(2)

DTIC FILE COPY

AD

TECHNICAL REPORT ARCCB-TR-88004

THE FORCES OF CONSTRAINT ON A PROJECTILE IN A RIFLED GUN BORE (PART 1)

DAVID F. FINLAYSON

JANUARY 1988

US ARMY ARMAMENT RESEARCH, DEVELOPMENT AND ENGINEERING CENTER
CLOSE COMBAT ARMAMENTS CENTER
BENÉT LABORATORIES
WATERVLIET, N.Y. 12189-4050

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED

DISCLAIMER

The findings in this report are not to be construed as an official Department of the Army position unless so designated by other authorized documents.

The use of trade name(s) and/or manufacturer(s) does not constitute an official indorsement or approval.

DESTRUCTION NOTICE

For classified documents, follow the procedures in DoD 5200.22-M, Industrial Security Manual, Section II-19 or DoD 5200.1-R, Information Security Program Regulation, Chapter IX.

For unclassified, limited documents, destroy by any method that will prevent disclosure of contents or reconstruction of the document.

For unclassified, unlimited documents, destroy when the report is no longer needed. Do not return it to the originator.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER ARCCB-TR-88004	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) THE FORCES OF CONSTRAINT ON A PROJECTILE IN A RIFLED GUN BORE (PART 1)		5. TYPE OF REPORT & PERIOD COVERED Final 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) David F. Finlayson		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS US Army ARDEC Benet Laboratories, SMCAR-CCB-TL Watervliet, NY 12189-4050		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS AMCMS No. 6126.24.1BL0.0 PRON No. 1A-7-2RCAP-NMSC
11. CONTROLLING OFFICE NAME AND ADDRESS US Army ARDEC Close Combat Armaments Center Picatinny Arsenal, NJ 07806-5000		12. REPORT DATE January 1988 13. NUMBER OF PAGES 34
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

Published in Journal of Ballistics, Volume 9, Number 3, 1987, pp. 2335-2378.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Spiral Wear, Muzzle Wear, Secondary Wear, Projectile Dynamics, Gun Dynamics, Balloting, Rifled Gun Tubes, Tube Curvature, Lateral Forces on Projectiles, Constraint Forces on Projectiles, Projectile Balance, Projectile Alignment.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The general equations of the six-degree of freedom problem of the lateral (constraint) forces on a projectile traveling in a gun bore are indicated. In general, the bore is not straight, the projectile is imbalanced, and the principal axis of inertia of the projectile is not aligned with the spin axis. Complete equations are derived for two special cases where the rifling is of constant pitch: the first case being that of an imbalanced and misaligned projectile traveling in a perfectly straight bore, and the second case being that of a perfectly balanced and aligned projectile traveling in a crooked bore.

Keywords

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

TABLE OF CONTENTS

	Page
LIST OF SYMBOLS	ii
INTRODUCTION	1
ASSUMPTIONS OF THE MODEL	2
COORDINATE SYSTEMS	3
KINEMATIC RELATIONSHIPS	4
COORDINATE TRANSFORMATION	7
FORCES ON THE PROJECTILE	10
Axisymmetric Band Drag Force	11
Lateral Force Friction Components	12
MOMENTS ABOUT THE BODY-FIXED AXES	14
THE EQUATIONS OF CONSTRAINT	20
REFERENCES	26
APPENDIX	27

LIST OF ILLUSTRATIONS

1.	Inertial (X,Y,Z), moving (x,y,z), and body-fixed (1,2,3) coordinate systems.	3
2.	Position vector for the projectile c.g. in the moving coordinate system.	5
3.	Forces in the developed circumferential (x-θ) plane.	11
4.	Components of the lateral force associated friction forces.	12
5.	Lateral force components in the moving coordinate system.	13
6.	Application points of the lateral constraint forces and their associated friction forces.	16
A-1.	Initial position of the body-fixed coordinate system in relation to the moving coordinate system.	27
A-2.	Auxiliary axes fixed in the body.	28

LIST OF SYMBOLS

A	-	Obturated projectile base area
B	-	Axial rotating band resistance force
a,b	-	Distance from projectile c.g. to the bourrelet and rotating band
e_m	-	Eccentricity of the center of mass of the projectile with respect to the bore center
$\vec{F}_F,\vec{F}_R$	-	Forces on the projectile at the bourrelet and rotating band
F_x	-	Total axial rotating band resistance force
$\vec{i},\vec{j},\vec{k}$	-	Unit vectors directed along the x, y, z axes
I_1,I_2,I_3	-	Principal mass moments of inertia of the projectile
I_A,I_T	-	Axial and transverse moments of inertia of the projectile
L_{nFy}, L_{nFz} L_{nRy}, L_{nRz} $n = 1,2,3$	-	Coefficients of the lateral force components in the moment equations
m_p,m_t	-	Masses of the projectile and tube
M_x	-	Projectile spinning-up torque
M_1,M_2,M_3	-	Moments about the body-fixed axes
N	-	Number of calibers per turn
P_b,P_c	-	Propelling gas pressure on the projectile base and gun chamber
$\ddot{\vec{Q}}_B$	-	Acceleration of the point on the tube axis coincident with the origin of the b coordinate system
$\vec{R}$	-	Position vector of the origin of the body-fixed coordinates relative to the inertial reference frame
r	-	Radius of the bore of the tube
$\vec{s}$	-	Position vector of the origin of the b coordinate system relative to the inertial reference frame
$s,\dot{s},\ddot{s}$	-	Travel, velocity, and acceleration of the projectile along the centerline of the bore
t	-	Time

$\vec{T}, \vec{N}, \vec{B}$ - Tangent, normal and binormal vectors of the Frenet-Serret Formulas

v_b, v_t - Velocities in the direction tangent to the bore of the tube of the projectile and the tube

κ, ρ, τ - Curvature, radius of curvature, and torsion of a curve in space

$\delta_{1x}, \delta_{1y}, \delta_{1z}$, $\delta_{2x}, \delta_{2y}, \delta_{2z}$, $\delta_{3x}, \delta_{3y}, \delta_{3z}$ - Angles between the directions of the body-fixed coordinate axes and the b coordinate axes

$\vec{\rho}$ - Position vector of the origin of the body-fixed coordinates relative to the origin of the b coordinate system

ψ - Angle between the tangent to the developed rifling curve and the bore centerline

θ - Angle of twist of rifling or rotation of the projectile

μ_B, μ_S - Coefficients of friction between the tube and the rotating band and the tube and the bourrelet

$\vec{\Omega}$ - Total angular velocity of the projectile

ω - Rotational speed of the projectile

INTRODUCTION

It is known that when projectiles are fired from gun tubes, the lateral forces of constraint may become so large as to cause engraving of the projectile body and excessive wear of the rifling of the tube - usually at the muzzle where the projectile velocity is highest, thus giving rise to the term muzzle wear. Normally, a projectile will, at some time during its travel down the bore of a cannon tube, come to bear against a particular rifling land and follow that land out to the muzzle; and when the same land is borne against with great enough force by a large enough number of projectiles, the result may be what is sometimes referred to as spiral wear. In any case, the result is undesirable and we wish to understand it so that it can be avoided.

All projectiles do not necessarily ride on the same land for the full length of travel down the bore, and when the balance or alignment of a projectile is such or becomes such that the shift from one land to another is violent enough to cause rebound, then the phenomenon called balloting may have begun. Since it is widely held that projectile balloting can have serious consequences, it has been the subject of much study, initially by Reno (ref 1), and Thomas (ref 2), and more recently by Perdreauville (ref 3), Chu and Soechting (ref 4), Walker (ref 5), and Soifer and Becker (ref 6). Although it is not the purpose of this study to go into the subject of balloting, there may be some application of the results to the problem of determining the necessary conditions of projectile balance and alignment and tube motion and curvature for balloting to occur.

The problem that is addressed here, however, is rather that of determining the combined effect of projectile balance and alignment and tube motion and curvature on the bearing (constraint) forces between the projectile and the tube

References are listed at the end of this report.

when stable projectile motion takes place. Darpas (ref 7) used some simplifying approximations in order to isolate the various effects and estimate their relative maximum magnitudes. His conclusion was that the dynamic unbalance of the projectile would most likely have the greatest effect. Dynamic unbalance would occur when a projectile becomes cocked in the bore, which as discussed by Gay (ref 8) and Montgomery (ref 9), is the more usual case. The following six-degree of freedom treatment of the problem is intended to establish the magnitudes of the various effects of specified projectile and tube conditions.

ASSUMPTIONS OF THE MODEL

The projectile is constrained by the inner wall of the tube and the rifling to move in the direction of and rotate about the axis of the tube. In general, the principal axis of inertia of the projectile is skewed with respect to the spin axis. The gas pressure driving force acts along the tube axis, the land driving force acts around the circumference of the bore and perpendicular to the rifling, and the lateral constraint forces act at the point of contact of the rotating band and bourrelet with the bore and in a direction perpendicular to the bore axis.

In addition, there are three frictional resistance forces normal to the land driving and lateral constraint forces and opposite to the direction of projectile motion at the point of contact. The band engraving resistance force which acts around the circumference of the bore and opposite to the direction of projectile motion is assumed to be known, as is the gas pressure, axial velocities of projectile and tube, mass, center of gravity (c.g.) location, axial and transverse moments of inertia of the projectile, and coefficients of friction between the rotating band and the tube, and between the projectile body and the tube.

COORDINATE SYSTEMS

Three coordinate systems are defined. Referring to Figure 1, the inertial reference frame with axes X, Y, and Z is designated the B coordinate system. The b coordinate system with axes x, y, and z originates on and moves along the tube axis (projectile spin axis) such that the center of mass of the projectile lies in the y-z plane and the x-axis is tangent to the tube axis. In general, the b coordinate system has rotational velocity relative to the inertial axes of $\vec{\omega}_{b/B}$ whose component in the direction of the spin axis = 0. Finally, there is the body-fixed coordinate system with origin at the center of gravity of the projectile and axes 1, 2, and 3, which are the principal axes of inertia of the projectile.

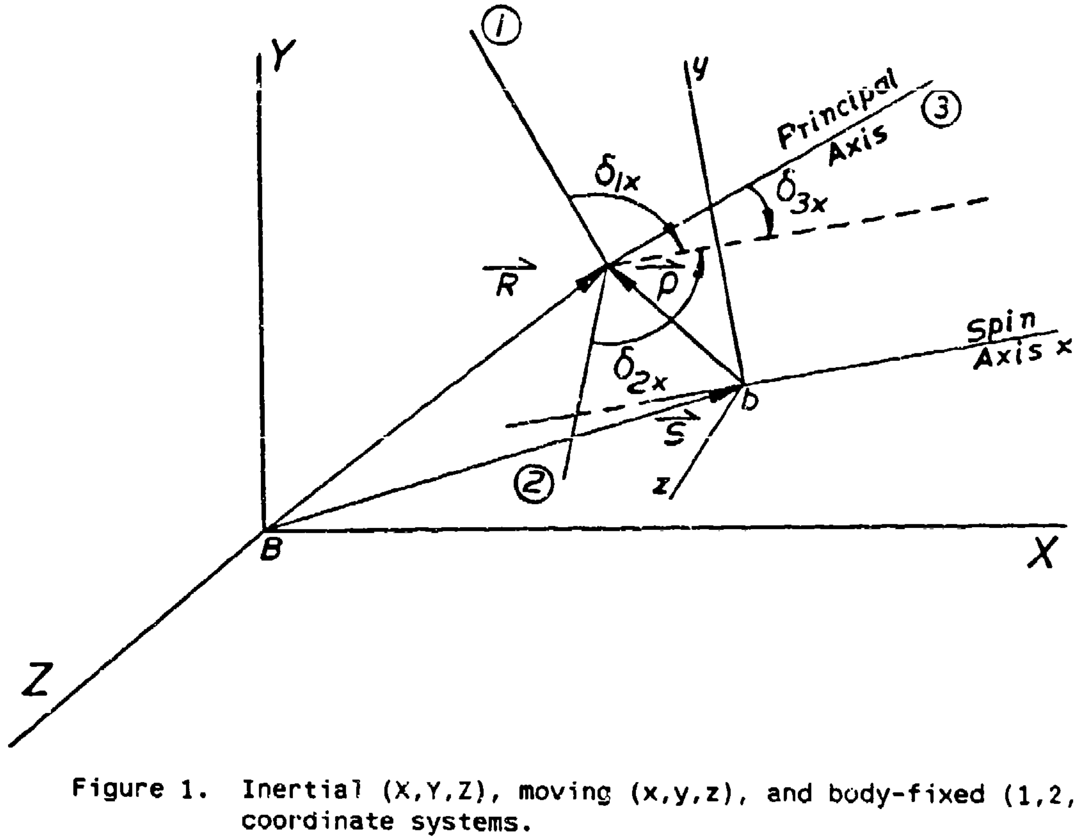

Figure 1. Inertial (X,Y,Z), moving (x,y,z), and body-fixed (1,2,3) coordinate systems.

