w.

We now sketch the proof of Theorem 2.2. Note that

$$\alpha_j + \alpha_j' = -\frac{\partial}{\partial x_j}(a'a) \equiv 0, \tag{2.21}$$

$$Q_j + Q_j' = 0, \tag{2.22}$$

In particular, the diagonal entries of Q_n are zero. The problem (2.14) with the conditions

$$\tilde{m}(\cdot, \lambda) \text{ is bounded} \quad \text{and} \quad \lim_{s \to -\infty} \tilde{m}(s, \lambda) = I \tag{2.23}$$

is exactly of the kind considered in [15]. Indeed $Q_{jj} \equiv 0$ and J_w is diagonal with distinct entries (since w is oblique). It follows from the results of [15] and the assumption (2.15) that (2.14), (2.23) has a unique solution $\tilde{m}$ which is bounded and holomorphic for $\lambda \in \mathbf{C} \setminus i\mathbf{R}$ and has a continuous limit on $\mathbf{R}^n \times i\mathbf{R}$. Moreover, $\tilde{m}$ is smooth with respect to s; hence our assumptions imply also that it is smooth with respect to y. These considerations give us many of the properties of m, which is defined by

$$m(y + sw, \lambda) = m(s, \lambda; w, y), \qquad y \perp w. \tag{2.24}$$

To show that m satisfies the full set of equations (2.8), we use the compatibility conditions (GWE). It is most convenient to choose new variables $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n)$ by an orthogonal change of coordinates in $\mathbf{R}^n$ chosen such that $\partial/\partial\tilde{x}_1 = \partial/\partial s$. The desired equations (2.8) take the form

$$\frac{\partial m}{\partial \tilde{x}_j} = \lambda[J_j', m] + Q_j' m \equiv R_j' m \tag{2.25}$$

for $j > 1$, and

$$\frac{\partial m}{\partial \tilde{x}_1} = \frac{\partial m}{\partial s} = \lambda[J_w, m] + Qm \equiv Rm.$$

The compatibility conditions (GWE) imply

$$\frac{\partial Q}{\partial \tilde{x}} + Q'Q' = \frac{\partial Q'}{\partial s} + Q'Q, \qquad j > 1; \tag{2.26}$$

$$[J', Q] = [J_w, Q'_j], \qquad j > 1. \tag{2.27}$$

The solution to (2.14) satisfies the integral equations (see [15])

$$\tilde{m}(s,\lambda) = I + \int_{\pm\infty}^{s} \phi((s-t)\lambda)[Q(t)\tilde{m}(t,\lambda)]\,dt, \tag{2.28}$$

where the limit $\pm\infty$ depends on the matrix entry and on the sign of $\operatorname{Re}\lambda$, while ϕ operates on matrices by

$$\phi(u)[B] = e^{uJ_w}Be^{-uJ_w}. \tag{2.29}$$

We utilize (2.27) (employing shorthand notation) to compute

$$\begin{aligned}
\frac{\partial m}{\partial x_j} - \lambda[J_j', m] &= \int^s \phi\left\{\frac{\partial Q}{\partial x_j}m + Q\frac{\partial m}{\partial \tilde{x}_j} - \lambda[J_j', Qm]\right\}dt \\
&= \int^s \phi\left\{\frac{\partial Q_j'}{\partial t}m + [Q_j', Q]m + Q\frac{\partial m}{\partial \tilde{x}_j}\right. \\
&\qquad \left. - \lambda[J_w, Q_j']m - \lambda Q[J_j', m]\right\}dt \\
&= \int^s \frac{d}{dt}\phi\{Q_j'm\}\,dt + \int^s \phi\left\{Q\left(\frac{\partial m}{\partial \tilde{x}_j} - \lambda[J_j', m] - Q_j'm\right)\right\}dt \\
&= Q_j'm + \int^s \phi\left\{Q\left(\frac{\partial m}{\partial \tilde{x}_j} - \lambda[J_j', m] - Q_j'm\right)dt.\right.
\end{aligned} \tag{2.30}$$

Thus

$$\frac{\partial m}{\partial \tilde{x}_j} - R_j'm = \int^s \phi\left[Q\left(\frac{\partial m}{\partial \tilde{y}_j} - R_j'm\right)\right]dt, \tag{2.31}$$

which implies (2.25). [Note that the asymptotic conditions were used in the calculation (2.30), to eliminate a boundary term in the integration.] This completes the proof of Theorem 2.2.

We turn now to the properties of the scattering data V. We introduce an automorphism of $2n \times 2n$ matrices:

$$B^a = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} B \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{2.32}$$

Theorem 2.3. *The scattering data V have the following properties*:

$$\text{each entry of } V - I \text{ belongs to the Schwartz space } \mathscr{S}(i\mathbf{R}); \tag{2.33}$$

$$V(-\lambda) = V(\lambda)' = \overline{V(\lambda)} = \left[V(\lambda)^{\sigma}\right]^{-1}. \tag{2.34}$$

Proof: (2.33) follows from results in [15]. To obtain the symmetries (2.34), observe first that J_j and Q_j are real and

$$J_j' = J_j = -J_j^{\sigma}, \qquad Q_j = -Q_j' = Q_j^{\sigma}. \tag{2.35}$$

It follows $\overline{m(x,\lambda)}$ satisfies the same equation as $m(s,\lambda)$ and that both $m(x,\lambda)^{\sigma}$ and $[m(x,\lambda)^{-1}]'$ satisfy the same equation as $m(x,-\lambda)$. The boundedness and asymptotic conditions are also satisfied, so

$$m(x,\bar{\lambda}) = \overline{m(x,\lambda)}, \tag{2.36}$$

$$m(x,\lambda) = \left[m(x,\lambda)^{-1}\right]' = m(x,\lambda)^{\sigma}. \tag{2.37}$$

Therefore

$$\begin{aligned} V(-\lambda) &= m_-(0,-\lambda)^{-1} m_+(0,-\lambda) \\ &= m_+(0,\lambda)'\left[m_-(0,\lambda)^{-1}\right]' = V(\lambda)', \end{aligned} \tag{2.38}$$

and *similarly for the remaining symmetries.*

Let us remark here that the construction of m by a Neumann series implies the estimates

$$\begin{aligned} \|m\| &\le (1-k)^{-1}, & \|m - I\| &\le k(1-k)^{-1} \\ \|m^{-1}\| &\le (1-k)^{-1}, & \|m^{-1} - I\| &\le k(1-k)^{-1}, \end{aligned} \tag{2.39}$$

where $k < 1$ is the constant of (2.15). It follows that

$$\|V - I\| \le 2k(1-k)^{-2}.$$

In particular,

$$\|V - I\| \le 1 \quad \text{if} \quad 0 \le k \le 2 - \sqrt{3}.$$

We conclude this section with a brief discussion of normalizations and the relationship of this treatment of the forward problem to that in [15]. The normalization (2.17) depends on the choice of a direction w; therefore the solution m and the associated scattering data V depend on w. In [15], with $n = 1$, the normalization was made at $-\infty$ and the resulting scattering data V had certain principal minors identically equal to 1. Here, the same considerations show that for a given direction w certain principal minors of the associated scattering data V are $\equiv 1$. In the absence of a single natural oblique direction, we have chosen to consider all possible scattering data and have not imposed conditions on principal minors. We return to this question at the end of Section 3.

3. The inverse problem for the GWE

Suppose $V: i\mathbf{R} \to \mathrm{SL}(2n, \mathbf{C})$ is a matrix-valued function which satisfies the conditions (2.33) and (2.34). Suppose also that

$$\|V(\lambda) - I\| < 1, \qquad \lambda \in i\mathbf{R}. \tag{3.1}$$

THEOREM 3.1. *For each $x \in \mathbf{R}^n$ there is a unique matrix-valued function $m(x, \cdot)$ which is bounded and holomorphic on $\mathbf{C} \setminus i\mathbf{R}$, with continuous limits $m_\pm$ on $i\mathbf{R}$, and which satisfies*

$$m_+(x, \lambda) = m_-(x, \lambda) e^{\lambda x \cdot J} V(\lambda) e^{-\lambda x \cdot J}, \qquad \lambda \in i\mathbf{R},$$
$$\lim_{|\lambda| \to \infty} m(x, \lambda) = I. \tag{3.2}$$

The function m is smooth on $\mathbf{R}^n \times (\mathbf{C} \setminus i\mathbf{R})$ and satisfies a system of equations

$$\frac{\partial m}{\partial x_j} = \lambda [J_j, m] + Q_j(x) m, \tag{3.3}$$

where $Q_j + Q_j' \equiv 1$ and Q_j is real,

$$Q_j(x) = U_2'^{-1} \begin{pmatrix} \alpha_j(x) & 0 \\ 0 & \gamma_j(x) \end{pmatrix} U_2'. \tag{3.4}$$

Moreover, the data $\{\alpha_j, \gamma_j\}$ are asymptotically flat in every oblique direction in $\mathbf{R}^n$.

This theorem essentially follows from results in [15]. One way to obtain the equations (3.3) is to note that the function $n_j = \partial m / \partial x_j - \lambda [J_j, m]$ also satisfies the Riemann-Hilbert condition (3.2), from which it follows that $Q_j = n_j m^{-1}$ is continuous across $i\mathbf{R}$. Therefore Q_j is entire; it is bounded, hence independent of λ, which gives (3.3). The symmetry conditions (2.34) imply that $m(x, \bar{\lambda})$, $[m(x, -\lambda)^{-1}]'$, and $m(x, -\lambda)^\sigma$ also solve the Riemann-Hilbert problem (3.2). By uniqueness, m has the symmetries (2.36) and (2.37). Therefore Q_j is real and has the symmetries (2.35), which in turn give (3.4). Finally, an oblique direction w corresponds to a diagonal matrix $J_w = \sum w_j J_j$ having distinct entries, and the results of [15] give rapid decrease of the data Q_j along lines in the direction w, as desired.

Remark: The data Q_j generally do *not* decrease rapidly in directions which are not oblique.

To connect this result to the GWE, we need one more step.

LEMMA 3.2. *There is a function* $a: \mathbf{R}^n \to \mathrm{SO}(n)$ *such that*

$$\alpha_j = -a^t \frac{\partial a}{\partial x_j}. \tag{3.5}$$

Proof: The compatibility relations for the system (3.3) imply

$$\frac{\partial \alpha_j}{\partial x_k} + \alpha_j \alpha_k = \frac{\partial \alpha_k}{\partial x_j} + \alpha_k \alpha_j. \tag{3.6}$$

These in turn are the compatibility relations for (3.5). If a solves (3.5) then $\partial(a^t a)/\partial x_j \equiv 0$, so we can guarantee that $a \in \mathrm{SO}(n)$ by choosing it to belong to $\mathrm{SO}(n)$ at a specified point or asymptotically in some oblique direction.

A solution of (3.5) is unique up to left multiplication by a fixed element of $\mathrm{SO}(n)$. If a is any such solution, we refer to $\{a, \gamma_j\}$ as *inverse data* for the function V.

THEOREM 3.3. *If* $\{a, \gamma_j\}$ *are inverse data for* V, *they satisfy the GWE.*

Proof: We simply reverse the procedure at the beginning of the preceding section. The function

$$\psi(x, \lambda) = \begin{pmatrix} a & 0 \\ 0 & I \end{pmatrix} U_2 m(x, \lambda) e^{\lambda x \cdot J} \tag{3.7}$$

satisfies the system (1.14), so (a, γ_j) satisfy the GWE.

Let us connect the inverse data explicitly to the asymptotics of m in λ. By [15], m has an asymptotic expansion

$$m(x, \lambda) \sim \sum_{\nu=0}^{\infty} m_\nu(x) \lambda^{-\nu}, \qquad \lambda \to \infty. \tag{3.8}$$

This expansion can be differentiated term by term, giving

$$\frac{\partial}{\partial x_j} m_\nu = Q_j m_\nu + \left[J_j, m_{\nu+1}\right]. \tag{3.9}$$

In particular, $m_0 \equiv I$ and so we obtain

$$Q_j(x) = -[J_j, m_1(x)]$$
$$= -\lim_{\lambda \to \infty} \lambda [J_j, m(x, \lambda)]. \tag{3.10}$$

This gives another method for deriving the symmetries (2.35) of Q from symmetries (2.36) and (2.37) of m.

As we noted at the end of Section 2, different functions V may occur as scattering data for the same inverse data unless some further normalization is imposed. Therefore to complete the analysis of the relationship between solutions of the GWE and scattering data, we need to know when two functions V_1, V_2 as above give rise to the same inverse data. Let m_1, m_2 be the associated solutions of (3.2). If the inverse data are the same, then by Proposition 2.1,

$$m_2(x, \lambda) \equiv m_1(x, \lambda)\Delta(\lambda), \qquad \lambda \in \mathbf{C} \setminus i\mathbf{R}, \tag{3.11}$$

where Δ is diagonal and holomorphic and has boundary values $\Delta_{\pm}$; moreover $\Delta(\lambda) \to I$ as $|\lambda| \to \infty$. Now Δ has the same symmetry properties as m, so Δ is the solution of a Riemann-Hilbert problem (2.3) for a diagonal V. Clearly V_1 and V_2 are related by

$$V_2 = (\Delta_-)^{-1} V_1 \Delta_+. \tag{3.12}$$

In particular, V gives trivial inverse data if and only if V is diagonal. Conversely, if V_2 and V_1 are related by (3.12), where $\Delta_{\pm}$ are the boundary values of the solution to (2.3) for a diagonal V, then V_1 and V_2 have the same inverse data.

4. A well-posed initial-value problem for the GWE

The result of the preceding two sections both suggest and solve an initial-boundary-value problem for the GWE. Let us say that a solution $\{a, \gamma_j\}$ of the GWE is *small* if there is some oblique direction such that the associated data $\{\alpha_j, \gamma_j\}$ are both small and asymptotically flat in that direction. As before, if w is a direction (unit vector) in $\mathbf{R}^n$ and y is orthogonal to w, we parametrize the line $L(w, y)$ by $s \to y + sw$. Without loss of generality we may translate the coordinates and take $y = 0$.

Theorem 4.1. *Suppose w is an oblique direction in $\mathbf{R}^n$. Suppose $\tilde{a}: L(w,0) \to$ SO(n) and $\tilde{\gamma}: L(w,0) \to M_n(\mathbf{R})$ are smooth mappings such that $\tilde{\alpha} = -\tilde{a}' \partial\tilde{a}/\partial s$ and $\tilde{\gamma}$ are Schwartz functions of s, $\tilde{\gamma}' + \tilde{\gamma} \equiv 0$, and*

$$\int_{-\infty}^{\infty} \|\tilde{\alpha}(s)\| \, ds < 3 - \sqrt{2}.$$

Then there is a unique small solution $\{a, \gamma_j\}$ *of the GWE such that*

$$\tilde{a}(s) \equiv a(sw),$$
$$\tilde{\gamma}(s) \equiv \sum w_j \gamma_j(sw). \tag{4.1}$$

Proof: Let $\tilde{m}$ be the solution of

$$\frac{\partial \tilde{m}}{\partial s}(s,\lambda) = \lambda[J_w, \tilde{m}] + Q\tilde{m}, \qquad \lim_{s\to\infty} \tilde{m}(s,\lambda) = I, \tag{4.2}$$

where

$$J_w = \sum w_j J_j \quad \text{and} \quad Q = U_2^{-1}\begin{pmatrix} \tilde{\alpha} & 0 \\ 0 & \tilde{\gamma} \end{pmatrix} U_2.$$

There is a mapping $V\colon i\mathbf{R} \to \mathrm{SL}(n,\mathbf{C})$ such that for $\lambda \in i\mathbf{R}$,

$$\tilde{m}_+(s,\lambda) = \tilde{m}_-(s,\lambda)e^{\lambda s J_w}V(\lambda)e^{-\lambda s J_w}. \tag{4.3}$$

Note the term $e^{\lambda s J_w}V(\lambda)e^{-\lambda s J_w}$ is the specialization to the line $L(w,0)$ of $e^{\lambda x \cdot J}V(\lambda)e^{-\lambda x \cdot J}$. Thus factorization of this latter function gives us an extension to $\mathbf{R}^n$ of $\tilde{m}$. V satisfies the hypotheses of Theorem 3.1, so there is an associated solution m of the Riemann-Hilbert problem (3.2) and

$$\tilde{m}(s,\lambda) \equiv m(sw,\lambda). \tag{4.4}$$

Let $\{a, \gamma_j\}$ be inverse data for V, normalized so that $a(s,w,y=0) = \tilde{a}(s)$. Because of (4.4) we obtain

$$\alpha(s) \equiv \sum w_j \alpha_j(sw),$$
$$\gamma(s) \equiv \sum w_j \gamma_j(sw). \tag{4.5}$$

The first identity implies

$$\frac{da}{as}a' = \frac{d\tilde{a}}{ds}a' \qquad \text{on } L(w,0),$$

so we obtain $a \equiv \tilde{a}$ on $L(w,0)$. This completes the proof of existence. Uniqueness follows from the fact that the scattering data associated to a small solution $\{a, \gamma_j\}$ and to the direction w are uniquely determined by m on $L(w,0)$ and therefore are uniquely determined by the functions $\tilde{\alpha}$ and $\tilde{\gamma}$ defined by (4.5). Therefore the scattering data are uniquely determined by the functions (4.1). The scattering data, in turn, determine γ_j and determine a up to left multiplication by a constant matrix. Since $a(s,w,0) = \tilde{a}(s)$ is prescribed, the proof is complete.

Remark: One can think of $V(\lambda)$ as the initial values for the function

$$V_1(\lambda, y) = e^{\lambda y \cdot J} V(\lambda, 0) e^{-\lambda y \cdot J}. \tag{4.6}$$

Replacing $V(\lambda)$ in (4.3) by $V_1(\lambda, y)$ gives the evolution of $\tilde{m}$ to all values of $\mathbf{R}^n$, which in turn corresponds to m. This is in analogy to the standard situation in IST problems.

5. The forward problem for the GSGE

Here we assume the GSGE and consider the associated spectral problem (1.17). Unlike the GWE, this problem cannot easily be transformed to a single standard form. Nevertheless we shall still associate a factorization problem of Riemann-Hilbert type with (1.17).

Once again we denote

$$U_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} I & -I \\ I & I \end{pmatrix}, \qquad J_j = \begin{pmatrix} e_j & 0 \\ 0 & -e_j \end{pmatrix}, \tag{5.1}$$

and we let # denote the automorphism

$$E^{\#} = U_2^{-1}\begin{pmatrix} u & 0 \\ 0 & I \end{pmatrix} U_2 E U_2^{-1} \begin{pmatrix} u & 0 \\ 0 & I \end{pmatrix} U_2, \tag{5.2}$$

where

$$u = \operatorname{diag}(+1, -1, -1, \ldots, -1) \in M_n. \tag{5.3}$$

In particular,

$$J_1^{\#} = J_1, \qquad J_j^{\#} = -J_j, \quad 1 < j \leq n. \tag{5.4}$$

We set

$$\tilde{\psi}(x, z) = U_2^{-1}\psi(x, z), \tag{5.5}$$

so that the spectral problem (1.17) becomes

$$\frac{\partial \tilde{\psi}}{\partial x_j} = \frac{z}{2}\tilde{A}_j\tilde{\psi} + \frac{1}{2z}\tilde{B}_j\tilde{\psi} + \tilde{C}_j\tilde{\psi}, \tag{5.6}$$

with

$$\tilde{A}_j = U_2^{-1}A_jU_2, \qquad \tilde{B}_j = U_2^{-1}B_jU_2, \qquad \tilde{C}_j = U_2^{-1}C_jU_2 \tag{5.7}$$

The trivial (unperturbed) solution $a \equiv I$, $\gamma_j \equiv 0$ of the GSGE has the associated equation

$$\frac{\partial \tilde{\psi}}{\partial x_j} = \tfrac{1}{2}\left(zJ_j + \frac{1}{z}J_j^*\right)\tilde{\psi} \equiv J_j(z)\tilde{\psi} \tag{5.8}$$

which has a solution $\exp[x \cdot J(z)]$. We view (5.6) as a perturbation of (5.8) and look for a solution in the form

$$\tilde{\psi}(x, z) = m(x, z)e^{x \cdot J(z)}. \tag{5.9}$$

The equations for m are then

$$\frac{\partial m}{\partial x_j} = \tfrac{1}{2}z\left[\tilde{A}_j m - mJ_j\right] + \frac{1}{2z}\left[\tilde{B}_j m - mJ_j^*\right] + \tilde{C}_j m. \tag{5.10}$$

As before, we normalize by

$$m(\cdot, z) \text{ is bounded.} \tag{5.11}$$

DEFINITION 5.1. The direction w in $\mathbf{R}^n$ is *principal* if $|w_1| > |w_j|$ for $1 < j \le n$.

Anticipating the argument below, let us consider

$$J_w(z) \equiv \sum w_j J_j(z) = w_1\delta(z)J_1 + \sum_{j=2}^{n} w_j\lambda(z)J_j. \tag{5.12}$$

This matrix is diagonal with entries $\pm w_1\delta(z)$, $\pm w_j\delta(z)$, $\pm w_j\lambda(z)$, $1 < j \le n$. The set of z in $\mathbf{C}$ such that two distinct diagonal entries have the same real part always contains the set

$$\Sigma = i\mathbf{R} \cup \{z : |z| = 1\}, \tag{5.13}$$

i.e. the union of the imaginary axis and the unit circle. It is equal to this set precisely when the direction w is oblique and principal.

DEFINITION 5.2. The data $\{a, \alpha_j, \gamma_j\}$, where again $\alpha_j = -a^t\dfrac{\partial a}{\partial x_j}$, are *small in the direction* w if for every $y \perp w$,

$$\int_{-\infty}^{\infty} \|Q(s, w, y)\|\, ds + \frac{1}{2}\int_{-\infty}^{\infty} \|a(y + sw) - 1\|\, ds \le k < 1. \tag{5.14}$$

Here again

$$Q = \sum w_j Q_j = \sum w_j \begin{pmatrix} \alpha_j & 0 \\ 0 & \gamma_j \end{pmatrix}.$$

We say that the data $\{a, \alpha_j, \gamma_j\}$ are *asymptotically flat in the direction* w if $\{\alpha_j, \gamma_j\}$ are asymptotically flat in the direction w.

THEOREM 5.1. *Suppose the data* $\{a, \alpha_j, \gamma_j\}$ *are small and asymptotically flat in some principal oblique direction* w. *Then for each* $z \in \mathbf{C}\setminus\Sigma$ *there is a unique* $m(\cdot, z)$ *which satisfies the system* (5.10), (5.11) *and such that for each* $y \perp w$

$$\lim_{s\to-\infty} m(y+sw, z) = I. \tag{5.15}$$

Moreover, m *is bounded*, $m(x,\cdot)$ *is holomorphic on* $\mathbf{C}\setminus\Sigma$, *and* $m(x,\cdot)$ *has continuous limits on* Σ *from each of the five components of* $\mathbf{C}\setminus\Sigma$.

To be specific, let us denote by m_+ the limit on Σ from the components $\{|z|>1, \operatorname{Re} z>0\}$ and $\{|z|<1, \operatorname{Re} z<0\}$, and denote by m_- the limits from the other two components.

COROLLARY 5.2. *There is a matrix valued function* $V: \Sigma\setminus\{\pm i\} \to SL(2n, \mathbf{C})$ *such that*

$$m_+(x,z) = m_-(x,z)e^{x\cdot J(z)}V(z)e^{-x\cdot J(z)}. \tag{5.16}$$

As before, we define V to be the scattering data associated to (a, γ_j) and the direction w. To prove Theorem 5.1, we make two transformations. First, let

$$\begin{aligned} m'(x,z) &= U_2^{-1}\begin{pmatrix} a' & 0 \\ 0 & I \end{pmatrix} U_2 m(x,z) \\ &= U^{-1}U_2 m(x,z). \end{aligned} \tag{5.17}$$

Then the system (5.10) becomes

$$\frac{\partial m'}{\partial x_j} = [J_j(z), m'] + Q_j' m', \tag{5.18}$$

where

$$Q_j'(x,z) = U_2^{-1}\begin{pmatrix} \alpha_j & 0 \\ 0 & \gamma_j \end{pmatrix} U_2 + \frac{1}{2z}\left[U^{-1}B_jU - J_j^*\right]. \tag{5.19}$$

Along a line $L(w, y)$, (5.18) leads to

$$\frac{\partial \tilde m}{\partial s} = [J_w(z), \tilde m] + Q'\tilde m,$$

$$\tilde m(\cdot, z) \text{ bounded}, \quad \lim_{s\to-\infty} \tilde m(s,z) = I. \tag{5.20}$$

where

$$Q'(s,z) = Q'(s,z;w,y) = \sum w_j Q_j'(y+sw,z). \tag{5.21}$$

Although this problem is not identical to that considered in [15], nevertheless the methods of [15] apply to give the existence of a unique solution $\tilde{m}(\cdot,z) = \tilde{m}(\cdot,z;w,y)$ for all $z \in \mathbf{C}\backslash\Sigma$ such that

$$\int_{-\infty}^{\infty} \|Q'(s,z)\|\,ds < 1. \tag{5.22}$$

The integral in (5.22) is majorized by that in (5.14) when $|z| \geq 1$. Changing to

$$m'(y+sw,z) \equiv \tilde{m}(s,z;w,y) \tag{5.23}$$

and arguing in Section 2, we see that $m = U_2^{-1}Um'$ has the desired properties for all $|z| \geq 1$. To obtain results for $|z| \leq 1$ we can either use a second transformation or take advantage of a symmetry. Note that

$$\begin{gathered} J_j(1/z) = J_j(z)^*, \\ \tilde{B}_j^* = \tilde{A}_j, \qquad \tilde{A}_j^* = \tilde{B}_j, \qquad \tilde{C}_j^* = \tilde{C}_j. \end{gathered} \tag{5.24}$$

Therefore $m(x,1/z)^*$ satisfies the conditions for $|z| \leq 1$. This completes our sketch of the proof of Theorem 6.1.

As for the GWE, one has symmetry properties in addition to (5.24), namely that $J_j, J_j^*, \tilde{A}_j, \tilde{B}_j, \tilde{C}_j$ are real and

$$\begin{aligned} J_j(z) &= J_j(z)' = -J_j(z)^\sigma, \\ \tilde{A}_j &= \tilde{A}_j' = -\tilde{A}_j^\sigma, \\ \tilde{B}_j &= \tilde{B}_j' = -\tilde{B}_j^\sigma, \\ \tilde{C}_j &= -\tilde{C}_j' = \tilde{C}_j^\sigma. \end{aligned} \tag{5.25}$$

Thus one has

$$\begin{gathered} m(x,z) = \left[m(x,z)^{-1}\right]' = m(x,z)^\sigma, \\ m(x,\bar{z}) = \overline{m(x,z)}, \qquad m(x,1/z) = m(x,z)^*. \end{gathered} \tag{5.26}$$

The symmetries of V are an immediate consequence.

Theorem 5.3. *The scattering data V have the symmetry properties*

$$V(-z) = V(z)^t = \left[V(z)^{-1}\right]^{\sigma},$$

$$V(\bar{z}) = \bar{V}(z), \qquad V\left(\frac{1}{z}\right) = \left[V(z)^{-1}\right]^{*}. \tag{5.27}$$

The analytical properties of V can also be deduced from the results of [15]. As given above, V is defined on each of the five components of $\Sigma\backslash\{\pm i\}$. We join the two unbounded components by compactifying at ∞ and set

$$\Sigma_1 = \{|z| = 1, \operatorname{Re} z > 0\},$$

$$\Sigma_2 = \{z + \bar{z} = 0, |z| > 1\},$$

$$\Sigma_3 = \{|z| = 1, \operatorname{Re} z < 0\},$$

$$\Sigma_4 = \{z + \bar{z} = 0, |z| < 1\}. \tag{5.28}$$

For convenience, we denote restrictions by

$$V_j = V|_{\Sigma_j}, \qquad j = 1,3,$$

$$V_j = V^{-1}|_{\Sigma_j}, \qquad j = 2,4. \tag{5.29}$$

Theorem 5.4. *Each V_j has a smooth extension to the closure of Σ_j. Each derivative of $V - I$ is $O(z^N)$ as $z \to 0$ and $O(z^{-N})$ as $z \to \infty$, for each integer $n \geq 0$. At $\pm i$ the V_j satisfy consistency conditions*

$$V_1 V_2 V_3 V_4(\pm i) = I. \tag{5.30}$$

More generally, for each integer $N \geq 0$ there are matrix-valued polynomials p_j of degree N such that

$$V_j(z - i) = \left[p_j(z - i)\right]^{-1} p_{j+1}(z - i) + O(z - i|^{N+1}) \quad \text{as} \quad z \to i, \tag{5.31}$$

with similar conditions at $-i$, where we take $P_5 = P_1$.

As motivation for the next section we note that the function m' in (5.18) extends to $\mathbf{C}\backslash\Sigma$ and is the solution of the Riemann-Hilbert factorization problem (5.16) which is characterized by

$$\lim_{z \to \infty} m'(x, z) = I. \tag{5.32}$$

6. The inverse problem for the GSGE

Let $V:\Sigma \to SL(2n,\mathbf{C})$ be a matrix-valued function satisfying the symmetry conditions in Theorem 5.3 and the smoothness, decay, and consistency conditions of Theorem 5.4. Suppose also that

$$\|V(\lambda)-I\| \le k', \qquad \lambda \in \Sigma, \tag{6.1}$$

where k' is a sufficiently small positive constant. Then by the methods of [15], for $x \in \mathbf{R}$ there is a unique function $m'(x,\cdot)$, holomorphic on $\mathbf{C}\setminus\Sigma$ with limits on Σ, such that

$$\begin{gathered} m'_+(x,z) = m'_-(x,z)e^{x\cdot J(z)}V(z)e^{-x\cdot J(z)}, \\ \lim_{|z|\to\infty} m'(x,z) = I. \end{gathered} \tag{6.2}$$

The function m' is smooth up to the boundary on $\mathbf{R}^n \times(\mathbf{C}\setminus\Sigma)$, and

$$m'(x,z) = I + O(z^{-1}), \qquad |z| \to \infty, \tag{6.3}$$

$$m'(x,z) \sim \sum_{\nu=0}^{\infty} m'_\nu(x)z^\nu, \qquad z \to 0. \tag{6.4}$$

Moreover, in any principal oblique direction w, for $y \perp w$ and $\lambda \in \mathbf{C}\setminus\Sigma$

$$\lim_{s\to\pm\infty} m'(y+sw,\lambda) = \Delta_\pm, \tag{6.5}$$

where $\Delta_\pm$ is diagonal. The convergence in (6.5) is $O(|s|^{-N})$ for every N, and the same is true for derivatives of m'. Also, m' and its inverse are bounded functions.

In view of these properties the functions

$$\left(\frac{\partial m'}{\partial x_j} - \left[J_j(z), m'\right]\right)(m')^{-1} \tag{6.6}$$

are holomorphic on $\mathbf{C}\setminus\Sigma$, continuous across Σ except at $z=0$, bounded at ∞, and $O(1/z)$ as $z \to 0$. For any fixed x such a function is affine in z^{-1}. Therefore m' satisfies a system of equations which we can write in the form

$$\frac{\partial m'}{\partial x_j} = \left[J_j(z), m'\right] + \frac{1}{2z}\left(B'_j - J_j^*\right)m + C'_j m', \tag{6.7}$$

where $B'_j = B'_j(x)$ and $C'_j = C'_j(x)$.

The asymptotic expansion (6.4) can be differentiated, and (6.7) implies in particular that

$$B'_j m'_0 = m'_0 J_j^*. \tag{6.8}$$

Now m_0' is asymptotically, and rapidly, diagonal in principal oblique directions, so in such directions

$$B_j' - J_j^* \to 0. \tag{6.9}$$

Because of the symmetries of V and the uniqueness of m' we obtain the symmetries

$$\begin{aligned} m'(x,-z) &= \left[m'(x,z)^{-1}\right]' = m'(x,z)^{\sigma}, \\ m'(x,\bar{z}) &= \overline{m'(x,z)}. \end{aligned} \tag{6.10}$$

These in turn imply that B_j' and C_j' are real, while

$$\begin{aligned} B_j' &= (B_j')' = -(B_j')^{\sigma}, \\ C_j' &= -(C_j')' = (C_j')^{\sigma}. \end{aligned} \tag{6.11}$$

Thus these matrices have the form

$$\begin{aligned} B_j' &= U_2^{-1}\begin{pmatrix} 0 & \beta_j \\ \beta_j' & 0 \end{pmatrix} U_2, \\ C_j' &= U_2^{-1}\begin{pmatrix} \alpha_j & 0 \\ 0 & \gamma_j \end{pmatrix} U_2. \end{aligned} \tag{6.12}$$

where β_j, α_j, γ_j are real and

$$\alpha_j + \alpha_j' = 0 = \gamma_j + \gamma_j'. \tag{6.13}$$

We can extract more information from (6.8) by exploiting the symmetries (6.10). These symmetries imply

$$m_0' = \bar{m}_0' = \left(m_0'^{-1}\right)' = m_0^{\sigma}, \tag{6.14}$$

so

$$m_0' = U_2^{-1}\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} U_2 \tag{6.15}$$

where f and g take values in $O(n)$. Let

$$m''(x,z) = m'\left(x, \frac{1}{z}\right)^*. \tag{6.16}$$

Then $(m_0'^{-1})^* m''$ satisfies (6.2), so

$$m'(x,z) = (m_0^{-1})^* m''(x,z). \tag{6.17}$$

Thus

$$m_0' = (m_0^{-1})^*. \tag{6.18}$$

so that

$$g = g'. \tag{6.19}$$

Since also $g^2 = g'g \equiv 1$, g has eigenvalues ± 1. Now g depends continuously on V and $g \equiv I$ when $V = I$. Thus g is symmetric with all eigenvalues $+1$; hence

$$g \equiv I. \tag{6.20}$$

Combining (6.8), (6.12), (6.15), and (6.20), we obtain

$$\beta_j = fue_j \tag{6.21}$$

Now (6.21) implies that for $j \neq k$,

$$\begin{aligned} J_j B_k &= U_2^{-1}\begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} U_2, \\ B_k J_j &= U_2^{-1}\begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} U_2. \end{aligned} \tag{6.22}$$

The compatibility relations for (6.7) include

$$\frac{\partial}{\partial x_k} C_j' + C_j' C_k + \tfrac{1}{4}(J_j B_k' + B_j J_k) = \frac{\partial}{\partial x_j} C_k' + C_k' C_j' + \tfrac{1}{4}(J_k B_j' + B_k' J_j). \tag{6.23}$$

In view of (6.22) and (6.12), (6.23) implies

$$\frac{\partial \alpha_j}{\partial x_k} + \alpha_j \alpha_k = \frac{\partial \alpha_k}{\partial x_j} + \alpha_k \alpha_j. \tag{6.24}$$

These are precisely the conditions for solving for a with

$$\alpha_j = -a' \frac{\partial a}{\partial x_j}. \tag{6.25}$$

We can require that $a \to I$ as $s \to -\infty$ along a family of principal oblique lines.

Then since α_j is skew symmetric (and real),

$$a : \mathbf{R}^n \to SO(n). \tag{6.26}$$

Definition 6.1. $\{a, \gamma_j\}$ is *inverse data* for the function V.

Theorem 6.1. *The inverse data* $\{a, \gamma_j\}$ *satisfy* GSGE.

Proof: Let

$$U(x) = \begin{pmatrix} a(x) & 0 \\ 0 & I \end{pmatrix} U_2, \tag{6.27}$$

and set

$$\psi(x, z) = U(x) m'(x, z) e^{x \cdot J(z)}. \tag{6.28}$$

Then the equations (6.6) become

$$\frac{\partial \psi}{\partial x_j} = \tfrac{1}{2} z A_j \psi + \frac{1}{2z} B_j \psi + C_j \psi, \tag{6.29}$$

where

$$A_j = U J_j U^{-1} = \begin{pmatrix} 0 & a e_j \\ e_j a' & 0 \end{pmatrix}, \tag{6.30}$$

$$B_j = U B_j' U^{-1} = \begin{pmatrix} 0 & b e_j \\ e_j b' & 0 \end{pmatrix}, \tag{6.31}$$

$$C_j = U C_j' U^{-1} - \frac{\partial U}{\partial x_j} U^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \gamma_j \end{pmatrix}, \tag{6.32}$$

and

$$b = afu. \tag{6.33}$$

To complete the proof we only need to prove

$$b = ua. \tag{6.34}$$

Let us write

$$E^\tau = \begin{pmatrix} u & 0 \\ 0 & I \end{pmatrix} E \begin{pmatrix} u & 0 \\ 0 & I \end{pmatrix}. \tag{6.35}$$

Then we want to prove

$$A_j = B_j^\tau. \tag{6.36}$$

To prove (6.36) we write the compatibility conditions for (6.29) in the notation of matrix-valued differential forms. Let

$$A = \sum A_j\,dx_j, \qquad B = \sum B_j\,dx_j, \qquad C = \sum C_j\,dx_j. \tag{6.37}$$

The compatibility conditions are

$$\begin{gathered} A \wedge A = 0 = B \wedge B, \\ dA = A \wedge C + C \wedge A, \qquad dB = A \wedge B + B \wedge A, \\ dC = C \wedge C + A \wedge B + B \wedge A. \end{gathered} \tag{6.38}$$

Since $C = C^\tau \equiv \Sigma C_j^\tau\,dx_j$, we have

$$d(A - B^\tau) = (A - B^\tau) \wedge C + C \wedge (A - B^\tau). \tag{6.39}$$

Now

$$\begin{aligned} A_j - B_j^\tau &= U\big(J_j - (B_j')^*\big)U^{-1} \\ &= U\big(J_j^* - B_j'\big)^* U^{-1}, \end{aligned} \tag{6.40}$$

and we know that $J_j^* - B_j$ vanishes asymptotically in certain directions. It follows from this fact and (6.39) that $A - B^\tau \equiv 0$.

Remarks:

(1) As for the GWE, the data $\{\alpha_j, \gamma_j)\}$ can be recovered from the asymptotics of m' as $z \to \infty$ as in (3.10). Thus the orthogonal matrix-valued function a is also determined implicitly by these asymptotics.

(2) The data $\{a, \alpha_j, \gamma_j\}$ are small in every principal oblique direction if the constant k' of (6.1) is small enough, and are asymptotically flat in every principal oblique direction.

(3) As for the GWE, two functions V_1 and V_2 give rise to the same inverse data if and only if

$$V_2 = (\Delta_-)^{-1} V_1 \Delta_+, \tag{6.41}$$

where Δ is the solution of the Riemann-Hilbert factorization problem (6.2) for a diagonal matrix-valued function on Σ. In particular, V gives the trivial solution of the GSGE if and only if V is diagonal.