Three force equations written with reference to the inertial coordinate system and three moment equations written with reference to the body-fixed coordinate system constitute the six equations used to solve for the six unknowns of the problem. The six unknowns are the two components of each of the two lateral forces of constraint (four unknowns), the driving torque around the spin axis (one unknown), and the acceleration in the direction of the spin axis (one unknown).

KINEMATIC RELATIONSHIPS

The relative velocity of the projectile and the tube is given by

$$v_b - v_t = \frac{ds}{dt} = \dot{s} \tag{1}$$

Since the rifling twist is a function of distance along the bore, we may write

$$r\theta = f(s) \ , \tag{2}$$

$$\omega = \dot{\theta} = \frac{1}{r} f'(s)\dot{s} \ , \tag{2a}$$

and

$$\dot{\omega} = \ddot{\theta} = \frac{1}{r} [f''(s)(\dot{s})^2 + f'(s)\ddot{s}] \ . \tag{2b}$$

For constant twist rifling we have

$$f(s) = \frac{\pi}{N} s \quad , \quad f'(s) = \frac{\pi}{N} \quad , \quad \text{and} \quad f''(s) = 0$$

so that

$$\omega = \dot{\theta} = \frac{\pi\dot{s}}{rN} \ , \tag{3a}$$

and

$$\dot{\omega} = \ddot{\theta} = \frac{\pi\ddot{s}}{rN} \ . \tag{3b}$$

From Figure 1 it can be seen that

$$\vec{R}_B = \vec{S}_B + \vec{\rho}_B \tag{4}$$

where the subscript B indicates reference to the B coordinate system. Differentiating twice with respect to time gives directly

$$\ddot{\vec{R}}_B = \ddot{\vec{S}}_B + \ddot{\vec{\rho}}_B \tag{5}$$

Texts on dynamics (ref 10) give the formula

$$\ddot{\vec{\rho}}_B = \ddot{\vec{\rho}}_b + \dot{\vec{\omega}}_{b/B} \times \vec{\rho}_b + 2\vec{\omega}_{b/B} \times \dot{\vec{\rho}}_b + \vec{\omega}_{b/B} \times \vec{\omega}_{b/B} \times \vec{\rho}_b \tag{6}$$

for rotating coordinate systems where the subscript b/B indicates rotation of the b coordinate system with respect to the B coordinate system. If e_m is the magnitude of the eccentricity of the projectile c.g. with respect to the spin axis, then $\vec{\rho}_b$, $\dot{\vec{\rho}}_b$, and $\ddot{\vec{\rho}}_b$ can be expressed as

$$\vec{\rho}_b = e_m\,(\cos\theta\vec{j} + \sin\theta\vec{k}) \;, \tag{7}$$

$$\dot{\vec{\rho}}_b = e_m\dot{\theta}\,(-\sin\theta\vec{j} + \cos\theta\vec{k}) \;, \tag{7a}$$

and

$$\ddot{\vec{\rho}}_b = e_m[-(\ddot{\theta}\,\sin\theta + \dot{\theta}^2\,\cos\theta)\vec{j} + (\ddot{\theta}\,\cos\theta - \dot{\theta}^2\,\sin\theta)\vec{k}] \tag{7b}$$

where θ is as shown in Figure 2.

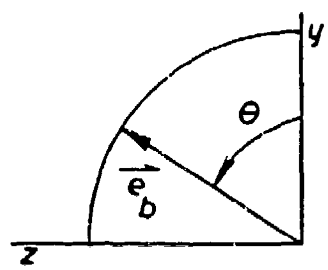

Figure 2. Position vector for the projectile c.g. in the moving coordinate system.

Since the spin axis is taken in general to be the tangent to a curve moving in space, it will be convenient to refer to the Frenet-Serret Formulas of differential geometry. These are

$$\frac{d\vec{B}}{ds} = -\tau\vec{N} \;, \quad \frac{d\vec{T}}{ds} = \kappa\vec{N} \;, \text{ and } \frac{d\vec{N}}{ds} = \tau\vec{B} - \kappa\vec{T} \tag{8a,b,c}$$

where

$$\vec{B} = \vec{T} \times \vec{N}. \quad , \quad \vec{T} = \frac{d\vec{R}}{ds} \text{ , and } \quad \kappa = \frac{1}{\rho} .$$

The velocity of a point moving along a curve in space is expressed as

$$\vec{v}(t) = \frac{d\vec{R}}{dt} = \frac{ds}{dt}\frac{d\vec{R}}{ds} . \tag{9}$$

Making use of the Frenet-Serret formulas allows the velocity to be written as

$$\vec{v}(t) = \frac{dS}{dt}\vec{T} , \tag{10}$$

and the acceleration to be written as

$$\vec{a}(t) = \frac{d^2s}{dt^2}\vec{T} + \left(\frac{ds}{dt}\right)^2 \kappa\vec{N} . \tag{11}$$

Now since $\ddot{\vec{s}}_B$ is the sum of the acceleration of the origin of the b coordinate system relative to its path and the acceleration of the point on the path coincident to the b coordinate system origin, we can write

$$\ddot{\vec{s}}_B = \ddot{\vec{Q}}_B + \frac{d^2s}{dt^2}\vec{T} + \left(\frac{ds}{dt}\right)^2 \kappa\vec{N} . \tag{12}$$

Also note that $\omega_{b/B}$ is the sum of the angular velocities due to rotation of the spin axis at the point coincident with the origin of the b coordinate system and motion of the coordinate system origin relative to its path. Therefore, if we designate by the subscript b', a coordinate system which is coincident to the b coordinate system but fixed to the spin axis, we can write

$$\vec{\omega}_{b/B} = \vec{\omega}_{b'/B} + \vec{\omega}_{b/b'} . \tag{13}$$

Recalling that $\vec{\omega}_{b/B}$ has no component in the direction of the spin axis, it follows that

$$\vec{\omega}_{b/b'} = \frac{ds}{dt}\vec{T} \times \frac{d\vec{T}}{ds} = \frac{ds}{dt}\vec{T} \times \kappa\vec{N} \tag{14}$$

so that

$$\vec{\omega}_{b/B} = \vec{\omega}_{b'/B} + \frac{ds}{dt}\vec{T} \times \kappa\vec{N} = \vec{\omega}_{b'/B} + \frac{ds}{dt}\kappa\vec{B} \tag{15}$$

Differentiating with respect to time gives

$$\dot{\vec{\omega}}_{b/B} = \dot{\vec{\omega}}_{b'/B} + \frac{d}{dt}\left[\frac{ds}{dt}\kappa\vec{B}\right] \tag{16}$$

which, after some manipulation, results in

$$\dot{\vec{\omega}}_{b/B} = \dot{\vec{\omega}}_{b'/B} + \left[\kappa\frac{d^2s}{dt^2} + \frac{d\kappa}{ds}\left(\frac{ds}{dt}\right)^2\right]\vec{B} - \kappa\tau\left(\frac{ds}{dt}\right)\vec{N} . \tag{17}$$

COORDINATE TRANSFORMATION

First of all, by observing that $\vec{T} = \vec{i}$, and $\vec{N} = N_y\vec{j} + N_z\vec{k}$, it follows that $\vec{B} = \vec{T}\times\vec{N} = N_y\vec{k} - N_z\vec{j}$. Also,

$$\ddot{\vec{Q}}_B = (\ddot{Q}_B)_x\vec{i} + (\ddot{Q}_B)_y\vec{j} + (\ddot{Q}_B)_z\vec{k} , \tag{18a}$$

$$\vec{\omega}_{b'/B} = (\omega_{b'/B})_y\vec{j} + (\omega_{b'/B})_z\vec{k} , \tag{18b}$$

and

$$\dot{\vec{\omega}}_{b'/B} = (\dot{\omega}_{b'/B})_y\vec{j} + (\dot{\omega}_{b'/B})_z\vec{k} . \tag{18c}$$

Therefore,

$$\ddot{\vec{s}}_B = \left[(\ddot{Q}_B)_x + \frac{d^2s}{dt^2}\right]\vec{i} + \left[(\ddot{Q}_B)_y + \left(\frac{ds}{dt}\right)^2\kappa N_y\right]\vec{j}$$

$$+ \left[(\ddot{Q}_B)_z + \left(\frac{ds}{dt}\right)^2\kappa N_z\right]\vec{k} , \tag{19a}$$

$$\vec{\omega}_{b/B} = \left[(\omega_{b'/B})_y - \frac{ds}{dt}\kappa N_z\right]\vec{j} + \left[(\omega_{b'/B})_z + \frac{ds}{dt}\kappa N_y\right]\vec{k} , \tag{19b}$$

and

$$\dot{\vec{\omega}}_{b/B} = \left\{(\dot{\omega}_{b'/B})_y - \left[\kappa\frac{d^2s}{dt^2} + \frac{d\kappa}{ds}\left(\frac{ds}{dt}\right)^2\right]N_z - \kappa\tau\left(\frac{ds}{dt}\right)^2 N_y\right\}\vec{j}$$

$$+ \left\{(\dot{\omega}_{b'/B})_z + \left[\kappa\frac{d^2s}{dt^2} + \frac{d\kappa}{ds}\left(\frac{ds}{dt}\right)^2\right]N_y - \kappa\tau\left(\frac{ds}{dt}\right)^2 N_z\right\}\vec{k} . \tag{19c}$$

The total angular velocity of the projectile is the vector sum of the projectile spin velocity and the angular velocity of the b coordinate system, or

$$\vec{\Omega} = \vec{\omega} + \vec{\omega}_{b/B} . \tag{20}$$

In the b coordinate system, the components of $\vec{\Omega}$ are then

$$\Omega_x = \omega \ , \tag{21a}$$

$$\Omega_y = (\vec{\omega}_{b'/B})_y + \frac{ds}{dt}\kappa N_z \ , \text{ and} \tag{21b}$$

$$\Omega_z = (\vec{\omega}_{b'/B})_z + \frac{ds}{dt}\kappa N_y \ . \tag{21c}$$

Similarly, the angular acceleration and its components are given by

$$\dot{\vec{\Omega}} = \dot{\vec{\omega}} + \dot{\vec{\omega}}_{b/B} \ , \tag{22}$$

$$\dot{\Omega}_x = \dot{\omega} \ , \tag{23a}$$

$$\dot{\Omega}_y = (\dot{\vec{\omega}}_{b'/B})_y - [\kappa \frac{d^2s}{dt^2} + \frac{d\kappa}{ds}(\frac{ds}{dt})^2]N_z - \kappa\tau(\frac{ds}{dt})^2 N_y \ , \tag{23b}$$

and

$$\dot{\Omega}_z = (\dot{\vec{\omega}}_{b'/B})_z + [\kappa \frac{d^2s}{dt^2} + \frac{d\kappa}{ds}(\frac{ds}{dt})^2]N_y - \kappa\tau(\frac{ds}{dt})^2 N_z \ . \tag{23c}$$

It remains now to express the projectile angular velocity and acceleration in terms of their components in the body-fixed coordinate system. For the angular velocity

$$\Omega_1 = \cos\delta_{1x}\Omega_x + \cos\delta_{1y}\Omega_y + \cos\delta_{1z}\Omega_z \ , \tag{24a}$$

$$\Omega_2 = \cos\delta_{2x}\Omega_x + \cos\delta_{2y}\Omega_y + \cos\delta_{2z}\Omega_z \ , \tag{24b}$$

and

$$\Omega_3 = \cos\delta_{3x}\Omega_x + \cos\delta_{3y}\Omega_y + \cos\delta_{3z}\Omega_z \ . \tag{24c}$$

Remembering that δ_{1x}, δ_{2x}, and δ_{3x} are assumed to remain constant, we get by differentiation

$$\dot{\Omega}_1 = \cos\delta_{1x}\dot{\Omega}_x + \frac{d}{dt}(\cos\delta_{1y})\Omega_y + \cos\delta_{1y}\dot{\Omega}_y + \frac{d}{dt}(\cos\delta_{1z})\Omega_z + \cos\delta_{1z}\dot{\Omega}_z \ , \tag{25a}$$

$$\dot{\Omega}_2 = \cos\delta_{2x}\dot{\Omega}_x + \frac{d}{dt}(\cos\delta_{2y})\Omega_y + \cos\delta_{2y}\dot{\Omega}_y + \frac{d}{dt}(\cos\delta_{2z})\Omega_z + \cos\delta_{2z}\dot{\Omega}_z \ , \tag{25b}$$

and

$$\dot{\Omega}_3 = \cos\delta_{3x}\dot{\Omega}_x + \frac{d}{dt}(\cos\delta_{3y})\Omega_y + \cos\delta_{3y}\dot{\Omega}_y$$

$$+ \frac{d}{dt}(\cos\delta_{3z})\Omega_z + \cos\delta_{3z}\dot{\Omega}_z \ . \qquad (25c)$$

Of the nine direction cosines appearing in the above equations, only two are actually independent. The other seven are expressible in terms of these two and are therefore eliminated as physical parameters. Because the conversion is lengthy, it is presented separately in the Appendix, and only the results are shown here.