7. A well-posed initial-value problem for the GSGE

With the same conventions as in section 4, one has the same conclusion:

Theorem 7.1. *Suppose w is a principal oblique direction in $\mathbf{R}^n$. Suppose $\tilde{a} = L(x,0) \to SO(n)$ and $\tilde{\gamma} = L(W,0) \to M_n(\mathbf{R})$ are smooth mappings such that $\tilde{\alpha} = -\tilde{a}' d\tilde{a}/ds$ and $\tilde{\gamma}$ are Schwartz functions $\tilde{\gamma} + \tilde{\gamma}' \equiv 0$, and*

$$\int_{-\infty}^{\infty} \|\tilde{\alpha}(s)\| ds < k_0,$$

where k_0 is a sufficiently small positive constant. Then there is a unique small solution $\{a, \gamma_j\}$ of the GSGE such that

$$\tilde{\alpha}(s) = a(sw),$$
$$\tilde{\gamma}(s) = \sum w_j \gamma_j(sw). \tag{7.1}$$

The proof is the same as the proof of the analogous result for the GWE in section 4, hence is omitted.

Acknowledgments

This work was supported in part by the National Science Foundation under grant number DMS-8202117 and DMS-8402637, the Office of Naval Research under grant number N00014-76-C-0867, the Air Force Office of Scientific Research under grant number AFOSR-84-0005, and CNPq, Brazil. We appreciate the hospitality of the Institute for Theoretical Physics at the University of California at Santa Barbara under NSF grant number PHY82-17853, supplemented by funds from the National Aeronautics and Space Administration, where much of this work was carried out and lectured upon in June 1985.

References

1. C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Phys. Rev. Lett.* 19:1095–1097 (1967).
2. M. J. Ablowitz and H. Segur, Solitons and the inverse scattering transform, *SIAM Appl. Math.* Phila. Pa. 4 (1981).
3. M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, *Phys. Rev. Lett* 30:1262–1264, 1973.
4. A. V. Bäcklund, *Concerning Surfaces with Constant Negative Curvature*, (translated by E. M. Coddington), New Era Printing Co., Lancaster, Pa., (1905).
5. L. Bianchi, di Geometria, in *L. Differenciale* (Nicola Zanichelli, Ed.), Bologna, 1927
6. S. S. Chern and C. L. Terng, *Rocky Mountain Math.* 10:105–124 (1980).
7. K. Tenenblat, and C. L. Terng, *Ann. of Math.* 111:477–490 (1980).
8. C. L. Terng, *Ann. of Math* 111:491–510 (1980).
9. K. Tenenblat, Bäcklund's theorem for submanifolds of space forms, and generalized wave equations, *Boletin da Sociedade Brasileira de Matematica* 16 (1985).
10. P. Winternitz, *Lecture Notes in Physics*, 189, Proceedings of the CIFMO School and Workshop held at Oaxtepec, Mexico (K. B. Wolf, Ed.), Springer, 1982.

11. M. J. ABLOWITZ and A. S. FOKAS, Comments on the inverse scattering transform and related nonlinear evolution equations in [10]; A. S. FOKAS and M. J. ABLOWITZ, The inverse scattering transform for multidimensional (2 + 1) problems, in [10].
12. A. I. NACHMAN and M. J. ABLOWITZ, *Stud. Appl. Math.* 71:243–250 (1984).
13. A. I. NACHMAN and M. J. ABLOWITZ, *Stud. Appl. Math.* 71:251–262 (1984).
14. R. BEALS and R. R. COIFMAN, Multidimensional inverse scattering and nonlinear PDE, *Proc. Sympos. Pure Math.* 43:45–70 (1985).
15. R. BEALS and R. R. COIFMAN, *Commun. Pure Appl. Math.*, 1984, pp. 39–90.

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(Received November 8, 1985)

Physica 18D (1986) 223-241
North-Holland, Amsterdam

MULTIDIMENSIONAL NONLINEAR EVOLUTION EQUATIONS AND INVERSE SCATTERING

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In this paper we will review some recent work done in the field of integrable nonlinear evolution equations and inverse scattering. We will concentrate on the basic underlying areas and refer interested readers to suitable references for complete details; specifically background material can be found in various texts on this subject (e.g. [1] by Ablowitz and Segur). More recent references will be given as necessary. The outline of the paper is as follows.

1) Introductory remarks.

2) A discussion of two separate but related issues. Namely, (a) solving certain nonlinear evolution equations in infinite space; and (b) inverse scattering. These are important problems having many physical applications. Moreover, they are related to each other by what we refer to as the Inverse Scattering Transform (IST).

3) At the end of the paper we will make some remarks on the possibility of solving nonlinear evolution equations in high dimensions (i.e. equations with more than two spatial and one time variable) by using the IST method as we now understand it.

1. Introduction

The prototype nonlinear evolution equations for our purposes will be the Korteweg-deVries (KdV) equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1}$$

in one spatial dimension, and the Kadomtsev-Petviashvili (KP) equation

$$(u_t - 6uu_x + u_{xxx})_x = -3\sigma^2 u_{yy} \tag{2}$$

in two spatial dimensions. (It turns out that the sign of σ^2 is critical: there being two cases labeled by $\mathrm{KP_I}$: $\sigma^2 = -1$; $\mathrm{KP_{II}} = \sigma^2 = 1$.)

Historically speaking, the KdV equation was the first equation solved (on the infinite line) by use of inverse scattering. Subsequently numerous other equations of physical interest in one spatial dimension were solved e.g. nonlinear Schrödinger, sine-Gordon, three-wave interaction, modified KdV, Boussinesq, ... These equations are all partial differential equations. In fact, there are other equations which are discrete in space and continuous in time (differential–difference) and equations discrete in both space and time which also may be solved by IST. One other class of equations in one spatial and one time dimension fit into this scheme, namely nonlinear singular integro-differential equations; with the prototype being the so-called

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Intermediate Long Wave equation [2a],

$$u_t + \frac{1}{\delta} u_x + 2uu_x + (Tu)_{xx} = 0, \quad Tu = \frac{1}{2\delta} \int_{-\infty}^{\infty} \coth \frac{\pi}{2\delta}(\xi - x) u(\xi)\, \mathrm{d}\xi. \tag{3}$$

As $\delta \to 0$, (3) tends to the KdV equation (with appropriate coefficients) and as $\delta \to \infty$ it tends to the so-called Benjamin–Ono equation

$$u_t + 2uu_x + (Hu)_{xx} = 0, \quad H_u = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(\xi)}{\xi - x}\, \mathrm{d}\xi. \tag{4}$$

The method to solve (4) was recently found and it has certain features in common with some two-dimensional problems – specifically $\mathrm{KP_I}$ (see [2b]).

It should also be remarked that some ode's can also be solved by similar methods; specifically the classical equations of Painlevé (see for example [3]). We will not dwell on this aspect any further in this lecture.

In two spatial one time dimension the KP equation is only one of the equations that can be solved in infinite space. However, an effective method was not realized until a short time ago. The important new idea of treating inverse scattering as a "$\bar{\partial}$ problem" (see [9a]) was used in [4] to solve $\mathrm{KP_{II}}$ and paved the way for the development of the IST for a wide class of equations in 2 + 1 dimensions (a review of this and related work can be found in [5a, b]). It should be mentioned that earlier work on $\mathrm{KP_I}$ had been done by Manakov [6a] and more recently by Fokas and Ablowitz [6b] and on the multidimensional three-wave equation by Cornille [7a] and Kaup [7b]. $\mathrm{KP_{II}}$ and others like it depart significantly from previous work and its study has led us to develop a general method to do inverse scattering in n spatial dimensions as we will indicate in this review (see [8a, b, c]).

The concept of treating inverse scattering as a "$\bar{\partial}$ problem" was originally discussed by Beals and Coifman in their study of first order systems of differential equations [9a]. Beals and Coifman have also recently considered multidimensional inverse scattering via $\bar{\partial}$ methods [9b].

It should be noted that important contributions in the study of multidimensional inverse scattering associated with the time-independent Schrödinger problem have been made by Faddeev [10] and Newton [11]. In one dimension we also note the important contributions of Shabat [12a], Mikhailov [12b] and Caudrey [12c]. Some of the work in this review is related to these studies although the methodology is different.

2. Inverse scattering and ther inverse scattering transform

The method of solution by IST begins with the study of two compatible linear operators (Lax pairs) (L depends on one or more "potentials" or functions which we call u)

$$Lv = \lambda v, \tag{5}$$

$$v_t = Mv, \tag{6}$$

connected by the compatibility condition

$$L_t + [L, M] = 0, \tag{7}$$

when the flow is isospectral, $\lambda_t = 0$. (7) is the nonlinear evolution equation to be solved. λ is a spectral parameter, which as it turns out loses significance in spatial dimensions greater than one. L is a spatial operator only; with time acting as a parameter. The parametric dependence in time is what allows us to study the question of inverse scattering separately and then after this task is completed allows us to solve the relevant nonlinear equation (7). For KdV the operators are

$$L = \frac{\partial^2}{\partial x^2} - u, \quad M = (4\lambda + 2u)\frac{\partial}{\partial x} - u_x. \tag{8}$$

The reader can now verify that (7) yields (1). It should be noted that there are generalizations of (5)–(7), but we shall not be concerned with that here.

The direct (or forward) scattering problem associated with L means given a potential, in a desired function class, and solve for eigenfunctions corresponding to suitable initial or boundary conditions. Usually, appropriate eigenfunctions are defined in terms of an integral equation (e.g. via Green's functions). From the eigenfunctions scattering coefficients, eigenvalues, etc. can be calculated. Call the set of all such data obtainable from the solution of (5) $\hat{S}$.

The inverse problem is as follows. Given some subset S of $\hat{S}$ (i) reconstruct the eigenfunctions and the potential; (ii) characterize the analytical, algebraic, and/or topological constraints on the data in order to find a potential in the desired function class.

In recent years significant strides forward have been made in regard to the solution of those inverse problems motivated by the study of nonlinear evolution equations. Examples in one dimension are

(i) $\dfrac{\mathrm{d}^n v}{\mathrm{d}x^n} + \sum\limits_{j=2}^{n} u^{(j)}(x)\dfrac{\mathrm{d}^{n-j}v}{\mathrm{d}x^{n-j}} = \lambda v, \quad u^{(j)}(x), v(x,\lambda)$ scalar [see 9c];

(ii) $\dfrac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{i}\lambda J v + qv, \quad v(x,\lambda), q(x) \in \mathbf{C}^{N\times N}, \quad J = \operatorname{diag}(J^1,\dots,J^N), (J^i \neq J^j, i \neq j)$ [see 9d].

In multidimensions examples are

(iii) $\sigma\dfrac{\partial v}{\partial y} + \Delta v - u(x,y)v = 0, \quad \sigma = \sigma_R + \mathrm{i}\sigma_I, \quad x \in \mathbf{R}^n, \quad y \in \mathbf{R}, \quad \Delta = \sum\limits_{l=1}^{n} \partial^2/\partial x_l^2$ [see 8a, 8c, 9b];

(iv) $-\Delta v + u(x)v = \lambda v$ [see 10, 11, 8a, 8c, 9c];

(v) $\dfrac{\partial v}{\partial y} + \sigma\sum\limits_{l=1}^{n} J_l\dfrac{\partial v}{\partial x_l} = qv, \quad \sigma = \sigma_R + \mathrm{i}\sigma_I, \quad x \in \mathbf{R}^n, \quad y \in \mathbf{R}; \quad v, q \in \mathbf{C}^{N\times N}, \quad J_l = \operatorname{diag}(J_l^1,\dots,J_l^N)$, $(J_l^i \neq J_l^j, i \neq j)$ [see 8b].

The inverse problem for (i) and (ii) may be written in a compact form. Namely solve

$$(\mu_+ - \mu_-)(x,k) = \mu_-(x,\alpha(k))V(x,k)$$

on Σ (Σ is an appropriate contour in the complex k-plane and V is a function depending explicitly on the scattering data and $\alpha(k)$ is problem dependent) with

$$\mu_\pm \xrightarrow[|k|\to\infty]{} I, \alpha(k), V(x,k) \text{ given on } \Sigma,$$

and

$$\mu_{\pm}(x,k) \text{ meromorphic in } k \in \mathbb{C}/\Sigma. \tag{9}$$

$\mu_{\pm}(x,k)$ has a finite number of poles with locations specified: $k_1,\dots,k_n$; and $\mathrm{Res}_{k=k_j}\mu_{\pm}(x,k)$ specified appropriately.

In (9), $\mu(x,k)$ is associated with an eigenfunction of the given operator. It is related to $v(x,k)$ by

$$v(x,k)=\mu(x,k)\,e^{\theta_{L_0}(x,k)},$$

where $\theta_{L_0}(x,k)$ is a concrete phase factor which depends on the unperturbed (potential zero) operator. The parametric dependence $\lambda=\lambda(k)$ is explicitly given (chosen for convenience).

(9) is a variant of the usual Riemann–Hilbert factorization problem. The standard situation involves finding $\mu_{\pm}$ analytic off Σ without any extra parameter such as x.

Corresponding to (i) and (ii) above, the second order case is classical and has been studied by numerous authors (a review of this appears in [1]). Although some work had been done for third order scalar operators nevertheless it has only been within the past few years that the solution to the general nth order case has been found. It should be noted that the matrix system (ii) above has also been studied in [12a–c]. A thorough analysis of the problems, including the case of complex diagonal elements of J appears in [9d].

To be concrete we shall given the results for the inverse problem associated with the one-dimensional time-independent Schrödinger equation: i.e. (i) above with $n=2$, $u(x)=-u^{(2)}(x)$. Let $\lambda(k)=-k^2$, then the scattering equation is

$$v_{xx}+(k^2-u)v=0,\quad -\infty<x<\infty,\quad v=\mu\,e^{-ikx}, \tag{10}$$

$$\mu_{xx}-2ik\mu_x-u\mu=0. \tag{11}$$

The relevant function class for $u(x)$ is $\int_{-\infty}^{\infty}(1+|x|)|u|\,dx<\infty$. $v(x,k)$ has solutions (Jost functions) which we denote by

$$\phi(x,k)\underset{x\to-\infty}{\simeq}e^{-ikx},\qquad \left.\begin{aligned}\bar\psi(x,k)&\underset{x\to+\infty}{\simeq}e^{-ikx},\\ \psi(x,k)&\underset{x\to+\infty}{\simeq}e^{ikx}.\end{aligned}\right\} \tag{12a}$$

Functions with "nice" analytical properties are obtained by multiplying by a suitable exponential factor:

$$M(x,k)=\phi\,e^{ikx}\underset{x\to-\infty}{\simeq}1,\qquad \left.\begin{aligned}\bar N(x,k)&=\bar\psi\,e^{ikx}\underset{x\to+\infty}{\simeq}1,\\ N(x,k)&=\psi\,e^{ikx}\underset{x\to+\infty}{\simeq}e^{2ikx}.\end{aligned}\right\} \tag{12b}$$

The relationship

$$\psi(x,k)=\bar\psi(x,-k) \tag{12c}$$

implies

$$N(x,k)=\bar N(x,-k)\,e^{2ikx}. \tag{12d}$$

Completeness of these eigenfunctions requires

$$M(x,k) = a(k)\bar{N}(x,k) + b(k)N(x,k),$$

or, using (12d),

$$\frac{M(x,k)}{a(k)} = \bar{N}(x,k) + r(k)\,e^{2ikx}\bar{N}(x,-k), \tag{12e}$$

where $r(k) = b(k)/a(k)$. The analyticity of $M(x,k)$, $\bar{N}(x,k)$ is deduced by studying the following integral equations:

$$M(x,k) = 1 + \int_{-\infty}^{\infty} G_+(x-x',k)\,u(x')\,M(x',k)\,dx', \tag{12f}$$

$$\bar{N}(x,k) = 1 + \int_{-\infty}^{\infty} G_-(x-x',k)\,u(x')\,\bar{N}(x',k)\,dx', \tag{12g}$$

where

$$G_\pm(x,k) = \frac{1}{2\pi}\int_{C_\pm} \frac{e^{i\xi x}}{\xi(\xi-2k)}\,d\xi, \tag{12h}$$

$C_\pm$ being the contour below $(+)$/above $(-)$ the singularities $\xi = 0$, $\xi = 2k$ inside the integral (12h). $G_\pm(x,k)$ is analytic for $\operatorname{Im} k \gtrless 0$ and vanishes as $|k| \to \infty$. $M(x,k)$, $\bar{N}(x,k)$ are therefore analytic for $\operatorname{Im} k > 0$, $\operatorname{Im} k < 0$ respectively and tend to unity as $|k| \to \infty$.

The scattering coefficient $a(k)$ is also analytic for $\operatorname{Im} k > 0$ and tends to unity as $|k| \to \infty$ (this can be deduced from the fact that $a(k)$ is a Wronskian of M, N). $a(k)$ can vanish at a finite number of locations in the upper half plane: $k = k_1, \ldots, k_n$, $\operatorname{Im} k > 0$. Calling

$$\mu_+(x,k) = \frac{M(x,k)}{a(k)}, \quad \mu_-(x,k) = \bar{N}(x,k), \tag{12i}$$

we see that (9e) is a special case of (9) where $\alpha(k) = -k$, $V(x,k) = r(k)e^{2ikx}$. The appropriate residue statement is

$$\operatorname*{Res}_{k=k_j}(\mu_+(x,k)) = c_j\,e^{2ik_jx}\mu_-(x,k_j), \tag{12j}$$

C_j being called the normalization constants.

It is worthwhile noting that when no poles (i.e. no eigenvalues or boundstates) appear, then the solvability of (12e) follows from the work of Gohberg and Krein [13] in which they prove the existence of uniqueness of the solution of the corresponding Riemann–Hilbert factorization problem (in a generic sense).

For completeness we list the integral equations for the eigenfunction and potential reconstruction:

$$N(x,k) = e^{2ikx}\left(1 - \sum_{j=1}^{n} \frac{c_j N_j(x)}{k+k_j} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{r(\xi)N(x,\xi)}{\xi + k + i0}\,d\xi\right), \tag{12k}$$

$$N_l(x) = e^{2ik_l x}\left(1 - \sum_{j=1}^{n} \frac{c_j N_j(x)}{k_l+k_j} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{r(\xi)N(x,\xi)}{\xi + k_l}\,d\xi\right), \tag{12l}$$

$$u(x) = \frac{\partial}{\partial x}\left(\sum_{j=1}^{n} 2ic_j N_j(x) - \frac{1}{\pi}\int_{-\infty}^{\infty} r(k)N(x,k)\,dk\right). \tag{12m}$$

The solution of the initial value problem for suitably decaying functions $u(x,k)$ of KdV is obtained by noting that $r(k,t) = r(k,0)e^{8ik^3t}$. This follows from the second linear operator M: see (6), (8). The reconstruction of $u(x,t)$ then follows from the inverse problem. In the general case, the data $V(x,k,t)$ in (9) also evolves simply in time (e.g. $V(x,k,t) = V(x,k,0)e^{\omega(k)t}$ when V, ω are scalars). Schematically, we have:

$$\underset{\text{(Direct problem)}}{u(x,0) \to \mu_\pm(x,k,t=0)} \to V(x,k,0) \to \underset{\text{(From } M \text{ operator)}}{V(x,k,t)} \to \underset{\text{(From inverse problem)}}{\mu_\pm(x,k,t)} \to u(x,t) \quad .$$

The method of solution is what is usually referred to as the Inverse Scattering Transform: IST. This program has been carried out for a surprisingly large number of physically interesting equations in one spatial dimension. In fact, the only equation in one spatial dimension mentioned above that does not have an associated inverse problem of the form (9) is the Benjamin–Ono equation (4). It shares with the KP_I equation an inverse problem of the nonlocal R–H form:

$$(\mu_+ - \mu_-)(x,k) = \int \mu_-(x,k')V(x,k',k)\,dk'. \tag{13}$$

Next, we shall discuss the KP equation and its associated scattering operator L.

$$\sigma v_y + v_{xx} - u(x,y)v = 0. \tag{14}$$

Note in (14) we have taken the eigenvalue $\lambda = 0$ without loss of generality (by the scaling property of v). Since the analysis for the generalization

$$\sigma v_y + \Delta v - u(x,y)v = 0, \tag{15}$$

where $\sigma = \sigma_R + i\sigma_I$, $\Delta = \sum_{j=1}^{n} \partial^2/\partial x_j^2$, $x \in \mathbf{R}^n$, $y \in \mathbf{R}$, is a natural extension of that in two dimensions, we shall discuss this case. Scattering parameters arise in (15) by looking for a function $\mu = \mu(x,y,k)$ where

$$v = \mu\, e^{ik\cdot x + k^2 y/\sigma}, \tag{16}$$

$$\sigma\mu_y + \Delta\mu + 2ik\cdot\nabla\mu - u\mu = 0, \tag{17}$$

and $k = k_R + ik_I \in \mathbf{C}^n$. We shall consider $\sigma_R \neq 0$, $\sigma_R < 0$.

We look for a solution $\mu(x, y, k)$ bounded for all x, y and $\mu \to 1$ as $|k| \to \infty$. The latter condition is a convenient normalization. If we should consider (17) for $\sigma = \pm 1$ in analogy to the $\mathrm{KP_{II}}$ scattering problem, we immediately notice that the dominant operator is the heat operator which is illposed as an initial value problem. Even though we pose a boundary problem, immediately we are led to believe that in this case there will be some type of unusual behavior. In fact in refs. [4, 8a] it is shown that the bounded function μ for $\sigma_R \neq 0$ may be analytic nowhere as a function of k. Specifically $\mu = \mu(x, y, k_R, k_I)$. In particular μ is constructed from the following equation. Given $u(x, y) \to 0$ sufficiently rapidly at ∞, the direct problem is

$$\mu = 1 + \tilde{G}(u\mu), \tag{18}$$

where

$$\tilde{G}f \equiv G * f \equiv \iint G(x - x', y - y', k_R, k_I) f(x', y')\,dx'\,dy'. \tag{19}$$

The Green's function G is obtained from

$$G(x, y, k_R, k_I) = C_{n+1} \iint \frac{e^{i(x\cdot\xi + y\eta)}}{i\sigma\eta - \xi^2 - 2k\cdot\xi}\,d\xi\,d\eta, \quad C_n \equiv \frac{1}{(2\pi)^n}, \tag{20a}$$

$$= \frac{\operatorname{sign}(y)}{\sigma} C_n \int e^{(y/\sigma)(\xi^2 + 2k\cdot\xi) + ix\cdot\xi}\, \Theta\left(-y\sigma_R\left(\xi^2 + 2\left(k_R + \frac{k_I\sigma_I}{\sigma_R}\right)\cdot\xi\right)\right) d\xi. \tag{20b}$$

where $\Theta(x) = \{1 \text{ for } x > 0,\ 0 \text{ for } x < 0\}$. In constructing (20) we have looked for a bounded Green's function, and have taken the Fourier transform in both x and y.

Taking the $\bar{\partial}$ derivative of (18) with respect to $\bar{k}_j$ we find $(\partial/\partial\bar{k}_j = \frac{1}{2}(\partial/\partial k_{R_j} + i\partial/\partial k_{I_j}))$:

$$\frac{\partial\mu}{\partial\bar{k}_j} = \frac{\partial\tilde{G}(u\mu)}{\partial\bar{k}_j} + \tilde{G}\left(u\frac{\partial\mu}{\partial\bar{k}_j}\right). \tag{21}$$

The first term in (21) is calculated directly using the definition of the Green's function (20).

$$\frac{\partial\tilde{G}(u\mu)}{\partial\bar{k}_j} = -\frac{C_n}{|\sigma_R|} \int e^{i\beta(x, y, k_R, k_I, \xi)} T(k_R, k_I, \xi)(\xi_j - k_{R_j})\,\delta(s(\xi))\,d\xi, \tag{22a}$$

where

$$T(k_R, k_I, \xi) \equiv \iint e^{-i\beta(x, y, k_R, k_I, \xi)} u(x, y)\mu(x, y, k_R, k_I)\,dx\,dy, \tag{22b}$$

$$\beta(x, y, k_R, k_I, \xi) = \left(x + 2y\frac{k_I}{\sigma_R}\right)\cdot(\xi - k_R), \tag{22c}$$

$$s(\xi) = s(\xi, k_R, k_I) \equiv \left(\xi + \frac{\sigma_I}{\sigma_R}k_I\right)^2 - \left(k_R + \frac{\sigma_I}{\sigma_R}k_I\right)^2 \tag{22d}$$

and $\delta(x)$ is the usual Dirac delta function. One can derive (22) either by taking the $\bar{\partial}$ derivative directly on

(20b) or on (20a) using the well-known fact

$$\frac{\partial}{\partial \bar{k}}\left(\frac{1}{\bar{k}-k_0}\right)=\pi\delta(k-k_0). \tag{22e}$$

From (22) one can readily calculate $\partial\mu/\partial\bar{k}_j$ (assuming (18) has no homogeneous solutions),

$$\frac{\partial\mu}{\partial\bar{k}_j}=\bar{T}_j\mu=-\frac{C_n}{|\sigma_R|}\int e^{i\beta(x,y,k_R,k_I,\xi)}T(k_R,k_I,\xi)(\xi_j-k_{R_j})\delta(s(\xi))\mu(x,y,\xi,k_I)\,d\xi. \tag{23}$$

(23) is found by noting that $\partial\mu/\partial\bar{k}_j$ is a suitable superposition over a fundamental solution $W(x,y,k_R,k_I,\xi)$ satisfying

$$W(x,y,k_R,k_I,\xi)=e^{i\beta(x,y,k_R,k_I,\xi)}+\tilde{G}(uW). \tag{24}$$

Using the symmetry condition on the Green's function,

$$e^{-i\beta(x,y,k_R,k_I,\xi)}G(x,y,k_R,k_I)=G(x,y,\xi,k_I),\quad \text{on}\quad s(\xi)=0, \tag{25}$$

allows us to find

$$W(x,y,k_R,k_I,\xi)=e^{i\beta(x,y,k_R,k_I,\xi)}\mu(x,y,\xi,k_I),\quad \text{on } s(\xi)=0, \tag{26}$$

and then (23) follows.

A special case of (23) is $n=1$ whereupon $\partial\mu/\partial\bar{k}_j$ depends locally on μ. For $n=1$, let $k_1\equiv k$; then (23) reduces to

$$\frac{\partial\mu}{\partial\bar{k}}=\frac{C_1}{|\sigma_R|}\operatorname{sgn}\left(k_R+\frac{\sigma_I}{\sigma_R}k_I\right)e^{i\beta(x,y,k_R,k_I,\xi_0)}T(k_R,k_I,\xi_0)\mu(x,y,\xi_0,k_I), \tag{27}$$

where $\xi_0=-k_R-(2\sigma_I/\sigma_R)k_I$. (27) is relevant to the solution of KP: KP_{II}: $\sigma_I=0$, $\sigma_R=-1$ (see [4]) and KP_I $\sigma_I=1$, $\sigma_R\to 0$ ($\sigma_R<0$) with the scaling $\hat{k}_I=k_I/\sigma_R$ (also see the discussion of the limit to the time-dependent Schrödinger equation later in this paper).

The above discussion is entirely within the context of the direct scattering problem. However, it suggests what the natural data might be for this problem. We shall call $T(k_R,k_I,\xi)$ the inverse data.

The inverse problem is: given $T(k_R,k_I,\xi)$ construct $u(x,y)$. However, it is immediately transparent that there is a serious redundancy question. Namely $T(k_R,k_I,\xi)$ is a function of $3n$ parameters with one restriction (the restriction is due to $\delta(s(\xi))$ in (23): i.e. T will be given as a function of $3n-1$ variables and we wish to construct a function $u(x,y)$ depending on $n+1$ variables. But for $n=1$, namely for the problem in two spatial dimensions the difficulty disappears. As (27) shows $T=T(k_R,k_I,\xi_0(k_R,k_I))$, hence T is a function of two parameters as is u.

Using (23) there are numerous reconstruction formulae for u available. However, serious restrictions on T must be imposed in order to obtain a function u depending only on x, y and vanishing at ∞. This is part of the characterization question, i.e. which inverse data $T(k_R,k_I,\xi)$ are "admissible".

One set of inversion formulae for μ is obtained from the generalized Cauchy formula

$$\mu(k)=\frac{1}{2\pi i}\oint_C\frac{\mu(l)}{l-k}\,dl+\frac{1}{\pi}\iint_R\frac{\partial\mu/\partial\bar{l}}{k-l}\,dl_R\,dl_I. \tag{28}$$

(Another, more symmetric inversion uses the Bochner–Martinelli formula but this is outside the scope of the present review.) Applying this to our problem where $\mu \to 1$, $|k| \to \infty$ (the first term is unity) we have

$$\mu(x, y, k_{\mathrm{R}}, k_{\mathrm{I}}) = 1 + \frac{1}{\pi} \iint \frac{\dfrac{\partial \mu}{\partial \bar{k}_j}(x, y, k'_{\mathrm{R}}, k'_{\mathrm{I}})}{k_j - k'_j} \, \mathrm{d}k'_{\mathrm{R}_j} \mathrm{d}k'_{\mathrm{I}_j}, \tag{29}$$

where we use the simplified notation $k'_{\mathrm{R}} \equiv (k_{\mathrm{R}_1}, \ldots, k'_{\mathrm{R}_j}, \ldots, k_{\mathrm{R}_n})$ and similarly for k'_{I}. (29) is a linear integral equation for (using 23)) the potential is constructed from

$$u(x, y) = \frac{2\mathrm{i}}{\pi} \frac{\partial}{\partial x_j} \iint \frac{\partial \mu}{\partial \bar{k}_j}(x, y, k'_{\mathrm{R}}, k'_{\mathrm{I}}) \, \mathrm{d}k'_{\mathrm{R}_j} \mathrm{d}k'_{\mathrm{I}_j}. \tag{30}$$

(30) is obtained by taking $k_j \to \infty$ in (18) and (29) and comparing the results.

It is clear that in general the right-hand side of (30) will be a function of $k_{\mathrm{R}_i}, k_{\mathrm{I}_i} : i = 1, 2, \ldots, j-1, j+1, \ldots, n$. One possible way of characterizing admissible data would be to require $T(k_{\mathrm{R}}, k_{\mathrm{I}}, \xi)$ to be such that the RHS of (30) be independent of these parameters, for all j. Such a requirement is analogous to what Newton refers to as the "miracle" in the time-independent problem (see [11]). However, in this formulation we can go further and give conditions directly on $T(k_{\mathrm{R}}, k_{\mathrm{I}}, \xi)$. The importance of characterizing $T(k_{\mathrm{R}}, k_{\mathrm{I}}, \xi)$ directly not only has to do with understanding on which manifolds of $k_{\mathrm{R}}, k_{\mathrm{I}}, \xi$ can one hope to reconstruct the potential, but also may indicate how one could in principle measure data so as to produce local potentials in a stable manner.

For $n > 1$ the compatibility condition $\partial^2\mu/\partial\bar{k}_i\partial\bar{k}_j = \partial^2\mu/\partial\bar{k}_j\partial\bar{k}_i$ $(i \neq j)$ leads to a nontrivial restriction on T; one which is nonlinear;

$$\mathscr{L}_{ij}(T) = N_{ij}(T), \tag{31a}$$

where

$$\mathscr{L}_{ij} = (\xi_j - k_{j_{\mathrm{R}}})\left(\frac{\partial}{\partial \bar{k}_i} + \frac{1}{2}\frac{\partial}{\partial \xi_i}\right) - (\xi_i - k_{i_{\mathrm{R}}})\left(\frac{\partial}{\partial \bar{k}_j} + \frac{1}{2}\frac{\partial}{\partial \xi_j}\right), \tag{31b}$$

$$N_{ij}[T](k, \xi) = \int \left[(\xi'_j - k_{j_{\mathrm{R}}})(\xi_i - \xi'_i) - (\xi'_i - k_{i_{\mathrm{R}}})(\xi_j - \xi'_j)\right] \delta(s(\xi')) T(k_{\mathrm{R}}, k_{\mathrm{I}}, \xi') T(\xi', k_{\mathrm{I}}, \xi) \, \mathrm{d}\xi'. \tag{31c}$$

In fact there is a change of variables which allows (31) to be put in a simplified form. Without loss of generality we may consider the equations (31) with $i = 1$, ($i \neq 1$, is obtained from $i = 1$ by straightforward manipulation) then introduce new variables $(\chi, w, w_0) \in \mathbf{C}^{n-1} \times \mathbf{R}^n \times \mathbf{R}$ which parameterize the sphere $s(\xi)$, $(\chi = (\chi_2, \ldots, \chi_n)$

$$\begin{aligned}
k_{1_{\mathrm{R}}} &= \sum_{j=2}^{n} w_j \chi_{j_{\mathrm{R}}} - \frac{w_1}{2} - \frac{\sigma_{\mathrm{I}} w_0 w_1}{2w^2}, \quad k_{j_{\mathrm{R}}} = -w_1 \chi_{j_{\mathrm{R}}} - \frac{w_j}{2} - \frac{\sigma_{\mathrm{I}} w_0 w_j}{2w^2}, \\
k_{1_{\mathrm{I}}} &= \sum_{2}^{n} w_j \chi_{j_{\mathrm{I}}} + \frac{\sigma_{\mathrm{R}} w_0 w_1}{2w^2}, \quad k_{j_{\mathrm{I}}} = -w_1 \chi_{j_{\mathrm{I}}} + \frac{\sigma_{\mathrm{R}} w_0 w_j}{2w^2}, \\
\xi_1 &= \sum_{2}^{n} w_j \chi_{j_{\mathrm{R}}} + \frac{w_1}{2} - \frac{\sigma_{\mathrm{I}} w_0 w_1}{2w^2}, \quad \xi_j = -w_1 \chi_{j_{\mathrm{R}}} + \frac{w_j}{2} - \frac{\sigma_{\mathrm{I}} w_0 w_j}{2w^2} \quad (j \geq 2).
\end{aligned} \tag{32}$$

Thus for $w_1 \neq 0$ there is a 1–1 map: $(k_R, k_I, \xi) \to (\chi, w, w_0)$ such that

$$w = \xi - k_R, \quad w = 2k_I \cdot (\xi - k_R)/\sigma_R, \tag{33a}$$

$$\frac{\partial}{\partial \bar{\chi}_j} = \mathscr{L}_{1j}, \tag{33b}$$

which for $i = 1$, $j = 2, \ldots, n$ yields

$$\frac{\partial T}{\partial \bar{\chi}_j} = N_{1j}(T)(\chi, w, w_0), \quad j = 2, \ldots, n. \tag{34}$$

Again using the generalized Cauchy formula we have

$$\mathscr{I} = T(\chi, w, w_0) - \frac{1}{\pi} \iint \frac{N_{1j}[T](\chi', w, w_0)}{\chi_j - \chi_j'} \, \mathrm{d}\chi'_{R_j} \, \mathrm{d}\chi'_{I_j} = \hat{u}(w, w_0), \tag{35}$$

where $\hat{u}(w, w_0) = \mathscr{F}(u(x, y))$ is the Fourier Transform of $u(x, y)$ with respect to w, w_0. The term $\hat{u}(w, w_0)$ is the boundary value of $T(\chi, w, w_0)$ as $\chi_j \to \infty$. This can be seen from the definition of $T(\chi, w, w_0)$ (22b) and the fact that from (32) $\chi_j \to \infty$ implies $k_j \to \infty$ and hence $\mu \to 1$. (35) leads both to admissibility criteria as well as reconstruction of $u(x, y)$. Given $T(k_R, k_I, \xi)$ one computes $\mathscr{I}$ by quadratures. We also reiterate the fact that the formula (35) assumes no homogeneous solutions to (18). We conjecture [8a] that if $\mathscr{I}$ is independent of χ and j and has suitable decay properties for large w, w_0, then T is *admissible*. The potential is recovered from

$$u(x, y) = \mathscr{F}^{-1}(\hat{u}(w, w_0)), \tag{36}$$

where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform. Moreover, we see that reconstruction follows purely by quadratures given $T(k_R, k_I, \xi)$ on $s(\xi) = 0$.

It turns out that the physically interesting cases of the time-dependent and time-independent Schrödinger equation in n dimensions fall out as special cases of the above result. In what follows we discuss these cases both as limits (reductions) of the above results and then briefly indicate how the formulae can be derived without recourse to any limit.

First consider the case $\sigma \to \mathrm{i}$, i.e. $\sigma_I = 1$, $\sigma_R \to 0 - (\sigma_R < 0)$; $\hat{k}_R, \hat{k}_I = k_I/\sigma_R$. Then $G(x, y, k_R, k_I) \to G_L(x, y, \hat{k}_R, \hat{k}_R)$ (in what follows we drop the symbol ˆ),

$$G_L(x, y, k_R, k_I) = -\mathrm{i}C_n \operatorname{sgn}(y) \int \mathrm{e}^{\mathrm{i}x \cdot \xi - \mathrm{i}y(\xi^2 + 2k_R \cdot \xi)} \Theta\left(y(\xi^2 + 2(k_R + k_I) \cdot \xi)\right) \mathrm{d}\xi. \tag{37}$$

(37) can be directly verified, i.e.

$$\mathscr{L}G_L(x, y, k_R, k_I) = \delta(x)\delta(y), \quad \mathscr{L} = \mathrm{i}\frac{\partial}{\partial y} + \Delta + 2\mathrm{i}k_R \cdot \nabla, \tag{38}$$

and hence $\mu \to \mu_L$ where μ_L satisfies

$$\mathscr{L}\mu_L = -u\mu_L \quad \text{and} \quad \mu_L(x, y, k_R, k_I) = 1 + \tilde{G}_L(u\mu_L). \tag{39a, b}$$

Acknowledgements

It is a pleasure to acknowledge the many fruitful and stimulating comments of Martin Kruskal. This work was partially supported by the Office of Naval Research under Grant Number N00014-76-C-0867, the National Science Foundation under Grant Number MCS-8202117, and the Air Force Office of Scientific Research under Grant Number 78-3674-D.