$$\cos\delta_{1y} = \sin\delta_{1x} \cos\frac{\pi s}{rN} \ , \qquad (26a)$$

$$\cos\delta_{1z} = \sin\delta_{1x} \sin\frac{\pi s}{rN} \ , \qquad (26b)$$

$$\cos\delta_{2y} = \frac{-\cos\delta_{1x}\cos\delta_{2x}\cos\frac{\pi s}{rN} - \sin\frac{\pi s}{rN}\cos\delta_{3x}}{\sin\delta_{1x}} \ , \qquad (26c)$$

$$\cos\delta_{2z} = \frac{-\cos\delta_{1x}\cos\delta_{2x}\sin\frac{\pi s}{rN} + \cos\frac{\pi s}{rN}\cos\delta_{3x}}{\sin\delta_{1x}} \ , \qquad (26d)$$

$$\cos\delta_{3y} = \frac{-\cos\delta_{1x}\cos\delta_{3x}\cos\frac{\pi s}{rN} + \sin\frac{\pi s}{rN}\cos\delta_{2x}}{\sin\delta_{1x}} \ , \qquad (26e)$$

$$\cos\delta_{3z} = \frac{-\cos\delta_{1x}\cos\delta_{3x}\sin\frac{\pi s}{rN} - \cos\frac{\pi s}{rN}\cos\delta_{2x}}{\sin\delta_{1x}} \ , \qquad (26f)$$

and

$$\cos^2\delta_{1x} + \cos^2\delta_{2x} + \cos^2\delta_{3x} = 1 \ . \qquad (27)$$

Performing the differentiation indicated on the direction cosines in the formulas for the angular acceleration components reveals the following:

$$\frac{d}{dt}(\cos\delta_{1y}) = -\cos\delta_{1z}\left(\frac{\pi\dot{s}}{rN}\right) , \tag{28a}$$

$$\frac{d}{dt}(\cos\delta_{1z}) = \cos\delta_{1y}\left(\frac{\pi\dot{s}}{rN}\right) , \tag{28b}$$

$$\frac{d}{dt}(\cos\delta_{2y}) = -\cos\delta_{2z}\left(\frac{\pi\dot{s}}{rN}\right) , \tag{28c}$$

$$\frac{d}{dt}(\cos\delta_{2z}) = \cos\delta_{2y}\left(\frac{\pi\dot{s}}{rN}\right) , \tag{28d}$$

$$\frac{d}{dt}(\cos\delta_{3y}) = -\cos\delta_{3z}\left(\frac{\pi\dot{s}}{rN}\right) , \tag{28e}$$

and

$$\frac{d}{dt}(\cos\delta_{3z}) = \cos\delta_{3y}\left(\frac{\pi\dot{s}}{rN}\right) . \tag{28f}$$

Therefore,

$$\dot{\Omega}_1 = \cos\delta_{1x}\dot{\Omega}_x + \cos\delta_{1y}[\dot{\Omega}_y + (\frac{\pi\dot{s}}{rN})\Omega_z] + \cos\delta_{1z}[\dot{\Omega}_z - (\frac{\pi\dot{s}}{rN})\Omega_y] , \tag{29a}$$

$$\dot{\Omega}_2 = \cos\delta_{2x}\dot{\Omega}_x + \cos\delta_{2y}[\dot{\Omega}_y + (\frac{\pi\dot{s}}{rN})\Omega_z] + \cos\delta_{2z}[\dot{\Omega}_z - (\frac{\pi\dot{s}}{rN})\Omega_y] , \tag{29b}$$

and

$$\dot{\Omega}_3 = \cos\delta_{3x}\dot{\Omega}_x + \cos\delta_{3y}[\dot{\Omega}_y + (\frac{\pi\dot{s}}{rN})\Omega_z] + \cos\delta_{3z}[\dot{\Omega}_z - (\frac{\pi\dot{s}}{rN})\Omega_y] . \tag{29c}$$

FORCES ON THE PROJECTILE

Three of the six equations needed for solution of the problem are available from the equation of motion

$$\sum_{i=1}^{n} \vec{F}_i = m_p\ddot{\vec{R}}_B . \tag{30}$$

Of the six forces acting on the projectile, one is the known gas pressure driving force and two are the unknown lateral constraint forces of the problem. The three remaining forces are frictional resistance forces which arise from the band pressure force, spinning torque, and constraint force applied between the projectile and tube in relative motion.

Axisymmetric Band Drag Force

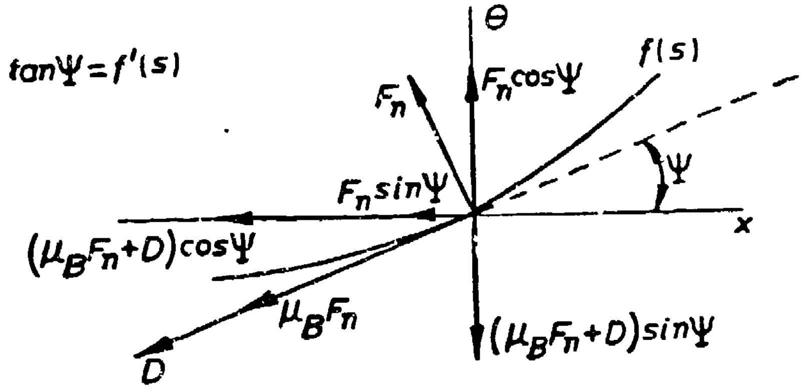

Figure 3. Forces in the developed circumferential (x-θ) plane.

Figure 3 is a development of the rifling curve onto the plane defined by the x and θ axes. The band pressure drag force is denoted by D, the normal land driving force is denoted by F_n, and the angle between the tangent to the rifling curve and the x-axis is denoted by ψ. If we assume that D is proportional to the width of the rotating band in the direction of its movement relative to the tube, then we can write

$$D = B/\cos\psi \; . \tag{31}$$

Summing forces in the x direction gives

$$F_x = - F_n \sin\psi - (\mu_B F_n + D)\cos\psi \; , \tag{32}$$

and since the sum of the forces in the θ direction are equal to the spinning torque divided by the radius of the bore, we have

$$\frac{M_x}{r} = F_n \cos\psi - (\mu_B F_n + D)\sin\psi \tag{33}$$

F_n can be eliminated between Eq. (32) and Eq. (33) by rearranging and dividing one by the other

$$\frac{-F_x - D\cos\psi}{\frac{M_x}{r} + D\sin\psi} = \frac{\sin\psi + \mu_B\cos\psi}{\cos\psi - \mu_B\sin\psi} \tag{34}$$

Dividing the numerator and denominator of the right side of the above equation by cos ψ gives

$$\frac{-F_x - B}{\frac{M_x}{r} + Bf'(s)} = \frac{f'(s) + \mu_B}{1 - \mu_B f'(s)} . \tag{35}$$

For constant pitch rifling $f'(s) = \frac{\pi}{N}$ so that

$$F_x = \left(\frac{M_x}{r} - \frac{B\pi}{N}\right)\left(\frac{\pi - \mu_B N}{\mu_B\pi - N}\right) - B . \tag{36}$$

Lateral Force Friction Components

The lateral force, $\vec{F}_R$, at the band and the lateral force, $\vec{F}_F$, at the bourrelet have frictional forces associated with them which have magnitude $\mu_B F_R$ and $\mu_S F_F$, respectively. These frictional forces act along the bore surface opposite in direction to the motion of the projectile and therefore have axial and tangential components as shown in Figure 4.

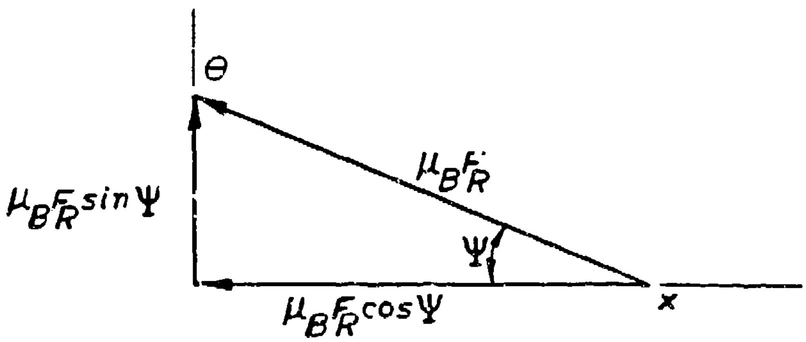

Figure 4. Components of the lateral force associated friction forces.

Again, for constant pitch rifling $\tan\psi = f'(s) = \frac{\pi}{N}$, so we have

$$\cos\psi = \frac{1}{\sqrt{\tan^2\psi + 1}} = \frac{1}{\sqrt{(\pi/N)^2 + 1}} , \tag{37}$$

and

$$\sin\psi = \cos\psi \tan\psi = \frac{\pi/N}{\sqrt{(\pi/N)^2 + 1}} . \tag{38}$$

The tangential component can now be broken up into y and z components as in Figure 5.

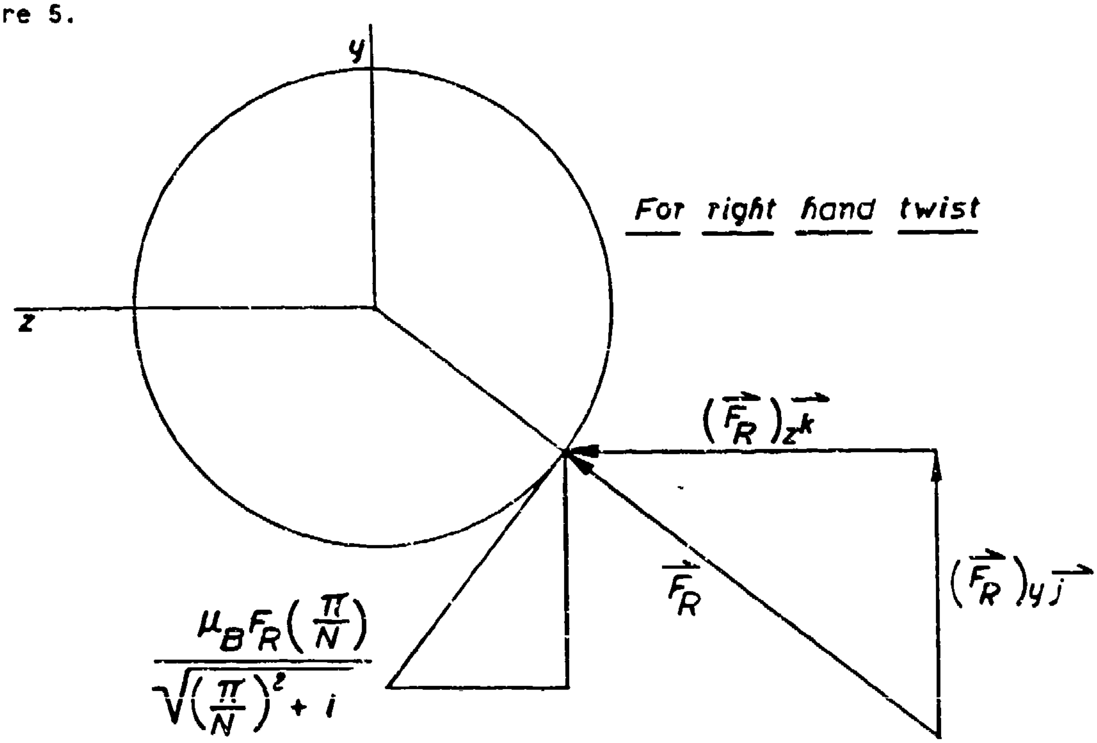

Figure 5. Lateral force components in the moving coordinate system.

From the similarity of the force triangles in Figure 5, the following is derived:

$$\left[\frac{\mu_B F_R(\pi/N)}{\sqrt{(\pi/N)^2 + 1}}\right]_y = -\frac{\mu_B F_R(\pi/N)}{\sqrt{(\pi/N)^2 + 1}} \frac{(\vec{F}_R)_z}{F_R} = -\frac{\mu_B(\vec{F}_R)_z(\pi/N)}{\sqrt{(\pi/N)^2 + 1}} , \tag{39a}$$

$$\left[\frac{\mu_B F_R(\pi/N)}{\sqrt{(\pi/N)^2 + 1}}\right]_z = \frac{\mu_B F_R(\pi/N)}{\sqrt{(\pi/N)^2 + 1}} \frac{(\vec{F}_R)_y}{F_R} = \frac{\mu_B(\vec{F}_R)_y(\pi/N)}{\sqrt{(\pi/N)^2 + 1}} , \tag{39b}$$

and similarly for the force, $\vec{F}_F$.

MOMENTS ABOUT THE BODY-FIXED AXES

In addition to the three equations derived from the equation of motion, we require three more equations to give us the six necessary for a solution. These additional equations are derived from Euler's equations which are written with respect to the body-fixed axes as follows:

$$M_1 = \dot{\Omega}_1 I_1 - \Omega_2\Omega_3(I_2 - I_3) , \tag{40a}$$

$$M_2 = \dot{\Omega}_2 I_2 - \Omega_3\Omega_1(I_3 - I_1) , \tag{40b}$$

and

$$M_3 = \dot{\Omega}_3 I_3 - \Omega_1\Omega_2(I_1 - I_2) . \tag{40c}$$

Note that the quantities I_1 and I_2 are the transverse moments of inertia of the projectile. Now since our concern is with repeatable conditions that occur with regularity, there is little point in differentiating between the two transverse moments of inertia of projectiles which are usually loaded with random angular orientation and are in any case nominally equal. Therefore, we shall say $I_1 = I_2 = I_T$ and redesignate the axial moment of inertia, I_3, as I_A. The Euler equations then become

$$M_1 = \dot{\Omega}_1 I_T - \Omega_2\Omega_3(I_T - I_A) \ , \tag{41a}$$

$$M_2 = \dot{\Omega}_2 I_T - \Omega_3\Omega_1(I_A - I_T) \ , \tag{41b}$$

and

$$M_3 = \dot{\Omega}_3 I_A \ . \tag{41c}$$

The moments about these axes are given by the triple product

$$M_n = \vec{M} \cdot \vec{u}_n = (\vec{r} \times \vec{F}) \cdot \vec{u}_n \ , \tag{42}$$

where $\vec{r}$ is the position vector from the origin of the body-fixed axes to the point of application of the force, $\vec{F}$, and the unit vectors, $\vec{u}_n$, for the body-fixed axes are expressed in terms of the direction cosines as follows:

$$\vec{u}_n = \cos\delta_{nx}\ \vec{i} + \cos\delta_{ny}\ \vec{j} + \cos\delta_{nz}\ \vec{k} \quad (n=1,2,3) \ . \tag{43}$$

For the axial forces, the unit force vector is $\vec{i}$ so that

$$\vec{r} = -\vec{\rho}_b = -e_m\left(\sin\frac{\pi s}{rN}\ \vec{k} + \cos\frac{\pi s}{rN}\ \vec{j}\right) \ , \tag{44}$$

$$\vec{r} \times \vec{i} = -e_m\left(\sin\frac{\pi s}{rN}\ \vec{j} - \cos\frac{\pi s}{rN}\ \vec{k}\right) \ , \tag{45}$$

$$(\vec{r} \times \vec{i}) \cdot \vec{u}_1 = 0 \ , \tag{46a}$$

$$(\vec{r} \times \vec{i}) \cdot \vec{u}_2 = e_m \frac{\cos\delta_{3x}}{\sin\delta_{1x}} \ , \tag{46b}$$

and

$$(\vec{r} \times \vec{i}) \cdot \vec{u}_3 = -e_m \frac{\cos\delta_{2x}}{\sin\delta_{1x}} \ . \tag{46c}$$

The unit moment vector for the rotating band torque is $\vec{i}$, so that for right-hand twist we have

$$\vec{i} \cdot \vec{u}_1 = \cos\delta_{1x} \ , \tag{47a}$$

$$\vec{i} \cdot \vec{u}_2 = \cos\delta_{2x} \ , \tag{47b}$$

and

$$\vec{i} \cdot \vec{u}_3 = \cos\delta_{3x} \ . \tag{47c}$$

The position vectors of the points of application of the lateral constraint forces and their associated forces are shown in Figure 6. The distances from the c.g. of the projectile along the x-axis to the rotating band and the bourrelet are given by b and a, respectively.

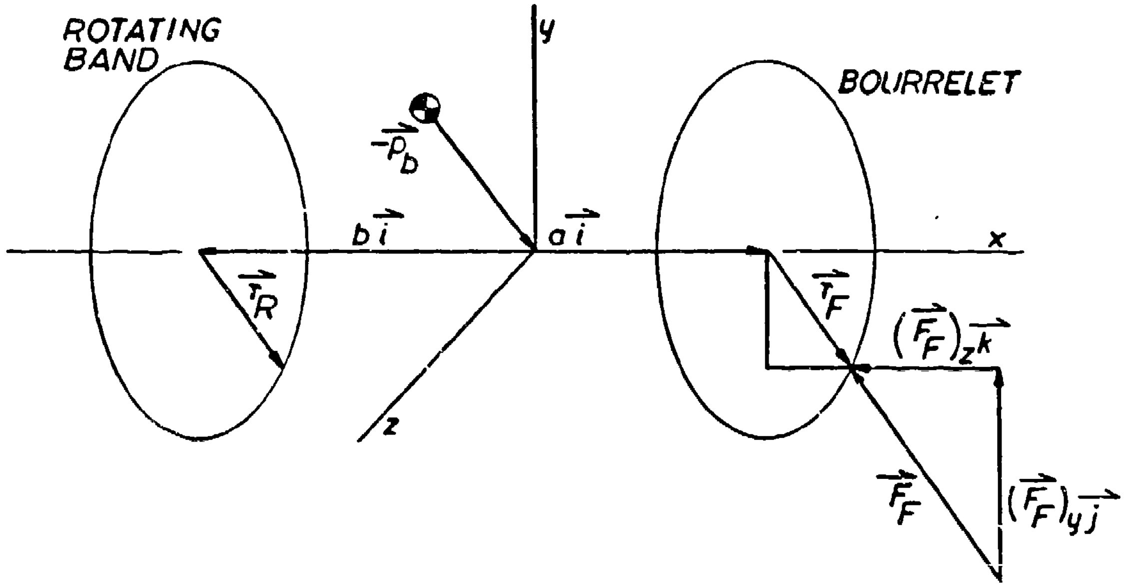

Figure 6. Application points of the lateral constraint forces and their associated friction forces.

From Figure 6 it can be seen that

$$\vec{r}_F = -\frac{r_F}{F_F}\left[(\vec{F}_F)_y\vec{j} + (\vec{F}_F)_z\vec{k}\right] , \tag{48a}$$

and

$$\vec{r}_R = -\frac{r_R}{F_R}\left[(\vec{F}_R)_y\vec{j} + (\vec{F}_R)_z\vec{k}\right] . \tag{48b}$$

The position vectors are then written as

$$a\vec{i} - \left[e_m \cos\frac{\pi s}{rN} + \frac{r_F}{F_F}(\vec{F}_F)_y\right]\vec{j} - \left[e_m \sin\frac{\pi s}{rN} + \frac{r_F}{F_F}(\vec{F}_F)_z\right]\vec{k} ,$$

and

$$-b\vec{i} - \left[e_m \cos\frac{\pi s}{rN} + \frac{r_R}{F_R}(\vec{F}_R)_y\right]\vec{j} - \left[e_m \sin\frac{\pi s}{rN} + \frac{r_R}{F_R}(\vec{F}_R)_z\right]\vec{k} .$$

Taking the vector product of the position vectors and the respective force vectors yields the moment vectors

$$e_m[(\vec{F}_F)_y \sin \frac{\pi s}{rN} - (\vec{F}_F)_z \cos \frac{\pi s}{rN}]\vec{i} - a(\vec{F}_F)_z\vec{j} + a(\vec{F}_F)_y\vec{k} ,$$

and

$$e_m[(\vec{F}_R)_y \sin \frac{\pi s}{rN} - (\vec{F}_R)_z \cos \frac{\pi s}{rN}]\vec{i} + b(\vec{F}_R)_z\vec{j} - b(\vec{F}_R)_y\vec{k} .$$

For the sake of convenience, make the following abbreviations:

$$\mu'_B = \frac{\mu_B}{\sqrt{(\pi/N)^2 + 1}} ,$$

and

$$\mu'_s = \frac{\mu_s}{\sqrt{(\pi/N)^2 + 1}} .$$

Using the above notation, the frictional forces associated with the lateral forces are

$$\mu'_s[-F_F\vec{i} - (\vec{F}_F)_z(\frac{\pi}{N})\vec{j} + (\vec{F}_F)_y(\frac{\pi}{N})\vec{k}] ,$$

and

$$\mu'_B[-F_R\vec{i} - (\vec{F}_R)_z(\frac{\pi}{N})\vec{j} + (\vec{F}_R)_y(\frac{\pi}{N})\vec{k}] .$$

And the vector products of the respective position vectors and the above forces are

$$- \mu'_s(\frac{\pi}{N})[e_m(\vec{F}_F)_y \cos \frac{\pi s}{rN} + e_m(\vec{F}_F)_z \sin \frac{\pi s}{rN} + rF_F]\vec{i}$$

$$- \mu'_s[a(\frac{\pi}{N})(\vec{F}_F)_y - e_mF_F \sin \frac{\pi s}{rN} - r(\vec{F}_F)_z]\vec{j}$$

$$- \mu'_s[a(\frac{\pi}{N})(\vec{F}_F)_z + e_mF_F \cos \frac{\pi s}{rN} + r(\vec{F}_F)_y]\vec{k} ,$$

and

$$- \mu'_B(\frac{\pi}{N})[e_m(\vec{F}_R)_y \cos \frac{\pi s}{rN} + e_m(\vec{F}_R)_z \sin \frac{\pi s}{rN} + rF_R]\vec{i}$$

$$+ \mu'_B[b(\frac{\pi}{N})(\vec{F}_R)_y + e_mF_R \sin \frac{\pi s}{rN} + r(\vec{F}_R)_z]\vec{j}$$

$$+ \mu'_B[b(\frac{\pi}{N})(\vec{F}_R)_z - e_mF_R \cos \frac{\pi s}{rN} - r(\vec{F}_R)_y]\vec{k} .$$

The moment of the lateral constraint forces and the associated friction forces about the origin of the body-fixed axis coordinate system is then expressed as:

$$\{e_m[(\vec{F}_F)_y \sin\frac{\pi s}{rN} - (\vec{F}_F)_z \cos\frac{\pi s}{rN} + (\vec{F}_R)_y \sin\frac{\pi s}{rN} - (\vec{F}_R)_z \cos\frac{\pi s}{rN}]$$

$$- \mu'_S(\tfrac{\pi}{N})[e_m(\vec{F}_F)_y \cos\frac{\pi s}{rN} + e_m(\vec{F}_F)_z \sin\frac{\pi s}{rN} + rF_F]$$

$$- \mu'_B(\tfrac{\pi}{N})[e_m(\vec{F}_R)_y \cos\frac{\pi s}{rN} + e_m(\vec{F}_R)_z \sin\frac{\pi s}{rN} + rF_R]\}\vec{i}$$

$$+ \{- a(F_F)_z + b(\vec{F}_R)_z - \mu'_S[a(\tfrac{\pi}{N})(\vec{F}_F)_y - e_mF_F \sin\frac{\pi s}{rN} - r(\vec{F}_F)_z]$$

$$+ \mu'_B[b(\tfrac{\pi}{N})(\vec{F}_R)_y + e_mF_R \sin\frac{\pi s}{rN} + r(\vec{F}_R)_z]\}\vec{j}$$

$$+ \{a(\vec{F}_F)_y - b(\vec{F}_R)_y - \mu'_S[a(\tfrac{\pi}{N})(F_F)_z + e_mF_F \cos\frac{\pi s}{rN} + r(\vec{F}_F)_y]$$

$$+ \mu'_B[b(\tfrac{\pi}{N})(\vec{F}_R)_z - e_mF_R \cos\frac{\pi s}{rN} - r(\vec{F}_R)_y]\}\vec{k}\ .$$