References

[1] M.J. Ablowitz and H. Segur, Solitons and the Inverse Scattering Transform, SIAM Stud. Appl. Math. (1981).
[2a] Y. Kodama, M. Ablowitz and J. Satsuma, J. Math. Phys. 23 (1982) 564–576.
[2b] A.S. Fokas and M.J. Ablowitz, Stud. Appl. Math. 68 (1983) 1–10.
[3] A.S. Fokas and M.J. Ablowitz, Comm. Math. Phys. 91 (1983) 381–403.
[4] M.J. Ablowitz, D. Bar Yaacov and A.S. Fokas, Stud. in Appl. Math. 69 (1983) 135–143.
[5a] M.J. Ablowitz and A.S. Fokas, Comments on the Inverse Scattering Transform and Related Nonlinear Evolution Equations, Lecture Notes in Physics, 189, Proc. CIFMO School and Workshop held at Oaxtepec, Mexico, 1982, K.B. Wolf, ed.
[5b] A.S. Fokas and M.J. Ablowitz, The Inverse Scattering Transform for Multidimensional (2 + 1) Problems, Lecture Notes in Physics, 189, Proc. CIFMO School and Workshop held at Oaxtepec, Mexico, 1982, K.B. Wolf, ed.
[6a] S.V. Manakov, Physica 3D (1981) 420.
[6b] A.S. Fokas and M.J. Ablowitz, Stud. Appl. Math. 69 (1983) 211–228.
[7a] H. Cornille, J. Math. Phys. 20 (1979) 1653.
[7b] D.J. Kaup, Physica 1D (1980) 45–67.
[8a] A.I. Nachman and M.J. Ablowitz, Stud. in Appl. Math. 71 (1984) 243–50.
[8b] A.I. Nachman and M.J. Ablowitz, Stud. Appl. Math 71 (1984) 251–262.
[8c] A.I. Nachman and M.J. Ablowitz, Multidimensional Inverse Scattering for the Time-dependent and Time-independent Schrödinger Operators, Preprint, (1985).
[9a] R. Beals and R.R. Coifman, Scattering, Transformations Spectrales et Equations d'evolution nonlineare I, II, et Seminaire Goulaouic-Meyer-Schwartz 1980–1981, exp. 22, 1981–1982 exp. 21, Ecole Polytechnique, Palaiseau.
[9b] R. Beals and R.R. Coifman, Multidimensional Scattering and Inverse Scattering, preprint (1984).
[9c] R. Beals, Inverse scattering for Ordinary Differential Operators on the Line, preprint (1982).
[9d] R. Beals and R.R. Coifman, Commun. Pure and Appl. Math 37 (1984).
[10] L.D. Faddeev, Dokl. Akad. Nauk SSSR 167:6a (1966); Soviet Phys. Dokl. 10:1033, 11:209 (1966); J. Soviet Math. 5:3340396 (1976).
[11] R.G. Newton, Phys. Rev. Lett. 43 (1970) 541–542; J. Math. Phys. 21 (1980) 1698–1715; 22 (1981) 2191–2200; 23 (1982) 693, 594.
[12a] A.B. Shabat, Func. Annal and Appl. 9 (1975) 75; Diff. eq. XV (1979) 1824.
[12b] A.V. Mikhailov, Physica 1D (1981) 73.
[12c] P.Caudrey, Physica 6D (1982) 51.
[13] I. Gohberg and M.G. Krein, Uspekhi Mat. Nauk. 13 (158) 2.
[14] A.S. Fokas and M.J. Ablowitz, Inverse Scattering for First Order Hyperbolic Systems and the N-Wave Interaction Equation in Multidimensions, preprint (1985).
[15] B.G. Konopelchenko, Phys. Lett. 93A (1983) 442.

NOTE ON SOLUTIONS TO A CLASS OF NONLINEAR SINGULAR INTEGRO-DIFFERENTIAL EQUATIONS

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Received 3 November 1986; accepted for publication 26 November 1986

Certain classes of nonlinear singular integro-differential equations are considered. These equations are mapped, via explicit transformations, to either ordinary differential equations or to linearizable partial differential equations.

In recent years considerable interest has focused on certain physically important nonlinear evolution equations which can be linearized. Many of these equations fall into the category of linearization via soliton theory and the inverse scattering transform (IST) (for a review of much of this work, see for example ref. [1]). Well-known equations are the Korteweg–de Vries equation (KdV)

$$u_t+2uu_x+u_{xxx}=0, \tag{1}$$

the sine-Gordon equation

$$u_{xt}=\sin u, \tag{2}$$

and the Kadomtsev–Petviashvili (KP) equation

$$(u_t+2uu_x+u_{xxx})_x=-3\sigma^2 u_{yy}. \tag{3}$$

Each of these equations has certain singular integro-differential analogs, the best known being the so-called Benjamin–Ono equation

$$u_t+2uu_x+(Hu)_{xx}=0. \tag{4}$$

Analogues of the sine-Gordon and of the KP equations include the sine-Hilbert equation,

$$(Hu)_t=\sin u, \tag{5}$$

and

$$(u_t+Hu_{xx}+2uu_x)_x=(Hu_{xx}+2uu_x)_y, \tag{6}$$

respectively. In the above, Hu is the Hilbert transform of u.

$$(Hu)(x)\doteq\frac{1}{\pi}\fint_{-\infty}^{\infty}\frac{u(\xi)}{\xi-x}\,d\xi, \tag{7}$$

where $\fint_{-\infty}^{\infty}$ denotes the Cauchy principal value integral. Although eqs. (4)–(6) are related in a certain sense to (1)–(3) respectively (for the relationship of (1) to (4) see ref. [12], the IST method for (4)–(6) has novel features. With respect to the IST method BO [3] has more similarities with KP [4] than with KdV (the IST for (5) has been recently considered in ref. [5]).

In this paper we consider other singular integro-differential analogs of (1)–(3). These analogs are more closely associated with (1)–(3): it is shown that via a direct transformation they can be mapped to (1)–(3). For example, the equations

$$u_t+u_{xxx}+2(uHu)_x=0, \tag{8}$$

$$[u_t+u_{xxx}+2(uHu)_x]_x=-3\sigma^2 u_{yy} \tag{9}$$

are mapped to KdV and KP (for the variable w), via the transformation

$$w=u+iv,\quad v=Hu, \tag{10}$$

where u is real and vanishes at infinity.

Various generalizations are possible:

(A) The transformation (10) can be used to map certain singular integro-differential equations to

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ODEs.

(B) Given a linearizable PDE, there exists an algorithmic procedure for obtaining a singular integro-differential analog similar to the above.

(C) The above results can be extended to allow complex valued functions u.

(D) The Hilbert operator can be replaced by other suitable operators, for example it can be replaced by the T operator

$$(Tu)(x) \doteq \frac{1}{2\delta} \int_{-\infty}^{\infty} \coth[(\pi/2\delta)(\xi - x)]u(\xi)\, d\xi, \tag{11}$$

δ constant.

This work was motivated by some recent results of Constantin, Lax, and Majda [6]; in particular these authors proposed the following equation as a model for the motion of vorticity for an inviscid incompressible fluid flow,

$$u_t = uHu. \tag{12}$$

They introduced the transformation (10) and showed that w satisfies the ODE,

$$w_t = -\tfrac{1}{2}iw^2. \tag{13}$$

We first consider (A). It should be noted that the above result can be obtained as follows. Operate on (12) with $(1+iH)$ and use

$$H(uHu) = \tfrac{1}{2}[(Hu)^2 - u^2], \tag{14}$$

which is a special case of the known formula

$$H(fHg) + H(gHf) = (Hf)(Hg) - fg. \tag{15}$$

The above result can easily be extended. Since as is known $H^2 = -1$ we have that $Hw = -iw$. Now w is the boundary value of a function analytic in the lower half plane (a "lower function"), vanishing at infinity. Hence, $Hw = -iw$, and more generally,

$$w \doteq u + iHu \Rightarrow Hw^n = -iw^n \quad (n > 0, \text{integer}), \tag{16}$$

$$He^w = -ie^w, \tag{17}$$

etc. This enables us to construct arbitrarily many reducible equations such as [(a) ODEs; (b) singular integro-differential equations]:

$$w_t = -\tfrac{1}{2}iw^2, \quad u_t = uHu, \tag{I.1a,b}$$

$$w_t = w^3, \quad u_t = u^3 - 3u(Hu)^2, \tag{I.2a,b}$$

$$w_t = ie^{-iw}, \quad u_t = e^{Hu}\sin u. \tag{I.3a,b}$$

(B) The extension of the above results to PDEs is straightforward once it is noted that the above considerations go through even if a linear operator is substituted for the time derivative in the above equations. For example, the following list is easily obtained [(a) PDEs; (b) singular integro-differential equations]:

$$w_t = w_{xx} - i(w^2)_x, \quad u_t = u_{xx} + 2(uHu)_x, \tag{II.1a,b}$$

$$w_t + w_{xxx} - i\alpha(w^2)_x + \beta(w^3)_x = 0, \tag{II.2a}$$

$$u_t + u_{xxx} + 2\alpha(uHu)_x + \beta[u^3 - 3u(Hu)^2]_x = 0, \tag{II.2b}$$

$$w_{xt} = ie^{-iw}, \quad u_{xt} = e^{Hu}\sin u, \tag{II.3a,b}$$

$$w_t + i[w_{xx} + (w^2)_x] = 0, \tag{II.4a}$$

$$u_t = (Hu)_{xx} + 2(uHu)_x. \tag{II.4b}$$

Eq. (II.1a) is essentially the Burgers equation and can be linearized via the Cole–Hopf transformation

$$w = -i(\ln f)_x. \tag{18}$$

Eq. (II.1b) arises in various population ecological models and to our knowledge, was first considered and solved via a dependent variable transformation and splitting into upper and lower functions by Satsuma [7]. In eqs. (II.2) α, β are real constants, and (II.2b) is an analog of the Gardner equation (a combination of KdV and modified KdV). Eq. (II.3b) is related to the Liouville equation (II.3a) and is known to be linearizable.

Let us consider the initial value problem for each of the above equations with u real. Given $u(x,0)$, initial values for $w(x,t)$ are obtained from $w(x,0) = u(x,0) + iHu(x,0)$, and the solution $u(x,t)$ is recovered from $u(x,t) = \operatorname{Re} w(x,t)$.

Generalizations to systems of equations as well as discrete analogs are immediate using these ideas, hence we shall not incorporate them into this discussion. Multidimensional analogs can also be readily constructed. For example, an analog to the KP equation (3) is

$$\frac{\partial}{\partial x}[u_t+u_{xxx}+(2uHu)_x]=-3\sigma^2 u_{yy}, \qquad (19)$$

$\sigma^2=$const and is linearized via the KP equation

$$\frac{\partial}{\partial x}[w_t+w_{xxx}-\mathrm{i}(w^2)_x]=-3\sigma^2 w_{yy}, \qquad (20)$$

which is formally a rescaled version of (3). Eq. (19) is (2+1)-dimensional. A (3+1)-dimensional equation can also be linearized via (20). Namely let $H_z u$ denote the Hilbert transform of $u(x, y, z, t)$ with respect to the variable z, i.e.,

$$H_z u=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(x,y,\xi,t)}{\xi-z}\,\mathrm{d}\xi. \qquad (21)$$

Then instead of KP we may consider a multi-dimensional analog of (19):

$$\frac{\partial}{\partial x}[u_t+u_{xxx}+2(uH_z u)_x]=-3\sigma^2 u_{yy}, \qquad (22)$$

and it is also mapped to the KP equation (20), via $w=u+\mathrm{i}H_z u$. Rational soliton solutions and nondecaying soliton solutions of KP are given in ref. [8]. The initial value problem for KP, with decaying initial data, is considered in refs. [4,9,10] (for a review see ref. [11]).

(C) Let us now consider complex u. For example,

$$\mathrm{i}u_t=u_{xx}+2(uHu)_x. \qquad (23)$$

In association with (23) define

$$w_\pm=u\pm\mathrm{i}Hu. \qquad (24)$$

Then

$$\mathrm{i}(w_\pm)_t=(w_\pm)_{xx}\mp\mathrm{i}(w_\pm^2)_x, \qquad (25)$$

which is linearized via

$$w_\pm=\mp\mathrm{i}(\ln f_\pm)_x. \qquad (26)$$

The initial values are obtained as before but the complex solution $u(x, t)$ is now recovered from

$$u(x,t)=\tfrac{1}{2}(w_+ + w_-). \qquad (27)$$

The above approach can also be used for dealing with complex initial values of the equations considered in (A), (B).

(D) As discussed above we deal with Hu by extending the function u to its upper and lower functions. Similarly we can deal with Tu by extending u to a function sectionally holomorphic in horizontal strips of thickness δ [12]. Operators associated with certain other geometries can also be considered (see ref. [13]). Actually one may replace, eqs. (12), and (10) say, by the more general system

$$U_t=UV,\quad U_x=V_y,\quad U_y=-V_x,\quad y\leqslant 0. \qquad (28)$$

Eqs. (12), (10) are special cases of the above, $u(x,t)=U(x,0,t)$, $v(x,t)=V(x,0,t)$. We note that these equations are mutually consistent. However, it should be stressed that eqs. (28) are not a (2+1)-dimensional system since the latter two equations in (28) are the Cauchy–Riemann equations and so $W=U+\mathrm{i}V=W(z,t)$, $z=x+\mathrm{i}y$. The transformation $W=U+\mathrm{i}V$ maps (28) to

$$W_t=-\tfrac{1}{2}\mathrm{i}W^2+C(t). \qquad (29)$$

This is derived as follows:

$$V_{yt}=U_{xt}=\frac{\partial}{\partial x}(UV),$$

$$V_{xt}=-U_{yt}=-\frac{\partial}{\partial y}(UV). \qquad (30)$$

Using the formula $g(x,y)=\int g_x\,\mathrm{d}x+g_y\,\mathrm{d}y$, from eqs. (28) and (30) we obtain

$$V_t=\int\left(-\frac{\partial}{\partial y}(UV)\,\mathrm{d}x+\frac{\partial}{\partial x}(UV)\,\mathrm{d}y\right)$$

$$=\int[(VV_x-UU_x)\,\mathrm{d}x+(V_yV-UU_y)\,\mathrm{d}y]$$

$$=\tfrac{1}{2}(V^2-U^2)-\mathrm{i}C(t).$$

Hence

$$W_t=-\tfrac{1}{2}\mathrm{i}W^2+C(t).$$

From the above discussion it follows that the results of (A)–(C) are also valid if one replaces H by T or by another suitable operator (see ref. [13]).

This work was partially supported by the Air Force Office of Scientific Research under Grant Number 78-3674-D, the Office of Naval Research under Grant

Number N00014-76-C-0867, and the National Science Foundation under Grant Number MCS-8202117.

References

[1] M.J. Ablowitz and H. Segur, Solitons and the inverse scattering transform, SIAM Studies in Applied Mathematics (Philadelphia, 1981).
[2] P.M. Santini, M.J. Ablowitz and A.S. Fokas, J. Math. Phys. 25 (1984) 892.
[3] A.S. Fokas and M.J. Ablowitz, Stud. Appl. Math. 68 (1983) 1.
[4] A.S. Fokas and M.J. Ablowitz, Stud. Appl. Math. 69 (1983) 211.
[5] P.M. Santini, M.J. Ablowitz and A.S. Fokas, On the solution of a class of nonlocal nonlinear evolution equations, INS #47, Clarkson University (1985), to be published in J. Math. Phys.
[6] P. Constantin, P.D. Lax and A. Majda, Commun. Pure Appl. Math. 38 (1985) 715.
[7] J. Satsuma, J. Phys. Soc. Japan 50 (1981) 1423.
[8] V.E. Zakharov and S.V. Manakov, Sov. Sci. Rev. C 1 (1979) 133.
[9] S.V. Manakov, Phys. D 3 (1981) 420.
[10] M.J. Ablowitz, D. Bar Yaacov and A.S. Fokas, Stud. Appl. Math. 69 (1983) 135.
[11] M.J. Ablowitz and A.S. Fokas, Comments on the inverse scattering transform and related nonlinear evolution equations, Lect. Notes in Phys. 189, Proc. CIFMO School & Workshop held at Oaxtepect, Mexico (1982), ed. K.B. Wolf; A.S. Fokas and M.J. Ablowitz, The inverse scattering transform for multidimensiona (2+1) problems, Lect. Notes in Phys. 189, Proc. CIFMO School and Workshop held at Oaxtepec, Mexico (1982), ed. K.B. Wolf.
[12] Y. Kodama, M.J. Ablowitz and J. Satsuma, J. Math. Phys. 23 (1982) 564.
[13] M.J. Ablowitz, A.S. Fokas, J. Satsuma and H. Segur, J. Phys. A 15 (1982) 781.

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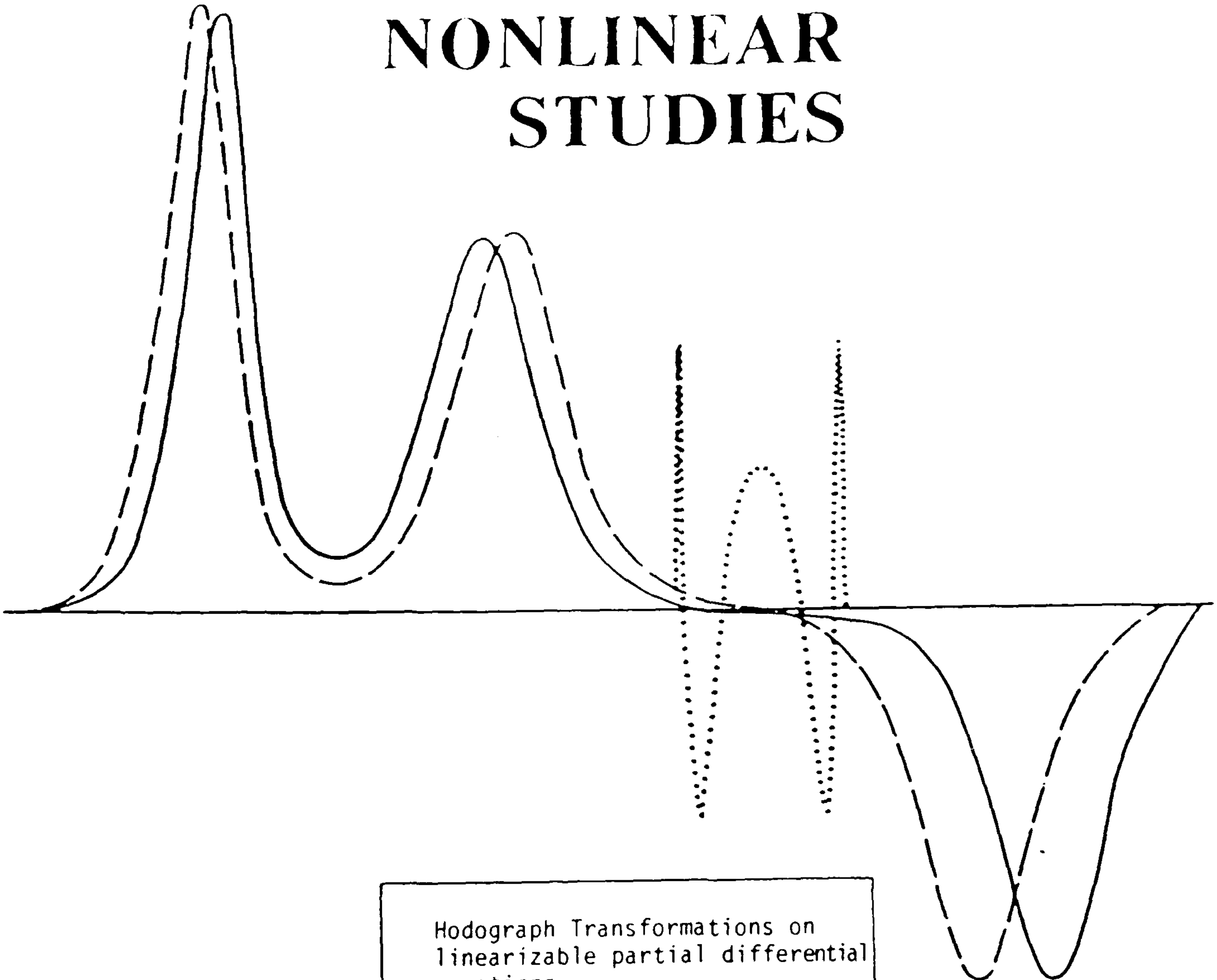

Hodograph Transformations on linearizable partial differential equations

by

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HODOGRAPH TRANSFORMATIONS ON LINEARIZABLE PARTIAL DIFFERENTIAL EQUATIONS

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Key words: hodograph transformation, inverse scattering, linearization, Painlevé property.

AMS Subject classification: 34A34, 35Q20, 58F07, 58G37.

Abbreviated title: Hodograph Transformations

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ABSTRACT

In this paper we develop an algorithmic method for transforming quasilinear partial differential equations of the form $u_t = g(u)u_{nx} + f(u,u_x,\ldots,u_{(n-1)x})$, $u_{mx} \equiv \partial^m u/\partial x^m$, where $dg/du \not\equiv 0$, into semilinear equations (i.e., equations of the above form with $g(u) = 1$). This crucially involves the use of hodograph transformations (i.e., transformations which involve the interchange of dependent and independent variables). Furthermore, we find the most general quasilinear equation of the above form which can be mapped via a hodograph transformation to a semilinear form.

This algorithm provides a method for establishing whether a given quasilinear equation is linearizable; i.e., is solvable in terms of either a linear partial differential equation or of a linear integral equation. In particular, we use this method to show how the Painlevé tests may be applied to quasilinear equations. This appears to resolve the problem that solutions of linearizable quasilinear partial differential equations, such as the Harry-Dym equation $u_t = (u^{-1/2})_{xxx}$, typically have movable fractional powers and so do not directly pass the Painlevé tests.

I. INTRODUCTION

Recently there has been considerable interest in the solution of certain physically significant, nonlinear partial differential equations. It turns out that the solutions of these equations may be expressed in terms of the solution of linear equations (either linear integral equations or linear partial differential equations). In 1967, Gardner, Greene, Kruskal and Miura [1] associated the solution of the Korteweg-de Vries (KdV) equation with the time independent Schrödinger equation and showed, using ideas from the theory of direct and inverse scattering, that the Cauchy problem for the KdV equation (for initial data on the line which decays sufficiently rapidly), could be solved in terms of the solution of a linear integral equation. Subsequently, this novelty was developed into a new method of mathematical physics, often referred to as the inverse scattering transform (I.S.T.), which has led to the solution of numerous evolution equations (see, for example, [2] for details). These nonlinear evolution equations have arisen in many branches of physics including water waves, stratified fluids, plasma physics, statistical mechanics and quantum field theory. Previous to the KdV equation, the first physically interesting nonlinear partial differential equation which was solved in terms of a linear partial differential equation was Burgers' equation

$$u_t = u_{xx} + 2uu_x, \tag{1.1}$$

which was mapped into the linear heat equation via the Cole-Hopf transformation [3].

Partial differential equations which can either be solved by an appropriate I.S.T. scheme or by a transformation to a linear partial differential equation are said to be linearizable. The most well known linearizable partial differential equations are of the form

$$u_t = u_{nx} + f(u,u_x,\ldots,u_{(n-1)x}),\ n \geq 2,\ u_{nx} \doteq \frac{\partial^n u}{\partial x^n} \tag{1.2}$$

Definition 1.1 A partial differential equation is said to be semilinear if it is of the form (1.2).

There also exist linearizable equations of the form

$$u_t = g(u)u_{nx} + f(u,u_x,\ldots,u_{(n-1)x}),\ n \geq 2, \tag{1.3}$$

where $dg/du \not\equiv 0$.

Definition 1.2 A partial differential equation is said to be quasilinear if it is of the form (1.3).

Well known examples of quasilinear linearizable equations include an equation studied in [4],

$$u_t = (u^{-2}u_x)_x + \alpha u^{-2}u_x, \tag{1.4}$$

where α is an arbitrary constant and the Harry-Dym equation (Kruskal [5])

$$u_t = 2(u^{-1/2})_{xxx}, \tag{1.5}$$

which is known to be linearizable [6] (see also [2b]).

Fokas and Yortsos [4] considered second order quasilinear partial differential equations using the symmetry approach of Fokas [7]. They showed that the most general equation of the form

$$u_t = g(u)u_{xx} + f(u,u_x), \tag{1.6}$$

which is linearizable is the equivalent to the equation (1.4), which via an extended hodograph transformation is mapped to the Burgers' equation. Similarly, it is known that the Harry-Dym equation (1.5) can be transformed either into the KdV equation (see, for example, [2b] or [8]), or the MKdV equation (see, for example, Kawamoto [9]). The notions of equivalence and hodograph transformations are defined below:

<u>Definition 1.3</u> Two partial differential equations are <u>equivalent</u> if one can be obtained from the other by a transformation involving the dependent variables $u = \phi(v)$ and/or the introduction of a potential variable ($u = v_x$ or $u_x = v$).

For example, the Burgers' equation is equivalent to the heat equation.

<u>Definition 1.4</u> A <u>pure hodograph transformation</u> is a transformation of the form

$$\tau = t, \quad \xi = u(x,t). \tag{1.7}$$

Definition 1.5 An extended hodograph transformation is a transformation of the form

$$\tau = t, \quad \xi = \int^{x} (u(x',t))dx'. \tag{1.8}$$

The above discussion naturally motivates the following questions: Equation (1.4) is a quasilinear analogue, via an extended hodograph transformation, of Burgers' equation. Similarly, the Harry-Dym equation (1.5) is a quasilinear analogue of the MKdV equation.

i) Is there an algorithmic method of finding a quasilinear analogue of any semilinear equation?

ii) Is the associated quasilinear equation unique?

iii) Conversely, given a quasilinear equation, is there an algorithmic method of finding whether it can be mapped to a semilinear equation as well as finding this semilinear equation?

In this paper we consider the above questions for semilinear and quasilinear equations (1.2) and (1.3) respectively. The answer to question i) is affirmative. Also, the associated quasilinear equation is unique, since extended and pure hodograph transformations yield equivalent quasilinear equations. Furthermore, we find the most general equation of the form (1.3) which can be mapped via an extended hodograph transformation to a semilinear form.

The above results are of some interest in establishing whether an equation is a candidate for linearization. Suppose that one is interested in investigating whether a given quasilinear equation is linearizable. We propose the following algorithmic procedure (see §III);

1. Put the equation into its potential canonical form

$$v_t = v_x^{-n} v_{nx} + H(v_x, v_{xx}, \ldots, v_{(n-1)x}), \tag{1.9}$$

by using the transformation $v_x = g^{-1/n}(u)$.

2. Apply a pure hodograph transformation to equation (1.9). If equation (1.9) is transformable to a semilinear equation, it will become

$$\eta_t = \eta_{n\xi} + \tilde{H}(\eta_\xi, \eta_{\xi\xi}, \ldots, \eta_{(n-1)x}. \tag{1.10}$$

3. Investigate whether equation (1.10) is linearizable. This is easier than investigating whether (1.2) is linearizable directly. The reason for this is twofold. First, for at least third order equations there is a complete classification of all linearizable equations. Within equivalence, there exist only six such equations (see below). Hence one needs to study if there exists an equivalence transformation to map equation (1.10) with n = 3, to one of these six canonical equations. Second, for equations with $n \geq 4$ one may investigate the question of linearization via the Painlevé test. The Painlevé approach is reviewed below. Here we only point out that quasilinear partial differential equations do not appear suitable for applying the Painlevé test. Ramani,

Dorizzi and Grammaticos [10] (see also [11] and the references therein) introduced the notion of "weak-Painlevé" in order to deal with equations such as the Harry-Dym equation which are linearizable after a change of variables. However, the higher KdV equation $u_t = u_{xxx} + u^3 u_x$, although not thought to be linearizable (since it has only three independent polynomial conservation laws of a certain type [12]), is also "weak-Painlevé" [13]. Therefore the "weak-Painlevé" concept does not distinguish between a linearizable and a non linearizable equation.

We point out that one often finds in the literature claims of "new" third order linearizable equations. These equations, using the notion of equivalence can be mapped via a pure hodograph transformation to one of the six canonical equations mentioned above.

The above algorithmic approach is useful provided that a given linearizable quasilinear equation can be mapped to a semilinear form. The above approach will fail if there exist linearizable quasilinear equations which can not be mapped to a semilinear from. It is shown in [4] that such equations do not exist for at least $n = 2$. The question of whether such equations exist for $n \geq 3$ remains open.

IA. Classification of third order equations

Svinolupov, Sokolov and Yamilov [14] have claimed that the only third order semilinear partial differential equations which are linearizable are equivalent to the following six equations:

$$u_t = u_{xxx} + \gamma u_x, \tag{1.11}$$

$$u_t = u_{xxx} + uu_x + \gamma u_x, \tag{1.12}$$

$$u_t = u_{xxx} + u^2 u_x + \gamma u_x, \tag{1.13}$$

$$u_t = u_{xxx} - \frac{1}{8}u_x^3 + (\alpha e^u + \beta e^{-u})u_x + \gamma u_x, \tag{1.14}$$

$$u_t = u_{xxx} - \frac{3}{2}u_x u_{xx}^2(1 + u_x^2)^{-1} - \frac{3}{2}P(u)(u_x^2 + 1)u_x + \gamma u_x, \tag{1.15}$$

$$u_t = u_{xxx} - \frac{3}{2}u_{xx}^2 u_x^{-1} + \alpha u_x^{-1} - \frac{3}{2}P(u)u_x^2 + \gamma u_x, \tag{1.16}$$

where

$$\left(\frac{dP}{du}\right)^2 = 4P^3 - \delta P - \epsilon , \tag{1.17}$$

and α, β, γ, δ and ϵ are arbitrary constants. Equation (1.11) is a linear partial differential equation which is sometimes referred to as the Airy equation in moving coordinates; equation (1.12) is the KdV equation, which was the first equation to be solved by I.S.T.[1]; equation (1.13) is the Modified KdV (MKdV) equation, also solvable by I.S.T. [15]; equation (1.14) is the Calogero-Degasperis-Fokas (CDF) equation [7],[16] equations (1.15) and (1.16) are as yet unnamed and involve the Weierstrass elliptic function P(u). We note that the CDF equation can be put into rational form: let $v = e^{u/2}$,

$$v_t = v_{xxx} - \frac{3}{2}(v_x^2/v)_x + (\alpha v^2 + \beta v^{-2} + \gamma)v_x . \qquad (1.18)$$

Alternatively, provided that $\alpha = \beta = -2\gamma$ (if $\alpha\beta \neq 0$, then one can rescale and translate the variables in (1.14) so that this holds), let $q = \sinh(u/2)$ to obtain

$$q_t = q_{xxx} - \frac{3}{2}[qq_x^2/(1 + q^2)]_x + 4\alpha q^2 q_x. \qquad (1.19)$$

(Equation (1.19) is sometimes referred to as the 'deformed MKdV' equation [17] or the modified MKdV [18], though it is equivalent to the CDF equation.) We also note that both equations (1.15) and (1.16) can be put into rational form by the substitution $v = P(u)$.

IB. The Painlevé Tests

The Painlevé ODE test, as formulated by Ablowitz, Ramani and Segur [19] and Hastings and McLeod [20] asserts that every ordinary differential equation which arises as a similarity reduction of a partial differential equation solvable by inverse scattering is of Painlevé type; that is, it has no movable singularities except poles, perhaps after a transformation of variables. Ablowitz, Ramani and Segur [19b] and McLeod and Olver [21] have given proofs of the Painlevé ODE test under certain restrictions. Subsequently, Weiss, Tabor and Carnevale [22] developed the Painlevé PDE test as a method of applying the Painlevé ODE test directly to a given partial differential equation, without having to study any similarity reductions (which may not exist anyway). A partial differential equation is said to possess the Painlevé property if its solutions are "single-valued" in the neighborhood of noncharacteristic movable singularity manifolds. These Painlevé tests have proved to be a useful criterion for the identification of linearizable partial differential equations. The method introduced by Weiss, Tabor and Carnevale (with simplifications due to Kruskal [23]), involves seeking solutions of a given partial differential equation in the form

$$u(x,t) = \phi^{p}(x,t)\sum_{j=0}^{\infty}u_j(t)\phi^{j}(x,t), \tag{1.20a}$$

with

$$\phi(x,t) = x + f(t), \tag{1.20b}$$

where $f(t)$ is an arbitrary, analytic function of t and $u_j(t)$, $j = 0,1,2,\ldots$, are analytic functions of t, in the neighborhood of a noncharacteristic movable singularity manifold defined by $\phi = 0$. Essentially, if a given partial differential equation possesses solutions of the form (1.20) where p is an integer and with the requisite number of arbitrary functions as required by the Cauchy-Kowalevski theorem, then the partial differential equation is said to pass the Painlevé PDE test.

However, the application of the Painlevé tests to quasilinear partial differential equations is not as straightforward. For example, consider the Harry-Dym equation (Kruskal [5])

$$u_t = 2(u^{-1/2})_{xxx}, \tag{1.21}$$

which is known to be linearizable [6] (see also [2b]). Then (1.21) does not directly (i.e., without a transformation of variables) pass the Painlevé PDE test since it has an expansion of the form

$$u(x,t) = \phi^{-4/3}(x,t) \sum_{j=0}^{\infty} u_j(t)\phi^{j/3}(x,t), \tag{1.22}$$

with $\phi(x,t) = x + f(t)$, in the neighborhood of a noncharacteristic movable singularity manifold defined by $\phi = 0$ and so has movable cube roots (see Weiss [24] for details). If an equation has an expansion of the form

$$u(x,t) = \phi^{p/r}(x,t) \sum_{j=0}^{\infty} u_j(t)\phi^{j/r}(x,t), \tag{1.23}$$

where p and r are integers determined from the leading order analysis, then the equation is said to be "weak-Painlevé". However, as was pointed out earlier, the non linearizable equation $u_t = u_{xxx} + u^3 u_x$ is also weak-Painlevé.

II. SECOND AND THIRD ORDER QUASILINEAR PARTIAL DIFFERENTIAL EQUATIONS

An extended hodograph transformation comprises of the change of variables $u \to v_x = \phi(u(x,t))$ followed by a pure hodograph transformation, and therefore these transfomations are simply related.

We first consider the pure hodograph transformation in more detail.

Let

$$t = \tau, \quad x = \eta(\xi, \tau) \tag{2.1}$$

Then using (1.7),

$$\partial_x = \xi_x \partial_\xi + \tau_x \partial_\tau = u_x \partial_\xi, \tag{2.2a}$$

$$\partial_t = \xi_t \partial_\xi + \tau_t \partial_\tau = u_t \partial_\xi + \partial_\tau. \tag{2.2b}$$

Therefore the Jacobian of this transformation is u_x. Similarly for the inverse transformation (2.1) we have

$$\partial_\xi = x_\xi \partial_x + t_\xi \partial_t = \eta_\xi \partial_x, \tag{2.3a}$$

$$\partial_\tau = x_\tau \partial_x + t_\tau \partial_t = \eta_\tau \partial_x + \partial_t. \tag{2.3b}$$

Under a pure hodograph transformation, derivatives transform as follows

$$u_x = \eta_\xi^{-1}, \quad u_t = -\eta_\tau \eta_\xi^{-1}, \tag{2.4a}$$

$$u_{xx} = -\eta_{\xi\xi} \eta_\xi^{-3}, \quad u_{xxx} = -\eta_{\xi\xi\xi} \eta_\xi^{-4} + 3\eta_{\xi\xi}^2 \eta_\xi^{-5}, \tag{2.4b}$$

or inversely

$$\eta_\xi = u_x^{-1}, \quad \eta_\tau = -u_t u_x^{-1} \tag{2.5a}$$

$$\eta_{\xi\xi} = -u_{xx} u_x^{-3}, \quad \eta_{\xi\xi\xi} = -u_{xxx} u_x^{-4} + 3u_{xx}^2 u_x^{-5}, \tag{2.5b}$$

Therefore the linear partial differential equation

$$u_t = u_{xxx}, \tag{2.6}$$

under a pure hodograph transforms to

$$\eta_\tau = \eta_{\xi\xi\xi}\eta_\xi^{-3} - 3\eta_{\xi\xi}^2\eta_\xi^{-4}. \tag{2.7}$$

Note that if one applies a pure hodograph transformation to a partial differential equation in potential form (that is an equation which does <u>not</u> depend explicitly on the dependent variable) which also does not depend explicitly on the independent variables, then the resulting equation is also in potential form with no explicit dependence on the independent variables. Therefore, before applying a pure hodograph transformation to a given partial differential equation, we shall put the equation into canonical potential form.

We now consider second order quasilinear partial differential equations.

IIA. SECOND ORDER QUASILINEAR PARTIAL DIFFERENTIAL EQUATIONS

The most general second order, quasilinear partial differential equation of the form

$$u_t = g(u)u_{xx} + f(u,u_x), \tag{2.8}$$

with $dg/du \neq 0$, which may be transformed via an extended hodograph transformation to a semilinear partial differential equation of the form

$$S_\tau = S_{\xi\xi} + G(S, S_\xi), \qquad (2.9)$$

is given by

$$u_t = g(u)u_{xx} + \left(\frac{gg''}{g'} - \frac{g'}{2}\right)u_x^2 + b'(u)u_x, \qquad (2.10)$$

where $' \equiv d/du$, and $g(u)$ and $b(u)$ are arbitrary functions which are twice and once differentiable, respectively. Furthermore, equation (2.9) is equivalent to the equation

$$v_t = v_x^{-2}v_{xx} + H(v_x), \qquad (2.11)$$

which is transformed via a pure hodograph transformation to

$$\eta_\tau = \eta_{\xi\xi} - \eta_\xi H(\eta_\xi^{-1}). \qquad (2.12)$$

<u>Proof</u>

In equation (2.8) we make the transformation

$$\tau = t, \quad \xi = F(x,t), \quad \eta(\xi,\tau) = u(x,t),$$

then (2.8) becomes

$$u_t = g(u)F_x^2 u_{\xi\xi} + (gF_{xx} - F_t)u_\xi + f(u, u_\xi F_x).$$

Now choose F such that

$$gF_x^2 = 1, \quad \text{i.e., } F_x = g^{-1/2}, \tag{2.13a}$$

$$F_t = A(u,u_x), \tag{2.13b}$$

where $A(u,u_x)$ is such that the compatibility of (2.13) (i.e., $F_{xt} = F_{tx}$) implies (2.8). Therefore

$$-\frac{1}{3}g^{-3/2}g'u_t = A_u u_x + A_{u_x} u_{xx}, \tag{2.14}$$

where $A_u = \partial A/\partial u$, $A_{u_x} = \partial A/\partial u_x$; using (2.8)

$$-\frac{1}{3}g^{-1/2}g'u_{xx} + gf(u,u_x) = A_u u_x + A_{u_x} u_{xx}, \tag{2.15}$$

Equating coefficients of u_{xx} to zero in (2.15), it is seen that

$$A(u,u_x) = -\frac{1}{3}g^{-1/2}g'u_x + a(u), \tag{2.16}$$

where $a(u)$ is an arbitrary function. Also from (2.15)

$$A_u u_x = -\frac{1}{3}g^{-3/2}g'f(u,u_x). \tag{2.17}$$

Therefore, from equations (2.16) and (2.17) we find that

$$f(u,u_x) = \left(\frac{gg''}{g'} - \frac{g'}{2}\right)u_x^2 + b'(u)u_x, \tag{2.18}$$

where b(u) is an arbitrary function. Hence, it follows that the most general equation of the form (2.17) which is transformed via the extended hodograph transformation

$$\tau = t, \quad \xi = \int^x g^{-1/2}(u(x',t))dx'$$

into a semilinear partial differential equation has the form

$$u_t = g(u)u_{xx} + \left(\frac{gg''}{g'} - \frac{g'}{2}\right)u_x^2 + b'(u)u_x. \tag{2.19}$$

We now wish to transform (2.19) into semilinear form. Our algorithm is to put (2.19) into a canonical (potential form) partial differential equation and then apply a pure hodograph transformation to convert the canonical equation into a semilinear equation. In (2.19) we make the transformation $g(u) = v_x^{-2}$ and obtain

$$v_t = v_x^{-2}v_{xx} + H(v_x), \tag{2.20}$$

where H is expressible in terms of b. Equation (2.20) is the canonical equation (since all equations of the form (2.19) are equivalent to (2.20)). It is essential that the ratio of the coefficients of v_{xx} and v_t in (2.20) is v_x^{-2} in order that the quasilinear equation is transformed into a semilinear one via a pure hodograph transformation.

Finally, applying a pure hodograph transformation to (2.20), we obtain

$$\eta_\tau = \eta_{\xi\xi} - \eta_\xi H(\eta^{-1}), \tag{2.21}$$

as required.

Therefore in summary, in order to determine which equations of the form

$$u_t = g(u)u_{xx} + f(u,u_x), \tag{2.22}$$

where $\frac{dg}{du} \neq 0$ and $f(u,u_x)$ is a rational function of u and u_x, are linearizable, it is sufficient to consider the canonical equation

$$v_t = v_x^{-2}v_{xx} + H(v_x), \tag{2.23}$$

where $H(v_x)$ is a rational function of v_x. Applying a pure hodograph transformation to (2.23) yields

$$\eta_\tau = \eta_{\xi\xi} - \eta_\xi H(\eta_\xi^{-1}) .$$

This can be put into non-potential form by making the transformation $w = \eta_\xi$, hence

$$w_\tau = w_{\xi\xi} + h(w)w_\xi , \tag{2.24}$$

where

$$h(w) = -\frac{d}{dw}[wH(1/w)]. \qquad (2.25)$$

It is shown in Appendix A that equation (2.24) can pass the Painlevé tests if and only if

$$h(w) = 2\alpha w + \beta,$$

where α and β are constants. Hence from (2.25),

$$H(w) = \alpha w^{-1} + \beta. \qquad (2.26)$$

Therefore, this suggests that the most general partial differential equation of the form (2.22) which is linearizable is equivalent to the equation

$$u_t = (u^{-2}u_x)_x + \alpha u^{-2}u_x. \qquad (2.27)$$

We use the word "suggests" because we are aware that the Painlevé tests have not yet been proven, though there is considerable evidence suggesting their validity. This completes the "proof" of the result first obtained by Fokas and Yortsos [4]. However, the method in the present paper is somewhat simpler than that used in [4] and is easily generalizable to higher order quasilinear partial differential equations.

IIB. THIRD ORDER QUASILINEAR PARTIAL DIFFERENTIAL EQUATIONS

Proposition 2.2

The most general third order, quasilinear partial differential equation of the form

$$u_t = g(u)u_{xxx} + f(u,u_x,u_{xx}), \quad \frac{dg}{du} \neq 0, \tag{2.28}$$

which may be transformed via an extended hodograph transformation to a semilinear partial differential equation of the form

$$S_\tau = S_{\xi\xi\xi} + G(S,S_\xi,S_{\xi\xi}), \tag{2.29}$$

is given by

$$u_t = g(u)u_{xxx} + B_u(u,u_x)u_x + B_{u_x}(u,u_x)u_{xx}$$

$$+ \left(\frac{g''}{g'} - \frac{4g'}{3g}\right)B(u,u_x)u_x + \left(\frac{gg''}{g'} - \frac{g'}{3}\right)u_x u_{xx}, \tag{2.30}$$

where $B_u := \partial B/\partial u$, $B_{u_x} := \partial B/\partial u_x$, prime denotes derivative with respect to u, and g(u) and $B(u,u_x)$ are arbitrary functions. Furthermore, equation (2.29) is equivalent to the equation

$$v_t = v_x^{-3}v_{xxx} + H(v_x,v_{xx}), \tag{2.31}$$

which is transformed via a pure hodograph transformation to

$$\eta_\tau = \eta_{\xi\xi\xi} - \eta_\xi H(\eta_\xi^{-1}, -\eta_{\xi\xi}\eta_\xi^{-3}). \tag{2.32}$$

<u>Proof</u>.

In equation (2.28) we make the transformation

$$\tau = t, \; \xi = F(x,t), \; \eta(\xi,\tau) = u(x,t),$$

then (2.28) becomes

$$\eta_\tau = g(u)F_x^3\eta_{\xi\xi\xi} + 3gF_xF_{xx}\eta_{\xi\xi} + (gF_{xxx} - F_t)\eta_\xi$$

$$+ f(\eta, \eta_\xi F_x, F_x^2\eta_{\xi\xi} + F_{xx}\eta_\xi^2).$$

Now choose F such that

$$gF_x^3 = 1, \text{ i.e., } F_x = g^{-1/3}, \tag{2.33a}$$

$$F_t = A(u,u_x,u_{xx}), \tag{2.33b}$$

where $A(u,u_x,u_{xx})$ is such that the compatibility of (2.33) (i.e., $F_{xt} = F_{tx}$) implies (2.28). Therefore

$$-\frac{1}{3}g^{-4/3}g'u_t = A_u u_x + A_{u_x}u_{xx} + A_{u_{xx}}u_{xxx},$$

or using (2.28)

$$- \tfrac{1}{3}g^{-1/3}g'u_{xxx} - \tfrac{1}{3}g^{-4/3}g'f(u,u_x,u_{xx})$$

$$= A_u u_x + A_{u_x} u_{xx} + A_{u_{xx}} u_{xxx} \qquad (2.34)$$

By collecting terms and equating the coefficient of u_{xxx} to zero in (2.34), it is seen that

$$A(u,u_x,u_{xx}) = -\tfrac{1}{3}g^{-1/3}g'u_{xx} + a(u,u_x), \qquad (2.35)$$

where $a(u,u_x)$ is an arbitrary function. Also

$$A_u u_x + A_{u_x} u_{xx} = -\tfrac{1}{3}g^{-4/3}g'u_x f(u,u_x,u_{xx}). \qquad (2.36)$$

Therefore, from equations (2.35) and (2.36) we find that

$$f(u,u_x,u_{xx}) = -3(g^{4/3}/g')[a_u u_x + a_u u_{xx}] + (\frac{gg''}{g'} - \frac{g'}{3})u_x^2 u_{xx},$$

$$= B_u(u,u_x)u_x + B_{u_x}(u,u_x)u_{xx} + (\frac{g''}{g'} - \frac{4g'}{3g})B(u,u_x)u_x$$

$$+ (\frac{gg''}{g'} - \frac{g'}{3})u_x^2 u_{xx}, \qquad (2.37)$$

where $B(u,u_x) := -3(g^{4/3}/g')a(u,u_x)$. Hence, it follows that the most general equation of the form (2.36) which is transformed via the extended hodograph transformation

$$\tau = t, \; \xi = \int^x g^{-1/3}(u(x',t))dx'$$

into a semilinear partial differential equation has the form

$$u_t = g(u)u_{xxx} + B_u(u,u_x)u_x + B_{u_x}(u,u_x)u_{xx}$$

$$+ \left(\frac{g''}{g'} - \frac{4g'}{3g}\right)B(u,u_x)u_x + \left(\frac{gg''}{g'} - \frac{g'}{3}\right)u_x u_{xx}, \tag{2.38}$$

In (2.38), make the transformation $g(u) = v_x^{-3}$, then we obtain

$$v_t = v_x^{-3}v_{xxx} + H(v_x,v_{xx}), \tag{2.39}$$

where $H(v_x,v_{xx})$ is expressible in terms of $B(u,u_x)$ and $g(u)$. Therefore, (2.39) is the canonical equation (again, since all equations of the form (2.38) are equivalent to (2.39)). Finally, applying a pure hodograph transformation to (2.39), we obtain

$$\eta_\tau = \eta_{\xi\xi\xi} - \eta_\xi H(\eta_\xi^{-1}\ ,\ -\eta_{\xi\xi}\eta_\xi^{-3})\ , \tag{2.40}$$

as required.

Thus proposition 2.2 provides an algorithmic method of transforming the quasilinear partial differential equation

$$u_t = g(u)u_{xxx} + f(u,u_x,u_{xx}) \tag{2.41a}$$

where

$$f(u,u_x,u_{xx}) = g(u)u_{xxx} + B_u(u,u_x)u_x + B_{u_x}(u,u_x)u_{xx}$$

$$+ \left(\frac{g''}{g'} - \frac{4g'}{3g}\right)B(u,u_x)u_x + \left(\frac{gg''}{g'} - \frac{g'}{3}\right)u_x u_{xx}, \qquad (2.41b)$$

into a semilinear partial differential equation; i.e.

1. Put equation (2.41) into the potential canonical form by making the transformation $v_x = g^{-1/3}(u)$; hence we obtain

$$v_t = v_x^{-3} v_{xxx} + H(v_x, v_{xx}) \qquad (2.42)$$

2. Apply a pure hodograph transformation to equation (2.42); hence we obtain

$$\eta_\tau = \eta_{\xi\xi\xi} - \eta_\xi H(\eta_\xi^{-1}, -\eta_{\xi\xi}\eta_\xi^{-3}). \qquad (2.43)$$

3. The resulting partial differential equation will be in potential form and usually one first puts the equation into nonpotential form by making the transformation $w = \eta_\xi$. Furthermore, if the resulting semilinear partial differential equation is linearizable, then it can be expected to be equivalent to one of the six partial differential equations given by Svinolupov, Sokolov and Yamilov [14], which are listed in §1 (equations (1.11)-(1.16)). Therefore it may be necessary to seek a change of dependent variables $w = \phi(Q)$ and write the resulting equation in non-potential form. An alternative approach is to apply the Painlevé tests directly

on the semilinear equation, provided that the nonlinear evolution equation is in rational form (i.e., H in (2.43) is a rational function of its arguments).

There are two remarks we wish to make about the above procedure.

1. It is important to first put equation (2.41) into canonical form by making the transformation $v_x = g^{-1/3}(u)$ before applying the pure hodograph transformation (otherwise the partial differential equation will remain quasilinear). To demonstrate this, consider the Harry-Dym equation

$$u_t = (u^{1/2})_{xxx}. \tag{2.44}$$

First put (2.44) into potential form by letting $v_x = u$, then

$$v_t = (v_x^{-1/2})_{xx}. \tag{2.45}$$

Applying a pure hodograph transformation to (2.45) gives

$$\eta_\tau = (\eta_\xi^{-1/2})_{\xi\xi},$$

which is just the same equation (i.e., the potential Harry-Dym equation is invariant under a pure hodograph transformation).

2. If the quasilinear partial differential equation is not in the special

form (2.41) then the transformation $v_x = g^{-1/3}(u)$ yields either a higher order or nonlocal partial differential equation. For example, consider the partial differential equation

$$u_t = u^{-3}u_{xxx}. \tag{2.46}$$

Then after making the transformation $v_x = u$ we obtain

$$v_{xt} = v_x^{-3}v_{xxxx},$$

or

$$v_t = v_x^{-3}v_{xxx} + 3\int^x v_x^{-4}v_{xx}v_{xxx}.$$

By considering several examples, we shall now demonstrate how the procedure developed above can be applied to determining whether a given third order quasilinear partial differential equation might be linearizable. In tnese examples, we apply the Painlevé tests to the semilinear equaticn to determine necessary conditions for the equation to be possibly linearizable. Furthermore, we show that when these condition are satisfied, then the equation is equivalent to a linearizable equation by exhibiting the requisite transformation. Since we are usinj the Painlevé tests in these examples to exclude several possibilit es, when we conclude below that an equation is "nonlinearizable" (tecause the above conditions are not satisfied), we mean "nonlinearizab e, subject to the validity of the Painlevé tests", i.e., in these cases the equation is "probably nonlinearizable."

Example 2.3

In this example we determine for which values of the constant α is the equation

$$u_t = u^3 u_{xxx} + \alpha u^2 u_x u_{xx}, \tag{2.47}$$

linearizable. Equation (2.47) was considered by Kawamoto [9], where we note that if $\alpha = 0$, then (2.47) is equivalent to the Harry-Dym equation $v_t + 2(v^{-1/2})_{xxx} = 0$ (set $u = v^{-1/2}$). In order to set (2.47) in canonical form we make the transformation $v_x = 1/u$, hence

$$v_t = v_x^{-3} v_{xxx} - \frac{1}{2}(\alpha + 3) v_x^{-4} v_{xx}^{\;2}. \tag{2.48}$$

Applying a pure hodograph transformation to (2.48) gives

$$\eta_\tau = \eta_{\xi\xi\xi} + \frac{1}{2}(\alpha - 3)\eta_{\xi\xi}^2 \; \eta_\xi^{-1} \tag{2.49}$$

We now apply a sequence of transformations to (2.49). First we put (2.49) into non-potential form by letting $w = \eta_\xi$, hence

$$w_\tau = w_{\xi\xi\xi} + \frac{1}{2}(\alpha - 3)(w_\xi^2/w)_\xi . \tag{2.50}$$

Then, in order to determine whether (2.50) is equivalent to one of the six linearizable equations given by Svinolupov, Sokolov and Yamilov [14] (equations (1.11)-(1.16)), we let $Q = \ln w$, hence

$$Q_\tau = Q_{\xi\xi\xi} + \alpha Q_\xi Q_{\xi\xi} + \frac{1}{2}(\alpha - 1)\, Q_\xi^3. \tag{2.51}$$

Finally, putting (2.51) into non-potential form

$$q_\tau = q_{\xi\xi\xi} + \alpha\,(q q_{\xi\xi} + q_\xi^2) + \frac{3}{2}(\alpha - 1) q^2 q_\xi . \tag{2.52}$$

(additionally it is simpler to apply Painlevé analysis on equation (2.52) rather than on (2.50)). It is shown in Appendix B that equation (2.52) can pass the Painlevé tests only if either $\alpha = 0$, $\alpha = 3/2$ or $\alpha = 3$. If $\alpha = 0$, then (2.52) is the MKdV equation, which is known to be linearizable [22]. If $\alpha = 3/2$ or $\alpha = 3$ (after rescaling q), then (2.52) is the second equation in the Burgers' hierarchy

$$q_\tau = q_{\xi\xi\xi} + \frac{3}{2}(q q_{\xi\xi} + q_\xi^2) + \frac{3}{4}\, q^2 q_\xi \tag{2.53}$$

(Olver [25]), which is reduced by the Cole-Hopf transformation

$$q_\tau = 2(\ln u)_\xi = 2u_\xi / u,$$

to the linear partial differential equation

$$u_\tau = u_{\xi\xi\xi}$$

(i.e., equation (2.53) is equivalent to (1.11)). Therefore we conclude that equation (2.47) is linearizable only for these three values of α.

Example 2.4

Consider the equation

$$u_t = [u_x(1 + u^2)^{-3/2}]_{xx} + 2\alpha u_x(1 + u^2)^{-3/2}, \tag{2.54}$$

where α is a constant. Note that if $\alpha = 0$, then (2.54) is an equation which was shown to be linearizable by Wadati, Konno and Ichikawa [6a]. To put (2.54) into canonical form we make the transformation $v_x = (1 + u^2)^{1/2}$, hence we obtain

$$v_t = v_x^{-3}v_{xxx} - \tfrac{3}{2}v_x^{-4}v_{xx}^2[(1 - 2v_x^2)/(1 - v_x^2)] - \alpha v_x^{-2}. \tag{2.55}$$

Applying a pure hodograph transformation to (2.55) gives

$$\eta_\tau = \eta_{\xi\xi\xi} + \alpha\eta_\xi^3 + \frac{3}{2}\,\frac{\eta_\xi\eta_{\xi\xi}}{1 - \eta_\xi^2}$$

which has the non-potential form ($w = \eta_\xi$)

$$w_\tau = w_{\xi\xi\xi} + 3\alpha w^2 w_\xi + \tfrac{3}{2}[ww_\xi^2/(1 - w^2)]_\xi . \tag{2.56}$$

Equation (2.56) is equivalent to equation (1.19) (after rescaling the variables), which is known as the 'deformed MKdV' equation [17] or 'modified MKdV' equation [18] and as shown in §1, is equivalent to the CDF equation (1.14) via the transformation $w = \cosh(q/2)$. Hence we obtain

$$q_\tau = q_{\xi\xi\xi} - \frac{1}{8}q_\xi^3 + 3\alpha\sinh^2(q/2)q_\xi ,$$

or

$$q_\tau = q_{\xi\xi\xi} - \frac{1}{8}q_\xi^3 + \frac{3}{4}\alpha(e^q - 2 + e^{-q})q_\xi . \tag{2.57}$$

If $\alpha = 0$ then (2.57) is the potential MKdV equation, while if $\alpha \neq 0$, then (2.57) is the CDF equation. Therefore equation (2.54) is linearizable for all values of α.

Example 2.5

Consider the equation

$$u_t + 2(u^{-1/2})_{xxx} + f'(u^{1/2})u_x = 0, \tag{2.58}$$

where f is a rational function and prime denotes differentiation with respect to the argument. The objective is to determine for which choices of f is (2.58) linearizable (note that if $f' \equiv 0$, then (2.58) is the Harry-Dym equation). First we put (2.58) into canonical form by making the transformation $v_x = u^{1/2}$; hence we obtain

$$v_t = v_x^{-3}v_{xxx} - \frac{3}{2}v_x^{-4}v_{xx}^2 - f(v_x). \tag{2.59}$$

Applying a pure hodograph transformation to (2.59) gives

$$\eta_\tau = \eta_{\xi\xi\xi} - \frac{3}{2}\eta_{\xi\xi}^2\eta_\xi^{-1} - \eta_\xi f(\eta_\xi^{-1}),$$

which has the non-potential form ($w = \eta_\xi$)

$$w_\tau = w_{\xi\xi\xi} - \frac{3}{2}(w_\xi^2/w)_\xi - g'(w)w_\xi, \tag{2.60}$$

where $g(w) := w\, f(1/w)$. It can be shown that (2.60) can pass the Painlevé tests if and only if

$$g(w) = \alpha w^3 + \beta w + \gamma w^{-1}, \tag{2.61}$$

hence

$$f(w) = \alpha w^{-2} + \beta + \gamma w^2, \tag{2.62}$$

where α, β and γ are arbitrary constants (see Appendix C for details). Note that equation (2.60) with $g(w)$ as given by (2.61) is just equation (1.18), which is equivalent to the CDF equation (1.14) if either $\alpha \neq 0$ or $\gamma \neq 0$ (let $w = e^{u/2}$); if $\alpha = \gamma = 0$ and $q = w_\xi/w$, then q satisfies the MKdV equation, hence equation (2.60) with $g(w)$ as given by (2.61) is linearizable. Therefore, we conclude that the most general equation of the form (2.58) which is linearizable is

$$u_t + 2(u^{-1/2})_{xxx} + 2\gamma u^{1/2}u_x - 2\alpha u^{-3/2}u_x = 0. \tag{2.63}$$

III. HIGHER ORDER QUASILINEAR PARTIAL DIFFERENTIAL EQUATIONS.

The method developed for second and third order quasilinear partial differential equations can easily be extended to higher order equations.

Proposition 3.1

The most general quasilinear partial differential equation of the form

$$u_t = g(u)u_{nx} + f(u,u_x,\ldots,u_{(n-1)x}), \quad u_{nx} \equiv \frac{\partial^n u}{\partial x^n}, \quad \frac{dg}{du} \neq 0 \tag{3.1}$$

which may be transformed via an extended hodograph transformation to a semilinear partial differential equation of the form

$$S_\tau = S_{n\xi} + G(S,S_\xi,\ldots,S_{(n-1)\xi}), \tag{3.2}$$

is given by

$$u_t = g(u)u_{nx} + \left(\frac{g''}{g'} - \frac{n+1}{n}\frac{g'}{g}\right)B(u,u_x,\ldots,u_{(n-2)x})u_x + B_u u_x + \sum_{r=2}^{n-1} B_{u_{(r-1)x}} u_{rx} + \left(\frac{gg''}{g'} - \frac{g'}{n}\right)u_x u_{(n-1)x}, \tag{3.3}$$

where prime denotes derivative with respect to u, and g(u) and $B(u,u_x,\ldots,u_{(n-2)x})$ are arbitrary functions. Furthermore, equation (3.2) is equivalent to the equation

$$v_t = v_x^{-n} v_{nx} + H(v_x,v_{xx},\ldots,v_{(n-1)x}), \tag{3.4}$$

which is transformed via a pure hodograph transformation to

$$\eta_t = \eta_{n\xi} + \tilde{H}(\eta_\xi,\eta_{\xi\xi},\ldots,\eta_{(n-1)\xi}). \tag{3.5}$$

Proof

The proof is analogous to those for Propositions 2.1 and 2.2 above and so we shall only sketch an outline. In equation (3.1) we make the transformation

$$\tau = t,\ \xi = F(x,t),\ \eta(\xi,\tau) = u(x,t),$$

and choose F such that

$$g F_x^n = 1, \text{ i.e., } F_x = g^{-1/n}, \tag{3.6}$$

$$F_t = A(u,u_x,\ldots,u_{(n-1)x}), \tag{3.7}$$

where $A(u,u_x,\ldots,u_{(n-1)x})$ is such that the compatibility of (3.6) $F_{xt} = F_{tx}$) implies (3.1). Therefore

$$-\frac{1}{n}g^{-1/n}g'u_{xxx} - \frac{1}{n}g^{-(n+1)/n}g'f(u,u_x,\ldots,u_{(n-1)x})$$
$$= A_u u_x + \sum_{r=2}^{n} A_{u_{(r-1)x}} u_{rx} . \tag{3.8}$$

Hence

$$A(u,u_x,\ldots,u_{(n-1)x}) = -\frac{1}{n}g^{-(n+1)/n}g'[gu_{(n-1)x}$$
$$+ B(u,u_x,\ldots,u_{(n-2)x})], \tag{3.9}$$

where $B(u,u_x,\ldots,u_{(n-2)x})$ is an arbitrary function. Therefore, from equation (3.9) we find that

$$f(u,u_x,\ldots,u_{(n-1)x}) = \left(\frac{g''}{g'} - \frac{n+1}{n}\frac{g'}{g}\right)B(u,u_x,\ldots,u_{(n-2)x})u_x$$

$$+ B_u u_x + \sum_{r=2}^{n-1} B_{u_{(r-1)x}} u_{rx} + \left(\frac{gg''}{g'} - \frac{g'}{n}\right)u_x u_{(n-1)x}, \tag{3.10}$$

Hence, it follows that the most general equation of the form (3.10) which is transformed via an extended hodograph transformation into a semilinear partial differential equation has the form (3.3) as required. Equation (3.4) is obtained from (3.3) by making the transformation $v_x = g^{-1/n}(u)$, where $H(v_x,\ldots,v_{(n-1)x})$ is expressible in terms of $B(u,u_x,\ldots,u_{(n-2)x})$ and $g(u)$ and therefore is the canonical equation. Finally, equation (3.5) is obtained by applying a pure hodograph transformation to (3.12).

Proposition 3.1 provides an algorithmic method of transforming the general quasilinear partial differential equation

$$u_t = g(u)u_{nx} + f(u,u_x,\ldots,u_{(n-1)x}) \tag{3.11a}$$

where

$$f(u,u_x,\ldots,u_{(n-1)x}) = \left(\frac{g''}{g'} - \frac{n+1}{n}\frac{g'}{g}\right)B(u,u_x,\ldots,u_{(n-2)x})u_x$$

$$+ B_u u_x + \sum_{r=2}^{n-1} B_{u_{(r-1)x}} u_{rx} + \left(\frac{gg''}{g'} - \frac{g'}{n}\right)u_x u_{(n-1)x}, \tag{3.11b}$$

into a semilinear partial differential equation as follows:

1. Put equation (3.11) into the potential canonical form by making the transformation $v_x = g^{-1/n}(u)$; hence we obtain

$$v_t = v_x^{-n} v_{nx} + H(v_x, v_{xx}, \ldots, v_{(n-1)x}). \tag{3.12}$$

2. Apply a pure hodograph transformation to equation (3.12); hence we obtain

$$\eta_t = \eta_{n\xi} + \tilde{H}(\eta_\xi, \eta_{\xi\xi}, \ldots, \eta_{(n-1)\xi}). \tag{3.13}$$

3. The resulting partial differential equation will be in potential form and usually one first puts the equation into nonpotential form by making the transformation $w = \eta_\xi$. It may also be convenient to seek a change of dependent variables $w = \phi\,(Q)$ (and then write the resulting equation in non-potential form if necessary) and then apply the Painlevé tests to the semilinear equation to determine if it is possibly linearizable. (For fourth and higher order semilinear partial differential equations, there is, at present, no equivalent theorem to the one given by Svinolupov, Sokolov and Yamilov [14] for third order equations.)

Example 3.1

In this example we consider the equation

$$u_t = u^{5/2} u_{5x}, \tag{3.14}$$

which was shown by Konopelchenko and Dubrovsky [26] to be the compatibility condition of the linear operators

$$L = u^{3/2} \partial_x^3,$$

$$M = 9u^{5/2}\partial_x^5 + \frac{45}{2} u^{3/2} u_x \partial_x^4 + 15u^{3/2} u_{xx} \partial_x^3 + \partial_t,$$

where $\partial_x \equiv \partial/\partial x$, $\partial_t \equiv \partial/\partial t$ (i.e., $LM - ML = 0$ if and only if u satisfies (3.13)).

We first put (3.14) into canonical form by making the transformation $v_x = u^{-1/2}$, hence we obtain

$$v_t = v_x^{-5} v_{5x} - 10v_x^{-6}(v_{2x}v_{4x} + v_{3x}^2) + 60v_x^{-7} v_{2x}^2 v_{3x} - 45v_x^{-8} v_{xx}^4.$$

Applying a pure hodograph transformation to the above equation we obtain

$$v_t = v_{5\xi} - 5v_{2\xi} v_{4\xi} v_\xi^{-1} + 5v_{2\xi}^2 v_{3\xi} v_\xi^{-2}, \tag{3.15}$$

which has the nonpotential form

$$w_\tau = w_{5\xi} - 5w^{-1}(w_\xi w_{4\xi} + w_{2\xi} w_{3\xi}) + 10w^{-2}(w_\xi^2 w_{3\xi} + w_\xi w_{2\xi}^2)$$

$$- 10w^{-3} w_\xi^3 w_{2\xi}. \tag{3.16}$$

We now let $Q = \ln w$, hence

$$Q_\tau = Q_{5\xi} + 5Q_{2\xi}Q_{3\xi} - 5Q_\xi Q_{2\xi}^2 - 5Q_\xi^2 Q_{3\xi} + Q_\xi^5,$$

which has the nonpotential form

$$q_\tau = q_{5\xi} + 5q_\xi q_{3\xi} + 5q_{2\xi}^2 - 5q_\xi^3 - 20qq_\xi q_{2\xi} - 5q^2 q_{3\xi} + 5q^4 q_\xi. \tag{3.17}$$

Equation (3.17) can be transformed into two linearizable fifth order equations. Fordy and Gibbons [27] show that if q satisfies (3.17) and u and v are defined by the Miura transformations

$$u = -q_\xi - q^2, \quad v = q_\xi - \tfrac{1}{2}q^2, \tag{3.18}$$

then u and v respectively satisfy the Sawada-Kotera equation [28] (sometimes referred to as the Caudrey-Dodd-Gibbon equation [29])

$$u_\tau = u_{5\xi} + 5uu_{3\xi} + 5u_\xi u_{2\xi} + 5u^2 u_\xi, \tag{3.19}$$

and the Kaup equation [30] (sometimes referred to as the Kuperschmidt equation, cf. [27])

$$v_\tau = v_{5\xi} + 10vv_{3\xi} + 25v_\xi v_{2\xi} + 20v^2 v_\xi. \tag{3.20}$$

Both equations (3.19) and (3.20) are known to be linearizable, see [31] and [30] respectively. This shows that equation (3.14) is the quasilinear analogue of equation (3.17), which is linearizable and so (3.14) should not be regarded as a "new" linearizable fifth order equation.

Example 3.2

The second equation in the Harry-Dym hierarchy is given by

$$u_t = u^3[u(uu_{xx} - \tfrac{1}{2}u_x^2)]_{xxx}$$

$$= u^5u_{5x} + 5u^4(u_xu_{4x} + u_{xx}u_{3x}) + \tfrac{5}{2}u^3u_x^2u_{3x} \tag{3.21}$$

(see [2b] or [32]). We first put (3.21) into canonical form by making the transformation $v_x = u^{-1}$, hence we obtain

$$v_t = v_x^{-5}v_{5x} - \tfrac{5}{2}v_x^{-6}(4v_{2x}v_{4x} - 3v_{3x}^2)$$

$$+ \frac{105}{2}v_x^{-7}v_{2x}^2v_{3x} - \frac{315}{8}v_x^{-8}v_{xx}^4. \tag{3.22}$$

Applying a pure hodograph transformation to (3.22) gives

$$\eta_\tau = \eta_{5\xi} - 5\eta_{2\xi}\eta_{4\xi}\eta_\xi^{-1} - \frac{5}{2}\eta_{\xi\xi}^2\eta_\xi^{-1}$$

$$+ \frac{25}{2}\eta_{2\xi}^2\eta_{3\xi}\eta_\xi^{-2} - \frac{45}{8}\eta_{\xi\xi}^4\eta_\xi^{-3} \tag{3.23}$$

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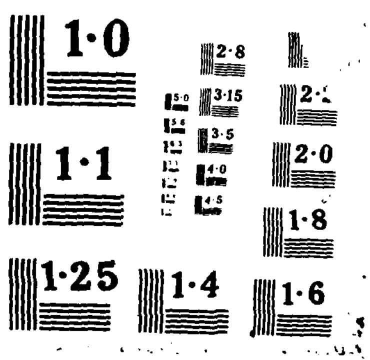
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which has the nonpotential form

$$w_\tau = w_{5\xi} - 5w^{-1}(w_\xi w_{4\xi} + 2w_{2\xi}w_{3\xi}) + \frac{35}{2} w^{-2}w_\xi^2 w_{3\xi}$$

$$+ \frac{55}{2} w_\xi w_{2\xi}^2 w^{-2} - \frac{95}{2} w^{-3} w_\xi^3 w_{2\xi} + \frac{135}{8} w_\xi^5 w^{-4}. \tag{3.24}$$

As in Example 3.1 above, we now let $Q = \ln w$, hence

$$Q_\tau = Q_{5\xi} - \frac{5}{2}(Q_\xi Q_{2\xi}^2 + Q_\xi^2 Q_{3\xi}) + \frac{3}{8}Q_\xi^5,$$

which has the nonpotential form

$$q_\tau = q_{5\xi} - \frac{5}{2}q_\xi^3 - 10qq_\xi q_{2\xi} - \frac{5}{2}q^2 q_{3\xi} + \frac{15}{8}q^4 q_\xi. \tag{3.25}$$

Equation (3.25) is the second equation in the MKdV hierarchy (see[25]).

This provides further evidence of the close relationship between the Harry-Dym equation and the MKdV equation. It is well known that the inverse scattering schemes for the MKdV equation and the Harry-Dym equation are related through a sequence of gauge transformations which also involve an interchange of independent and dependent variables [34] (see also [35]). Since the recursion operator for the Harry-Dym equation is well known (cf. [2b], [32]), then using a theorem due to Fokas and Fuchssteiner [36], it can be shown that these recursion operators (or hereditary symmetries in the terminology of [36]) are related by a Bäcklund transformation.

IV. DISCUSSION

In this paper we have discussed the relationship between quasilinear and semilinear partial differential equations. In particular, an algorithmic procedure was developed for finding the quasilinear (semilinear) analogue of a given semilinear (quasilinear) equation (if it exists). Furthermore, the associated quasilinear (semilinear) equation is unique up to equivalence. This procedure provides a simple algorithmic method for determining whether a given quasilinear partial differential equation might be linearizable. Consequently, several quasilinear partial differential equations which might appear initially to be "new" linearizable equations are actually equivalent to the quasilinear analogue of a semilinear equation which is known to be integrable.

For example, Abellanas and Galindo [37] showed that the quasilinear equation

$$u_t = (\alpha u^2 + 2\beta u + \gamma)^{3/2} u_{xxx}, \tag{4.1}$$

where α, β, γ are constants, possesses a bihamiltonian structure and hence an infinite number of nontrivial conservation laws. Note that equation (4.1) contains as special cases both the Harry-Dym equation

$$u_t = u^3 u_{xxx}, \tag{4.2}$$

and an equation considered by Bruschi and Ragnisco [38]

$$u_t = u^{3/2} u_{xxx}, \tag{4.3}$$

Applying the method developed in the present paper shows that (4.1) is transformed into either the MKdV equation (if $\alpha \neq 0$) or the linear equation $\eta_\tau = \eta_{\xi\xi\xi}$ (if $\alpha = 0$ and $\beta \neq 0$). (Bruschi and Ragnisco [38] showed that (4.3) can be transformed via an extended hodograph transformation to the linear equation.)

In two recent papers, Mikhailov and Shabat [39] have determined necessary conditions for the existence of nontrivial conservation laws for systems of equations of the form

$$\underline{u}_t = A(\underline{u})\underline{u}_{xx} + \underline{f}(\underline{u},\underline{u}_x), \tag{4.4}$$

where

$$\underline{u} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad A(\underline{u}) = \begin{pmatrix} a(u,v) & b(u,v) \\ c(u,v) & d(u,v) \end{pmatrix},$$

$$\underline{f}(\underline{u},\underline{u}_x) = \begin{pmatrix} f(u,v,u_x,v_x) \\ g(u,v,u_x,v_x) \end{pmatrix}.$$

(This is analogous to the work of Svinolupov, Sokolov and Yamilov [14] who also used the existence of nontrivial conservation laws as the criterion in their determination of which third order semilinear equations are linearizable.) In order to determine their necessary conditions, Mikhailov and Shabat [39] first transformed the quasilinear equation (4.4) into the semilinear canonical form

$$\underline{\eta}_\tau = \sigma_3 \underline{\eta}_{\xi\xi} + H(\underline{\eta}, \underline{\eta}_\xi), \tag{4.5}$$

where

$$\underline{\eta} = \begin{pmatrix} \eta \\ \theta \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$H(\underline{\eta}, \underline{\eta}_\xi) = \begin{pmatrix} h(\eta, \theta, \eta_\xi, \theta_\xi) \\ k(\eta, \theta, \eta_\xi, \theta_\xi) \end{pmatrix}. \tag{4.6}$$

This transformation was achieved by first transforming (4.4) into the form

$$\underline{U}_t = g(\underline{U})\sigma_3 \underline{U}_{xx} + \underline{F}(\underline{U}, \underline{U}_x), \tag{4.7}$$

where

$$\underline{U} = \begin{pmatrix} U \\ V \end{pmatrix}, \quad \underline{F}(\underline{U}, \underline{U}_x) = \begin{pmatrix} F(U, V, U_x, V_x) \\ G(U, V, U_x, V_x) \end{pmatrix}$$

(so equations (4.4) and (4.7) are equivalent), and then applying an extended hodograph transformation to (4.7).

We note that it would be useful to extend the method outlined in earlier sections to quasilinear nonlinear evolution equations in two spatial and one temporal dimensions. Due to the presence of more independent variables, there is more flexibility in the hodograph transformation.

Finally, we make a remark regarding the application of the Painlevé tests. These tests have proved to be a useful criterion for the indentification of linearizable (semilinear) partial differential equations; however, there is one major restriction in their application. Since the Painlevé tests require that a linearizable partial differential equation possesses the Painlevé property possibly after a change of variables, then one may first have to make a change of variables before applying the tests. An open question is: Which transformations are allowable in the application of the Painlevé tests? (i.e., which transformations does one have to check?). We believe that by using pure hodograph transformations and the notion of equivalence, the answer to this question might be found.

APPENDIX A

In this appendix we show that the partial differential equation

$$u_t = u_{xx} + h(u)u_x, \tag{A.1}$$

where $h(u)$ is a rational function of u can pass the Painlevé tests if and only if $h(u)$ is a linear function of u. In (A.1) consider the traveling wave solution $u(x,t) = u(z)$, $z = x-ct$, where c is a constant. Then $u(z)$ satisfies

$$u'' + h(u)u' + cu' = 0. \tag{A.2}$$

Integrating yields

$$u' + H(u) + cu = A, \tag{A.3}$$

where $\frac{dH}{du} = h(u)$ and A is a constant. It is known that the only equation of the form

$$u' = R(u),$$

where $R(u)$ is a rational function of u, which is of Painlevé type is the Riccati equation

$$u' = \alpha_2 u^2 + \alpha_1 u + \alpha_0,$$

where α_2, α_1 and α_0 are constants (see Hille [40] or Ince [41] for a proof). Therefore (A.3) is of Painlevé type if and only if H(u) is a quadratic function of u, so necessarily

$$h(u) = \alpha u + \beta, \tag{A.4}$$

where α and β are constants. If h(u) has the special form (A.4), then equation (A.1) is either (i) equivalent to Burgers' equation if $\alpha \neq 0$, or (ii) a linear equation if $\alpha = 0$. Hence (A.1) can pass the Painlevé tests if and only if h(u) is a linear function of u, as required.