The components of this moment vector in the direction of the body-fixed coordinate axes are determined by taking the scalar products with the unit vectors of the body-fixed axes $\cos\delta_{nx}\vec{i} + \cos\delta_{ny}\vec{j} + \cos\delta_{nz}\vec{k}$ (n = 1,2,3) and adding to the axial force moment and the spinning-up torque for substitution into the Euler equations. These components of the moment vector take the form

$$L_{nFy}(\vec{F}_F)_y + L_{nFz}(\vec{F}_F)_z + L_{nRy}(\vec{F}_R)_y + L_{nRz}(\vec{F}_R)_z \quad (n = 1,2,3)$$

where L_{nFy}, L_{nFz}, L_{nRy}, and L_{nRz} are functions of the known parameters of the problem only. For the sake of brevity in writing the equations of constraint, the coefficients are given here as:

$$L_{1Fy} = e_m \cos\delta_{1x}[\sin\frac{\pi s}{rN} - \mu'_S(\tfrac{\pi}{N}) \cos\frac{\pi s}{rN}]$$

$$- \mu'_S a(\tfrac{\pi}{N}) \sin\delta_{1x}\cos\frac{\pi s}{rN} + (a-\mu'_S r) \sin\delta_{1x} \sin\frac{\pi s}{rN}\ , \tag{49a}$$

$$L_{1Fz} = -e_m \cos\delta_{1x}[\cos\frac{\pi s}{rN} + \mu'_S(\tfrac{\pi}{N}) \sin\frac{\pi s}{rN}]$$

$$+ (a-\mu'_S r) \sin\delta_{1x} \cos\frac{\pi s}{rN} + \mu'_S a(\tfrac{\pi}{N}) \sin\delta_{1x} \sin\frac{\pi s}{rN}\ , \tag{49b}$$

$$L_{1Ry} = e_m \cos\delta_{1x} \left[\sin\frac{\pi s}{rN} - \mu'_B\left(\frac{\pi}{N}\right)\cos\frac{\pi s}{rN}\right]$$

$$+ \mu'_B b\left(\frac{\pi}{N}\right)\sin\delta_{1x}\cos\frac{\pi s}{rN} - (b+\mu'_B r)\sin\delta_{1x}\sin\frac{\pi s}{rN}\ , \tag{49c}$$

$$L_{1Rz} = -e_m \cos\delta_{1x}\left[\cos\frac{\pi s}{rN} + \mu'_B\left(\frac{\pi}{N}\right)\sin\frac{\pi s}{rN}\right]$$

$$-(b+\mu'_B r)\sin\delta_{1x}\cos\frac{\pi s}{rN} - \mu'_B b\left(\frac{\pi}{N}\right)\sin\delta_{1x}\sin\frac{\pi s}{rN}\ , \tag{49d}$$

$$L_{2Fy} = e_m \cos\delta_{2x}\left[\sin\frac{\pi s}{rN} - \mu'_S\left(\frac{\pi}{N}\right)\cos\frac{\pi s}{rN}\right]$$

$$+ \mu'_S a\left(\frac{\pi}{N}\right)\left(\cos\delta_{1x}\cos\delta_{2x}\cos\frac{\pi s}{rN} + \sin\frac{\pi s}{rN}\cos\delta_{3x}\right)/\sin\delta_{1x}$$

$$- (a-\mu'_S r)\left(\cos\delta_{1x}\cos\delta_{2x}\sin\frac{\pi s}{rN} - \cos\frac{\pi s}{rN}\cos\delta_{3x}\right)/\sin\delta_{1x}\ , \tag{49e}$$

$$L_{2Fz} = -e_m \cos\delta_{2x}\left[\cos\frac{\pi s}{rN} + \mu'_S\left(\frac{\pi}{N}\right)\sin\frac{\pi s}{rN}\right]$$

$$- (a-\mu'_S r)\left(\cos\delta_{1x}\cos\delta_{2x}\cos\frac{\pi s}{rN} + \sin\frac{\pi s}{rN}\cos\delta_{3x}\right)/\sin\delta_{1x}$$

$$+ \mu'_S a\left(\frac{\pi}{N}\right)\left(\cos\delta_{1x}\cos\delta_{2x}\sin\frac{\pi s}{rN} - \cos\frac{\pi s}{rN}\cos\delta_{3x}\right)/\sin\delta_{1x}\ , \tag{49f}$$

$$L_{2Ry} = e_m \cos\delta_{2x}\left[\sin\frac{\pi s}{rN} - \mu'_B\left(\frac{\pi}{N}\right)\cos\frac{\pi s}{rN}\right]$$

$$- \mu'_B b\left(\frac{\pi}{N}\right)\left(\cos\delta_{1x}\cos\delta_{2x}\cos\frac{\pi s}{rN} + \sin\frac{\pi s}{rN}\cos\delta_{3x}\right)/\sin\delta_{1x}$$

$$+ (b+\mu'_B r)\left(\cos\delta_{1x}\cos\delta_{2x}\sin\frac{\pi s}{rN} - \cos\frac{\pi s}{rN}\cos\delta_{3x}\right)/\sin\delta_{1x}\ , \tag{49g}$$

$$L_{2Rz} = -e_m \cos\delta_{2x}\left[\cos\frac{\pi s}{rN} + \mu'_B\left(\frac{\pi}{N}\right)\sin\frac{\pi s}{rN}\right]$$

$$+ (b+\mu'_B r)\left(\cos\delta_{1x}\cos\delta_{2x}\cos\frac{\pi s}{rN} + \sin\frac{\pi s}{rN}\cos\delta_{3x}\right)/\sin\delta_{1x}$$

$$- \mu'_B b\left(\frac{\pi}{N}\right)\left(\cos\delta_{1x}\cos\delta_{2x}\sin\frac{\pi s}{rN} - \cos\frac{\pi s}{rN}\cos\delta_{3x}\right)/\sin\delta_{1x}\ , \tag{49h}$$

$$L_{3Fy} = e_m \cos\delta_{3x}[\sin \frac{\pi s}{rN} - \mu'_s(\frac{\pi}{N}) \cos \frac{\pi s}{rN}]$$

$$+ \mu'_s a(\frac{\pi}{N})(\cos\delta_{1x} \cos\delta_{3x} \cos \frac{\pi s}{rN} - \sin \frac{\pi s}{rN} \cos\delta_{2x})/\sin\delta_{1x}$$

$$- (a-\mu'_s r)(\cos\delta_{1x} \cos\delta_{3x} \sin \frac{\pi s}{rN} + \cos \frac{\pi s}{rN} \cos\delta_{2x})/\sin\delta_{1x} , \qquad (49i)$$

$$L_{3Fz} = -e_m \cos\delta_{3x}[\cos \frac{\pi s}{rN} + \mu'_s(\frac{\pi}{N}) \sin \frac{\pi s}{rN}]$$

$$- (a-\mu'_s r)(\cos\delta_{1x} \cos\delta_{3x} \cos \frac{\pi s}{rN} - \sin \frac{\pi s}{rN} \cos\delta_{2x})/\sin\delta_{1x}$$

$$+ \mu'_s a(\frac{\pi}{N})(\cos\delta_{1x} \cos\delta_{3x} \sin \frac{\pi s}{rN} + \cos \frac{\pi s}{rN} \cos\delta_{2x})/\sin\delta_{1x} , \qquad (49j)$$

$$L_{3Ry} = e_m \cos\delta_{3x}[\sin \frac{\pi s}{rN} - \mu'_B(\frac{\pi}{N}) \cos \frac{\pi s}{rN}]$$

$$- \mu'_B b(\frac{\pi}{N})(\cos\delta_{1x} \cos\delta_{3x} \cos \frac{\pi s}{rN} - \sin \frac{\pi s}{rN} \cos\delta_{2x})/\sin\delta_{1x}$$

$$+ (b+\mu'_B r)(\cos\delta_{1x} \cos\delta_{3x} \sin \frac{\pi s}{rN} + \cos \frac{\pi s}{rN} \cos\delta_{2x})/\sin\delta_{1x} , \qquad (49k)$$

and

$$L_{3Rz} = -e_m \cos\delta_{3x}[\cos \frac{\pi s}{rN} + \mu'_B(\frac{\pi}{N}) \sin \frac{\pi s}{rN}]$$

$$+ (b+\mu'_B r)(\cos\delta_{1x} \cos\delta_{3x} \cos \frac{\pi s}{rN} - \sin \frac{\pi s}{rN} \cos\delta_{2x})/\sin\delta_{1x}$$

$$- \mu'_B b(\frac{\pi}{N})(\cos\delta_{1x} \cos\delta_{3x} \sin \frac{\pi s}{rN} + \cos \frac{\pi s}{rN} \cos\delta_{2x})/\sin\delta_{1x} . \qquad (49l)$$

THE EQUATIONS OF CONSTRAINT

We are now in a position to write the six equations of constraint. This will be done here for the special cases of a perfectly straight tube ($\kappa=0$), and a perfectly balanced and aligned projectile ($e_m = \delta_{3x} = 0$). It is not necessary to make either one of these simplifications in order to obtain the equations, but by making one or the other, we avoid the very tediously complex equations of the general case. Furthermore, this arrangement permits the study of the isolated effects.

In both cases it will be assumed that the only motion of the tube is that of free recoil in the axial direction. Observing that $(\ddot{\vec{Q}}_B)_x = \dot{v}_t$, and that for free recoil the only forces acting on the tube are the gas pressure force and the mutually opposite tube to projectile forces, we have

$$\dot{v}_t = \frac{1}{m_t}\left[B - \left(\frac{M_x}{r} - \frac{B\pi}{N}\right)\left(\frac{\pi - \mu_B N}{\mu_B \pi - N}\right) + \mu'_B F_R + \mu'_s F_F - P_c A\right] \tag{50}$$

For the straight tube case (κ=0) where $(\ddot{\vec{Q}}_B)_y = (\ddot{\vec{Q}}_B)_z = (\vec{\omega}_{b'/B})_y = (\vec{\omega}_{b'/B})_z = (\dot{\vec{\omega}}_{b'/B})_y = (\dot{\vec{\omega}}_{b'/B})_z = v$,

$$\ddot{\vec{R}}_B = \ddot{\vec{S}}_B + \ddot{\vec{\rho}}_B = [(\ddot{Q}_B)_x + \ddot{s}]\vec{i}$$

$$- e_m\left[\frac{\pi\ddot{s}}{rN}\sin\frac{\pi s}{rN} + \left(\frac{\pi\dot{s}}{rN}\right)^2\cos\frac{\pi s}{rN}\right]\vec{j}$$

$$+ e_m\left[\frac{\pi\ddot{s}}{rN}\cos\frac{\pi s}{rN} - \left(\frac{\pi\dot{s}}{rN}\right)^2\sin\frac{\pi s}{rN}\right]\vec{k}\ . \tag{51}$$

The force equations then become

$$\left(P_b + \frac{m_p}{m_t}P_c\right)A + \left(1 + \frac{m_p}{m_t}\right)\left[\left(\frac{M_x}{r} - \frac{B\pi}{N}\right)\left(\frac{\pi - \mu_B N}{\mu_B\pi - N}\right)\right.$$

$$\left. - B - \mu'_B F_R - \mu'_s F_F\right] = m_p\ddot{s}\ , \tag{52}$$

$$(\vec{F}_F)_y - \mu'_s\left(\frac{\pi}{N}\right)(\vec{F}_F)_z + (\vec{F}_R)_y - \mu'_B\left(\frac{\pi}{N}\right)(\vec{F}_R)_z$$

$$= -m_p e_m\left[\frac{\pi\ddot{s}}{rN}\sin\frac{\pi s}{rN} + \left(\frac{\pi\dot{s}}{rN}\right)^2\cos\frac{\pi s}{rN}\right]\ , \tag{52a}$$

and

$$(\vec{F}_F)_z + \mu'_s\left(\frac{\pi}{N}\right)(\vec{F}_F)_y + (\vec{F}_R)_z + \mu'_B\left(\frac{\pi}{N}\right)(\vec{F}_R)_y$$

$$= m_p e_m\left[\frac{\pi\ddot{s}}{rN}\cos\frac{\pi s}{rN} - \left(\frac{\pi\dot{s}}{rN}\right)^2\sin\frac{\pi s}{rN}\right]\ . \tag{52b}$$

Also,

$$\Omega_x = \omega = \frac{\pi\dot{s}}{rN} \quad , \quad \Omega_y = 0 \quad , \quad \Omega_z = 0 \ ,$$

$$\dot{\Omega}_x = \dot{\omega} = \frac{\pi\ddot{s}}{rN} \quad , \quad \dot{\Omega}_y = 0 \quad , \quad \dot{\Omega}_z = 0 \ .$$

The moment equations are then:

$$\cos\delta_{1x}M_x$$

$$+ L_{1Fy}(\vec{F}_F)_y + L_{1Fz}(\vec{F}_F)_z + L_{1Ry}(\vec{F}_R)_y + L_{1Rz}(\vec{F}_R)_z$$

$$= \cos\delta_{1x}(\frac{\pi\ddot{s}}{rN})I_T - \cos\delta_{2x}\cos\delta_{3x}(\frac{\pi\dot{s}}{rN})^2(I_T-I_A) \ , \tag{53a}$$

$$\cos\delta_{2x}M_x + e_m(\frac{\cos\delta_{3x}}{\sin\delta_{1x}})[P_bA + (\frac{M_x}{r} - \frac{B\pi}{N})(\frac{\pi - \mu_B N}{\mu_B\pi - N}) - B]$$

$$+ L_{2Fy}(\vec{F}_F)_y + L_{2Fz}(\vec{F}_F)_z + L_{2Ry}(\vec{F}_R)_y + L_{2Rz}(\vec{F}_R)_z$$

$$= \cos\delta_{2x}(\frac{\pi\ddot{s}}{rN})I_T - \cos\delta_{3x}\cos\delta_{1x}(\frac{\pi\dot{s}}{rN})^2(I_A-I_T) \ , \tag{53b}$$

and

$$\cos\delta_{3x}M_x - e_m(\frac{\cos\delta_{2x}}{\sin\delta_{1x}})[P_bA + (\frac{M_x}{r} - \frac{B\pi}{N})(\frac{\pi - \mu_B N}{\mu_B\pi - N}) - B]$$

$$+ L_{3Fy}(\vec{F}_F)_y + L_{3Fz}(\vec{F}_F)_z + L_{3Ry}(\vec{F}_R)_y + L_{3Rz}(\vec{F}_R)_z$$

$$= \cos\delta_{3x}(\frac{\pi\ddot{s}}{rN})I_A \ . \tag{53c}$$

For the balanced and aligned projectile case ($e_m = \delta_{3x} = 0$) where

$$(\ddot{\vec{Q}}_B)_y = (\ddot{\vec{Q}}_B)_z = (\vec{\omega}_{b'/B})_y = (\vec{\omega}_{b'/B})_z = (\dot{\vec{\omega}}_{b'/B})_y = (\dot{\vec{\omega}}_{b'/B})_z = 0 \ ,$$

$$\ddot{\vec{R}}_B = \ddot{\vec{S}}_B + \ddot{\vec{\rho}}_B$$

$$= [(\ddot{\vec{Q}}_B)_x + \ddot{s}]\vec{i} + [k(\dot{s})^2N_y]\vec{j} + [k(\dot{s})^2N_y]\vec{k} \ . \tag{54}$$

The force equations are therefore

$$(P_b + \frac{m_p}{m_t}P_c)A + (1 + \frac{m_p}{m_t})[(\frac{M_x}{r} - \frac{B\pi}{N})(\frac{\pi - \mu_B N}{\mu_B\pi - N})$$

$$- B - \mu'_B F_R - \mu'_s F_F] = m_p\ddot{s} \ , \tag{55}$$

$$(\vec{F}_F)_y - \mu'_S(\tfrac{\pi}{N})(\vec{F}_F)_z + (\vec{F}_R)_y - \mu'_B(\tfrac{\pi}{N})(\vec{F}_R)_z$$

$$= m_p\kappa(\dot{s})^2N_y \; , \qquad (55a)$$

and

$$(\vec{F}_F)_z + \mu'_S(\tfrac{\pi}{N})(\vec{F}_F)_y + (\vec{F}_R)_z + \mu'_B(\tfrac{\pi}{N})(\vec{F}_R)_y$$

$$= m_p\kappa(\dot{s})^2N_z \; .$$

Also, $\delta_{3x} = 0 \Rightarrow \delta_{1x} = \delta_{2x} = 90°$ so that $\sin\delta_{1x} = \sin\delta_{2x} = \cos\delta_{3x} = 1$, and $\cos\delta_{1x} = \cos\delta_{2x} = \sin\delta_{3x} = 0$.