APPENDIX B

In this appendix we show that the partial differential equation

$$q_t = q_{xxx} + (qq_{xx} + q_x^2) + \frac{3}{2}(\alpha - 1)q^2 q_x, \tag{B.1}$$

where α is a constant, can pass the Painlevé tests if and only if α takes one of the three values 0, 3/2, 3. We first note that if $\alpha = 0$ then (B.1) is the MKdV equation, which is known to be linearizable [15] and pass the Painlevé PDE test [22]. Now we shall assume that $\alpha \neq 0$ and we consider the time-independent solution $q(x,t) = y(x)$ of (B.1), then $y(x)$ satisfies

$$y''' + \alpha[yy'' + (y')^2] + \frac{3}{2}(\alpha - 1)y^2y' = 0. \tag{B.2}$$

which can be integrated once, yielding

$$y'' + \alpha yy' + \frac{1}{2}(\alpha - 1)y^3 = A, \tag{B.3}$$

where A is an arbitrary constant. Now make the transformation $y = 3w/\alpha$, giving

$$w'' + 3ww' + \frac{9}{2}(\alpha - 1)\alpha^{-2}w^3 = B, \tag{B.4}$$

where $B := \alpha A/3$. Ince [43, p332] shows that the equation

$$w'' + 3ww' + \gamma w^3 = B, \tag{B.5}$$

where γ and $B(\neq 0)$ are constants, is of Painlevé type if and only if $\gamma = 1$ (the case $B = 0$ is discussed below). Hence (B.4) (and hence also (B.3)) is of Painlevé type if and only if

$$\frac{9}{2}(\alpha - 1) = \alpha^2,$$

i.e.,

$$(\alpha - 3)(\alpha - \frac{3}{2}) = 0. \tag{B.6}$$

If $\alpha = 3/2$ or $\alpha = 3$ (after rescaling q by a factor of 2) then (B.1) is the second equation in the Burger's hierarchy

$$q_t = q_{xxx} = \frac{3}{2}(qq_{xx} + q_x^2) + \frac{3}{4}q^2q_x, \tag{B.7}$$

(Olver [25]), which is reduced by the Cole-Hopf transformation

$$q = 2(\ln u)_x = 2u_x/u,$$

to the linear partial differential equation

$$u_t = u_{xxx}.$$

If B = 0 in (B.5), then there exist two choices of γ such that the equation is of Painlevé type, $\gamma = 1$ or $\gamma = -9$. If $\gamma = -9$, then

$$\frac{9}{2}(\alpha - 1) = -9\alpha^2,$$

i.e.,

$$(\alpha + 1)(\alpha - \tfrac{1}{2}) = 0. \tag{B.8}$$

If $\alpha = -1$ or $\alpha = 1/2$ (after rescaling q by a factor of 1/2), then (B.1) is

$$q_t = q_{xxx} - (qq_{xx} + q_x^2) - 3q^2q_x. \tag{B.9}$$

If we seek a solution of (B.9) in the form

$$q(x,t) = \phi^p \sum_{j=0}^{\infty} q_j(t)\phi^j(x,t), \tag{B.10}$$

with $\phi = x + f(t)$, in the neighborhood of the noncharacteristic singularity manifold defined by $\phi = 0$, then leading order analysis shows that

$p = -1$ and there are two choices for q_0, $q_0 = -1$ and $q_0 = 2$. Equating coefficients of powers of [illegible] determines the recursion relations defining $q_j(t)$, for $j \geq 1$. For to the choice $q_0 = -1$, the resonances are -1, 3, 3 (the resonances are the values of j at which arbitrary functions arise in the expansion (B.10) and for each positive resonance there is a compatibility condition which must be identically satisfied). A double resonance indicates that the expansion (B.10) does not represent the general solution (logarithmic terms must be introduced into the expansion (B.10) so that it represents the general solution). For the choice $q_0 = 2$, the resonances are -1, 3, 6; the compatibility condition corresponding to the resonance $j = 6$ is not identically satisfied which indicates that logarithmic terms again must be introduced into the expansion (B.10). Therefore (B.9) does not pass the Painlevé PDE test.

We therefore conclude that equation (B.1) can pass the Painlevé tests if and only if α takes one of the three values 0, 3/2, 3, as required.

APPENDIX C

In this appendix we show that the partial differential equation

$$w_t = w_{xxx} - \frac{3}{2}(w_x^2/w)_x + g(w)w_x, \qquad \text{(C.1)}$$

where $g(w)$ is a rational function, can pass the Painlevé tests if and only if

$$g(w) = \text{[illegible]}\,w^3 + \text{[illegible]}\,w + \text{[illegible]}\,w^{-1}, \qquad \text{(C.2)}$$

where α, β and γ are constants. First, consider the time-independent solution $w(x,t) = y(x)$, then y satisfies

$$y''' = \frac{3}{2}[(y')^2/y]' - g(y)y', \tag{C.3}$$

where $' := d/dx$. Integrating (C.3) gives

$$y'' = \frac{3}{2}(y')^2/y - G(y) + A, \tag{C.4}$$

where $\frac{dG}{dy} = g(y)$ and A is a constant. Multiplying $y^{-3}y'$ and integrating again yields

$$\frac{1}{2}y^{-3}(y')^2 = -\int^y v^{-3}G(v)dv - \frac{A}{2}y^{-2} + B, \tag{C.5}$$

where B is another constant. It is well known that the equation

$$(y')^2 = R(y), \tag{C.6}$$

where $R(y)$ is a rational function, is of Painlevé type if and only if $R(y)$ is a polynomial of degree not exceeding 4 (see Hille [40] or Ince [41] for a proof). Hence equation (C.5) is of Painlevé type if and only if

$$-\int^y v^{-3}G(v)dv - \frac{A}{2}y^{-2} + B = y^{-3}(a_4y^4 + a_3y^3 + a_2y^2 + a_1y + a_0), \tag{C.7}$$

where α_4, α_3, α_2, α_1 and α_0 are constants. Solving (C.7) for g(y) yields

$$g(y) = -3\alpha_4 y^2 + \alpha_2 - 3\alpha_0 y^{-2}. \tag{C.8}$$

If g(y) has the special form (C.8), then equation (C.1) is equation (1.18) which is equivalent to the CDF equation and which is known to pass the Painlevé PDE test [42]. Hence we have the required result.

ACKNOWLEDGMENT

P.A. Clarkson would like to thank the U.K. Science and Engineering Research Council for the support of a Postdoctoral Research Fellowship. We thank Y. Yortsos for many interesting discussions. This work was supported in part by the National Science Foundation under grant number DMS-8202117, the Office of Naval Research under grant number N00014-76-C-0867, and the Air Force Office of Scientific Research under grant number AFOSR-84-0005.

REFERENCES.

[1] C.S. Gardner, J.M. Greene, M.D. Kruskal and R.M. Miura, Method for solving the Korteweg-de Vries equation, Phys. Rev. Lett., 19, (1967), pp. 1095-1097.

[2a] M.J. Ablowitz and H. Segur, Solitons and the Inverse Scattering Transform, SIAM, Philadelphia, 1981.

[2b] F. Calogero and A. Degasperis, Spectral Transform and Solitons I, North-Holland, Amsterdam, 1982.

[3a] E. Hopf, The partial differential equation $u_t + uu_x = \mu u_{xx}$, Comm. Pure Appl. Math., 3, (1950), pp. 201-250.

[3b] J.D. Cole, On a quasilinear parabolic equation occuring in aerodynamics, Quart. Appl. Math., 9, (1951), pp. 225-236.

[4] A.S. Fokas and Y.C. Yortsos, On the exactly solvable equation $S_t = [(\beta S+\gamma)^{-2}S_x]_x + \alpha(\beta S+\gamma)^{-2}S_x$ occuring in two-phase flow in porous media, SIAM J. Appl. Math., 42, (1982), pp. 318-332.

[5] M.D. Kruskal, Nonlinear wave equations, in Dynamical Systems, Theory and Applications, ed. J. Moser, Lect. Notes Phys., 38, (1975), pp. 310-354, Springer-Verlag, New York.

[6a] M. Wadati, K. Konno and Y.H. Ichikawa, New integrable nonlinear evolution equations, J. Phys. Soc. Japan, 47, (1979), pp. 1698-1700.

[6b] T. Shimuzu and M. Wadati, A new integrable nonlinear evolution equation, Prog. Theor. Phys., 63, (1980), pp. 808-820.

[7] A.S. Fokas, A symmetry approach to exactly solvable evolution equations, J. Math. Phys., 21, (1980), pp. 1318-1325.

[8] D. Levi, O. Ragnisco and A. Sym, The Bäcklund transformation for nonlinear evolution equations which exhibit exotic solitons, Phys. Lett., 100A, (1984), pp. 7-10.

[9] S. Kawamoto, An exact transformation from the Harry Dym equation to the Modified KdV equation, J. Phys. Soc. Japan, 54, (1985), pp. 2055-2056.

[10] A. Ramani, B. Dorizzi and B. Grammaticos, Painlevé conjecture revisited, Phys. Rev. Lett., 49, (1982), pp. 1539-1541.

[11] A.F. Ranada, A. Ramani, B. Dorizzi and B. Grammaticos, The weak-Painlevé property as a criterion for the integrability of dynamical systems, J. Math. Phys., 26, (1985), pp. 708-710.

[12] R.M. Miura, The Korteweg-de Vries equation: a survey of results, SIAM Rev., 18, (1976), pp. 412-459.

[13] J. Weiss, Bäcklund transformations and the Painlevé property, J. Math. Phys., (1986), pp. 1293-1305.

[14a] S.I. Svinolupov and V.V. Sokolov, Evolution equations with nontrivial conservation laws, Func. Anal. Appl., 16, (1982), pp. 317-319.

[14b] S.I. Svinolupov, V.V. Sokolov and R.I. Yamilov, On Bäcklund transformations for integrable equations, Sov. Math. Dokl., 28, (1983), pp. 165-168.

[15a] M. Wadati, The modified Korteweg-de Vries equation, J. Phys. Soc. Japan, 32, (1972), pp. 1681.

[15b] M.J. Ablowitz, D.J. Kaup, A.C. Newell and H. Segur, The inverse scattering transform - Fourier analysis for nonlinear problems, Stud. Appl. Math., 53, (1974), pp. 249-315.

[16] F. Calogero and A. Degasperis, Reduction technique for matrix nonlinear evolution equations solvable by the spectral transform, J. Math. Phys., 22, (1981), pp. 23-31.

[17a] B.A. Kuperschmidt, On the nature of the Gardner transformation, J. Math. Phys., 22, (1981), pp. 449-451.

[17b] R. Dodd and A. Fordy, The prolongation structure of quasi-polynomial flows, Proc. R. Soc. Lond. A, 385, (1983), pp. 389-429.

[18] F. Calogero and A. Degasperis, A modified modified Korteweg-de Vries equation, Inverse Problems, 1, (1985), pp. 57-66.

[19a] M.J. Ablowitz, A. Ramani and H. Segur, Nonlinear evolution equations and ordinary differential equations of Painlevé type, Lett. Nuovo Cim., 23, (1978), pp. 333-338.

[19b] M.J. Ablowitz, A. Ramani and H. Segur, A connection between nonlinear evolution equations and ordinary differential equations of P-type. I, J. Math. Phys., 21, (1980), pp. 715-721.

[20] S.P. Hastings and J.B. McLeod, A boundary value problem associated with the second Painlevé transcendent and the Korteweg-de Vrise equation, Arch. Rat. Mech. Anal., 73, (1980), pp. 31-51.

[21] J.B. McLeod and P.J. Olver, The connection between partial differential equations soluble by inverse scattering and ordinary differential equations of Painlevé type, SIAM J. Math. Anal., 14, (1983), pp. 488-506.

[22] J. Weiss, M. Tabor and G. Carnevale, The Painlevé property for partial differential equations, J. Math. Phys., 24, (1983), pp. 522-526.

[23] M.D. Kruskal, private communication.

[24] J. Weiss, The Painlevé property for partial differential equations. II: Bäcklund transformations, Lax pairs, and the Schwarzian derivative, J. Math. Phys., 24, (1983), pp. 1405-1413.

[25] P.J. Olver, Evolution equations posessing infinitely many symmetries, J. Math. Phys., 18, (1977), pp. 1212-1215.

[26] B.G. Konopelchenko and V.G. Dubrovsky, Some new integrable evolution equations in 2+1 dimensions, Phys. Lett., 102A, (1984), pp. 15-17.

[27] A.P. Fordy and J.D. Gibbons, Some remarkable nonlinear transformations, Phys. Lett., 75A, (1980), p. 325.

[28] K. Sawada and T. Kotera, A method for finding N-soliton solutions of the KdV and KdV-like equation, Prog. Theo. Phys., 51, (1974), pp. 1355-1367.

[29] P.J. Caudrey , R.K. Dodd and J.D. Gibbon, A new hierarchy of Korteweg-de Vries equations, Proc. Roy. Soc. London A, 351, (1976), pp. 407-422.

[30] D.J. Kaup, On the inverse scattering problem of the class $\psi_{xxx} + 6Q\,\psi_x + 6R\,\psi = \lambda\psi$, Stud. Appl. Math., 62, (1980), pp. 189-216.

[31a] R.K. Dodd and J.D. Gibbon, The prolongation structure of a higher order Korteweg-de Vries equation, Proc. Roy. Soc. London A, 358, (1977), pp. 287-296.

[31b] Satsuma J. and Kaup D.J., A Bäcklund transformation for a higher order Korteweg-de Vries equation, J. Phys. Soc. Japan, 43, (1978), pp. 692-697.

[32] C. Rogers and M.C. Nucci, On reciprocal Bäcklund transformations and the Korteweg-de Vries hierarchy, Physica Scrip., 33, (1986), pp. 289-292.

[33] M. Wadati, K. Konno and Y.H. Ichikawa, A generalization of the inverse scattering method, J. Phys. Soc. Japan, 46, (1979), pp. 1965-1966.

[34a] Y. Ishimori, A relationship between the Ablowitz-Kaup-Newell-Segur and Wadati-Konno-Ichikawi schemes of the inverse scattering method, J. Phys. Soc. Japan, 50, (1981), pp. 3036-3041.

[34b] M. Wadati and K. Sogo, Gauge transformations in soliton theory, J. Phys. Soc. Japan, 52, (1983), pp. 394-398.

[35] C. Rogers and P. Wong, On reciprocal Bäcklund transformations of inverse scattering schemes, Physica Scrip., 30, (1984), pp.

10-14.

[36] A.S. Fokas and B. Fuchssteiner, Bäcklund transformation for hereditary symmetries, Nonlinear theory, Meth. Appl., 5, (1980), pp. 423-432.

[37] L. Abellanas and A. Galindo, A Harry dym class of bihamiltonian evolution equations, Phys. Lett., 107A, (1985), pp. 159-160.

[38] M. Bruschi and O. Ragnisco, On the solutions of a new class of nonlinear evolution equations, Phys. Lett., 102A, (1984) 327-328.

[39] A.V. Mikhailov and A.B. Shabat, Integrability conditions for systems of two equations of the form $\underline{u}_t = A(\underline{u})\underline{u}_{xx} + \underline{F}(\underline{u},\underline{u}_x)$, I & II, Theo. Math. Phys., 62, (1985), 107-122; 66, (1986), 31-43.

[40] E. Hille, Ordinary Differential Equations in the Complex Domain, Wiley, New York, 1976.

[41] E.L. Ince, Ordinary Differential Equations, Dover, New York, 1956.

[42] L. Hlavaty, Painlevé analysis of the Calogero-Degasperis-Fokas equation, Phys. Lett., 113A, (1985), pp. 177-178.

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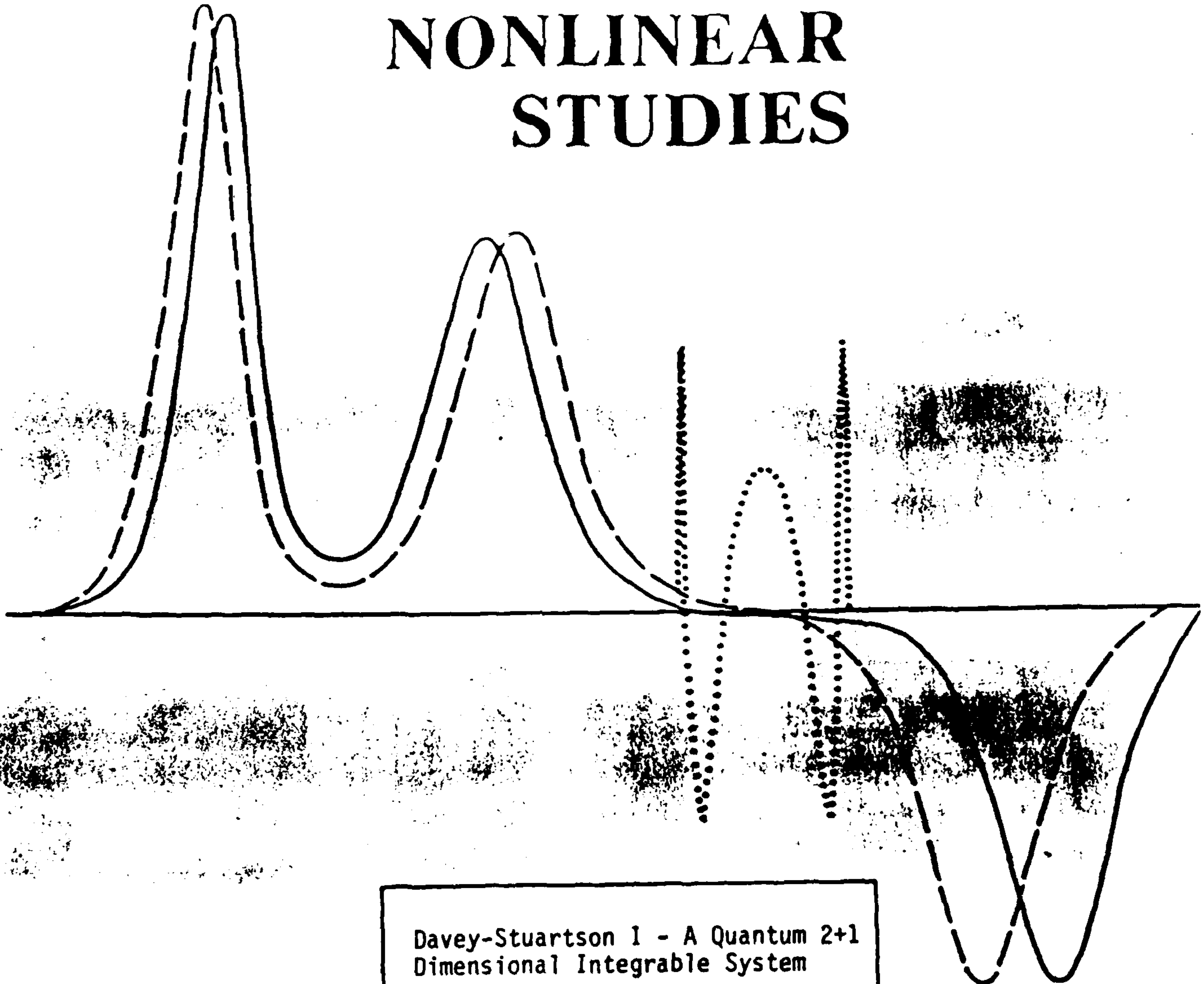

Davey-Stuartson I - A Quantum 2+1 Dimensional Integrable System

by

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MAY 1987

Davey-Stuartson I - A Quantum 2+1 Dimensional Integrable System

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We introduce a quantum version of the Davey-Stuartson I system, an exactly integrable, non-local, non-relativistic field theory in 2+1 dimensions. Quantum commutation relations between elements of the scattering matrix of the underlying linear problem are calculated and are consistent with the classical result of zero phase shift for the lump type solitons. These commutation relations can be used to demonstrate the existence of an infinite set of commuting operators, and to exactly diagonalize the Hamiltonian.

PAC numbers: 11.10.Lm

The field of exactly integrable systems, once largely confined to the study of phenomena in two dimensions (or equivalently, 1 space + 1 time dimension) has recently seen exciting progress in the understanding of problems in higher numbers of dimensions. The classical inverse scattering transform (CIST)[1] has been extended[2] and used to solve exactly a number of non-linear evolution equations in 2+1d, including the Kadomtsev-Petviashvili (KP) equation and the Davey-Stuartson (DS) equation, both of which admit localized lump-type as well as extended string-type soliton solutions. Indeed, more recently, recursion operators have been found for a general class of equations, including KP and DS.[3] There now also exists an integrable quantum system in three dimensions, obtained by A.B. Zamolodchikov[4] and R. Baxter[5] by solving the tetrahedron equations, a 3d analogue of the Yang-Baxter equations. The Zamolodchikov-Baxter solution can be interpreted as a model for the scattering of straight strings in 2+1d, or as a model of interacting random surfaces on a lattice in 3d.

Here we use an alternative approach to search for new quantum integrable systems in higher dimensions. Instead of attempting to find another solution of the tetrahedron equations, we exploit our knowledge of existing classical systems and investigate a quantum analogue of the DS system. Davey-Stuartson is an obvious choice because it reduces in the 1+1d limit to the well-known nonlinear Schrödinger (NLS) equation, whose quantum version, the δ-function Bose gas model[6], or quantum NLS model[7] is one of the best understood integrable quantum systems.

In this letter we calculate Poisson bracket relations between elements of the scattering matrix of the underlying linear problem for DS. These relations allow one to identify the action-angle variables of the classical problem. We then formally repeat the calculation by replacing the conjugate variables by operators and Poisson brackets by commutators and find the commutation relations between elements of the scattering matrix of the corresponding quantum problem. We thus obtain an algebra which is a higher dimensional analog of the Yang-Baxter algebra (in its infinite line version.) As is the case in 1+1 dimensions[7], from this algebra we can demonstrate that the Hamiltonian associated with DS is a member of an infinite set of commuting operators, and can be exactly diagonalized.

We first discuss the classical case. We will be concerned with the hyperbolic version of the DS equation, a non-linear partial differential equation for a complex-valued function $q = q(x,y,t)$,

$$i\frac{\partial q}{\partial t} = -\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)q + iA_1 q - iqA_2, \tag{1}$$

where $$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)A_1 = \frac{-i}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)(qr),$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)A_2 = \frac{i}{2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)(rq), \tag{2}$$

with $r = \pm q^*$ (q^* denoting the complex conjugate of q).

This time evolution equation for q can be generated by a non-local Hamiltonian (which will depend on the choice made for A_1 and A_2) via the Hamiltonian formulation of classical mechanics, where q and r are the conjugate variables.

As is the case for all nonlinear PDE's solvable by the CIST, (1) appears as the compatibility condition for two underlying linear equations,

$$\frac{\partial}{\partial x}\psi = J\frac{\partial}{\partial y}\psi + Q\psi \tag{3a}$$

$$\frac{\partial}{\partial t}\psi = A\psi + iQ\frac{\partial}{\partial y}\psi + iJ\frac{\partial^2}{\partial y^2}\psi, \tag{3b}$$

where $$Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4}$$

$$A = \begin{pmatrix} A_1 & \frac{i}{2}(q_x + q_y) \\ \frac{-i}{2}(r_x - r_y) & A_2 \end{pmatrix}, \tag{5}$$

and $\psi = \psi(x,y,t)$ is a 2x2 solution matrix.

The first of these equations, (3a), can be viewed simply as a linear scattering problem in which q plays the role of the potential. (3a) for suitable choice of boundary conditions, can be rewritten as a system of linear integral equations,

$$\tilde{\psi}_{ij}(\xi, \kappa, \lambda) = \delta_{ij} e^{2i(\kappa_R + \lambda)J_j \xi_j} + \iint d\xi' \, G^L_{ij}(\xi - \xi', \kappa)(Q(\xi')\tilde{\psi}(\xi', \kappa, \lambda))_{ij}, \tag{6}$$

where $\xi_1 = (x+y)$ and $\xi_2 = (x-y)$ with ξ denoting the coordinate pair (ξ_1, ξ_2), $\kappa = \kappa_R + i\kappa_I$ is a complex parameter, λ is a real parameter, the indices i, j can each take on values 1 or 2 (where we use the notation $\bar{1} \equiv 2$ and $\bar{2} \equiv 1$,) and all integrations are over infinite space. Also, for convenience of notation we use $\tilde{\psi}_{ij}(\xi, \kappa, \lambda, t) = \psi_{ij}(\xi, \kappa, \lambda, t)e^{i(\kappa_R+\lambda)^2 J_j t}$ and we shall suppress the argument, t.

We choose the Greens function

$$G^L_{ij}(\xi, \kappa) = G_i(\xi, \hat{\kappa}_{ij})$$

$$= \int \frac{d\ell}{2\pi} e^{2i(\hat{\kappa}^R_{ij}+\ell)J_i\xi_i} \Big(\Theta(\xi_1+\xi_2)\,\Theta(-J_i\ell) - \Theta(-\xi_1-\xi_2)\,\Theta(J_i\ell)\Big) \tag{7a}$$

$$= \delta(\xi_{\bar{i}})\,\theta(\xi_i) - \int \frac{d\ell}{2\pi}\,\theta(J_i\ell)e^{2i(\hat{\kappa}^R_{ij}+\ell)J_i\xi_i} \tag{7b}$$

$$= -\delta(\xi_{\bar{i}})\,\theta(-\xi_i) + \int \frac{d\ell}{2\pi}\,\theta(-J_i\ell)e^{2i(\hat{\kappa}^R_{ij}+\ell)J_i\xi_i} \tag{7c}$$

with $\hat{\kappa}_{ij} = \hat{\kappa}^R_{ij} + i\hat{\kappa}^I_{ij}$, $\hat{\kappa}^R_{ij} = \kappa_I + J_iJ_j(\kappa_R - \kappa_I)_1$

and $\hat{\kappa}^I_{ij} = \kappa_I$.

$G^L(\xi, \kappa)$ is obtained by taking the appropriate limit of the Greens function of the more general D-bar problem.[8]

We also will find it useful to define a solution, ζ, of an adjoint linear problem,

$$\tilde{\zeta}_{ik}(\xi, \hat{\kappa}_{kj}, \lambda') = \delta_{ik} e^{-2i(\hat{\kappa}^R_{kj}+\lambda')J_k\xi_k} + \iint d\xi' \sum_{\ell=1}^{2} \tilde{\zeta}_{i\ell}(\xi', \hat{\kappa}_{\ell j}, \lambda')Q_{\ell k}(\xi')G_k(\xi'-\xi, \tag{8}$$

Of fundamental interest in both the classical and the quantum problem is the "scattering matrix" or the "scattering data" of (6), which we define to be

$$T_{ij}(\kappa, \lambda, \lambda') = \iint d\xi \, e^{-2i(\hat{\kappa}^R_{ij}+\lambda')J_i\xi_i} (Q(\xi)\, \tilde{\psi}(\xi, \kappa, \lambda))_{ij} \,. \tag{9}$$

For certain choices of the parameters κ, λ and λ', T can be shown to have a very simple time dependence, and is thus used in the CIST to "reconstruct" the potential $q(x,y,t)$ at arbitrary times, for appropriately given initial conditions.

We can calculate Poisson bracket relations between elements of T, where we define canonical Poisson brackets

$$\{f, g\} = i\iint d\xi \left[\frac{\delta f}{\delta q(\xi)} \frac{\delta g}{\delta r(\xi)} - \frac{\delta f}{\delta r(\xi)} \frac{\delta g}{\delta q(\xi)}\right] . \tag{10}$$

We find, by use of the linear integral equations, (6) and (8), that

$$\{T_{\alpha\beta}(\kappa, \lambda, \lambda'), T_{\gamma\delta}(\tau, \mu, \mu')\}$$
$$= \sum_{a=1}^{2} \iint d\xi \; \tilde{\zeta}_{\alpha a}(\xi, \hat{\kappa}_{a\beta}, \lambda')\, \tilde{\psi}_{\bar{a}\beta}(\xi, \kappa, \lambda) \quad \tilde{\zeta}_{\gamma\bar{a}}(\xi, \tau_{\bar{a}\delta}, \mu')\, \tilde{\psi}_{a\delta}(\xi, \tau, \mu) \tag{11}$$

The solution $\tilde{\psi}$ and its adjoint $\tilde{\zeta}$ satisfy $\sum_{k=1}^{2} \frac{\partial}{\partial\xi_k} \tilde{\zeta}_{ik}(\xi, \hat{\kappa}_{kj}, \lambda')$ $\tilde{\psi}_{kj}(\xi, \tau, \mu) = 0$.

This identity can be used to rewrite the integrand appearing in (11) as follows:

$$\{T_{\alpha\beta}(\kappa,\ \lambda,\ \lambda'),\quad T_{\gamma\delta}(\tau,\ \mu,\ \mu') =$$

$$- \sum_{a=1}^{2} \int d\xi_a \int d\xi_a'\ \Theta(J_a(\xi_a - \xi_a'))\ \tilde{\zeta}_{\alpha a}(\xi_a,\ \xi_{\bar{a}},\ \hat{\kappa}_{a\beta},\ \lambda')\ \tilde{\psi}_{a\beta}(\xi_a',\ \xi_{\bar{a}},\ \kappa,\ \lambda)$$

$$\cdot\ \zeta_{\gamma a}(\xi_a',\ \xi_{\bar{a}},\ \hat{\tau}_{a\delta},\ \mu')\ \tilde{\psi}_{a\delta}(\xi_a,\ \xi_{\bar{a}},\ \tau,\ \mu)\Bigg|_{\xi_{\bar{a}} = -\infty}^{\xi_{\bar{a}} = +\infty}$$

$$+ \sum_{a=1}^{2} J_a \int d\xi_a\ \tilde{\zeta}_{\alpha a}(\xi,\ \hat{\kappa}_{a\beta},\ \lambda')\ \tilde{\psi}_{a\delta}(\xi,\ \tau,\ \mu)\Bigg|_{J_{\bar{a}}\xi_{\bar{a}} = \infty}$$

$$\cdot \int d\xi_{\bar{a}}\ \tilde{\zeta}_{\gamma\bar{a}}(\xi',\ \hat{\tau}_{\bar{a}\delta},\ \mu')\tilde{\psi}_{\bar{a}\beta}(\xi',\ \kappa,\ \lambda)\Bigg|_{J_a\xi_a' = -\infty} \qquad (12)$$

In order to evaluate (12) it is necessary to find asymptotic expressions for $\tilde{\psi}$ and $\tilde{\zeta}$. However, these can be found easily by using (6) and (8) and noting that it is possible to write G^L in the two alternative forms (7b) or (7c). Then

$$\lim_{\xi_k \to \pm\infty} \psi_{kj}(\xi,\ \kappa,\ \lambda) = \delta_{kj} e^{2i(\kappa_R + \lambda)J_k\xi_k}$$

$$\pm \int \frac{d\ell}{2\pi}\ \Theta(\mp J_k \ell) e^{2i(\hat{\kappa}_{kj}^R + \ell)J_k\xi_k}\ T_{kj}(\kappa,\ \lambda,\ \ell) \qquad (13)$$

and

$$\lim_{\xi_k \to \pm\infty} \tilde{\zeta}_{ik}(\xi, \hat{\kappa}_{kj},\ \lambda') = \delta_{ik} e^{-2i(\hat{\kappa}_{kj}^R + \lambda')J_k\xi_k}$$

$$\mp \int \frac{d\ell}{2\pi}\ T_{ik}(\hat{\kappa}_{kj},\ \ell,\ \lambda')\ \Theta(\pm J_k \ell) e^{-2i(\hat{\kappa}_{kj}^R + \ell)J_k\xi_k} \qquad (14)$$

Inserting (13) and (14) into (12), we arrive at an expression for $\{T_{\alpha\beta}(\kappa, \lambda, \lambda'), T_{\gamma\delta}(\tau, \mu, \mu')\}$ purely in terms of T's.

Instead of writing down the a lengthy expression, which contains terms up to quartic in T, we instead give results in two interesting limiting cases. First, letting $\lambda = \lambda' = \mu = \mu' = 0$ and $T(\kappa, 0, 0) \equiv T^L(\kappa)$, we recover the scattering data of the hyperbolic limit of the D-bar problem, and making use of an identity easily derived from (6), find Poisson bracket relations

$$\{T^L_{11}(\kappa), T^L_{11}(\tau)\} = \{T^L_{12}(\kappa), T^L_{12}(\tau)\} = \{T^L_{11}(\kappa), T^L_{22}(\tau)\} = 0 \tag{15a}$$

$$\{T^L_{12}(\kappa), T^L_{21}(\tau)\} = (2\pi)^2 \, \delta(\kappa_R+\tau_R-\kappa_I-\tau_I) \, \delta(\kappa_I-\tau_I) \tag{15b}$$

$$\{T^L_{11}(\kappa), T^L_{12}(\tau)\} = \left[\frac{i}{\kappa_R-\hat{\tau}^R_{12}-i\epsilon} + 2\pi \, \delta(\kappa_R-\hat{\tau}^R_{12}) \, \theta(\kappa_I-\tau_I)\right] T^L_{12}(\tau), \tag{15c}$$

as well as a number of other similar relations.

Alternatively, we can take the limit $\kappa_I \to +\infty$, $\kappa_R \to +\infty$, $T(\kappa, \lambda, \lambda') \to T^+(\theta, \theta')$, where $\theta = \kappa_R + \lambda$, $\theta' = \kappa_R + \lambda'$ are kept finite. In this way, we recover the scattering data associated with a solution to (3a), $\mu^+_{ij}(\xi, \theta) \equiv \psi_{ij}(\xi, \theta)e^{-2i\theta J_i\xi_j + i\theta^2 J_j t}$ analytic in the upper-half θ plane, which is used in the Riemann-Hilbert approach to CIST. We find

$$\begin{aligned}\{S^+_{\alpha\beta}(\theta, \theta'), S^+_{\gamma\delta}(\phi, \phi')\} &= S^+_{\alpha\beta}(\phi, \theta') \, S^+_{\gamma\delta}(\theta, \phi') \\ &+ \delta_{\beta\delta} \, J_\beta \int \frac{d\sigma}{2\pi i(\sigma-i\epsilon)} \, S^+_{\alpha\delta}(\phi+\sigma, \theta') \, S^+_{\delta\beta}(\theta-\sigma, \phi') \\ &- \delta_{\alpha\delta} \, J_\beta \int \frac{\sigma}{2\pi i(\sigma-i\epsilon)} \, S^+_{\alpha\delta}(\phi, \theta'-\sigma) \, S^+_{\delta\beta}(\theta, \phi'+\sigma),\end{aligned} \tag{16}$$

where we've defined $S^{+}_{\alpha\beta}(\Theta, \Theta') = 2\pi\ \delta(\Theta - \Theta')\delta_{\alpha\beta} + J_{\beta}\ T^{+}_{\alpha\beta}(\Theta, \Theta')$. Similarly, the limit $\kappa_I \to -\infty$, $\kappa_R \to +\infty$ give us the scattering data associated with a solution analytic in the lower-half Θ plane.

The calculation of commutation relations for the quantum DS problem is formally similar to the Poisson bracket calculation, with Poisson brackets $\{q(\xi), r(\xi')\} = i\delta(\xi_1-\xi_1')\ \delta(\xi_2-\xi_2')$ replaced by commutators $[q(\xi), r(\xi')] = i\delta(\xi_1-\xi_1')\delta(\xi_2-\xi_2')$, etc. Now elements of the solution matrix, ψ, and of the scattering matrix, T, are treated as operators, and care must be taken throughout the calculation to maintain proper ordering. For the quantum problem defined by the ordering appearing in (1), (2), (3) and (6), the quantum results are given by (15) and (16) with $\{\ ,\ \}$ replaced by $[\ ,\ ]$. Note that we do not treat the normal ordered problem.

The classical results, (15), can be used to demonstrate that the coefficients appearing in a $(1/\kappa_R)$ expansion of $T^{L}_{11}(\kappa)$ form an infinite set of constants of the motion, and to identify, by suitable rescaling, the canonical action-angle variables of the problem. The corresponding quantum results show that $T^{L}_{11}(\kappa)$ generates an infinite set of commuting operators, including as a member, the Hamiltonian of the DS system. Furthermore, these operators can be exactly diagonalized by normalized eigenstates formed by $\prod_i T^{L}_{12}(\kappa_i)$ acting on an appropriate reference state. This quantum theory appears to have a trivial S-matrix, consistent with the fact that the classical lump-type soliton solutions of DS experience no phase shift asymptotically when they interact.

The results, (16) for the Riemann-Hilbert formulation of the problem, have a very different form, and do not immediately allow one to identify the action-angle variables. Never-the-less, T^{+} and T^{-} are related to T and T^L through nonlinear integral equations. In the classical problem, T^-_{12} and T^-_{21} are known to evolve simply in time to have angle variable structure. The quantum results corresponding to (16) reduce to the well-known Yang-Baxter algebra (in its infinite line version) in the 1+1d limit $\mu^{\pm}(x, y, \theta) \to \mu^{\pm}(x, \theta)$ and $S^{\pm}(\theta, \theta') \to \delta(\theta - \theta')\, S^{\pm}(\theta)$.

This work was partially supported by the National Science Foundation under grant numbers DMS-8507658 and DMS-8202117, the Office of Naval Research under grant number N00014-76-C-0867, and the Air Force Office of Scientific Research under grant number AFOSR-84-0005.

One of the authors, C.L. Schultz, would like to thank H.H. Chen for his hospitality and helpful discussions at the Laboratory for Plasma and Fusion Energy Studies, University of Maryland, where part of this work was completed.

References

1. V. E. Zakharov and A. B. Shabat, Sov. Phys. JETP 34, 62 (1972), and M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, Phys. Rev. Lett. 30, 1262 (1973) and Phys. Rev. Lett. 31, 125 (1973).

2. See A. S. Fokas and M. J. Ablowitz, "Lectures on the Inverse Scattering Transform form Multi-dimensional (2+1) Problems", INS #28, and references therein.

3. P. M. Santini and A. S. Fokas, INS (preprint) #65 (1987).

4. A. B. Zamolodchikov, Commun. Math. Phys. 79, 489 (1981).

5. R. J. Baxter, Commun. Math. Phys. 88, 185 (1983), Phys. Rev. Lett. 53, 1795 (1984).

6. E. H. Lieb and W. Liniger, Phys. Rev. 130, 1605 (1963).

7. See H. B. Thacker, Rev. Mod. Phys. 53, 253 (1981) and L. D. Faddeev, Sov. Sci. Rev. Math. Phys. C1, 107 (1980), and references therein.

8. A. I. Nachman and M. J. Ablowitz, Stud. Appl. Math. 71, 251 (1984).

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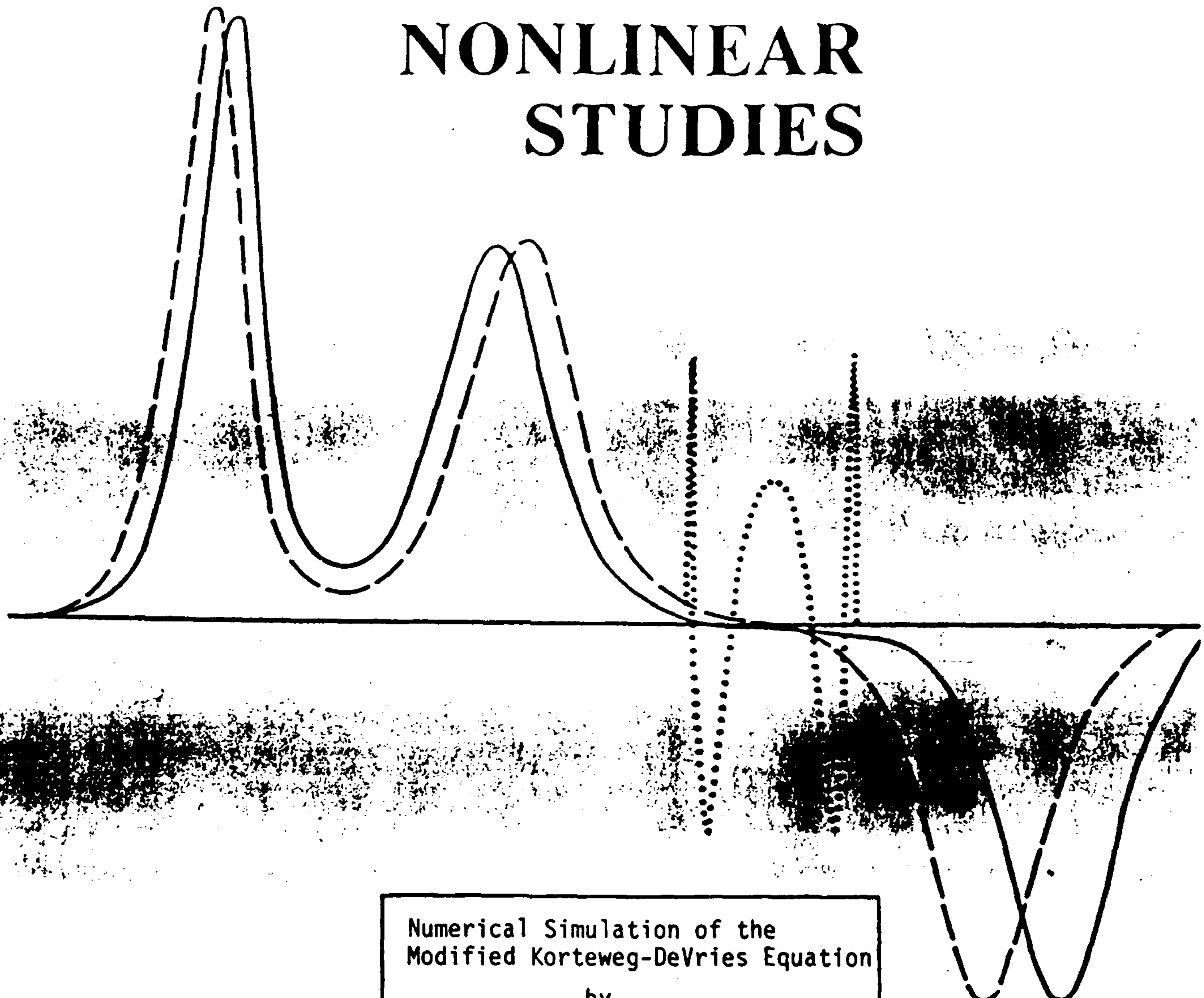

Numerical Simulation of the Modified Korteweg-DeVries Equation

by

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February 1987

NUMERICAL SIMULATION OF THE MODIFIED KORTEWEG-DE VRIES EQUATION

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Abstract

Proposed schemes for the numerical simulation of the Modified Korteweg-de Vries (MKdV) equation are implemented and compared to other known numerical methods. These schemes are constructed by methods related to the inverse scattering transform (IST). In this paper a summary of their performance using both solitons and nonsoliton initial values as they were applied to the MKdV equation will be presented. Results for nonsoliton initial values are quite novel.

1. Introduction

The Modified Korteweg-de Vries (MKdV) equation describes a wide class of physical phenomena (e.g., acoustic waves in certain anharmonic lattices [1] and Alfén waves in a collisionless plasma [2]).

In (1984) we derived nonlinear partial difference equations which have as limiting forms the Korteweg-de Vries (KdV) and the MKdV equations [3]. These difference equations have a number of special properties [4] and are constructed by methods related to the inverse scattering transform (IST). We have also implemented similar schemes for the nonlinear Schrödinger (NLS) equation (Ablowitz-Ladik) and the KdV equation and compared them with known numerical schemes [5,6]. Experiments have shown that the IST schemes for the NLS and KdV equations compare very favorably with the other known numerical methods. Recently we have implemented and compared the proposed schemes which were developed in [3] with certain other known numerical methods for the MKdV equation (1.1a) [10].

$$u_t \pm 6u^2 u_x + u_{xxx} = 0\,. \qquad (1.1a,b)$$

The following numerical methods were applied to the MKdV equation: (i) a proposed global scheme, (ii) a proposed local scheme, (iii) an implicit scheme, (iv) a split step Fourier method (Tappert), and (v) a pseudospectral method (Fornberg and Whitham).

Our approach for comparison was to (a) fix the accuracy (L_∞) for computations beginning at $t = 0$ and ending at $t = T$; (b) leave other parameters free (e.g., Δt, or Δx), and compare the computing time required to attain such accuracy for various choices of the parameters.

In the above equation (1.1a) one and two soliton solutions with various values of amplitudes were used as initial conditions, and periodic boundary conditions were imposed. The numerical solution is compared to the exact solution, and in addition, two of the conserved quantities are computed, namely $\int u^2 dx$, and $\int (u^4 - (u_x)^2)dx$.

2. The Representation of the MKdV Equation (1.1) Using Numerical Methods

(i) The proposed global scheme which is based on the IST is (Taha and Ablowitz, [3]).

$$\begin{aligned}
\Delta^m R_n^m = {} & R_{n+2}^m \underset{-}{A}^{(4)} - R_{n+2}^{m+1} \gamma_{n+1} \underset{-}{D}^{(4)} + R_{n+1}^m S_{n+1} \\
& - R_{n+1}^{m+1} P_n - \Big[R_{n-2}^{m+1} \underset{-}{A}^{(4)} - R_{n-2}^m \gamma_{n-2} \underset{-}{D}^{(4)} \\
& + R_{n-1}^{m+1} S_{n-2} - R_{n-1}^m P_{n-1} \Big] + R_n^m \left(\underset{-}{D}^{(0)} \mp \sum_{l=-\infty}^{n} T_l \right), \\
& - R_n^{m+1} \left(\underset{-}{A}^{(0)} \mp \sum_{l=-\infty}^{n-1} T_l \right),
\end{aligned} \qquad (2.1)$$

where

$$\begin{aligned}
T_l = {} & R_l^{m+1} \Big\{ R_{l-2}^{m+1} \underset{-}{A}^{(4)} - R_{l-2}^m \gamma_{l-2} \underset{-}{D}^{(4)} + R_{l-1}^{m+1} S_{l-2} \\
& - R_{l-1}^m P_{l-1} \Big\} - R_l^m \Big\{ R_{l+2}^m \underset{-}{A}^{(4)} - R_{l+2}^{m+1} \gamma_{l+1} \underset{-}{D}^{(4)} \\
& + R_{l+1}^m S_{l+1} - R_{l+1}^{m+1} P_l \Big\},
\end{aligned}$$

$$S_n = \underset{-}{A}^{(2)} + \underset{-}{A}^{(4)} F_n + \underset{-}{D}^{(4)} \sum_{j=-\infty}^{n} H_j,$$

$$P_n = \left(\underset{-}{D}^{(2)} + \sum_{j=-\infty}^{n} \left[\underset{-}{A}^{(4)} E_j + \underset{-}{D}^{(4)} G_j \right] \eta_j \right) \gamma_n,$$

$$\gamma_n = \prod_{i=-\infty}^{n} \left(\delta_i^{m+1} / \delta_i^m \right), \quad \delta_i^m = 1 \pm R_i^m,$$

$$\eta_n = \gamma_n^{-1} / \delta_n^m, H_n = \mp \left\{ R_n^m R_{n+1}^{m+1} \delta_n^{m+1} - R_{n-1}^m R_n^{m+1} \delta_n^m \right\} \beta_{n-1},$$

$$\beta_n = \gamma_n / \delta_{n+1}^m, F_n = \pm \left[R_{n+1}^{m+1} R_n^{m+1} - \sum_{j=-\infty}^{n} \Delta^m \left(R_j^m R_{j+1}^m \right) \right],$$

$$G_n = \pm \left(R_n^{m+1} R_{n+1}^{m+1} - R_n^m R_{n-1}^m \right) \gamma_{n-1} \delta_n^{m+1},$$

$$E_n = \pm \left(R_n^m R_{n-1}^{m+1}\delta_n^{m+1} - R_{n+1}^m R_n^{m+1}\delta_n^m\right),$$

$$A_\cdot^{(2)} = -\frac{2}{3}A_\cdot^{(0)} + \frac{1}{2}\alpha,\ D_\cdot^{(2)} = -\frac{2}{3}A_\cdot^{(0)} - \frac{l}{2}\alpha,$$

$$A_\cdot^{(4)} = \frac{1}{6}A^{(0)} - \frac{1}{4}\alpha, D_\cdot^{(4)} = \frac{1}{6}A_\cdot^{(0)} + \frac{1}{4}\alpha,$$

$$\alpha = \frac{\Delta t}{(\Delta x)^3}, A_\cdot^{(0)} = \text{arbitrary constant},$$

$R = \Delta x\, u$, and $|n| < p$ (half the length of the interval of interest, and $m > 0$). This scheme is implemented with the value of $A_\cdot^{(0)} = \frac{3}{2}\alpha$, and using the sweeping/iteration technique presented by the authors [5,6].

(ii) The proposed local scheme which is derived from equation (2.1) with $A_\cdot^{(0)} = \frac{3}{2}\alpha$ is

$$\frac{u_n^{m+1} - u_n^m}{\Delta t} = \frac{u_{n-1}^{m+1} - 3u_n^{m+1} + 3u_{n+1}^{m+1} - u_{n+2}^{m+1}}{2(\Delta x)^3}$$
$$+ \frac{u_{n-2}^m - 3u_{n-1}^m + 3u_n^m - u_{n+1}^m}{2(\Delta x)^3}$$
$$\mp \frac{1}{2\Delta x}\left[u_{n+2}^{m+1}\left\{\left(u_{n+1}^m\right)^2 + \left(u_n^m\right)^2\right\}\right.$$
$$- u_{n-2}^m\left\{\left(u_{n-1}^{m+1}\right)^2 + \left(u_n^{m+1}\right)^2\right\}$$
$$+ \frac{u_{n+1}^{m+1}}{2}\left[u_n^m u_{n+1}^m + u_n^{m+1}u_{n+1}^{m+1} + 2u_{n-1}^m u_n^m\right]$$
$$- \frac{u_{n-1}^m}{2}\left[u_n^m u_{n-1}^m + u_n^{m+1}u_{n-1}^{m+1} + 2u_n^{m+1}u_{n+1}^{m+1}\right]$$
$$+ \frac{u_n^m}{2}\left(u_n^{m+1}u_{n+1}^{m+1} + u_n^m u_{n+1}^m\right)$$
$$- \frac{u_n^{m+1}}{2}\left(u_{n-1}^{m+1}u_n^{m+1} + u_{n-1}^m u_n^m\right)$$
$$\left. - 3\left[\left(u_n^m\right)^2 u_{n+1}^{m+1} - \left(u_n^{m+1}\right)^2 u_{n-1}^m\right]\right] \tag{2.2}$$

This scheme is implemented using the sweeping/iteration technique.

(iii) An implicit scheme (Kruskal, 1981) [7]:

$$\frac{u_n^{m+1} - u_n^m}{\Delta t} = \frac{u_{n-1}^{m+1} - 3u_n^{m+1} + 3u_{n+1}^{m+1} - u_{n+2}^{m+1}}{2(\Delta x)^3}$$
$$+ \frac{u_{n-2}^m - 3u_{n-1}^m + 3u_n^m - u_{n+1}^m}{2(\Delta x)^3}$$
$$\mp \frac{1}{2(\Delta x)^3}\left\{\theta\left[\left(u^3\right)_{n+1}^{m+1} - \left(u^3\right)_{n-1}^{m+1}\right.\right.$$
$$\left. + \left(u^3\right)_{n+1}^m - \left(u^3\right)_{n-1}^m\right]$$
$$+ 3(1-\theta)\left[\left(u^2\right)_n^{m+1}\left(u_{n+1}^{m+1} - u_{n-1}^{m+1}\right)\right.$$
$$\left.\left. + \left(u^2\right)_n^m\left(u_{n+1}^m - u_{n-1}^m\right)\right]\right\} \tag{2.3}$$

This scheme is also implemented using the sweeping/iteration technique. Several values of θ are employed and experimentally we find that $\theta = \frac{2}{3}$ gives the best results.

(iv) Split step Fourier method (Tappert [8])

For convenience the spatial period was normalized to $[0,2\pi]$, then Eq. (1.1) becomes

$$u_t \pm 6\frac{\pi}{p}u^2 u_X + \frac{\pi^3}{p^3}u_{XXX} = 0, \tag{2.4}$$

where p is half the length of the interval of interest, and $X = (x + p)\pi/p$.

In order to apply the split step Fourier method for Eq. (2.4) we (a) advance the solution using only the nonlinear part

$$u_t \pm 6\frac{\pi}{p}u^2 u_x = 0. \tag{2.5}$$

This can be approximated by using an implicit method such as

$$\bar{u}_n^{m+1} = u_n^m \mp \frac{\Delta t}{12\Delta x}\frac{\pi}{p}\{[8(\bar{u}^3)_{n+1}^{m+1} - 8(\bar{u}^3)_{n-1}^{m+1}$$
$$- (\bar{u}^3)_{n+2}^{m+1} + (\bar{u}^3)_{n+2}^{m+1}] + [8(u^3)_{n+1}^m$$
$$- 8(u^3)_{n-1}^m - (u^3)_{n+2}^m + (u^3)_{n-2}^m]\} \tag{2.6}$$

where $\bar{u}$ is a solution of Eq. (2.5); (b) advance the solution according to

$$u_t + \frac{\pi^3}{p^3}u_{XXX} = 0 \tag{2.7}$$

by means of the discrete Fourier transform

$$u(X_j, t + \Delta t) = F^{-1}(e^{i(k^3\pi^3/p^3)\Delta t}F(\bar{u}(X_j,t))), \tag{2.8}$$

(V) Pseudospectral Method by Fornberg and Whitham [9].

The pseudospectral method for Eq. (2.4) is

$$u(X, t + \Delta t) - u(X, t - \Delta t) \pm 12i\frac{\pi}{p}\Delta t\, u^2(X,t)F^{-1}(kF(u))$$
$$- 2iF^{-1}\{Sin[\frac{\pi^3k^3}{p^3}\Delta t]F(u)\} = 0. \tag{2.9}$$

Our numerical experiments indicate (for the range of amplitudes we considered) that

(1) The proposed global scheme, based on IST, proved to be faster than all of the methods we considered. It is worth noting that this proposed global scheme behaves much better than the othr utilized schemes either when better accuracy is required or for large amplitudes.

(2) The pseudospectral method becomes competitive with the IST global scheme when both high accuracy and large amplitudes are involved.

(3) The implicit scheme behaves better than the proposed local scheme and the pseudospectral method for low amplitudes, and it is much better than the split step (Tappert) method.

(4) The proposed local schemes behaves better than the pseudospectral method for small amplitudes for the 1-soliton case, and becomes competitive with the implicit scheme for large amplitudes.

(5) The split step Fourier method behaves much slower than all of the methods we considered. We note that the proposed local scheme did not perform as well as its global version. We intend to study this situation further.

Very recently we implemented the proposed global scheme for the MKdV equation (1.1b) and compared it to the pseudospectral method, since our earlier experiments indicate that the pseudospectral method is the most competitive scheme for the MKdV equation (1.1a). In Eq. (1.1b) the following initial conditions is considered.

$$u(x,0) = \frac{2}{(1+x^2)^2} \tag{2.10}$$

Periodic boundary conditions on the interval [-20,20] are imposed. Our approach for comparison is to (a) compute two of the conserved quantities at each time step, namely $c_1 = \int u^2 dx$, and $c_2 = \int (u^4 + (u_x)^2) dx$ for computations beginning at $t = 0$ and ending at $t = T$; (b) leave other parameters free (e.g. Δt or Δx), and compare the computing time required to attain a relative error in the conserved quantities c_1 and c_2 smaller than some tolerance. From the experiments we conducted we have found that (a) the stability condition of the pseudospectral method applied to Eq. (1.1b) is more restricted than for Eq. (1.1a): $\frac{\Delta t}{(\Delta x)^3} < 0.045$, compared to $\frac{\Delta t}{(\Delta x)^3} < 0.152$. Hence Δt must be taken smaller. (b) the proposed global scheme is much *faster* than the pseudospectral *method* to attain relative errors: $E_1 = \left|\frac{c_1 - c_1^*}{c_1}\right| < 0.1\%$, and $E_2 = \left|\frac{c_2 - c_2^*}{c_2}\right| < 0.2\%$, where c_1 is the exact value of $\int u^2 dx$ and c_1^* is the calculated one, and c_2 is the exact value of $\int (u^4 + (u_x)^2) dx$ and c_2^* is the calculated one.

3. Conclusion

The proposed schemes which are constructed by methods related to the IST can be used to find numerical solutions of nonlinear evolution equations with initial conditions other than solitons. It is worth noting that this work can be extended to cover other so-called soliton equations.

Acknowledgements

This research was partially supported by the Research Foundation of the University of Georgia (by way of a Faculty Research Grant and a Michael Award), and by AFSOR grant #84-0005, NSF grant #DMS-8301325.

REFERENCES

[1] N. J. Zabusky, "A synergetic approach to problems of Nonlinear Wave Propagation and Interaction in Nonlinear Partial Differential Equations", (W F. Ames, Ed.), pp. 223-258, New York, 1967 (b) N. Zabusky, Computational Synergetics and mathematics innovation, J. Comp. Phys. 43 (1981), p. 195.

[2] A. Scott, F. Chu, and D. McLaughlin, "The soliton: A new concept in applied sciences", Proceedings of the IEEE, Vol. 61, No. 10, (1973), p. 1443.

[3] T. R. Taha and M. J. Ablowitz, "Analytical and Numerical Aspects of Certain Nonlinear Evolution Equations. I. Analytical", J. Comp. Phys. 55, No. 2 (1984), p. 192.

[4] M. Ablowitz and H. Segur, "Solitons and the inverse scattering transform", (SIAM, Philadelphia, 1981).

[5] T. R. Taha and M. J. Ablowitz, "Analytical and Numerical Aspects of Certain Nonlinear Evolution Equations. II. Numerical, Nonlinear Schrödinger Equation", J. Comp. Phys. 55, No. 2 (1984), p. 203.

[6] T. R. Taha and M. J. Ablowitz, "Analytical and Numerical Aspects of Certain Nonlinear Evolution Equations. III. Numerical, Korteweg-de Vries Equation", J. Comp. Phys. Vol. 55, No. 2 (1984), p. 231.

[7] M. D. Kruskal, private communication, (1981).

[8] F. Tappert, Lect. Appl. Math. Am. Math. Soc. 15 (1974), p. 215.

[9] B. Fornberg and G. B. Whitham, Phil. Trans. Roy. Soc. 289 (1978), p. 373.

[10] T. R. Taha and M. J. Ablowitz, "Analytical and numerical aspects of certain nonlinear evolution equations. IV. Numerical, MKdV equation, preprint.

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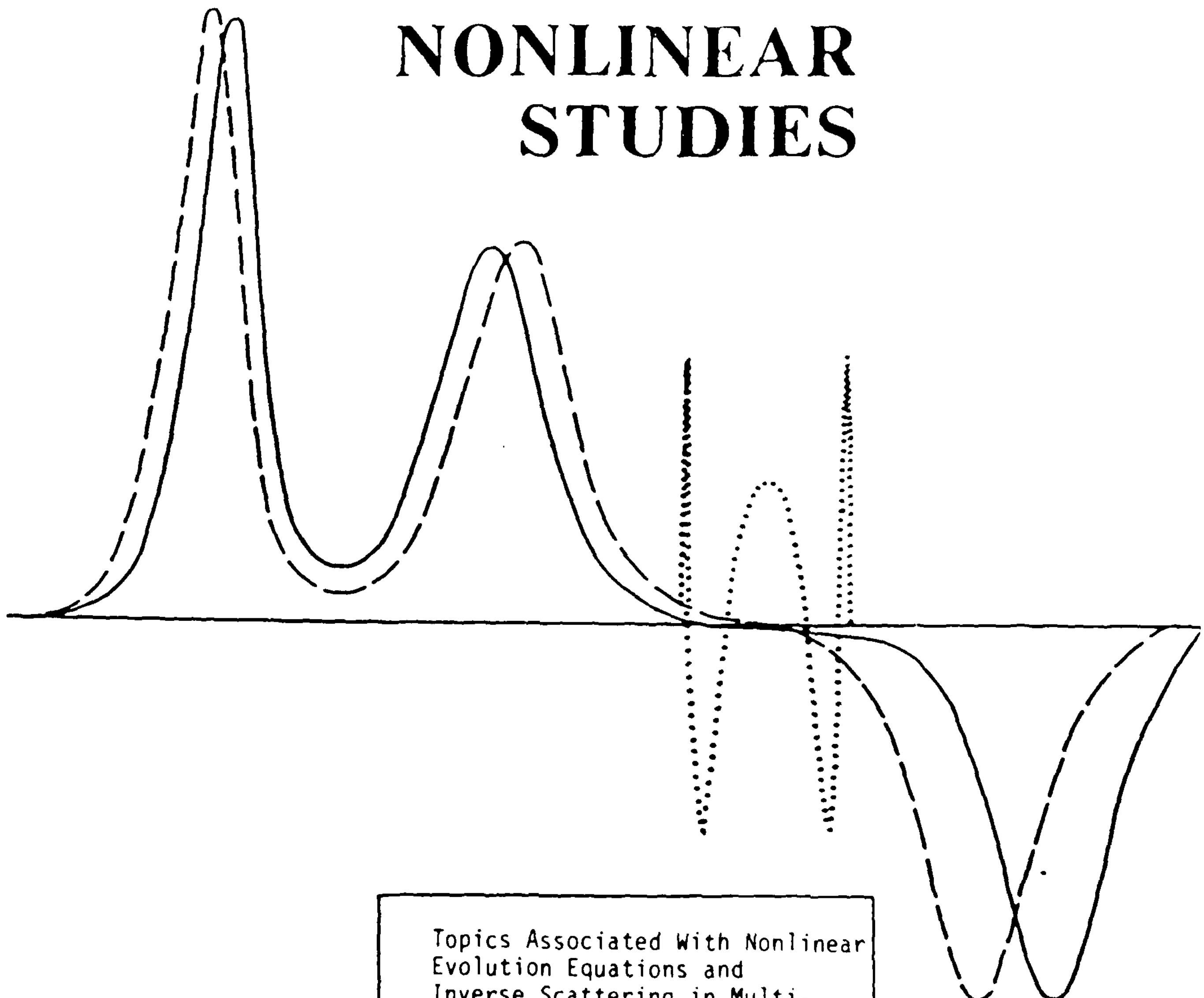

Topics Associated With Nonlinear Evolution Equations and Inverse Scattering in Multi-dimensions

by

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March 1987

TOPICS ASSOCIATED WITH NONLINEAR EVOLUTION EQUATIONS AND INVERSE SCATTERING IN MULTIDIMENSIONS

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Abstract

In recent years the basic structure required to implement the inverse scattering transform in 1+1 and 2+1 dimensions has been clarified and extended. Aspects involved with fully multidimensional problems have also been treated. In particular the inverse scattering associated with various multidimensional operators and generalizations of the Sine-Gordon and self-dual Yang-Mills equations have been studied. A review of some of this work will be discussed in this review.

The Inverse Scattering Transform (I.S.T.) is a method to solve certain nonlinear evolution equations. There has been wide ranging interest in this method for many reasons. A review of earlier work can be found in [1]. A surprisingly large number of physically interesting nonlinear equations can be solved via IST; there are many applications in physics including: surface waves, internal waves, lattice dynamics, plasma physics, nonlinear optics, particle physics and relativity. Mathematically speaking the field is also quite rich, with nontrivial results in the areas of analysis, group theory, algebra, differential and algebraic geometry being used by various researchers. From our point of view, IST allows us to solve the Cauchy problem for these nonlinear systems. We shall concentrate on questions in infinite space. All of the nonlinear equations discussed below arise as the compatibility condition of certain linear equations, one of which is identified as a scattering (direct and inverse scattering is required) problem and the other(s) serves to fix the "time evolution" of the scattering data.

In one spatial dimension the prototype problem is the (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1}$$

The KdV equation is compatible with

$$v_{xx} + u(x,t)v = \lambda v \tag{2}$$

$$v_t = (\gamma + u_x)v - (4\lambda + 2u)v_x \tag{3}$$

i.e. $v_{xxt} = v_{txx}$ implies (1). Equation (2) is the time independent Schrodinger scattering problem, λ the eigenvalue (γ = const. in (3)). The solution of (1) on the line: $-\infty < x < \infty$ for initial values $u(x,t=0)$ vanishing sufficiently rapidly at infinity is obtained by studying the associated direct and inverse scattering problem of (2) and using (3) to fix the time evolution of the scattering data. It turns out that the inverse problem amounts to solving a matrix Riemann-Hilbert boundary value problem (RHBVP) whose jump discontinuity depends explicitly on the scattering data. Calling $\lambda=-k^2, v(x,k)=\mu(x,k)e^{-ikx}$ the RHBVP takes the following form,

$$(\mu_+-\mu_-)(x,t,k) = \mu_-(x,t,\alpha(k))\, V(x,t,k) \text{ on } \Sigma$$

$$\mu_\pm \to 1, \quad |k|\to\infty \tag{4}$$

where

$V(x,t,k) = r(k,t)\, e^{2ikx}$, $\alpha(k) = -k$, $\Sigma=\{k: k\in \mathbb{R}\}$, and $\mu_\pm$ are the limiting boundary values, as $\operatorname{Im} k\to 0\pm$, of meromorphic functions in the upper (+) lower (-) half plane. (4) may be converted into a linear integral equation by taking a minus projection and the potential is reconstructed via

$$u(x,t) = -\frac{1}{\pi}\frac{\partial}{\partial x}\int_C \mu(k,x,t,-k)\, V(x,t,k)dk \tag{5}$$

where the contour is taken above all poles of $r(k,t)$; of which there is at most a finite number, $k_j = i\kappa_j$, $\kappa_j>0$ $j = 1,\cdots N$. The scattering data: the reflection coefficient, $r(k,t)$ evolves simply in time

$$r(k,t) = r(k,0)\, e^{8ik^2t} \tag{6}$$

The above scheme may be extended so as to solve a surprisingly large number of interesting nonlinear evolution equations. There are two scattering problems of particular interest in one dimension:

(i) Scalar scattering problems:

$$\frac{d^n v}{dx^n} + \sum_{j=2}^{n} u_j(x)\frac{d^{n-j}v}{dx^{n-j}} = \lambda v,$$

$$v(x,k),\quad u_j \in \mathbb{C}$$

(ii) First order systems - generalized AKNS

$$\frac{dv}{dx} = ikJv + qv$$

$$v(x,k), q(x) \in \mathbb{C}^{N\times N},\ J = \operatorname{diag}(J^1,\cdots J^n)$$

$$J^i \neq J^j,\ i\neq j$$

$$q^{ii} = 0.$$

Via an appropriate transformation the inverse problem associated with (i), (ii) can be expressed as a matrix RHBVP of the form (4). The potentials u_j, q can be shown to satisfy nonlinear evolution equations by appending to (i) and (ii), suitable linear time evolution equations. One then finds that the scattering data $V(x,t,k)$ evolves simply in time. Well known solvable nonlinear equations include the Boussinesq, modified KdV, sine-Gordon, nonlinear Schrodinger, and three wave interaction equations. The reader may wish to consult for example [2a-e] for a detailed discussion of some of this material.

It is most significant that these concepts can be generalized to 2 spatial plus one time dimension. Here the prototype equation is the Kadomtsev-Petviashvili (K-P) equation:

$$(u_t + 6uu_x + u_{xxx})_x = -3\sigma^2 u_{yy} \tag{7}$$

which is the compatibility equation between the following linear problems:

$$\sigma v_y + v_{xx} + u(x,y,t)v = 0 \tag{8}$$

$$v_t + 4v_{xxx} + 6uv_x + 3\left(u_x - \sigma \int_{-\infty}^{x} u_y \, dx'\right)v + \gamma v = 0 \tag{9}$$

(γ = const.). We shall consider the question of solving (7) for $u(x,y,0)$ decaying sufficiently rapidly in the plane $r^2 = x^2 + y^2 \to \infty$. Physically speaking, both cases $\sigma^2 = -1$ (KPI) $\sigma^2 = +1$ (KPII) are of interest. Whereas KPI can be related to a RHBVP of a certain type (nonlocal; see ref.[3]) KPII turns out to require new ideas. Letting

$$v = \mu(x,y,k)e^{ikx + k^2 y/\sigma}$$

$\sigma = \sigma_R + i\sigma_I$, $\sigma_R \neq 0$. Then there exist functions μ bounded for all x,y satisfying $\mu \to 1$ as $|k| \to \infty$. However such a function turns out to be nowhere analytic in k, rather it depends nontrivially on both the real and imaginery parts of $k (k = k_R + ik_I)$. $\mu = \mu(x,y,k_R,k_I)$.

In fact μ satisfies a generalization of a RHBVP - namely a $\bar{\partial}$ (DBAR) problem where μ satisfies,

$$\frac{\partial\mu}{\partial\bar{k}} = \mu(x,y,\xi_0,k_I)\,V(x,y,k_R,k_I) \tag{10}$$

where $\frac{\partial}{\partial\bar{k}} = \frac{1}{2}\left(\frac{\partial}{\partial k_R} + i\frac{\partial}{\partial k_I}\right)$ and V has the structure

$$V(x,y,k_R,k_I) = \frac{\operatorname{sgn}(k_0)\,e^{i\beta(x,y,k_R,k_I,\xi_0)}}{2\pi|\sigma_R|}\,T(k_R,k_I)$$

$$\beta(x,y,k_R,k_I,\xi_0) = \left(x + 2y\frac{k_I}{\sigma_R}\right)(\xi_0 - k_R) = -2\left(x + 2y\frac{k_I}{\sigma_R}\right)k_0$$

$$\xi_0 = -k_R - \frac{2\sigma_I}{\sigma_R}k_I,\quad k_0 = k_R + \frac{\sigma_I}{\sigma_R}k_I \tag{11}$$

(10-11) may be converted into a linear integral equation by employing the generalized Cauchy formula. $T(k_R,k_I)$ is viewed as the "nonphysical" data, (i.e. inverse scattering data or inverse data) and the potential is reconstructed via

$$u(x,y) = \frac{2i}{\pi}\frac{\partial}{\partial x}\iint \mu(x,y,\xi_0,k_I)V(x,y,k_R,k_I)\,dk_R\,dk_I. \tag{12}$$

The basic ideas used in order to derive these equations is as follows. We convert the equation for $\mu = \mu(x,y,k)$:

$$\sigma\mu_y + \mu_{xx} + 2ik\mu_x - u(x,y)\mu = 0 \tag{13}$$

into an integral equation

$$\mu(x,y,k) = 1 + \tilde{G}(u,\mu) \tag{14}$$

where

$$\tilde{G}(f) = G*f = \iint G(x-x',y-y',k)\,f(x',y')\,dx'dy', \tag{15}$$

the Green's function kernel being givenby $(k=k_R+ik_I)$:

$$\begin{aligned} G(x,y,k_R,k_I) &= \frac{1}{(2\pi)^2}\frac{e^{i(\xi x+\gamma y)}}{(i\sigma\eta-\xi^2-2k\xi)}\,d\xi\,d\gamma \\ &= \frac{\operatorname{sgn}(y)}{2\pi\sigma}\int d\xi\, e^{ix\xi + \xi(\xi+2k)y/\sigma} \\ &\quad\cdot\,\theta\,(-y\sigma_R(\xi^2+2\xi k_0))\,d\xi \end{aligned} \tag{16}$$

where $k_0 = k_R = -\frac{\sigma_I}{\sigma_R} k_I$ and $\upsilon(x) = \{1 x>0,\ 0 x<0\}$ (16)

The $\bar{\partial}$ derivative of the Green's function is especially simple,

$$\frac{\partial G}{\partial \bar{k}}(x,y,k_R,k_I) = \frac{\mathrm{sgn}(k_0)}{2\pi|\sigma_R|} e^{i\beta(x,y,k_R,k_I)} \tag{17}$$

when

$$\partial/\partial\bar{k} = \frac{1}{2}\left(\frac{\partial}{\partial k_R} + i\,\frac{\partial}{\partial k_I}\right) \text{ and}$$

$$\beta(x,y,k_R,k_I) = -2\left(x+2y\frac{k_I}{\sigma_R}\right)k_0.$$

Taking the $\bar{\partial}$ derivative of (14)

$$\frac{\partial \mu}{\partial \bar{k}}(x,y,k_R,k_I) = \iint \frac{\partial G}{\partial \bar{k}}(x-x',y-y',k_R,k_I)u(x',y')\mu(x',y',k_R,k_I)dx'dy'$$
$$+ \iint G(x-x',y-y',k_R,k_I)u(x',y')\frac{\partial \mu}{\partial \bar{k}}(x',y',k_R,k_I)dx'dy' \tag{18}$$

and using (17) shows that

$$\frac{\partial \mu}{\partial \bar{k}} = \frac{\mathrm{sgn}(k_0)}{\pi\ \sigma} T(k_R,k_I)\ w(x,y,k_R,k_I) \tag{19}$$

where $T(k_R,k_I) = \iint e^{-i\beta(x,y,k_R,k_I)}u(x,y)\mu(x,y,k_R,k_I)dxdy$ and $w(x,y,k_R,k_I)$ satisfies:

$$w(x,y,k_R,k_I) = e^{i\beta(x,y,k_R,k_I)} + \iint G(x-x',y-y',k_R,k_I)$$
$$u(x',y')w(x',y',k_R,k_I)dx'dy'. \tag{20}$$

Multiplying (20) by $e^{-i\beta(x,y,k_R,k_I)}$ and employing the following symmetry condition on the Green's function

$$e^{-i\beta(x,y,k_R,k_I)}G(x,y,k_R,k_I)$$
$$= G(x,y,\xi_0,k_I) \tag{21}$$

where $\xi_0 = -k_0 - \frac{\sigma_I}{\sigma_R}k_I$, yields

$$w(x,y,k_R,k_I) = e^{i\beta(x,y,k_R,k_I)}\ \mu(x,y,\xi_0,k_I) \tag{22}$$

whereupon (10-11) follow. The eigenfunction μ is recovered with the generalized Cauchy formula

$$\mu(x,y,k_R,k_I) = 1 + \frac{1}{\pi}\iint \frac{\frac{\partial\mu}{\partial\bar{k}}(x,y,k_R',k_I')}{k-k'}\,dk_R',dk_I' \tag{23}$$

noting that using (10-11), (23) becomes a linear integral equation for μ. The potential $u(x,y)$ is recovered by taking $k\to\infty$ in (13) or (14) and (23).

For the K-P the evolution of the data obeys ($\gamma = 4ik^3$ in (9)

$$\frac{\partial T}{\partial t} = (8ik_0)(6kk_0 - 4k_0^2 - 3k^2)T \tag{24}$$

where $k_0 = k_R + \frac{\sigma_I k_I}{\sigma_R}$, $k = k_R + ik_I$.

Special cases include $\sigma = \sigma_R + i\sigma_I$:

(a) KP_{II}; $\sigma = -1$: $\sigma_R = -1, \sigma_I = 0$

$$\frac{\partial T}{\partial t} = 8ik_R(3k_I^2 - k_R^2)T \tag{25}$$

(b) KP_I; $\sigma = i$: $\sigma_R \to 0-$, $\sigma_I = 1$, $\hat{k}_I = k_I/\sigma_R$

$$\frac{\partial T}{\partial t} = -8i(k_R+\hat{k}_I)(k_R^2+2k_R\hat{k}_I+4\hat{k}_I^2)T \tag{26}$$

These formulae allow us in principle to solve the Cauchy problem for K-P and in particular the limit (ii) discussed above allows us to give an alternative solution for KP_I via $\bar{\partial}$ and not via a nonlocal RHBVP.

Similar ideas apply to higher order scalar problems

(iii) $$\sigma\frac{\partial v}{\partial y} + \frac{\partial^n v}{\partial x^n} + \sum_{j=2}^{n} u_j(x)\frac{\partial^{n-j}v}{\partial x^{n-j}} = 0$$

where: $v, u_j \in \mathbb{C}$ and to first order systems

(iv) $$\sigma\frac{\partial v}{\partial y} + J\frac{\partial v}{\partial x} + q(x,y)v = 0$$

where: $v, q \in \mathbb{C}^{N\times N}$, $J = \mathrm{diag}(J^1,\ldots,J^N)$, $J^i \neq J^j$, $i \neq j$ with $q^{ii} = 0$.

Interested readers may consult refernce 4a, and review 4b for more details.

The notion of $\bar\partial$ extends to higher dimensional scattering and inverse scattering problems. However as we shall mention, despite the fact that the inverse scattering problem is essentially tractable there does not appear to be any local nonlnear evolution equations in dimensions greater than 2 + 1 associated with multidimensional generalizations of (iii) or (iv).

Our prototype scattering problem will be

$$\sigma v_y + \Delta v + u(x,y)v = 0$$

$$\Delta = \sum_{\ell=1}^{n} \frac{\partial^2}{\partial x_\ell^2}, \qquad x \in \mathbb{R}^n, \quad y \in \mathbb{R} . \tag{27}$$

Letting

$$v = \mu(x,y,k)e^{ik\cdot x + k^2 y/\sigma}$$

$$k = k_R + ik_I, \quad k \in \mathbb{C}^n$$

$$k\cdot x = \sum_1^n k_j x_j, \quad \sigma = \sigma_R + i\sigma_I .$$

Then there exist functions μ bounded for all x, y satisfying $\mu \to 1$, as $|k_j| \to \infty$, $j = 1,\dots,n$. When $\sigma_R \neq 0$ μ turns out to be nonanalytic in each of the variables k, i.e. $\mu = \mu(x,y,k_{R_1},\dots k_{R_n}, k_{I_1},\dots,k_{I_n})$ and satisfies a $\bar\partial$ problem linear in μ, in each of the variables k_j; i.e. we shall show that μ satisfies an equation of the form,

$$\frac{\partial \mu}{\partial \bar k_j} = \tilde T_j(\mu); \qquad j = 1,\dots,n \tag{28}$$

where $\tilde T_j$ is an appropriate linear integral operator.