From the above and the formulas relating the direction cosines, we get:

$$\cos\delta_{1y} = \cos\frac{\pi s}{rN} \quad , \quad \cos\delta_{1z} = \sin\frac{\pi s}{rN} \; ,$$

$$\cos\delta_{2y} = -\sin\frac{\pi s}{rN} \quad , \quad \cos\delta_{2z} = \cos\frac{\pi s}{rN} \; ,$$

$$\cos\delta_{3y} = 0 \quad , \quad \cos\delta_{3y} = 0 \; .$$

The moment equations therefore become

$$L_{1Fy}(\vec{F}_F)_z + L_{1Fz}(\vec{F}_F)_z + L_{1Ry}(\vec{F}_R)_y + L_{1Rz}(\vec{F}_R)_z$$

$$= \Big\{[\kappa\ddot{s} + \frac{d\kappa}{ds}(\dot{s})^2](N_y \sin\frac{\pi s}{rN} - N_z \cos\frac{\pi s}{rN})$$

$$- \kappa\tau(\dot{s})^2(N_y \cos\frac{\pi s}{rN} + N_z \sin\frac{\pi s}{rN})$$

$$+ \kappa\dot{s}(\frac{\pi \dot{s}}{rN})(N_y \cos\frac{\pi s}{rN} - N_z \sin\frac{\pi s}{rN})\Big\} I_T$$

$$- \kappa\dot{s}(\frac{\pi \dot{s}}{rN})(N_y \cos\frac{\pi s}{rN} + N_z \sin\frac{\pi s}{rN})(I_T - I_A) \; , \qquad (56a)$$

$$L_{2Fy}(\vec{F}_F)_y + L_{2Fz}(\vec{F}_F)_z + L_{2Ry}(\vec{F}_R)_y + L_{2Rz}(\vec{F}_R)_z$$

$$= \Big\{[\kappa\ddot{s} + \frac{d\kappa}{ds}(\dot{s})^2](N_y \cos\frac{\pi s}{rN} + N_z \sin\frac{\pi s}{rN})$$

$$+ \kappa\tau(\dot{s})^2(N_y \sin\frac{\pi s}{rN} - N_z \cos\frac{\pi s}{rN})$$

$$- \kappa\dot{s}(\frac{\pi \dot{s}}{rN})(N_y \sin\frac{\pi s}{rN} + N_z \cos\frac{\pi s}{rN})\Big\} I_T$$

$$- \kappa\dot{s}(\frac{\pi \dot{s}}{rN})(N_y \sin\frac{\pi s}{rN} - N_z \cos\frac{\pi s}{rN})(I_T - I_A) \; , \qquad (56b)$$

and

$$M_x + L_{3Fy}(\vec{F}_F)_y + L_{3Fz}(\vec{F}_F)_z + L_{3Ry}(\vec{F}_R)_y + L_{3Rz}(\vec{F}_R)_z = \frac{\pi \ddot{s}}{rN} I_A \; . \qquad (56c)$$

A brief note and suggestion concerning the solution of the equations would seem to be in order here. In the first force equations, there appear the terms F_R and F_F which as nonlinear functions

$$F_R = \sqrt{(\vec{F}_R)_y^2 + (\vec{F}_R)_z^2} \; ,$$

and

$$F_F = \sqrt{(\vec{F}_F)_y^2 + (\vec{F}_F)_z^2}$$

of the unknowns would seem to present a difficulty. This difficulty should be fairly easy to circumvent by first substituting estimates for F_R and F_F and solving the equations for $\ddot{s}$. Estimates of F_R and F_F can be based on their immediately previous up-bore calculation ($F_R = F_F = 0$ at $s = 0$) and, as inspection of the equations show, are not critical. Then, after integrating to get $\dot{s}$ and substitution of its value and that of s where required in the remaining equations, the simultaneous solution for $(\vec{F}_F)_y$, $(\vec{F}_F)_z$, $(\vec{F}_R)_y$, $(\vec{F}_R)_z$, and M_x is obtained. F_R and F_F are then calculated and compared to the original estimate. If too different from the estimate, the calculated values can be used and the above repeated. Following this solution outline, the equations have been programmed for computer solution for a spectrum of gun and ammunition combinations, as will be reported in a subsequent report.

Also, it should be noted, in arriving at these demonstration equations, the terms that couple the dynamic response of the tube and the constraint forces were dismissed by assumption. In reality, of course, the force of the projectile bearing against the tube causes motion and flexure of the tube which in turn have an effect on the forces which are their cause. For some studies this may not be important. But for others, it would be a requirement to solve the equations of constraint and the tube dynamics equations simultaneously. This suggests the utility the equations of constraint may have in tube dynamics studies.

REFERENCES

1. F. V. Reno, "The Motion of the Axis of a Spinning Shell Inside the Bore of a Gun," USA Ballistic Research Laboratory Report No. 320, Aberdeen Proving Ground, MD, February 1943.

2. L. H. Thomas, "The Motion of the Axis of a Spinning Shell Inside the Bore of a Gun," USA Ballistic Research Laboratory Report No. 544, Aberdeen Proving Ground, MD, May 1945.

3. F. J. Perdreauville, "Analysis of the Lateral Motion of a Projectile in the Gun Tube," Sandia Laboratories Report No. SC-RR-71 0071, Albuquerque, NM, March 1971.

4. S. H. Chu and F. K. Soechting, "Transverse Motion of Accelerating Shell," Picatinny Arsenal Technical Report 4314, Dover, NJ, June 1972.

5. E. H. Walker, "Yawing and Balloting Motion of a Projectile in the Bore of a Gun with Application to Gun Tube Dynamics," USA Ballistic Research Laboratory Memorandum Report No. 2411, Aberdeen Proving Ground, MD, September 1974.

6. M. T. Soifer and R. S. Becker, "Equations of Projectile Motion in a Flexible Gun Tube," USA Ballistic Research Laboratory Contractor Report BRL-CR-536, S&D Dynamics, Inc., October 1984.

7. J. G. Darpas (Translated by H. P. Hitchcock), "Transverse Forces on Projectiles Which Rotate in the Barrel," USA Ballistic Research Laboratory Memorandum Report No. 1208, Aberdeen Proving Ground, MD, March 1959.

8. H. P. Gay, "Notes on the Yawing Motion of a Projectile in the Bore," USA Ballistic Research Laboratory Memorandum Report No. 2259, Aberdeen Proving Ground, MD, January 1973.

9. R. S. Montgomery, "Muzzle Wear of Cannon," <u>Wear</u>, Volume 33, 1975, p. 359.

10. S. W. Groesberg, <u>Advanced Mechanics</u>, John Wiley & Sons, Inc., New York, 1968, p. 32.

APPENDIX

DERIVATION OF THE DIRECTION COSINE RELATIONSHIPS

Since we shall be concerned here only with axially symmetric projectiles, it will be permissible to arbitrarily establish the initial position of one of the transverse axes of inertia with respect to the b coordinate system. Thus, the number of rotational degrees of freedom of the body-fixed coordinate system with respect to the b coordinate system is reduced from three to two. Because of the simplification in analysis that is affected, we shall select the 1-axis to lie initially in the x-y plane as shown in Figure A-1.

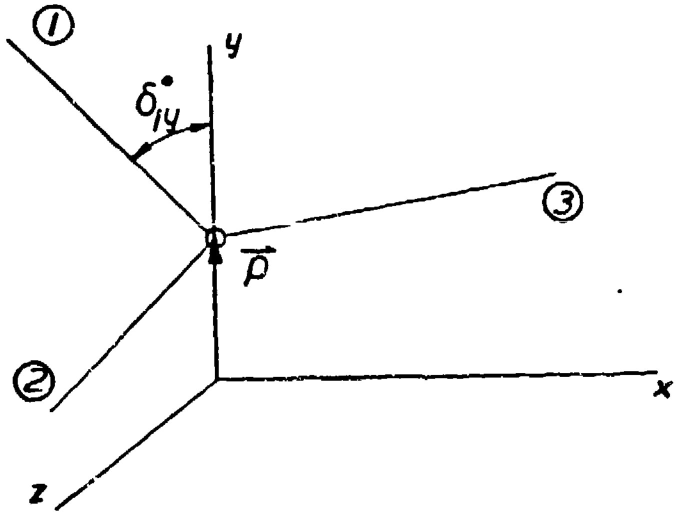

Figure A-1. Initial position of the body-fixed coordinate system in relation to the moving coordinate system.

The direction cosines of the initial position of the 1-axis are related according to

$$\cos^2\delta^{\circ}_{1x} + \cos^2\delta^{\circ}_{1y} + \cos^2\delta^{\circ}_{1z} = 1 \ . \qquad \text{(A-1)}$$

Now since $\delta^{\circ}_{1x} = \delta_{1x}$ = constant, and $\delta^{\circ}_{1z} = 90^{\circ}$, we have

$$\cos^2\delta^{\circ}_{1y} = 1 - \cos^2\delta_{1x} = \sin^2\delta_{1x} \ . \qquad \text{(A-2)}$$

Figure A-2 shows an axis parallel to the 1-axis and an axis originally parallel to the y-axis, but fixed in the body and therefore rotated through the angle θ, the angle of rotation of the projectile.

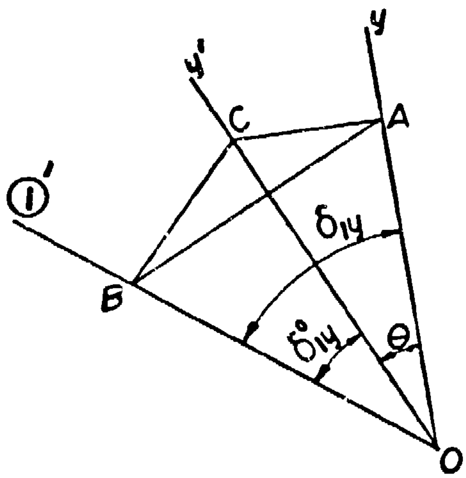

Figure A-2. Auxiliary axes fixed in the body.