The basic idea in order to derive (28) follows a similar format to the two dimensional case described earlier. From the definition of $\mu(x,y,k)$ below (27) we see that it satisfies

$$\sigma\mu_y + \Delta\mu + 2ik\cdot\nabla\mu - \mu(x,y) = 0. \tag{29}$$

We convert to an integral equation

$$\mu = 1 + \tilde{G}(u\,\mu) \tag{30}$$

where the Green's function kernel is given by

$$G(x,y,k_R,k_I) = \frac{1}{(2\pi)^{n+1}} \iint \frac{e^{i(x\cdot\xi+y\eta)}}{i\sigma y-\xi^2-2k\cdot\xi} d\xi dy$$

$$= \frac{\mathrm{sgn}(y)}{\sigma} \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi+\frac{y}{\sigma}(\xi^2+2k\cdot\xi)}. \tag{31}$$

$$\cdot\ \Theta\left(-y\sigma_R(\xi^2+2\left(k_R+\frac{\sigma_I k_I}{\sigma_R}\right)\cdot\xi\right)d\xi. \tag{32}$$

Taking the $\bar{\partial}$ derivative of (30)

$$\frac{\partial\mu}{\partial\bar{k}_j} = \frac{\partial\tilde{G}}{\partial\bar{k}}(u\mu) + \tilde{G}\left(u\frac{\partial\mu}{\partial\bar{k}_j}\right), \tag{33}$$

and using

$$\frac{\partial G}{\partial\bar{k}_j}(x,y,k_R,k_I) = -\frac{1}{(2\pi)^n}|\sigma_R|\int e^{i\beta(x,y,k_R,k_I,\xi)}$$

$$\cdot(\xi_j-k_{Rj})\delta(\rho(\xi))d\xi \tag{34}$$

where

$$\beta(x,y,k_R,k_I\xi) = \left(x+2y\frac{k_I}{\sigma_R}\right).(\xi\cdot k_R)$$

$$\rho(\xi) = \left(\xi+\frac{\sigma_I}{\sigma_R}k_I\right)^2 - \left(k_R+\frac{\sigma_I}{\sigma_R}k_I\right)^2 \tag{35}$$

shows that

$$\frac{\partial\mu}{\partial\bar{k}_j} = -\frac{1}{(2\pi)^n}\frac{1}{|\sigma_R|}\int T(k_R,k_I\xi)(\xi_j-k_{Rj})\delta(\rho(\xi))$$

$$\cdot\ w(x,y,k_R,k_I,\xi)d\xi \tag{36}$$

where

$$T(k_R,k_I,\xi) = \int e^{-i\beta(x,y,k_R,k_I,\xi)} u(x,y)\, \mu(x,y,k_R,k_I)\,dxdy \qquad (37)$$

and w satisfies

$$w(x,y,k_R,k_I,\xi) = e^{i\beta(x,y,k_R,k_I\xi)} + \tilde{G}(uw). \qquad (38)$$

Multiplying (37) by $e^{-i\beta}$ and using the symmetry condition

$$e^{-i\beta(x,y,k_R,k_I,\xi)}\, G(x,y,k_R,k_I) = G(x,y,\xi,k_I) \qquad (39)$$

yields

$$w(x,y,k_R,k_I,\xi) = e^{-i\beta(x,y,k_R,k_I,\,\xi)}\mu(x,y,\xi,\, k_I) \qquad (40)$$

and hence (36) gives

$$\frac{\partial\mu}{\partial\bar{k}_j} = \tilde{T}_j(\mu) = -\frac{1}{(2\pi)^n}\frac{1}{|\sigma_R|}\int T(k_R,k_I,\xi)(\xi_j-k_{Rj})$$

$$\cdot\ \delta(\rho(\xi))\, e^{i\beta(x,y,k_R,k_I,\xi)}\mu(x,y,\xi,k_I)d\xi. \qquad (41)$$

We see that $\tilde{T}_j$ is an integral operator which depends on a scalar scattering function $T = T(k_R,k_I,\xi)\xi$ being effectively (n-1) integration parameters (due to the delta function in (41) in the nonlocal operator $\tilde{T}_j$).

One can use a generalized Cauchy formula such as (23) in order to obtain a linear integral equation to reconstruct μ. However due to the redundancy of the data discussed below, we find that an alternative method is more useful. The inverse problem is redundant, i.e. we are given $T(k_R,k_I,\xi)$ (3n-1 parameters) and we must reconstruct a local potential $u(x,y)$ (n+1 parameters). A serious issue is how to characterize admissible inverse data T, i.e. data that really arises from a local potential (small generic changes in $T(k_R,k_I,\xi)$ cannot be expected to arise from a local potential $u(x,y)$). Insight into this question is obtained by noting that T must satisfy a nonlinear constraint, one which is obtained by requiring $\partial^2\mu/\partial\bar{k}_i\partial\bar{k}_j = \partial^2\mu/\partial\bar{k}_j\partial\bar{k}_i$ $(i \neq j)$. the form of this constraint is given by

$$\mathcal{L}_{ij}(T) = \tilde{N}_{ij}[T] \tag{42}$$

where $\mathcal{L}_{ij}$ is a linear operator and $\tilde{N}_{ij}$ a nonlinear (quadratic) nonlocal operator. These operators are given by

$$\mathcal{L}_{ij} = (\xi_j - k_{jR})\left(\frac{\partial}{\partial \bar{k}_i} + \frac{1}{2}\frac{\partial}{\partial \xi_i}\right) - (\xi_i - k_{iR})\left(\frac{\partial}{\partial \bar{k}_j} + \frac{1}{2}\frac{\partial}{\partial \xi_j}\right) \tag{43}$$

$$\tilde{N}_{ij}(T) = \int [(\xi_j' - k_{jR})(\xi_i - \xi_i') - (\xi_i' - k_{iR})(\xi_j - \xi_j')]$$

$$\cdot \delta(\rho(\xi')\; T\,(k_R, k_I, \xi)\; T(\xi', k_I, \xi)d\xi'. \tag{44}$$

There is, in fact, an explicit transformation of variables

$$(k_R, k_I, \xi) \to (\chi, w_o, w) \in \mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}^n$$

which simplifies (42). Namely,

$$k_{R1} = \sum_{j=2}^{n} w_j \chi_{Rj} - \frac{w_1}{2} - \frac{\sigma_I w_0 w_1}{2w^2}\ ,$$

$$k_{Rj} = -w_1 \chi_{Rj} - \frac{w_j}{2} - \frac{\sigma_I w_0 w_j}{2w^2},\ (j \geq 2)$$

$$k_{i1} = \sum_{j=2}^{n} w_j \chi_{Ij} + \frac{\sigma_R w_0 w_1}{2w^2}\ ,$$

$$k_{ij} = -w_1 \chi_{Ij} + \frac{\sigma_R w_0 w_j}{2w^2}\ ,\ (j \geq 2)$$

$$\xi_1 = \sum_{j=2}^{n} w_j \chi_{Rj} + \frac{w_1}{2} - \frac{\sigma_I w_0 w_1}{2w^2}\ ,$$

$$\xi_j = -w_1 \chi_{Rj} + \frac{w_j}{2} - \frac{I w_0 w_j}{2w^2}\ .\ (j \geq 2) \tag{45}$$

transforms (42) into:

$$\frac{\partial T}{\partial \tilde{x}_j} = \tilde{N}_{ij}(T)(\chi,w_0,w) \; . \; j=2,---n \tag{46}$$

using the generalized Cauchy formula (23) we have

$$I_j[T](\chi,w,w_0) = T(\chi,w,w_0) - \frac{1}{\pi}\iint \frac{\tilde{N}_{ij}(T)(\tilde{\chi},w,w_0)}{\chi - \tilde{\chi}} \, d\chi_R' d\chi_I'$$

$$= \hat{u}(w,w_0) \tag{47}$$

where

$$\tilde{x} = (x_2,x_3,---x_j',---x_n)$$

$$\hat{u}(w_0,w) = \iint e^{-i(yw_0+x\cdot w)}u(x,y)dxdy \tag{48}$$

We have used the fact that when $w_0 = 2k_I\cdot(\xi-k_R)/\sigma_R$ and $w = \xi-k_R$ are kept fixed, $T(\chi,w,w_0) \to \hat{u}(w,w_0)$ (The Fourier Transform of u(x,y)) for large $\chi_j(w_1\neq 0)$; this is the analogue of the Born approximation.

We expect that for suitably "small" u (i.e. no homogeneous solutions to the relevant integral quations) if I is independent of χ,j and decays sufficiently fast for $|w|,|w_0|\to\infty$, then $T(k_R,k_I,\xi)$ is admissable. Moreover (47) gives a formula to reconstruct the potential by quadratures. Limits to case $\sigma = i$ and reductions to stationary potentials $u(x,y) = u(x)$ can be carried out. Details can be found in Ref. [5a,b]. It should also be noted that in recent work Nachman and Lavine [5c] have extended theaw ideas to situations where there are homogeneous solutions to the relevant integral equations. (42) also suggests why simple local nonlinear evolution equations have not been associated with equation (27). Namely in the previous lower dimensional (2+1 and 1+1) problems the time evolution of the scattering data obeyed a particularly simple equation, (e.g. $\frac{\partial T}{\partial t} = \omega(k_R,k_I)T$). However in this case such a simple flow will not be maintained - due to the nonlinear constraint (42).

These ideas can be generalized to first order systems:

$$\text{(v)} \qquad \frac{\partial v}{\partial y} + \sigma \sum_{j=1}^{n} J_j \frac{\partial v}{\partial x_j} = qv$$

$$v,\ q \in \mathbb{C}^{N\times N},\quad J_j = \mathrm{diag}(J_j^1,\ldots,J_j^N)$$

$$J_j^k \neq J_j^\ell, \qquad k \neq \ell\ .$$

with many similar results obtained 6a,b,c; though there are some important differences as well: see ref. [6c]. Again the scattering data satisfies a nonlinear constraint. In general, there is no compatible local nonlinear evolution equation associated with (v). However when certain restrictions are put on J_j then the constraint equation becomes linear and the so-called N wave interaction equations are compatible with the system (v). Nachman and Ablowitz [6a] showed that at most, the system would be 3+1 dimensional, and Fokas[6b] showed that indeed the system is reducible to 2+1 dimensions by a transformation of independent variables (characteristic variables). In [6c] Fokas studies the inverse scattering of (v). For $\sigma = i$ he finds an equation similar to (42). However its integrated form shows that in order for the potential to be reconstructed one must solve a reduced system of equations of the form (v): i.e. for N = 2. This is in contrast to the scalar problem where reconstruction is via quadratures.

Beals and Coifman haven an alternative but similar formulation [7a,b] for multidimensional scalar problems.

There is an n-dimensional problem which also fits within the framework of IST: The so-called generalized wave and generalized sine-Gordon equation (GWE and GSGE). These equations arise in the context of differential geometry and serve to extend the classical results of Bäcklund for the sine-Gordon equation to n-dimensions [8]. The n-dimensional Bäcklund transformation is given by:

$$dX + XA^{t}X = A - XB, \tag{49}$$

where

$$dX = \sum_{j=1}^{n} \frac{\partial X}{\partial x_j}\, dx_j,$$

$$A_{ij} = \beta_i(z) a_{ij} dx_j,$$

$$B_{ij} = \frac{1}{a_{1i}} \frac{\partial a_{1j}}{\partial x_i} dx_j - \frac{1}{a_{1j}} \frac{\partial a_{1i}}{\partial x_j} dx_i, \quad 1 \le i,\ j \le n, \tag{50}$$

and $a = \{a_{ij}\} \in \mathbb{R}^{n \times n}$. Equations (49-50) reduce to the Bäcklund transformation for the generalized sine-Gordon equation (GSGE) when

$$\beta_i(z) = (z^2 + (2\delta_{i1} - 1))/2z, \tag{51}$$

and for the generalized wave equation (GWE) when

$$\beta_i(z) = -(1-z^2)/2z \equiv \lambda(z). \tag{52}$$

The compatibility condition required for the existence of solutions to these Bäcklund transformations results in a system of second-order partial differential equations for an orthogonal $n \times n$ matrix $a = \{a_{ij}\}$ in (49) which is a function of n independent variables $a = a(x_1, x_2, \ldots, x_n)$. The equation has the form

$$\frac{\partial}{\partial x_i}\left(\frac{1}{a_{1i}} \frac{\partial a_{1j}}{\partial x_i}\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{a_{1k}} \frac{\partial a_{1i}}{\partial x_j}\right)$$

$$+ \sum_{k \neq i,j} \frac{1}{a_{1k}^2} \frac{\partial a_{1i}}{\partial x_k} \frac{\partial a_{1j}}{\partial x_k} = \epsilon a_{1i} a_{1j}, \quad i \neq j,$$

$$\frac{\partial}{\partial x_k}\left(\frac{1}{a_{1j}} \frac{\partial a_{1i}}{\partial x_j}\right) = \frac{1}{a_{1k} a_{1j}} \frac{\partial a_{1i}}{\partial x_k} \frac{\partial a_{1k}}{\partial x_j}, \quad i, j, k \text{ distinct},$$

$$\frac{\partial a_{jk}}{\partial x_k} = \frac{\partial a_{ji}}{a_{1i}} \frac{\partial a_{1k}}{\partial x_i}, \quad i \neq k, \tag{53}$$

where $\epsilon = 1$ for the GSGE and $\epsilon = 0$ for the GWE.

We observe that when $n = 2$ and $\kappa = 1$ (GSGE), the orthogonal matrix $a = \{a_{ij}\}$ given by

$$a = \begin{pmatrix} \cos\frac{1}{2}u & \sin\frac{1}{2}u \\ -\sin\frac{1}{2}u & \cos\frac{1}{2}u \end{pmatrix} \tag{54}$$

for the function $u = u(x,t)$ reduces the GSGE to the classical sine-Gordon equation ($\kappa = -1$),

$$u_{tt} - u_{xx} - \kappa\sin u = 0. \tag{55}$$

On the other hand when $n = 2$ and $\kappa = 0$, then with (54) the GWE reduces to the wave equation (55). When $n \geq 3$ the generalization of the wave equations discussed here is nonlinear.

The Bäcklund transformations (49) described above are in fact matrix Riccati equations. Linearizations of such a system can be performed in a striaghtforward manner. Introducing the transformation

$$X = UV^{-1}, \tag{56}$$

where U, V and $n \times n$ matrix functions of $x_1,\ldots,x_n$, the following linear system is deduced:

$$\begin{pmatrix} dU \\ dV \end{pmatrix} = \begin{pmatrix} 0 & A \\ A^t & B \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} \tag{57}$$

with the components of A, B given by (50). Compatibility ensures that the orthogonal matrix $a = \{a_{ij}\}$ satisfies the GSGE with (51) and GWE with (52). Alternatively, if we call

$$\begin{pmatrix} U \\ V \end{pmatrix} = \psi \ , \tag{58}$$

the following linear system of 2n o.d.e.'s are obtained:

$$\frac{\partial\psi}{\partial x_j} = \lambda\tilde{A}_j\psi + C_j\psi, \tag{59}$$

where $\tilde{A}_j$, C_j are $2n \times 2n$ matrices with the block structure

$$\tilde{A}_j = \begin{pmatrix} 0 & \tilde{a}_j \\ \tilde{a}_j^t & 0 \end{pmatrix}, \quad C_j = \begin{pmatrix} \gamma_j & 0 \\ 0 & \tilde{\gamma}_j \end{pmatrix}. \tag{60}$$

Here $\tilde{a}_j$, $\tilde{\gamma}_j$ are $n \times n$ matrices having the following structure:

$$\tilde{a}_j = (\frac{\delta}{\lambda} - 1)e_1 a_j + a_j,$$

$$a_j = a e_j \tag{61}$$

where $e_j = \{e_j\}_{ik}$ is the unit matrix

$$\{e_j\}_{ik} = \begin{cases} 1 & i = k = j, \\ 0 & \text{otherwise}, \end{cases} \tag{62}$$

and in component form γ_j takes the form

$$(\gamma_j)_{k\ell} = (1-\delta_{kj})\frac{1}{a_{\ell k}} \frac{\partial a_{\ell j}}{\partial x_k} \delta_{\ell j} - (1-\delta_{\ell j})\frac{1}{a_{1\ell}} \frac{\partial a_{1j}}{\partial x_\ell} \delta_{kj}. \tag{63}$$

In (61) a is the orthogonal matrix $\mathbb{R}^n \to SO(n)$ associated with the GWE when $\delta = \lambda$ and with the GSGE when $\delta = \frac{1}{2}(z + 1/z)$, $\lambda = \frac{1}{2}(z - 1/z)$, and γ_j is the matrix (63): $\mathbb{R}_n \to M_n(\mathbb{R})$, $\gamma_j + \gamma_j^t = 0$. Equations (53) arise as the compatibility condition associated with (58). More explicitly, for the GWE the scattering problem takes the form $[\psi = \psi(x,\lambda)]$

$$\frac{\partial \psi}{\partial x_j} = \lambda A_j \psi + C_j \psi \tag{64}$$

with

$$A_j = \begin{pmatrix} 0 & a_j \\ a_j^t & 0 \end{pmatrix}, \tag{65}$$

and C_j given by (60,63).

For the GSGE the scattering problem for $\psi = \psi(x,z)$ takes the form

$$\frac{\partial \psi}{\partial x_j} = \delta(z) \begin{pmatrix} 0 & e_1 a_j \\ a_j^t e_1 & 0 \end{pmatrix} \psi$$

$$+ \lambda(z) \begin{pmatrix} 0 & (I-e_1)a_j \\ a_j^t(I-e_1) & 0 \end{pmatrix} \psi + C_j \psi, \tag{66}$$

$\delta(z)$, $\lambda(z)$, C_j given above, or equivalently

$$\frac{\partial\psi}{\partial x_j} = \frac{z}{2} A_j \psi + \frac{z}{2} B_j \psi + C_j \psi, \tag{67}$$

where

$$B_j = \begin{pmatrix} 0 & u a_j \\ a_j^t u & 0 \end{pmatrix}, \quad u = \mathrm{diag}(+1, -1, \ldots, -1). \tag{68}$$

In [8] it is shown how these linear problems may be viewed as a direct and inverse scattering problem for the GWE and GSGE. Namely the direct and inverse problem may be solved for matrix potentials, depending on the orthogonal matrix a, tending to the identity sufficiently fast in certain "generic" directions. It should be noted that solving the n-dimensional GWE and GSGE reduces to the study of the scattering and inverse scattering associated with a coupled system of n one-dimensional o.d.e.'s. This is in marked contrast to other attempts described earlier to isolate solvable (local) multidimensional nonlinear evolution equation which are compatibility conditions of two Lax-type operators, e.g.,

$$L\psi = \lambda\psi \tag{69}$$

$$\psi_t = M\psi \tag{70}$$

where L is a partial differential operator with the variable t entering only parametrically. Although as we have seen nonlinear evolution equations in three independent variables can be associated with such Lax pairs (e.g. the K-P, Davey-Stewartson, three wave interaction equations, etc.) little progress via this route has been made in more than three dimensions. As discussed earlier one has to overcome a serious constraint inherent in the scattering/inverse scattering theory for higher dimensional partial differential operators in order to be able to isolate associated solvable nonlinear equations, i.e. the scattering data generally satisfies a nonlinear equation (eq. (42)). The analysis associated with the GWE and GSGE avoids these difficulties since the GWE and GSGE problems are simply a compatible set of nonlinear one-dimensional o.d.e.'s.

The results in ref. [8] demonstrate that the initial value problem is posed with given data along lines and not on (n-1) dimensional manifolds.

Similar ideas apply to certain n-dimensional extensions of the so-called anti-self-dual Yang-Mills equations (SDYM) [9]. In two complex variables the self-dual Yang Mills equations take the form (see [10])

$$\frac{\partial}{\partial \bar{x}_1}\left(\Omega^{-1}\frac{\partial \Omega}{\partial x_1}\right) + \frac{\partial}{\partial \bar{x}_2}\left(\Omega^{-1}\frac{\partial \Omega}{\partial x_2}\right) = 0, \tag{71}$$

where Ω is a positive matrix valued function of $(x_1,x_2)\in\mathbb{C}^2$.
Alternatively SDYM takes the form

$$\frac{\partial A_1}{\partial \bar{x}_1} + \frac{\partial A_2}{\partial \bar{x}_2} = 0 \tag{72}$$

$$\frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1} + [A_1,A_2] = 0, \tag{73}$$

where

$$A_j = -\Omega^{-1}\frac{\partial \Omega}{\partial x_j} \tag{74}$$

The SDYM may be obtained via the compatibility condition of the following linear system

$$\begin{aligned} \frac{\partial m}{\partial x_1} - z\frac{\partial m}{\partial \bar{x}_2} &= A_1 m \\ \frac{\partial m}{\partial x_2} + z\frac{\partial m}{\partial \bar{x}_1} &= A_2 m \end{aligned} \tag{75}$$

multidimensional extensions may be obtained. For example, consider the linear system

$$D_z^j m(x,z) = A_j(x)m(x,z),\ j = 1,\cdots n \tag{76}$$

$$D_z^j = \frac{\partial}{\partial x_j} + zs_j\frac{\partial}{\partial \bar{x}_{j+1}} \tag{77}$$

and

$$x_{n+1} = x_1\ ,\ s_j = (-1)^j.$$

Compatibility (commutativity) implies:

$$D_z^i A_j - D_z^j A_i + [A_i, A_j] = 0 \tag{78}$$

$$\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} + [A_i, A_j] = 0 \tag{79}$$

$$s_j \frac{\partial A_i}{\partial x_{j+1}} - s_i \frac{\partial A_j}{\partial \bar{x}_{i+1}} = 0. \tag{80}$$

A potential Ω may be introduced as before: (81)

$$A_j = \Omega^{-1} \frac{\partial \Omega}{\partial x_j}$$

to obtain

$$s_j \frac{\partial}{\partial \bar{x}_{j+1}} \left(\Omega^{-1} \frac{\partial \Omega}{\partial x_i}\right) - s_i \frac{\partial}{\partial \bar{x}_{i+1}} \left(\Omega^{-1} \frac{\partial \Omega}{\partial x_j}\right) = 0 . \tag{82}$$

Clearly when n=2 this system reduces to the classical SDYM equation.

Solutions to these equations may be constructed via the $\bar{\partial}$ method. Define

$$D_z^j = L_1^j + z L_2^j \tag{83}$$

with

$$L_1^j = \frac{\partial}{\partial x_j}, \quad L_2^j = s_j \frac{\partial}{\partial \bar{x}_{j+1}} \quad .$$

We shall show that the $\bar{\partial}$ integral equation

$$m(x,z) = I + \frac{1}{2\pi i} \iint \frac{(mV)(x,\zeta)}{\zeta - z} \, d\zeta \wedge d\bar{\zeta} \tag{84}$$

satisfies (76). Operating on (84) with D_z^j yields,

$$D_z^j m = \frac{1}{2\pi i}\int \frac{(L_1^j m)V + m(L_1^j V)}{\zeta - z} d\zeta \wedge d\bar{\zeta} + J \tag{85}$$

where

$$J = \frac{1}{2\pi i}\int \frac{z L_2^j (mV)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}$$

$$= -\frac{1}{2\pi i}\int L_2^j (mV) d\zeta \wedge d\bar{\zeta}$$

$$+ \frac{1}{2\pi i}\int \frac{\zeta L_2^j (mV)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}. \tag{86}$$

Putting (85), (86) together gives

$$D_z^j m = \tilde{A}_j + \frac{1}{2\pi i}\int \frac{(D_\zeta^j m)V + m(D_\zeta^j V)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta} \tag{87}$$

where

$$\tilde{A}_j(x) = -\frac{1}{2\pi i}\int L_2^j (mV) d\zeta \wedge d\zeta = -\frac{1}{2\pi i}\, s_j \frac{\partial}{\partial \bar{x}_{j+1}} \int (mV) d\zeta \wedge d\bar{\zeta}\ . \tag{88}$$

We shall require $V(x,z)$ to satisfy

$$D_z^j V = 0 \tag{89}$$

in which case using (84) in (87) by writing

$$\tilde{A}_j = \tilde{A}_j\left(m - \frac{1}{2\pi i}\int \frac{mV}{\zeta - z} d\zeta \wedge d\bar{\zeta}\right) \tag{90}$$

we find

$$(D_z^j m - \tilde{A}_j m) = \frac{1}{2\pi i} \int \frac{((D_\zeta^j m) - \tilde{A}_j(x)m)V}{\zeta - z} d\zeta \wedge d\bar{\zeta}. \tag{91}$$

For V suitably chosen (84) has a unique solution in which case

$$D_z^j m - \tilde{A}_j m = 0. \tag{92}$$

Thus $\tilde{A}_j = A_j$ and solutions of the extended SDYM are obtained.

The condition (89) is satisfied if we take $V(x,z) = V(u(x),z)$, with $u_j(x) = zx_j + s_{j+1}\bar{x}_{j+1}$ and V holomorphic in the u_j. Then

$$D_z^j V = \left(\frac{\partial}{\partial x_j} + zs_j\frac{\partial}{\partial \bar{x}_{j+1}}\right) V(u_1,\ldots u_n,z)$$

$$= \sum_{\ell=1}^{n} V'(u_\ell,z)(z\delta_{j\ell} + s_j s_{j+1} z\delta_{j\ell}) = 0 \tag{93}$$

by virtue of $s_j = (-)^j$. In ref. [9] other examples of multidimensional extensions of SDYM and a rigorous derivation of the foregoing is given.

ACKNOWLEDGEMENTS

I am most pleased to acknowledge the many crucial contributions of my colleagues: A.S. Fokas, D. Bar Yaacov, A.I. Nachman, R. Beals and K. Tenenblat. This work was supported in part by the National Science Foundation under grant number DMS-8202117, the Office of Naval Research under grant number N00014-76-C-0867, and the Air Force Office of Scientific Research under grant number AFOSR-84-0005

RERFERENCES

[1] Ablowitz, M.J. and Segur, H., "Solitons and the Inverse Scattering Transform", SIAM Appl. Math., Phila., PA, 4 (1981).

[2a] Ablowitz, M.J. and Fokas, A.S., "Comments on the Inverse Scattering Transform and Related Nonlinear Evolution Equations", Lect. Notes in Phys., 189, Proc. CIFMO School and Workshop held at Oaxtepec, Mexico (K.B. Wolf, ed.), Springer, 1982.

[2b] Beals, R. and Coifman, R., Commun. Pure Appl. Math., 1984, pp. 39-90.

[2c] Beals, R., The Inverse Problems for Ordinary Differential Operators on the Line; to appear, Amer. J. of Math.

[2d] Shabat, A.B., Func. Annal and Appl. 9, (1975); Diff. eq. XV (1979) 1824.

[2e] Caudrey, P., Physica 6D (1982) 51.

[3a] Manakov, S.V., Phys. 3D, 420 (1981).

[3b] Fokas, A.S. and Ablowitz, M.J. Stud. Appl. Math. 69, 211 (1983).

[4a] Ablowitz, M.J., BarYaacov, D., and Fokas, A.S., Stud. Appl. Math. 69, 135, (1983).

[4b] Fokas, A.S. and Ablowitz, M.J., The Inverse Scattering Transform for Multidimensional (2+1) Problems, Lect. Notes in Phys., 189, Proc. CIFMO School and Workshop held at Oaxtepec, Mexico, 1982, K.B. Wolf, ed.

[5a] Nachman, A.I. and Ablowitz, M.J., Stud. Appl. Math. 71, 243-250, (1984).

[5b] Ablowitz, M.J. and Nachman, A.I., Physica 18D, 223 (1986).

[5c] Nachman, A.I. and Lavine, R., On the Inverse Scattering Transform for the n-Dimensional Schrodinger Operator, to be published as Proceedings of conference on Nonlinear Evolution Equations, Solitons and the Inverse Scattering Transform, held at Mathematische Forschungsistitut, Oberwolfach W. Germany.

[6a] Nachman, A.I. and Ablowitz, M.J., Stud. Appl. Math. 71, 251-262, (1984).

[6b] Fokas, A.S., Phys. Rev. Lett., 57, No. 2, 159 (1986).

[6c] Fokas, A.S., J. Math. Phys. 27 (7), 1737-1746 (1986).

[7a] Beals, R., and Coifman, R., Multidimensional Inverse Scattering on Nonlinear PDE, Proc. Symposium Pure Math, 43, 45 (1985).

[7b] Beals, R. and Coifman, R., Physica 18D, 242, (1986).

[8] Ablowitz, M.J., Beals, R., and Tenenblat, K., Stud. Appl. Math., 74, 177-203 (1986).

[9] Ablowitz, M.J., Costa, D.G., and Tenenblat, K., Solutions of Multidimensional Extensions of the Anti-Self-Dual Yang-Mills Equations, preprint, INS #66, 1986.

[10] Pohlmeyer, K., Commun. Math. Phys. 72, 37-47, (1980).

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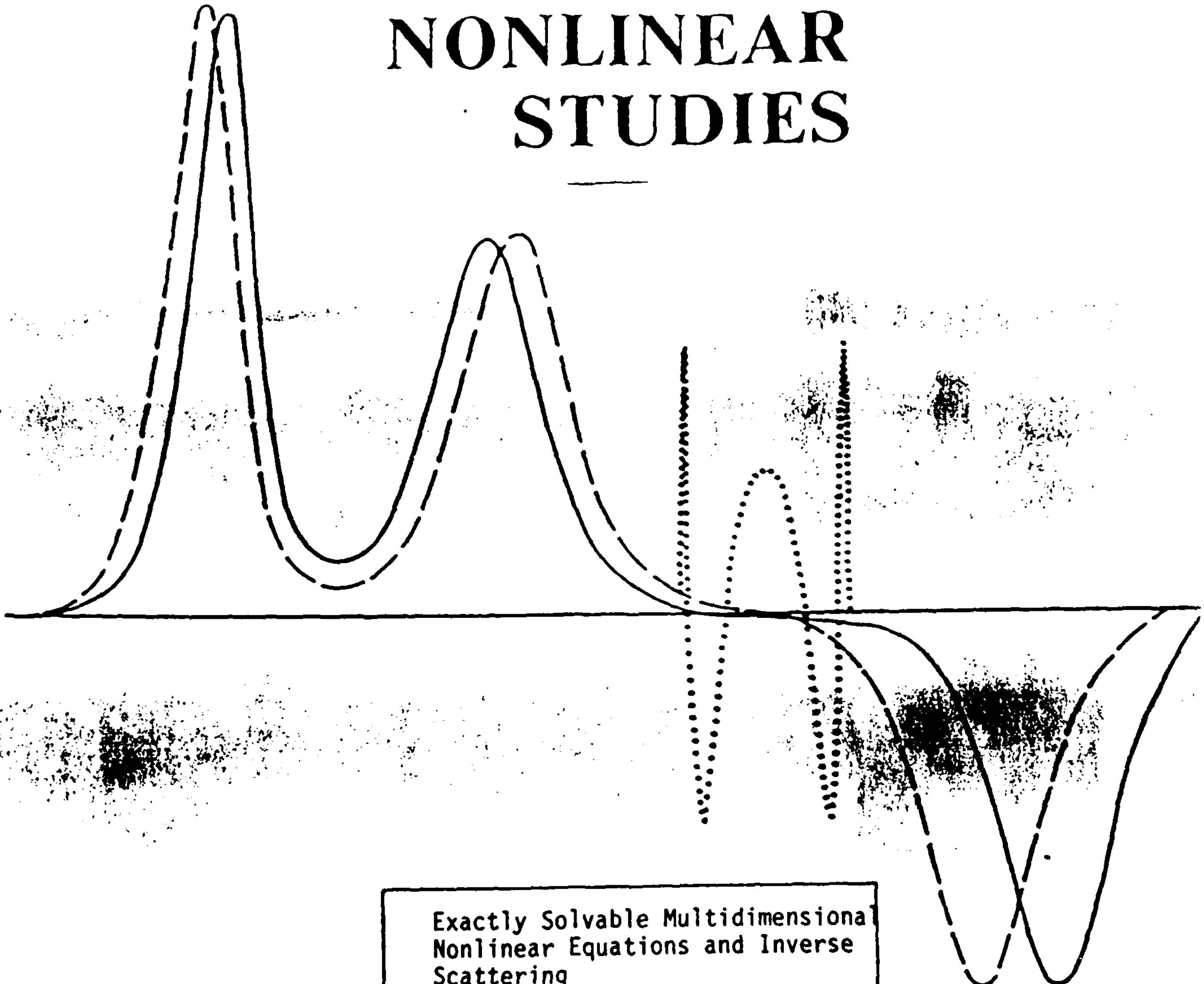

Exactly Solvable Multidimensional Nonlinear Equations and Inverse Scattering

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December, 1986

EXACTLY SOLVABLE MULTIDIMENSIONAL NONLINEAR EQUATIONS AND INVERSE SCATTERING

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ABSTRACT

A review of recent results associated with exactly solvable multidimensional nonlinear systems and related questions of direct and inverse scattering is given.

In this lecture a review of some recent results associated with exactly solvable multidimensional nonlinear systems will be given. The motivation for much of this work has come via what is commonly referred to as the Inverse Scattering Transform (I.S.T.; as a reference see, for example, [1]). IST is a method to solve certain nonlinear equations by associating them with appropriate compatible linear equations, one of which is identified as a scattering problem and the others(s) serves to fix the "time evolution" of the scattering data.

In one spatial dimension the prototype problem is the (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1}$$

The KdV equation is compatible with

$$v_{xx} + u(x,t)v = \lambda v \tag{2}$$

$$v_t = (\gamma+u_x)v - (4\lambda+2u)v_x \tag{3}$$

i.e. $v_{xxt} = v_{txx}$ implies (1). Equation (2) is the Schrödinger scattering problem, λ the eigenvalue (γ= const. in (3)). The solution of (1) on the line: $-\infty<x<\infty$ for initial values $u(x,t=0)$ vanishing sufficiently rapidly at infinity is obtained by studying the associated direct and inverse scattering problem of (2) and using (3) to fix the time evolution of the scattering data. It turns out that the inverse problem amounts to solving a matrix Riemann-Hilbert boundary value problem (RHBVP) whose jump discontinuity depends explicitly on the scattering data. Calling $\lambda=-k^2, v(x,k)=\mu(x,k)e^{-ikx}$ the RHBVP takes the following form,

$$(\mu_+ - \mu_-)(x,t,k) = \mu_-(x,t,\alpha(k))\, V(x,t,k) \text{ on } \Sigma$$
$$\mu_\pm \to 1, \ |k| \to \infty \tag{4}$$

where

$V(x,t,k) - r(k,t)\, e^{2ikx}$, $\alpha(k) = -k$, $\Sigma=\{k: k\in \mathbb{R}\}$, and $\mu_\pm$ are the limiting boundary values as $\text{Im}\,k \to 0\pm$ of meromorphic functions in the upper (+) lower (-) half plane. (4) may be converted into a linear integral equation by taking a minus projection and the potential is reconstructed via

$$u(x,t) = -\frac{1}{\pi}\frac{\partial}{\partial x}\int_C \mu(k,x,t,-k)\ V(x,t,k)dk \qquad (5)$$

where the contour is taken above all poles of $r(k,t)$; of which there is at most a finite number, $k_j = i\kappa_j$, $\kappa_j>0$ $j = 1,---N$. The scattering data: the reflection coefficient, $r(k,t)$ evolves simply in time

$$r(k,t) = r(k,0)\ e^{8ik^2t} \qquad (6)$$

The above scheme may be extended so as to solve a surprisingly large number of interesting nonlinear evolution equations. There are two scattering problems of particular interest in one dimension:

(i) Scalar scattering problems:

$$\frac{d^n v}{dx^n} + \sum_{j=2}^{n} u_j(x)\ \frac{d^{n-j}v}{dx^{n-j}} = \lambda v,$$

$$v(x,k),\quad u_j \in \mathbb{C}$$

(ii) First order systems - generalized AKNS

$$\frac{dv}{dx} = i\ k\ J\ v + q\ v$$

$$v(x,k), q(x) \in \mathbb{C}^{N\times N},\ J = \mathrm{diag}\ (J^1,---J^n)$$

$$J^i \neq J^j, i\neq j$$

$$q^{ii} = 0.$$

Via an appropriate transformation the inverse problem associate with (i), (ii) can be expressed as a matrix RHBVP of the form (4). The potentials u_j,q can be shown to satisfy nonlinear evlution equations by appending to (i), (ii) suitable linear time evolution equations. One then finds that the scattering data $V(x,t,k)$ evolves simply in time. Well known solvable nonlinear equations include the Boussinesq, modified KdV, sine-Gordon, nonlinear Schrodinger, and three wave interaction equations. The reader may wish to consult for example[2a-e] for a detailed discussion of some of this material.

It is most significant that these concepts can be generalized to 2 spatial plus one time dimension. Here the prototype euqation is the Kadomtsev-Petviashvili (K-P) equation:

$$(u_t + 6uu_x + u_{xxx})_x = -3\sigma^2 u_{yy} \qquad (7)$$

which is the compatibility equation between the following linear problems:

$$v_y + v_{xx} + u(x,y,t)v = 0 \qquad (8)$$

$$v_t + 4v_{xxx} + 6uv_x + 3(u_x - \sigma \int_{-\infty}^{x} u_y dx')v + \gamma v = 0 \qquad (9)$$

(γ = const.). We shall consider the question of solving (7) for $u(x,y,0)$ decaying sufficiently rapidly in the plane $r^2 = x^2 + y^2 \to \infty$. Physically speaking, both cases $\sigma^2 = -1$ (KPI) $\sigma^2 = +1$ (KPII) are of interest. Whereas KPI can be related to a RHBVP of a certain type (nonlocal; see ref.[3]) KPII turns out to require new ideas. Letting

$$v = \mu(x,y,k)e^{ikx + k^2 y/\sigma}$$

$\sigma = \sigma_R + i\sigma_I$, $\sigma_R \neq 0$. Then there exist functions μ bounded for all x,y satisfying $\mu \to 1$ as $|k| \to \infty$. However such a function turns out to be nowhere analytic in k, rather it depends nontrivially on both the real and imaginery parts of $k = (k_R + ik_I)$. $\mu = \mu(x,y,k_R,k_I)$.