In the figure the plane defined by triangle AOC is perpendicular to the plane defined by triangle BOC, and the plane defined by triangle ABC is perpendicular to line $\bar{O}_{y'}$. Therefore, the angles ACB, ACO, and BCO are right angles and if we set $\overline{OC} = 1$, then $\overline{CA} = \tan\theta$, $\overline{BC} = \tan\delta^{\circ}_{1y}$, $\overline{OA} = 1/\cos\theta$, and $\overline{OB} = 1/\cos\delta^{\circ}_{1y}$. From the law of cosines

$$\cos\delta_{1y} = \frac{(\overline{OB})^2 + (\overline{OA})^2 - (\overline{BC})^2 - (\overline{CA})^2}{2(\overline{OB})(\overline{OA})} . \tag{A-3}$$

Now since

$$\tan^2\theta = \frac{1 - \cos^2\theta}{\cos^2\theta} ,$$

and

$$\tan^2\delta^{\circ}_{1y} = \frac{1 - \cos^2\delta^{\circ}_{1y}}{\cos^2\delta^{\circ}_{1y}} ,$$

we get

$$\cos\delta_{1y} = \frac{\cos^{\circ}\delta_{1y}\cos\theta}{2}\left[\frac{1 - 1 + \cos^2\delta^{\circ}_{1y}}{\cos^2\delta^{\circ}_{1y}} + \frac{1 - 1 + \cos^2\theta}{\cos^2\theta}\right]$$

$$= \cos\delta^{\circ}_{1y}\cos\theta = \pm\sin\delta_{1x}\cos\theta . \tag{A-4}$$

The positive sign is appropriate here and so for constant twist rifling

$$\cos\delta_{1y} = \sin\delta_{1x}\cos\frac{\pi s}{rN} . \tag{A-5}$$

Also

$$\cos^2\delta_{1z} = 1 - \cos^2\delta_{1x} - \sin^2\delta_{1x}\cos^2\frac{\pi s}{rN} \qquad \text{(A-6)}$$

so that

$$\cos\delta_{1z} = \sin\delta_{1x}\sin\frac{\pi s}{rN} . \qquad \text{(A-7)}$$

The four remaining direction cosine variables can now be determined with two equations of the normalization condition and two equations of the orthogonality condition:

$$\cos^2\delta_{2x} + \cos^2\delta_{2y} + \cos^2\delta_{2z} = 1 , \qquad \text{(A-8a)}$$

$$\cos^2\delta_{3x} + \cos^2\delta_{3y} + \cos^2\delta_{3z} = 1 , \qquad \text{(A-8b)}$$

$$\cos\delta_{1x}\cos\delta_{2x} + \cos\delta_{1y}\cos\delta_{2y} + \cos\delta_{1z}\cos\delta_{2z} = 0 , \qquad \text{(A-9a)}$$

$$\cos\delta_{1x}\cos\delta_{3x} + \cos\delta_{1y}\cos\delta_{3y} + \cos\delta_{1z}\cos\delta_{3z} = 0 . \qquad \text{(A-9b)}$$

Solving for $\cos\delta_{2z}$ in Eq. (A-9a) gives

$$\cos\delta_{2z} = -\frac{\cos\delta_{1x}\cos\delta_{2x} + \cos\delta_{1y}\cos\delta_{2y}}{\cos\delta_{1z}} . \qquad \text{(A-10)}$$

Substitution into Eq. (A-8a) and rearrangement then gives

$$\begin{aligned} &\cos^2\delta_{1y}\cos^2\delta_{2y} + \cos^2\delta_{2y}\cos^2\delta_{1z} \\ &+ 2\cos\delta_{1x}\cos\delta_{2x}\cos\delta_{1y}\cos\delta_{2y} \\ &+ \cos^2\delta_{2x}\cos^2\delta_{1z} + \cos^2\delta_{1x}\cos^2\delta_{2x} - \cos^2\delta_{1z} = 0 . \end{aligned} \qquad \text{(A-11)}$$

If we let $a = \cos^2\delta_{1y} + \cos^2\delta_{1z}$, $b = 2\cos\delta_{1x}\cos\delta_{2x}\cos\delta_{1y}$, and $c = \cos^2\delta_{2x}$ $\cos^2\delta_{1z} + \cos^2\delta_{1x} + \cos^2\delta_{2x} - \cos^2\delta_{1z}$, then by the quadratic formula

$$\cos\delta_{2y} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} .$$

$$\cos\delta_{2y} = \frac{-\cos\delta_{1x}\cos\delta_{2x}\cos\delta_{1y} \pm \cos\delta_{1z}\sqrt{\cos^2\delta_{1z}+\cos^2\delta_{1y}-\cos^2\delta_{2x}}}{\cos^2\delta_{1y} + \cos^2\delta_{1z}} . \qquad \text{(A-12)}$$

Now since

$$\cos^2\delta_{1x} + \cos\delta^2{}_{1y} + \cos\delta^2{}_{1z} = 1 = \cos^2\delta_{1x} + \cos^2\delta_{2x} + \cos^2\delta_{3x}$$

we have

$$\cos^2\delta_{1z} + \cos^2\delta_{1y} - \cos^2\delta_{2x} = \cos^2\delta_{3x} \; . \tag{A-13}$$

The expression under the radical in Eq. (A-12) is thus simplified so we now have

$$\cos\delta_{2y} = \frac{-\cos\delta_{1x}\cos\delta_{2x}\cos\delta_{1y} \pm \cos\delta_{1z}\cos\delta_{3x}}{\cos^2\delta_{1y} + \cos^2\delta_{1z}} \; . \tag{A-14}$$

Substitution into the expression for $\cos\delta_{2z}$ and rearrangement now gives

$$\cos\delta_{2z} = \frac{-\cos\delta_{1x}\cos\delta_{2x}\cos\delta_{1z} \pm \cos\delta_{1y}\cos\delta_{3x}}{\cos^2\delta_{1y} + \cos^2\delta_{1z}} \tag{A-15}$$

The $\pm$ signs in these expressions for the direction cosines can be resolved by expressing mathematically the fact that the 1, 2, 3, coordinate system is a right-handed cartesian system. That is, if the 1 axis is crossed into the 2-axis the result is the 3-axis.

$$\begin{aligned}
&(\cos\delta_{1x}\vec{i} + \cos\delta_{1y}\vec{j} + \cos\delta_{1z}\vec{k}) \times (\cos\delta_{2x}\vec{i} + \cos\delta_{2y}\vec{j} + \cos\delta_{2z}\vec{k}) \\
&= \cos\delta_{1x}\cos\delta_{2y}\vec{k} - \cos\delta_{1x}\cos\delta_{2z}\vec{j} - \cos\delta_{1y}\cos\delta_{2x}\vec{k} \\
&+ \cos\delta_{1y}\cos\delta_{2z}\vec{i} + \cos\delta_{1z}\cos\delta_{2x}\vec{j} - \cos\delta_{1z}\cos\delta_{2y}\vec{i} \\
&= \cos\delta_{3x}\vec{i} + \cos\delta_{3y}\vec{j} + \cos\delta_{3z}\vec{k} \; .
\end{aligned} \tag{A-16}$$

Equating the coefficients of the same unit vectors of both sides of the equation gives for the $\vec{i}$ unit vector

$$\cos\delta_{1y}\cos\delta_{2z} - \cos\delta_{1z}\cos\delta_{2y} = \cos\delta_{3x} \; . \tag{A-17}$$

Substitution of Eqs. (A-14) and (A-15) and rearrangement gives the following:

$$\begin{aligned}
&-\cos\delta_{1y}\cos\delta_{1x}\cos\delta_{2x}\cos\delta_{1z} \pm \cos^2\delta_{1y}\cos\delta_{3x} \\
&+\cos\delta_{1z}\cos\delta_{1x}\cos\delta_{2x}\cos\delta_{1y} \mp \cos^2\delta_{1z}\cos\delta_{3x} \\
&= \cos\delta_{3x}(\cos^2\delta_{1y} + \cos^2\delta_{1z}) \; .
\end{aligned} \tag{A-18}$$

For equality to exist in the above equation, it is clear that the signs must be as follows:

$$\cos\delta_{2y} = \frac{-\cos\delta_{1x}\cos\delta_{2x}\cos\delta_{1y} - \cos\delta_{1z}\cos\delta_{3x}}{\cos^2\delta_{1y} + \cos^2\delta_{1z}} \; , \tag{A-19}$$

and

$$\cos\delta_{2z} = \frac{-\cos\delta_{1x}\cos\delta_{2x}\cos\delta_{1z} + \cos\delta_{1y}\cos\delta_{3x}}{\cos^2\delta_{1y} + \cos^2\delta_{1z}} . \tag{A-19b}$$

In the same way as above, it can be shown that

$$\cos\delta_{3y} = \frac{-\cos\delta_{1x}\cos\delta_{3x}\cos\delta_{1y} + \cos\delta_{1z}\cos\delta_{2x}}{\cos^2\delta_{1y} + \cos^2\delta_{1z}} , \tag{A-20}$$

and

$$\cos\delta_{3z} = \frac{-\cos\delta_{1x}\cos\delta_{3x}\cos\delta_{1z} - \cos\delta_{1y}\cos\delta_{2x}}{\cos^2\delta_{1y} + \cos^2\delta_{1z}} .$$

Now since

$$\cos\delta_{1y} = \sin\delta_{1x}\cos\frac{\pi s}{rN} ,$$

and

$$\cos\delta_{1z} = \sin\delta_{1x}\sin\frac{\pi s}{rN}$$

for constant pitch rifling we also have

$$\cos^2\delta_{1y} + \cos^2\delta_{1z} = \sin^2\delta_{1x}\left(\cos^2\frac{\pi s}{rN} + \sin^2\frac{\pi s}{rN}\right) \tag{A-21}$$

so that finally we have the direction cosine relationships shown as Eqs. (26a) through (26f).

TECHNICAL REPORT INTERNAL DISTRIBUTION LIST

	NO. OF COPIES
CHIEF, DEVELOPMENT ENGINEERING BRANCH	
ATTN: SMCAR-CCB-D	1
-DA	1
-DC	1
-DM	1
-DP	1
-DR	1
-DS (SYSTEMS)	1
CHIEF, ENGINEERING SUPPORT BRANCH	
ATTN: SMCAR-CCB-S	1
-SE	1
CHIEF, RESEARCH BRANCH	
ATTN: SMCAR-CCB-R	2
-R (ELLEN FOGARTY)	1
-RA	1
-RM	1
-RP	1
-RT	1
TECHNICAL LIBRARY ATTN: SMCAR-CCB-TL	5
TECHNICAL PUBLICATIONS & EDITING UNIT ATTN: SMCAR-CCB-TL	2
DIRECTOR, OPERATIONS DIRECTORATE ATTN: SMCWV-OD	1
DIRECTOR, PROCUREMENT DIRECTORATE ATTN: SMCWV-PP	1
DIRECTOR, PRODUCT ASSURANCE DIRECTORATE ATTN: SMCWV-QA	1

<u>NOTE</u>: PLEASE NOTIFY DIRECTOR, BENET LABORATORIES, ATTN: SMCAR-CCB-TL, OF ANY ADDRESS CHANGES.

TECHNICAL REPORT EXTERNAL DISTRIBUTION LIST

	NO. OF COPIES		NO. OF COPIES
ASST SEC OF THE ARMY RESEARCH AND DEVELOPMENT ATTN: DEPT FOR SCI AND TECH THE PENTAGON WASHINGTON, D.C. 20310-0103	1	COMMANDER ROCK ISLAND ARSENAL ATTN: SMCRI-ENM ROCK ISLAND, IL 61299-5000	1
ADMINISTRATOR DEFENSE TECHNICAL INFO CENTER ATTN: DTIC-FDAC CAMERON STATION ALEXANDRIA, VA 22304-6145	12	DIRECTOR US ARMY INDUSTRIAL BASE ENGR ACTV ATTN: AMXIB-P ROCK ISLAND, IL 61299-7260	1
COMMANDER US ARMY ARDEC ATTN: SMCAR-AEE SMCAR-AES, BLDG. 321 SMCAR-AET-O, BLDG. 351N SMCAR-CC SMCAR-CCP-A SMCAR-FSA SMCAR-FSM-E SMCAR-FSS-D, BLDG. 94 SMCAR-MSI (STINFO) PICATINNY ARSENAL, NJ 07806-5000	 1 1 1 1 1 1 1 1 2	COMMANDER US ARMY TANK-AUTMV R&D COMMAND ATTN: AMSTA-DDL (TECH LIB) WARREN, MI 48397-5000	1
		COMMANDER US MILITARY ACADEMY ATTN: DEPARTMENT OF MECHANICS WEST POINT, NY 10996-1792	1
		US ARMY MISSILE COMMAND REDSTONE SCIENTIFIC INFO CTR ATTN: DOCUMENTS SECT, BLDG. 4484 REDSTONE ARSENAL, AL 35898-5241	2
DIRECTOR US ARMY BALLISTIC RESEARCH LABORATORY ATTN: SLCBR-DD-T, BLDG. 305 ABERDEEN PROVING GROUND, MD 21005-5066	1	COMMANDER US ARMY FGN SCIENCE AND TECH CTR ATTN: DRXST-SD 220 7TH STREET, N.E. CHARLOTTESVILLE, VA 22901	1
DIRECTOR US ARMY MATERIEL SYSTEMS ANALYSIS ACTV ATTN: AMXSY-MP ABERDEEN PROVING GROUND, MD 21005-5071	1	COMMANDER US ARMY LABCOM MATERIALS TECHNOLOGY LAB ATTN: SLCMT-IML (TECH LIB) WATERTOWN, MA 02172-0001	2
COMMANDER HQ, AMCCOM ATTN: AMSMC-IMP-L ROCK ISLAND, IL 61299-6000	1		

NOTE: PLEASE NOTIFY COMMANDER, ARMAMENT RESEARCH, DEVELOPMENT, AND ENGINEERING CENTER, US ARMY AMCCOM, ATTN: BENET LABORATORIES, SMCAR-CCB-TL, WATERVLIET, NY 12189-4050, OF ANY ADDRESS CHANGES.