In fact μ satisfies a generalization of a RHBVP - namely a $\bar{\partial}$ (DBAR) problem where μ satisfies,

$$\frac{\partial \mu}{\partial \bar{k}} = \mu(x,y,\xi_0,k_I)\ V(x,y,k_R,k_I) \qquad (10)$$

where $\frac{\partial}{\partial \bar{k}} = \frac{1}{2}(\frac{\partial}{\partial k_R} + i\frac{\partial}{\partial k_I})$ and V has the structure

$$V(x,y,k_R,k_I) = \frac{\mathrm{sgn}(k_0)e^{i\beta(x,y,k_R,k_I,\xi_0)}}{\pi|\sigma R|} T(k_R,k_I)$$

$$\beta(x,y,k_R,k_I,\xi_0) = (x + 2y\frac{k_I}{\sigma_R})(\xi_0 - k_R) = -2(x + 2y\frac{k_I}{\sigma_R})k_0$$

$$\xi_0 = -k_R - \frac{2\sigma_I}{\sigma_R}k_I,\quad k_0 = k_R + \frac{\sigma_I}{\sigma_R}k_I \qquad (11)$$

(11) may be converted into a linear integral equation by employing the generalized Cauchy formula. $T(k_R,k_I)$ is viewed as the ("nonphysical" data, i.e. inverse scattering data: i.e. inverse data) and the potential is reconstructed via

$$u(x,y) = \frac{2i}{\pi}\frac{\partial}{\partial x}\iint \mu(x,y,\xi_0,k_I)V(x,y,k_R,k_I)\,dk_R\,dk_I. \tag{12}$$

For K-P the evolution of the data obeys ($\gamma = 4ik^3$ in (9))

$$\frac{\partial T}{\partial t} = (8ik_0)(6kk_0 - 4k_0^2 - 3k^2)T \tag{13}$$

where $k_0 = k_R + \dfrac{\sigma_I k_I}{\sigma_R}$, $\quad k = k_R + ik_I$.

Similar ideas apply to higher order scalar problems

$$\text{(iii)} \qquad \sigma\frac{\partial v}{\partial y} + \frac{\partial^n v}{\partial x^n} + \sum_{j=2}^{n} u_j(x)\frac{\partial^{n-j}v}{\partial x^{n-j}} = 0$$

where: $v,\ u_j \in \mathbb{C}$ and to first order systems

$$\text{(iv)} \qquad \sigma\frac{\partial v}{\partial y} + J\frac{\partial v}{\partial x} + q(x,y)v = 0$$

where: $v, q \in \mathbb{C}^{N\times N}$, $J = \mathrm{diag}(J^1,\ldots,J^N)$, $J^i \neq J^j$, $i \neq j$ with $q^{ii} = 0$. Interested readers many consult reference[4a,b] for associated details.

The notion of $\bar\partial$ extends to higher dimensional scattering and inverse scattering problems. However as we shall mention, despite the fact that the inverse scattering problem is essentially tractable there does not appear to be any local nonlnear evolution equations in dimensions greater than 2 + 1 associated with multidimensional generalizations of (iii) or (iv).

Our prototype scattering problem will be

$$\sigma v_y + \Delta v + u(x,y)v = 0$$
$$\Delta = \sum_{\ell=1}^{n}\frac{\partial^2}{\partial x_\ell^2}, \qquad x \in \mathbb{R}^n, \quad y \in \mathbb{R}\,. \tag{14}$$

Letting

$$v = \mu(x,y,k)e^{ik\cdot x + k^2 y/\sigma}$$
$$k = k_R + ik_I, \qquad k \in \mathbb{C}^n$$
$$k\cdot x = \sum_1^n k_j x_j, \qquad \sigma = \sigma_R + i\sigma_I.$$

Then there exist functions μ bounded for all x, y satisfying $\mu \to 1$, as $|k_j| \to \infty$, $j = 1,\ldots,n$. When $\sigma_R \neq 0$ μ turns out to be nonanalytic in each of the variables k, i.e. $\mu = \mu(x,y,k_{R_1},\ldots k_{R_n}, k_{I_1},\ldots,k_{I_n})$ and satisfies a $\bar{\partial}$ problem linear in μ, in each of the variables $\bar{k}_j$; i.e. μ satisfies an equation of the form,

$$\frac{\partial \mu}{\partial \bar{k}_j} = \tilde{T}_j(\mu); \qquad j = 1, \ldots, n \tag{15}$$

where $\tilde{T}_j(\mu)$ is an appropriate linear integral operator which depends only on one scalar scattering function T : $\tilde{T}_j = \tilde{T}_j[T]$, $T=T(k_R, k_I, \xi)$ ξ being $(n-1)$ integration parameters in the nonlocal operator $\tilde{T}_j$. The inverse problem is redundant, i.e. we are given $T(k_R, k_I, \xi)$ $(3n-1)$ parameters) and we must reconstruct a local potential $u(x,y)$ $(n+1$ parameters). A serious issue is how to characterize admissible inverse data T, i.e. data that really arises from a local potential (small generic changes in $T(k_R, k_I, \xi)$ cannot be expected to arise from a local potential $u(x,y)$). Insight into this question is obtained by requiring $\partial^2\mu/\partial\bar{k}_i\partial\bar{k}_j = \partial^2\mu/\partial\bar{k}_j\partial\bar{k}_i$ $(i \neq j)$. The form of this constraint is given by

$$\mathcal{L}_{ij}(T) = \tilde{N}_{ij}[T] \tag{16}$$

where $\mathcal{L}_{ij}$ is a linear operator and $\tilde{N}_{ij}$ a nonlinear (quadratic) nonlocal operator. Details can be found in[5a,b]. Equation (16) can be integrated and this integrated version may be used to reconstruct $u(x,y)$ as well as give a characterization for admissible scattering data: $T(k_R, k_I, \xi)$. However (16) also indicates why simple local nonlinear evolution equations have not been associated with equation (8). Namely in the previous lower dimensional (2+1 and 1+1) problems the time evolution of the scattering data obeyed a particularly simple equation, (e.g. $\frac{\partial T}{\partial t} = \omega(k_R, k_I)T$. However in this case such a simple flow will not be maintained - due to the nonlinear constraint (16).

These ideas can be generalized to first order systems:

(v) $$\frac{\partial v}{\partial y} + \sigma \sum_{j=1}^{n} J_j \frac{\partial v}{\partial x_j} = qv$$

$v, q \in \mathbb{C}^{N\times N}$, $J_j = \mathrm{diag}(J_j^1, \ldots, J_j^N)$

$J_j^k \neq J_j^\ell$, $\quad k \neq \ell$.

with similar results obtained[6a,b]. Again the scattering data satisfies a nonlinear constraint. In general, there is no compatible local nonlinear evolution equation associated with (v). However when certain restrictions are put on J_j then the constraint equation becomes linear and the so-called N wave interaction equations are compatible with the system (v). Nachman and Ablowitz[6a] showed that at most, the system would be 3+1 dimensional, and Fokas[6b] showed that indeed the system is reducible to 2+1 dimensions by a transformation of independent variables (characteristic variables).

Beals and Coifman have given an alternative but similar formulation[7a,b] in the scalar case.

There is an n-dimensional problem which also fits within the framework of IST: The so-called generalized wave and generalized sine-Gordon euqation (GWE and GSGE). These equations arise in the context of differential geometry and serve to extend the classical results of Bäcklund for the sine-Gordon equation to n-dimensions[8]. The n-dimensional Bäcklund tranformation is given by:

$$dX + XA^tX = A - XB, \tag{17}$$

where

$$dX = \sum_{j=1}^{n} \frac{\partial X}{\partial x_j} dx_j,$$

$$A_{ij} = \beta_i(z)a_{ij}dx_j,$$

$$B_{ij} = \frac{1}{a_{1i}} \frac{\partial a_{1j}}{\partial x_i} dx_j - \frac{1}{a_{1j}} \frac{\partial a_{1i}}{\partial x_j} dx_i, \quad 1 \le i, j \le n, \tag{18}$$

and $a = \{a_{ij}\} \in R^{n\times n}$. Equations (17-18) reduce to the Bäcklund transformation for the generalized sine-Gordon equation (GSGE) when

$$\beta_i(z) = (z^2 + (2\delta_{i1} - 1))/2z, \tag{19}$$

and for the generalized wave equation (GWE) when

$$\beta_i(z) = -(1-z^2)/2z \equiv \lambda(z). \tag{20}$$

The compatibility condition required for the existence of solutions to these Bäcklund transformations results in a system of second-order partial differential equations for an orthogonal $n \times n$ matrix $a = \{a_{ij}\}$ in (17) which is a function of n independent variables $a = a(x_1, x_2, \ldots, x_n)$. The equation has the form

$$\frac{\partial}{\partial x_i}\left(\frac{1}{a_{1i}} \frac{\partial a_{1j}}{\partial x_i}\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{a_{1k}} \frac{\partial a_{1i}}{\partial x_j}\right)$$

$$+ \overline{\sum_{k \neq i,j}} \frac{1}{a_{1k}^2} \frac{\partial a_{1i}}{\partial x_k} \frac{\partial a_{1j}}{\partial x_k} = \epsilon a_{1i} a_{1j}, \quad i \neq j,$$

$$\frac{\partial}{\partial x_k} \left(\frac{1}{a_{1j}} \frac{\partial a_{1i}}{\partial x_j}\right) \quad \frac{1}{a_{1k} a_{1j}} \frac{\partial a_{1i}}{\partial x_k} \frac{\partial a_{1k}}{\partial x_j}, \quad i,\ j,\ k \ \text{distinct},$$

$$\frac{\partial a_{jk}}{\partial x_k} = \frac{\partial a_{ji}}{a_{1i}} \frac{\partial a_{1k}}{\partial x_i}, \qquad i \neq k, \tag{21}$$

where $\epsilon = 1$ for the GSGE and $\epsilon = 0$ for the GWE.

We observe that when $n = 2$ and $\epsilon = 1$ (GSGE), the orthogonal matrix $a = \{a_{ij}\}$ given by

$$a = \begin{pmatrix} \cos \frac{1}{2} u & \sin \frac{1}{2} u \\ -\sin \frac{1}{2} u & \cos \frac{1}{2} u \end{pmatrix} \tag{22}$$

for the function $u = u(x,t)$ reduces the GSGE to the classical sine-Gordon equation ($\kappa = -1$),

$$u_{tt} - u_{xx} - \kappa \sin u = 0. \tag{23}$$

On the other hand when $n = 2$ and $\kappa = 0$, then with (22) the GWE reduces to the wave equation (23). When $n \geq 3$ the generalization of the wave equations discussed here is nonlinear.

The Bäcklund transformations (17) described above are in fact matrix Riccati equations. Linearizations of such a system can be performed in a striaghtforward manner (see for example[9]). Introducing the transformation

$$X = UV^{-1}, \tag{24}$$

where U, V and $n \times n$ matrix functions of $x_1, \ldots, x_n$, the following linear system is deduced:

$$\begin{pmatrix} dU \\ dV \end{pmatrix} = \begin{pmatrix} 0 & A \\ A^t & B \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} \tag{25}$$

with the components of A, B given by (18). Compatibility ensures that the orthogonal matrix $a = \{a_{ij}\}$ satisfies the GSGE with (19) and GWE with (20). Alternatively, if we call

$$\begin{pmatrix} U \\ V \end{pmatrix} = \psi ,$$

the following linear system of 2n o.d.e.'s are obtained:

$$\frac{\partial \psi}{\partial x_j} = \lambda \tilde{A}_j \psi + C_j \psi, \tag{26}$$

where $\tilde{A}_j$, C_j are $2n \times 2n$ matrices with the block structure

$$\tilde{A}_j = \begin{pmatrix} 0 & \tilde{a}_j \\ \tilde{a}_j^t & 0 \end{pmatrix}, \quad C_j = \begin{pmatrix} 0 & 0 \\ 0 & \gamma_j \end{pmatrix}. \tag{27}$$

Here $\tilde{a}_j$, $\tilde{\gamma}_j$ are $n \times n$ matrices having the following structure:

$$\begin{aligned} \tilde{a}_j &= (\frac{\delta}{\lambda} - 1)e_1 a_j + a_j, \\ a_j &= a e_j \end{aligned} \tag{28}$$

where $e_j = \{e_j\}_{ik}$ is the unit matrix

$$\{e_j\}_{ik} = \begin{cases} 1 & i = k = j, \\ 0 & \text{otherwise}, \end{cases} \tag{29}$$

and in component form λ_j takes the form

$$(\lambda_j)_{k\ell} = (1-\delta_{kj})\frac{1}{a_{\ell k}} \frac{\partial a_{\ell j}}{\partial x_k} \delta_{\ell j} - (1-\delta_{\ell j})\frac{1}{a_{1j}} \frac{\partial a_{1j}}{\partial x_\ell} \delta_{kj}. \tag{30}$$

In (28) a is the orthogonal matrix $\mathbb{R}^n \to SO(n)$ associated with the GWE when $\delta = \lambda$ and with the GSGE when $\delta = \frac{1}{2}(z + 1/z)$, $\lambda = \frac{1}{2}(z - 1/z)$, and γ_j is the matrix (30): $\mathbb{R}_n \to M_n(\mathbb{R})$, $\gamma_j + \gamma_j^t = 0$. Equations (21) arise as the compatibility condition associated with (26). More explicitly, for the GWE the scattering problem takes the form $[\psi = \psi(x,\lambda)]$

$$\frac{\partial \psi}{\partial x_j} = \lambda A_j \psi + C_j \psi \tag{31}$$

with

$$A_j = \begin{pmatrix} 0 & a_j \\ a_j^t & 0 \end{pmatrix}, \tag{32}$$

and C_j given by (27,30).

For the GSGE the scattering problem for $\psi = \psi(x,z)$ takes the form

$$\frac{\partial \psi}{\partial x_j} = \delta(z) \begin{pmatrix} 0 & e_1 a_j \\ a_j^t e_1 & 0 \end{pmatrix} \psi$$

$$+ \lambda(z) \begin{pmatrix} 0 & (I-e_1)a_j \\ a_j^t(I-e_1) & 0 \end{pmatrix} \psi + C_j \psi, \tag{33}$$

$\delta(z)$, $\lambda(z)$, C_j given above, or equivalently

$$\frac{\partial \psi}{\partial x_j} = \frac{z}{2} A_j \psi + \frac{z}{2} B_j \psi + C_j \psi, \tag{34}$$

where

$$B_j = \begin{pmatrix} 0 & u a_j \\ a_j^t u & 0 \end{pmatrix}, \quad u = \mathrm{diag}(+1, -1, \ldots, -1). \tag{35}$$

In[8] it is shown how these linear problems may be viewed as a direct and inverse scattering problem for the GWE and GSGE. Namely the direct and inverse problem may be solved for matrix potentials, depending on the orthogonal matrix a, tending to the identity sufficiently fast in certain "generic" directions. It should be noted that solving the n-dimensional GWE and GSGE reduces to the study of the scattering and inverse scattering associated with a coupled system of n one-dimensional o.d.e.'s. This is in marked contrast to other attempts described earlier to isolate solvable (local) multidimensional nonlinear evolution equation which are compatibility conditions of two Lax-type operators, e.g.,

$$L\psi = \lambda\psi \tag{36}$$

$$\psi_t = M\psi \tag{37}$$

where L is a partial differential operator with the variable t entering only parametrically. Although as we have seen nonlinear evolution equations in three independent variables can be associated with such Lax pairs (e.g. the K-P, Davey-Stewartson, three wave interaction equations, etc.) little progress via this route has been made in more than three dimensions. As discussed earlier one has to overcome a serious constraint inherent in the scattering/inverse scattering theory for higher dimensional partial differential operators in order to be able to isolate associated solvable nonlinear equations, i.e. the scattering data generally satisfies a nonlinear equation (e.g. (16)). The analysis associated with the GWE and GSGE avoids these difficulties since the GWE and GSGE problems are simply a compatible set of nonlinear one-dimensional o.d.e.'s. The results in [8] demonstrate that the initial value problem is posed with given data along lines and not on (n-1) dimensional manifolds.

Similar ideas apply to certain n-dimensional extensions of the so-called anti-self-dual Yang-Mills equations (SDYM). In[9] it is shown that these multi-dimensional nonlinear equations are associated with compatible two-dimensional linear systems. Broad classes of solutions may be calculated by the $\bar{\partial}$ method. Since the overall compatible linear systems are coupled two-dimensional equations, the scattering data does not satisfy the nonlinear constraint discussed earlier.

Finally we remark that there is a class of nonlocal equations which can be reduced to exactly solvable equations. In the context of multidimensional nonlinear equations perhaps the most interesting example is

$$(u_t + u_{xxx} + 2(uH_z u)_x)_x = -3\sigma^2 u_{yy}, \tag{38}$$

where

$$(H_z u)(x,y,z,t) = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{u(x,y,\xi,t)}{\xi - z}\, d\xi \tag{39}$$

and $\fint_{-\infty}^{\infty}$ denotes the Cauchy principal value integral. (38) is reduced to the K-P euqation

$$(w_t + w_{xxx} - i(w^2)_x)_x = -3\sigma^2 w_{yy} \tag{40}$$

via the transformation

$$w = u + iH_z u. \tag{41}$$

Details and other examples are given in [10].

ACKNOWLEDGEMENTS

I am most pleased to acknowledge the many crucial contributions of my colleagues: A.S. Fokas, D. Bar Yaacov, A.I. Nachman, R. Beals and K. Tenenblat. This work was supported in part by the National Science Foundation under grant numer DMS-8202117, the Office of Naval Research under grant number N00014-76-C-0867, and the Air Force Office of Scientific Research under grant number AFOSR-84-0005.

REFERENCES

[1] Ablowitz, M.J. and Segur, H., "Solitons and the Inverse Scattering Transform", SIAM Appl. Math., Phila., PA, 4 (1981).

[2a] Ablowitz, M.J. and Fokas, A.S., "Comments on the Inverse Scattering Tranform and Related Nonlinear Evolution Equations", Lect. Notes in Phys., 189, Proc. CIFMO School and Workshop held at Oaxtepec, Mexico (K.B. Wolf, ed.), Springer, 1982.

[2b] Beals, R. and Coifman, Commun. Pure Appl. Math., 1984, pp. 39-90.

[2c] Beals, R., The Inverse Problems for Ordinary Differential Operators on the Line; to appear, Amer. J. of Math.

[2d] Shabat, A.B., Func. Annal and Appl. 9, (1975) 75; Diff. eq. XV (1979) 1824.

[2e] Caudrey, P., Physica 6D (1982) 51.

[3a] Fokas, A.S. and Ablowitz, M.J., Stud. Appl. Math. 69, 211 (1983).

[3b] Manakov, S.V., Phys. 3D, 420 (1981).

[4a] Ablowitz, M.J., BarYaacov, D., and Fokas, A.S., Stud. Appl. Math. 69, 135, (1983).

[4b] Fokas, A.S. and Ablowitz, M.J., The Inverse Scattering Transform for Multidimensional (2+1) Problems, Lect. Notes in Phys., 189, Proc. CIFMO School and Workshop held at Oaxtepec, Mexico, 1982, K.B. Wolf, ed.

[5a] Nachman, A.I. and Ablowitz, M.J., Stud. Appl. Math. 71, 243-250 (1984).

[5b] Ablowitz, M.J. and Nachman, A.I., Physica 18D, 223 (1986).

[6a] Nachman, A.I. and Ablowitz, M.J., Stud. Appl. Math. 71 251-262 (1984).

[6b] Fokas, A.S., Phys. Rev. Lett., 57, No. 2, 159 (1986).

[7a] Beals, R., and Coifman, R.R., Multidimensional Inverse Scattering on Nonlinear PDE, Proc. Symposium Pure Math., 43, 45 (1985).

[7b] Beals, R. and Coifman, R.R., Physica 18D, 242 (1986).

[8] Ablowitz, M.J., Beals, R., and Tenenblat, K., Stud. Appl. Math., 74, 177-203 (1986).

[9] Ablowitz, M.J., Costa, D.G., and Tenenblat, K., Solutions of Multidimensional Extensions of the Anti-Self-Dual Yang-Mills Equations, preprint, INS #66, 1986.

[10] Ablowtiz, M.J., Fokas, A.S., and Kruskal, M.D., Note on Solutions to a Class of Nonlinear Singular Integro-Differential Equations, preprint, INS #64, 1986, to be published Phys. Lett.

NOV 4 1987

An example of $\bar\partial$ problem arising in a finite difference context: Direct and inverse problem for the discrete analog of the equation $\psi_{xx}+u\psi=\sigma\psi_y$

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(Received 7 January 1986; accepted for publication 18 November 1986)

The direct and inverse spectral problem for the discrete analog of the equation $\psi_{xx}+u\psi=\sigma\psi_y$ is solved in the framework of "$\bar\partial$" theory. The time evolution of the spectral data for the simplest nonlinear differential-difference equations associated to this linear problem is derived.

I. INTRODUCTION

In recent years, there has been considerable interest in the study of exactly solvable nonlinear evolution equations by the method of the inverse scattering transform (IST). The results for one-dimensional partial differential equations and their discrete analogs is by now classical and covered in tests on the subject.[1] On the other hand, the work done on IST for 2 + 1 dimensions has only been satisfactorily understood within the past few years.[2] The prototype problem studied is the Kadomtsev–Petviashvili (KP) equation:

$$(u_t + 6uu_x + u_{xxx})_x = -3\sigma^2 u_{yy}, \tag{1.1}$$

together with its associated linear problem

$$\sigma\psi_y = \psi_{xx} + u\psi. \tag{1.2}$$

There are two critical choices of the parameter σ: $\sigma = i$ (KPI); $\sigma = 1$ (KPII).

Manakov[3] showed that KPI fits within the context of Riemann–Hilbert (RH) theory (i.e., it leads to a nonlocal RH problem). The second case, KPII, was found to lie outside RH theory. It required essential use of the notion of "$\bar\partial$" (DBAR) problem. We recall that Beals and Coifman,[4] in their elegant work on systems of ordinary differential equations, noted that the RH problem was, in fact, a special case of the more general notion of a $\bar\partial$ problem. The $\bar\partial$ problem gives a simple and powerful method by which the underlying inverse spectral problem for the KPII equation (and other analogous equations, like Davey–Stewardson II, modified KPII, ...) can be solved.

In this paper a discrete analog of (1.2) for the case $\sigma = 1$ is investigated. To our knowledge this is the first consideration of a discrete multidimensional scattering problem via $\bar\partial$ theory. One very important observation is that fully discrete spectral problems virtually always require the use of a $\bar\partial$ approach. The reason for this has to do with the fact that discretizations are generally unstable ("ill-posed") as partial difference equations in $\mathbb{Z}^2$ [in analogy with problem (1.2) with $\sigma = 1$ for both x and y finite].

Of course, the corresponding one-dimensional discrete problem (i.e., the finite-difference analog of the Schrödinger equation) has been investigated via RH methods by a number of authors.[5] In particular, we refer to the thesis of Sanda L. Chitlaru-Briggs who not only considered this one-dimensional problem, but went beyond to study multidimensional problems such as the one under scrutiny in this paper. Unfortunately, her life was prematurely cut short and her study had to be ended. This article is dedicated to Sanda L. Chitlaru-Briggs.

II. THE DIRECT PROBLEM

We investigate the linear problem

$$\psi(n-1,m) + B(n,m)\psi(n,m) + A(n,m)\psi(n+1,m) = 2\psi(n,m+1), \tag{2.1}$$

where $(n,m)\in\mathbb{Z}^2$, and the "potentials" $B, A-1$ vanish sufficiently fast as n and (or) m go to infinity.

This problem, as well as the simplest evolution equations associated with it, has been already introduced in Ref. 6.

It is easy to see that, when $B = 0$, the continuum limit of (2.1) is just Eq. (1.2) for $\sigma\in\mathbb{R}$; to perform this limit, set $A(n,m) = \exp[\Delta(V(n+1,m+1) - V(n,m))]$, $x = n\Delta$, $y = (\sigma/2)m\Delta^2$, and let $\Delta\to 0$: one recovers (1.2) with $u = V_x$.

To handle Eq. (2.1), we introduce a function μ, defined as

$$\mu(n,m;z) = \psi(n,m)[\psi^{(0)}(n,m;z)]^{-1}, \tag{2.2}$$

where $\psi^{(0)}$ is a special solution of the "bare" problem associated to (2.1) (i.e., the one corresponding to $B = 0, A = 1$), given by

$$\psi^{(0)}(n,m;z) = z^{-n}((z+z^{-1})/2)^m. \tag{2.3}$$

The function μ will then satisfy the following equation:

$$z\mu(n-1,m;z) + B(n,m)\mu(n,m;z) + z^{-1}A(n,m)\mu(n+1,m;z) = (z+z^{-1})\mu(n,m+1;z). \tag{2.4}$$

Requiring that, as a function of z, μ satisfies the boundary condition

$$\lim_{|z|\to\infty}\mu(n,m;z) = 1, \tag{2.5}$$

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Eq. (2.4) is equivalent to the summation equation

$$\mu(n,m;z) = 1 - \sum_{n',m'=-\infty}^{+\infty} G(n-n',m-m';z) \times [B(n',m')\mu(n',m';z) + z^{-1}(A(n',m')-1)\mu(n'+1,m';z)], \tag{2.6}$$

where the Green's function G is defined as

$$G(n,m;z) = \frac{1}{(2\pi i)^2}\oint_{|z_1|=1}\frac{dz_1}{z_1}\oint_{|z_2|=1}\frac{dz_2}{z_2} \times z_1^n z_2^m \widehat{G}(z_1,z_2;z) \tag{2.7}$$

with

$$\widehat{G}(z_1,z_2;z) = \left[\frac{z}{z_1}+\frac{z_1}{z}-z_2\left(z+\frac{1}{z}\right)\right]^{-1}. \tag{2.8}$$

From its very definition, it turns out that G enjoys the following symmetry properties:

$$G(n,m;z) = -G(n,m;-z), \tag{2.9a}$$

$$G(n,m;z) = -(-1)^{n+m}G(n,m;z), \tag{2.9b}$$

$$G(n,m;\bar z) = \omega_1^{-n}\omega_2^{-m}G(n,m;z), \tag{2.9c}$$

where $\pm\omega_1$, $\pm\omega_2$, defined as

$$\omega_1 = z/\bar z, \tag{2.10a}$$

?

$$\omega_2 = (\bar z + 1/2)/(z+1/z), \tag{2.10b}$$

are the simple pole singularities of G, as a function of z_1 and z_2 on the integration contours. As in the corresponding continuum linear problem (1.2), these singularities are integrable, and by performing the integration with respect to z_2, we get for G the following expression, which clearly shows that G is not an analytic function of $z = r\exp(i\varphi)$ [in Eq. (2.11), $z_1 = \exp(i\vartheta_1)$],

$$\begin{aligned} G(n,m;z) = {} & \frac{1}{2\pi i}(z+z^{-1})^{-1}\oint_{|z_1|=1}\frac{dz_1}{z_1}z_1^n\left(\frac{z_1}{z}+\frac{z}{z_1}\right)^{m-1} \\ & \times\Big\{\Theta(1-m)-\Theta\left(\frac{\pi}{2}+\varphi\right)\Theta(-\varphi)[\Theta(\vartheta_1+\pi)\Theta(2\varphi-\vartheta_1)+\Theta(\vartheta_1)\Theta(\pi+2\varphi-\vartheta_1)] \\ & -\Theta(\pi+\varphi)\Theta\left(-\varphi-\frac{\pi}{2}\right)[\Theta(-\vartheta_1)\Theta(\vartheta_1-2\varphi-\pi)+\Theta(\pi-\vartheta_1)\Theta(\vartheta_1-2\varphi-2\pi)] \\ & -\Theta(\varphi)\Theta\left(\frac{\pi}{2}-\varphi\right)[\Theta(\pi-\vartheta_1)\Theta(\vartheta_1-2\varphi)+\Theta(-\vartheta_1)\Theta(\vartheta_1-2\varphi+\pi)] \\ & -\Theta\left(-\frac{\pi}{2}+\varphi\right)\Theta(\pi-\varphi)[\Theta(\vartheta_1)\Theta(2\varphi-\pi-\vartheta_1)+\Theta(\vartheta_1+\pi)\Theta(2\varphi-2\pi-\vartheta_1)]\Big\}. \end{aligned} \tag{2.11}$$

The "departure from analyticity" of the function G is measured by its "$\bar\partial$" derivative, whose expression is the following one:

$$\frac{\partial G}{\partial \bar z} = c(z,\bar z)[1-(-1)^{n+m}]\omega_1^n\omega_2^m, \tag{2.12}$$

where

$$c(z,\bar z) = (i/\pi)\,\mathrm{sgn}(\sin 2\varphi)(\bar z^2+1)^{-1}. \tag{2.13}$$

Equation (2.12) can be either derived from (2.11) by means of the standard formula

$$\frac{\partial}{\partial \bar z} = \frac{1}{2\bar z}\left[r\frac{\partial}{\partial r}+i\frac{\partial}{\partial\varphi}\right], \tag{2.14a}$$

or, directly from (2.7), taking into account the distribution formula

$$\frac{\partial}{\partial \bar z}\left(\frac{1}{z-z_0}\right) = \pi\delta(z-z_0). \tag{2.14b}$$

As in the continuum case, the existence of a connection formula between μ and its "$\bar\partial$" derivative plays an essential role in the method. In our case, it has the following expression:

$$\frac{\partial}{\partial\bar z}\mu(n,m;z) = \alpha(z)\mu(n,m;\bar z) + (-1)^{n+m}\beta(z)\mu(n,m;-\bar z). \tag{2.15}$$

The "spectral data" $\alpha(z)$, $\beta(z)$ are related to the potentials through the formulas

$$\alpha(z) = c(z,\bar z)\sum_{n,m=-\infty}^{+\infty}\omega_1^{-n}\omega_2^{-m} \times [B(n,m)\mu(n,m;z) + z^{-1}(A(n,m)-1)\mu(n+1,m;z)], \tag{2.16a}$$

$$\beta(z) = -c(z,\bar z)\sum_{n,m=-\infty}^{+\infty}(-1)^{n+m}\omega_1^{-n}\omega_2^{-m} \times [B(n,m)\mu(n,m;z) + z^{-1}(A(n,m)-1)\mu(n+1,m;z)]. \tag{2.16b}$$

To prove formulas (2.15), (2.16) it is sufficient to perform the "$\bar\partial$" derivative of the summation equation (2.6), taking into account Eq. (2.12) and the symmetry properties (2.9), and then to notice that the lhs and the rhs of (2.15) satisfy the same nonhomogeneous summation equation.

III. THE INVERSE PROBLEM

The main tool for solving the inverse problem, namely for reconstructing the potentials $A(n,m)$ and $B(n,m)$ from the spectral data $\alpha(z)$ and $\beta(z)$, is provided by the generalized Cauchy formula

$$f(\bar z) = \frac{1}{2\pi i}\iint_D \frac{\partial f/\partial \bar\zeta}{\zeta - z}\, d\zeta \wedge d\bar\zeta + \frac{1}{2\pi i}\oint_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \tag{3.1}$$

where D is a suitable domain in the z plane and $d\zeta \wedge d\bar\zeta = 2i d\zeta_R d\zeta_I (\zeta = \zeta_R + i\zeta_I = \rho \exp(i\chi))$.

Identifying f with μ, choosing D as the whole complex z plane, and taking into account Eqs. (2.5) and (2.15), formula (3.1) yields the following *linear* integral equation for μ:

$$\mu(n,m;z) = 1 + \frac{1}{2\pi i}\iint_D d\zeta \wedge d\bar\zeta \,\frac{[\alpha(\zeta)\mu(n,m;\bar\zeta) + (-1)^{n+m}\beta(\zeta)\mu(n,m;-\bar\zeta)]}{\zeta - z}. \tag{3.2}$$

Once, through the solution of (3.2), μ is known in the whole complex z plane, one can easily reconstruct the potentials through the formulas

$$B(n,m) = \mu^{(1)}(n,m+1) - \mu^{(1)}(n-1,m), \tag{3.3a}$$

$$A(n,m) = 1 + \mu^{(2)}(n,m+1) - \mu^{(2)}(n-1,m) + \mu^{(1)}_{(n,m)}(\mu^{(1)}_{(n-1,m)} - \mu^{(1)}_{(n,m+1)}), \tag{3.3b}$$

where $\mu^{(1)}$ and $\mu^{(2)}$ are the leading terms in the asymptotic expansion of μ around infinity, namely [illegible]

$$\mu^{(1)}(n,m) = \lim_{|z|\to\infty} [illegible]$$

$$\mu^{(2)}(n,m) = \lim_{|z|\to\infty} z^2 [illegible] - [\mu^{(1)}(n,m)/z]]. \tag{3.4b}$$

In terms of the spectral data, they read

$$\mu^{(1)}(n,m) = -\frac{1}{2\pi i}\iint_D d\zeta \wedge d\bar\zeta\, [\alpha(\zeta)\mu(n,m;\bar\zeta) + (-1)^{n+m}\beta(\zeta)\mu(n,m;-\bar\zeta)], \tag{3.5a}$$

$$\mu^{(2)}(n,m) = -\frac{1}{2\pi i}\iint_D d\zeta \wedge d\bar\zeta\, \{\zeta\, [\alpha(\zeta)\mu(n,m;\bar\zeta) + (-1)^{n+m}\beta(\zeta)\mu(n,m;-\bar\zeta)]\}. \tag{3.5b}$$

IV. SOME ASSOCIATED EVOLUTION EQUATIONS AND THE CORRESPONDING TIME EVOLUTION OF THE SPECTRAL DATA

The simplest nonlinear differential-difference equations [illegible] IST scheme [illegible]

$$[illegible] = [illegible]\, z^{-1}\psi(n,m) + G^{(0)}(n,m)\psi(n+1,m), \tag{4.1a}$$

$$G^{(0)}(n,m) := \prod_{j=0}^{\infty} \frac{A(n-j,m+j)}{A(n-1-j,m+j)},$$

$$\psi_t(n,m) = -z^{-2}\psi(n,m) - \tfrac{1}{2} G^{(1)}(n,m) \times [\psi(n,m) - 2\psi(n+1,m+1)], \tag{4.1b}$$

$$G^{(1)}(n,m) := \prod_{j=0}^{\infty} \frac{A(n-j,m+1+j)}{A(n-1-j,m+j)},$$

$$\psi_t(n,m) = -(8z^2)^{-1}(z+z^{-1})^2\psi(n,m) - [\tfrac{1}{2} G^{(2)}(n,m) + \tfrac{1}{2} A(n+1,m+2)G^{(3)}(n,m)] \times \psi(n,m) + G^{(2)}(n,m)\psi(n+1,m+1) + G^{(3)}(n,m)\psi(n+1,m+3),$$

$$G^{(2)}(n,m) := \prod_{j=0}^{\infty} \frac{A(n-j,m+1+j)}{A(n-1-j,m+j)}\, \frac{1}{[illegible]} \sum [illegible] (n-1-j,m+2+j)G^{(3)}(n-2-j,m [illegible]) - A(n-j,m+3+j)[illegible] + j)]\Big[\prod \frac{A(n-j-s,m [illegible] j+s)}{A(n-1-j-s,m [illegible] j+s)}\Big]^{-1}, \tag{4.1c}$$

$$G^{(3)}(n,m) := \prod_{j=0}^{\infty} \frac{A(n+1-j,m [illegible])}{A(n-j,m+j)}.$$

The corresponding evolution equations read

$$A_t(n,m) = B(n+1,m)G^{(0)}(n,m+1) - B(n,m)G^{(0)}(n,m),$$
$$B_t(n,m) = G^{(0)}(n,m+1) - G^{(0)}(n-1,m), \tag{4.2a}$$

$$A_t(n,m) = -\tfrac{1}{2}A(n,m)[G^{(1)}(n-1,m) - G^{(1)}(n+1,m)],$$
$$B_t(n,m) = 0, \tag{4.2b}$$

$$A_t(n,m) = -\tfrac{1}{2}A(n,m)[G^{(2)}(n-1,m) - G^{(2)}(n+1,m) + \tfrac{1}{2}A(n,m+2)G^{(3)}(n-1,m) - \tfrac{1}{2}A(n+2,m+2)G^{(3)}(n+1,m)], \tag{4.2c}$$
$$B_t(n,m) = 0.$$

Equation (4.2a) is clearly a two-dimensional version of the Toda lattice,[7] which is immediately recovered, by assuming that A and B do not depend on m.

Equation (4.2b) is in turn a two-dimensional version of the infinite Volterra system,[8] and finally Eq. (4.2c) is a differential-difference analog of the KPII equation.

The evolution of the spectral data is derived from formulas (4.1) by letting $n,m \to \infty$ and comparing the "$\bar\partial$" derivative of Eqs. (4.1) with the time derivative of (2.15). To perform this comparison one has to take into account that, as it can be seen from (2.6), for large n and m μ goes to a constant value as z approaches 0.

The corresponding results are the following.

(i) For Eq. (2a)

$$\alpha_t(z) = (z^{-1} - \bar{z}^{-1})\alpha(z);$$

$$\beta_t(z) = -(z^{-1} + \bar{z}^{-1})\beta(z).$$

(ii) For Eq. (2b)

$$\alpha_t(z)/\alpha(z) = \beta_t(z)/\beta(z) = z^{-2} - \bar{z}^{-2}.$$

(iii) For Eq. (2c)

$$\alpha_t(z)/\alpha(z) = \beta_t(z)/\beta(z)$$
$$= \tfrac{1}{4}[(\bar{z} + \bar{z}^{-1})^2(1 - \bar{z}^{-2}) - (z + z^{-1})^2(1 - z^{-2})].$$

A more systematic investigation of the class of evolution equation associated with the linear problem (2.1) is contained in Ref. 9, where the bi-Hamiltonian structure of this class is explicitly derived.

[1] See, for example, M. J. Ablowitz and H. Segur, SIAM Stud. Appl. Math. 4 (1981); F. Calogero and A. Degasperis *Spectral Transform and Solitons I* (North-Holland, Amsterdam, 1982).

[2] A. S. Fokas and M. J. Ablowitz, "Lectures on the inverse scattering transform for multidimensional (2 + 1) problems," in *Nonlinear Phenomena*, in *Lecture Notes in Physics*, Vol. 189, edited by K. Wolf (Springer, Berlin, 1983), pp. 137–183, M. J. Ablowitz, D. Bar Yaacov, and A. S. Fokas, Stud. Appl. Math. 69, 135 (1983).

[3] S. V. Manakov, Physica D 3, 420 (1981).

[4] R. Beals and R. R. Coifman, Comm. Pure Appl. Math. 37, 39 (1984).

[5] S. V. Manakov, Zh. Eksp. Teor. Fiz. 67, 543 (1974); M. Bruschi, S. V. Manakov, O. Ragnisco, and D. Levi, J. Math. Phys. 21, 2749 (1980).

[6] D. Levi, L. Pilloni, and P. M. Santini, J. Phys. A: Math. Gen. 14, 1567 (1981).

[7] M. Toda, Prog. Theor. Phys. 45, 174 (1970); H. Flaschka, Prog. Theor. Phys. 51, 703 (1974).

[8] V. Volterra, *Lecons sur la theorie mathematique de la lutte pour la vie* (Gauthier-Villars, Paris, 1931); R. Hirota and J. Satsuma, J. Phys. Soc. Jpn. 40, 891 (1976).

[9] O. Ragnisco and P. M. Santini, "Recursion operator and bi-Hamiltonian structure of integrable multidimensional lattices," preprint, Dipartimento di Fisica, Università di Roma, 1987.

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