TECHNICAL REPORT EXTERNAL DISTRIBUTION LIST (CONT'D)

	NO. OF COPIES		NO. OF COPIES
COMMANDER US ARMY LABCOM, ISA ATTN: SLCIS-IM-TL 2800 POWDER MILL ROAD ADELPHI, MD 20783-1145	1	COMMANDER AIR FORCE ARMAMENT LABORATORY ATTN: AFATL/MN EGLIN AFB, FL 32542-5434	1
COMMANDER US ARMY RESEARCH OFFICE ATTN: CHIEF, IPO P.O. BOX 12211 RESEARCH TRIANGLE PARK, NC 27709-2211	1	COMMANDER AIR FORCE ARMAMENT LABORATORY ATTN: AFATL/MNF EGLIN AFB, FL 32542-5000	1
DIRECTOR US NAVAL RESEARCH LAB ATTN: MATERIALS SCI & TECH DIVISION CODE 26-27 (DOC LIB) WASHINGTON, D.C. 20375	1 1	METALS AND CERAMICS INFO CTR BATTELLE COLUMBUS DIVISION 505 KING AVENUE COLUMBUS, OH 43201-2693	1

NOTE: PLEASE NOTIFY COMMANDER, ARMAMENT RESEARCH, DEVELOPMENT, AND ENGINEERING CENTER, US ARMY AMCCOM, ATTN: BENET LABORATORIES, SMCAR-CCB-TL, WATERVLIET, NY 12189-4050, OF ANY ADDRESS CHANGES.

SUPPLEMENTARY

INFORMATION

21 August 1989

ERRATA SHEET
(Change Notice)

C1 TO: TECHNICAL REPORT ARCCB-TR-88004

THE FORCES OF CONSTRAINT ON A
PROJECTILE IN A RIFLED GUN BORE (PART 1)

by

David F. Finlayson

Please remove pages 5 through 8 from above Technical Report and insert new pages enclosed. Equation (10) on page 6 and Equations (15) and (19c) on page 7 have been corrected.

US ARMY ARMAMENT RESEARCH, DEVELOPMENT,
AND ENGINEERING CENTER
CLOSE COMBAT ARMAMENTS CENTER
BENET LABORATORIES
WATERVLIET, N.Y. 12189-4050

$$\ddot{\vec{R}}_B = \ddot{\vec{S}}_B + \ddot{\vec{\rho}}_B \tag{5}$$

Texts on dynamics (ref 10) give the formula

$$\ddot{\vec{\rho}}_B = \ddot{\vec{\rho}}_b + \dot{\vec{\omega}}_{b/B} \times \vec{\rho}_b + 2\vec{\omega}_{b/B} \times \dot{\vec{\rho}}_b + \vec{\omega}_{b/B} \times \vec{\omega}_{b/B} \times \vec{\rho}_b \tag{6}$$

for rotating coordinate systems where the subscript b/B indicates rotation of the b coordinate system with respect to the B coordinate system. If e_m is the magnitude of the eccentricity of the projectile c.g. with respect to the spin axis, then $\vec{\rho}_b$, $\dot{\vec{\rho}}_b$, and $\ddot{\vec{\rho}}_b$ can be expressed as

$$\vec{\rho}_b = e_m\ (\cos\theta\vec{j} + \sin\theta\vec{k})\ , \tag{7}$$

$$\dot{\vec{\rho}}_b = e_m\dot{\theta}\ (-\sin\theta\vec{j} + \cos\theta\vec{k})\ , \tag{7a}$$

and

$$\ddot{\vec{\rho}}_b = e_m[-(\ddot{\theta}\ \sin\theta + \dot{\theta}^2\ \cos\theta)\vec{j} + (\ddot{\theta}\ \cos\theta - \dot{\theta}^2\ \sin\theta)\vec{k}] \tag{7b}$$

where θ is as shown in Figure 2.

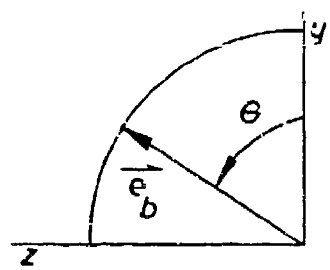

Figure 2. Position vector for the projectile c.g. in the moving coordinate system.

Since the spin axis is taken in general to be the tangent to a curve moving in space, it will be convenient to refer to the Frenet-Serret Formulas of differential geometry. These are

$$\frac{d\vec{B}}{ds} = -\tau\vec{N}\ ,\quad \frac{d\vec{T}}{ds} = \kappa\vec{N}\ ,\ \text{and}\quad \frac{d\vec{N}}{ds} = \tau\vec{B} - \kappa\vec{T} \tag{8a,b,c}$$

where

$$\vec{B} = \vec{T} \times \vec{N} \quad , \quad \vec{T} = \frac{d\vec{R}}{ds} \ , \text{ and } \quad \kappa = \frac{1}{\rho} \ .$$

The velocity of a point moving along a curve in space is expressed as

$$\vec{v}(t) = \frac{d\vec{R}}{dt} = \frac{ds}{dt}\frac{d\vec{R}}{ds} \quad . \tag{9}$$

Making use of the Frenet-Serret formulas allows the velocity to be written as

$$\vec{v}(t) = \frac{ds}{dt}\vec{T} \ , \tag{10}$$

and the acceleration to be written as

$$\vec{a}(t) = \frac{d^2s}{dt^2}\vec{T} + \left(\frac{ds}{dt}\right)^2 \kappa\vec{N} \ . \tag{11}$$

Now since $\ddot{\vec{s}}_B$ is the sum of the acceleration of the origin of the b coordinate system relative to its path and the acceleration of the point on the path coincident to the b coordinate system origin, we can write

$$\ddot{\vec{s}}_B = \ddot{\vec{Q}}_B + \frac{d^2s}{dt^2}\vec{T} + \left(\frac{ds}{dt}\right)^2 \kappa\vec{N} \ . \tag{12}$$

Also note that $\omega_{b/B}$ is the sum of the angular velocities due to rotation of the spin axis at the point coincident with the origin of the b coordinate system and motion of the coordinate system origin relative to its path. Therefore, if we designate by the subscript b', a coordinate system which is coincident to the b coordinate system but fixed to the spin axis, we can write

$$\vec{\omega}_{b/B} = \vec{\omega}_{b'/B} + \vec{\omega}_{b/b'} \ . \tag{13}$$

Recalling that $\vec{\omega}_{b/B}$ has no component in the direction of the spin axis, it follows that

$$\vec{\omega}_{b/b'} = \frac{ds}{dt}\vec{T} \times \frac{d\vec{T}}{ds} = \frac{ds}{dt}\vec{T} \times \kappa\vec{N} \tag{14}$$

so that

$$\vec{\omega}_{b/B} = \vec{\omega}_{b'/B} + \frac{ds}{dt}\,\vec{T} \times \kappa\vec{N} = \vec{\omega}_{b'/B} + \frac{ds}{dt}\,\kappa\vec{B} \tag{15}$$

Differentiating with respect to time gives

$$\dot{\vec{\omega}}_{b/B} = \dot{\vec{\omega}}_{b'/B} + \frac{d}{dt}\left[\frac{ds}{dt}\,\kappa\vec{B}\right] \tag{16}$$

which, after some manipulation, results in

$$\dot{\vec{\omega}}_{b/B} = \dot{\vec{\omega}}_{b'/B} + \left[\kappa\,\frac{d^2s}{dt^2} + \frac{d\kappa}{ds}\left(\frac{ds}{dt}\right)^2\right]\vec{B} - \kappa\tau\left(\frac{ds}{dt}\right)\vec{N}\ . \tag{17}$$

COORDINATE TRANSFORMATION

First of all, by observing that $\vec{T} \equiv \vec{i}$, and $\vec{N} = N_y\vec{j} + N_z\vec{k}$, it follows that $\vec{B} = \vec{T} \times \vec{N} = N_y\vec{k} - N_z\vec{j}$. Also,

$$\ddot{\vec{Q}}_B = (\ddot{Q}_B)_x\vec{i} + (\ddot{Q}_B)_y\vec{j} + (\ddot{Q}_B)_z\vec{k}\ , \tag{18a}$$

$$\vec{\omega}_{b'/B} = (\omega_{b'/B})_y\vec{j} + (\omega_{b'/B})_z\vec{k}\ , \tag{18b}$$

and

$$\dot{\vec{\omega}}_{b'/B} = (\dot{\omega}_{b'/B})_y\vec{j} + (\dot{\omega}_{b'/B})_z\vec{k}\ . \tag{18c}$$

Therefore,

$$\ddot{\vec{s}}_B = \left[(\ddot{Q}_B)_x + \frac{d^2s}{dt^2}\right]\vec{i} + \left[(\ddot{Q}_B)_y + \left(\frac{ds}{dt}\right)^2\kappa N_y\right]\vec{j} + \left[(\ddot{Q}_B)_z + \left(\frac{ds}{dt}\right)^2\kappa N_z\right]\vec{k}\ , \tag{19a}$$

$$\vec{\omega}_{b/B} = \left[(\omega_{b'/B})_y - \frac{ds}{dt}\,\kappa N_z\right]\vec{j} + \left[(\omega_{b'/B})_z + \frac{ds}{dt}\,\kappa N_y\right]\vec{k}\ , \tag{19b}$$

and

$$\dot{\vec{\omega}}_{b/B} = \left\{(\dot{\omega}_{b'/B})_y - \left[\kappa\,\frac{d^2s}{dt^2} + \frac{d\kappa}{ds}\left(\frac{ds}{dt}\right)^2\right]N_z - \kappa\tau\left(\frac{ds}{dt}\right)^2 N_y\right\}\vec{j} + \left\{(\dot{\omega}_{b'/B})_z + \left[\kappa\,\frac{d^2s}{dt^2} + \frac{d\kappa}{ds}\left(\frac{ds}{dt}\right)^2\right]N_y - \kappa\tau\left(\frac{ds}{dt}\right)^2 N_z\right\}\vec{k}\ . \tag{19c}$$

The total angular velocity of the projectile is the vector sum of the projectile spin velocity and the angular velocity of the b coordinate system, or

$$\vec{\Omega} = \vec{\omega} + \vec{\omega}_{b/B}\ . \tag{20}$$

In the b coordinate system, the components of $\vec{\Omega}$ are then

$$\Omega_x = \omega \ , \tag{21a}$$

$$\Omega_y = (\vec{\omega}_{b'/B})_y + \frac{ds}{dt} \kappa N_z \ , \text{ and} \tag{21b}$$

$$\Omega_z = (\vec{\omega}_{b'/B})_z + \frac{ds}{dt} \kappa N_y \ . \tag{21c}$$

Similarly, the angular acceleration and its components are given by

$$\dot{\vec{\Omega}} = \dot{\vec{\omega}} + \dot{\vec{\omega}}_{b\ B} \ , \tag{22}$$

$$\dot{\Omega}_x = \dot{\omega} \ , \tag{23a}$$

$$\dot{\Omega}_y = (\dot{\vec{\omega}}_{b'/B})_y - [\kappa \frac{d^2s}{dt^2} + \frac{d\kappa}{ds} (\frac{ds}{dt})^2] N_z - \kappa\tau(\frac{ds}{dt})^2 N_y \ , \tag{23b}$$

and

$$\dot{\Omega}_z = (\dot{\vec{\omega}}_{b'/B})_z + [\kappa \frac{d^2s}{dt^2} + \frac{d\kappa}{ds} (\frac{ds}{dt})^2] N_y - \kappa\tau(\frac{ds}{dt})^2 N_z \ . \tag{23c}$$

It remains now to express the projectile angular velocity and acceleration in terms of their components in the body-fixed coordinate system. For the angular velocity

$$\Omega_1 = \cos\delta_{1x}\Omega_x + \cos\delta_{1y}\Omega_y + \cos\delta_{1z}\Omega_z \ , \tag{24a}$$

$$\Omega_2 = \cos\delta_{2x}\Omega_x + \cos\delta_{2y}\Omega_y + \cos\delta_{2z}\Omega_z \ , \tag{24b}$$

and

$$\Omega_3 = \cos\delta_{3x}\Omega_x + \cos\delta_{3y}\Omega_y + \cos\delta_{3z}\Omega_z \ . \tag{24c}$$

Remembering that δ_{1x}, δ_{2x}, and δ_{3x} are assumed to remain constant, we get by differentiation

$$\dot{\Omega}_1 = \cos\delta_{1x}\dot{\Omega}_x + \frac{d}{dt} (\cos\delta_{1y})\Omega_y + \cos\delta_{1y}\dot{\Omega}_y$$

$$+ \frac{d}{dt} (\cos\delta_{1z})\Omega_z + \cos\delta_{1z}\dot{\Omega}_z \ , \tag{25a}$$

$$\dot{\Omega}_2 = \cos\delta_{2x}\dot{\Omega}_x + \frac{d}{dt} (\cos\delta_{2y})\Omega_y + \cos\delta_{2y}\dot{\Omega}_y$$

$$+ \frac{d}{dt} (\cos\delta_{2z})\Omega_z + \cos\delta_{2z}\dot{\Omega}_z \ , \tag{25b}$$