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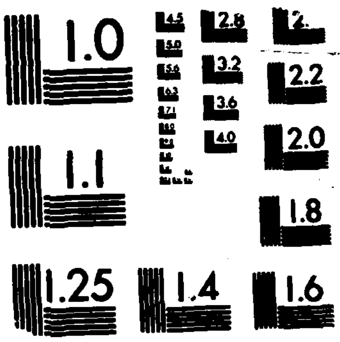

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Exact Tests for Variance Component Models With Unequal Cell Frequencies in the Last Stage

by

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This paper presents a method to construct exact tests for the variance components in a general unbalanced random model. It is assumed that the imbalance affects only the last stage of the design with no missing cells.

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EXACT TESTS FOR VARIANCE COMPONENT MODELS WITH UNEQUAL CELL FREQUENCIES IN THE LAST STAGE

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Abstract: Khuri and Littell (1987) derived exact tests for testing hypotheses concerning the variance components in an unbalanced random two-way model. The method used in the development of these tests can be extended to more general unbalanced random models. In this article, such an extension is established for models that are unbalanced only with respect to the last stage of their associated designs. A numerical example is given to illustrate the implementation of the proposed methodology.

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Key words and phrases: Unbalanced random models; Hypothesis testing; Power of exact tests; Nested and crossed classification models.

1. Introduction

The traditional analysis of data from an unbalanced random model uses approximate F tests that are based on sums of squares, which, in general, are neither independent nor distributed as scaled chi-square variates. The true critical values and power functions of these tests are unknown and depend on variance components other than those under consideration. Furthermore, the tests, particularly those that depend on Satterthwaite's procedure, can be quite unreliable. This was demonstrated by Tietjen (1974), in the case of the unbalanced random two-fold nested model, and by Cummings and Gaylor (1974).

The method developed by Khuri and Littell (1987) produced exact tests for the variance components in an unbalanced random two-way classification with interaction model. This method is based on a particular transformation that reduces the analysis of the unbalanced model to that of a balanced one. In this article, a demonstration is given of the applicability of the same kind of transformation to any unbalanced random model provided that the imbalance occurs only in the last stage of the associated design.

2. Notation and preliminaries

The model for a general unbalanced design whose imbalance is caused by unequal cell frequencies in the last stage can be written in the form

$$y_\theta = \sum_{i=0}^{\nu} \gamma_{\theta_i(\bar{\theta}_i)} + \epsilon_\theta, \tag{2.1}$$

where $\theta = (k_1, k_2, \ldots, k_s)$ is a complete set of subscripts that identify a typical response y. The term, $\gamma_{\theta_i(\bar{\theta}_i)}$, denotes the i^{th} effect in the model, where $\bar{\theta}_i$ and θ_i are, respectively, the corresponding sets of rightmost and nonrightmost bracket subscripts (see Section 2 in Khuri (1982)). By definition, $\theta_i = \bar{\theta}_i = \phi$, the empty set, for i=0 and the corresponding γ is the grand mean, usually denoted by μ. It is assumed that $\gamma_{\theta_i(\bar{\theta}_i)}$, for $i=1,2,\ldots,\nu$, and ϵ_θ are independent and normally distributed random variables with zero means and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_\nu^2$, and σ_ϵ^2, respectively.

Since the design is balanced except for its last stage, the ranges of subscripts $k_1, k_2, \ldots, k_s$ can be expressed in the form

$$k_j = \begin{matrix} 1, 2, \ldots, a_j, & \text{for } j=1,2,\ldots,s-1 \\ 1, 2, \ldots, n_\tau, & \text{for } j=s, \end{matrix} \tag{2.2}$$

where τ is a subset of θ consisting of the first s-1 subscripts, that is,

$$\tau = (k_1, k_2, \ldots, k_{s-1}). \tag{2.3}$$

Except for ϵ_θ, all the effects on the right side of (2.1) are indexed by the subscripts in τ. Let T be the set of all (s-1)-tuples as in (2.3), and let c denote the number of elements in T, that is,

$$T = \left\{ \tau = (k_1, k_2, \ldots, k_{s-1}) : k_j = 1, 2, \ldots, a_j; \quad j = 1, 2, \ldots, s-1 \right\} \tag{2.4}$$

$$c = \prod_{j=1}^{s-1} a_j. \tag{2.5}$$

It is assumed that $n_\tau \geq 1$ for all $\tau \epsilon T$ and, for reasons to be seen later, that

$$N > 2c-1, \tag{2.6}$$

where $N = \sum_{\tau \epsilon T} n_\tau$ is the total number of observations. Let $\psi_i = (\theta_i : \bar{\theta}_i)$ be the set of subscripts associated with the i^{th} effect, which results from combining the elements of θ_i and $\bar{\theta}_i$ $(i=0,1,\ldots,\nu)$. The complement of ψ_i with respect to τ is denoted by ψ_i^c $(i=0,1,\ldots,\nu)$.

We note that if it were not for ϵ_θ, the model in (2.1) would be of the same form as that of a balanced model (see Khuri (1982) for a general representation of a balanced model). This fact will be quite useful in the development of the exact tests in Section 3.

The model in (2.1) can be written in matrix form as

$$\underset{\sim}{y} = \sum_{i=0}^{\nu} \underset{\sim}{X}_i \underset{\sim}{\beta}_i + \underset{\sim}{\epsilon}, \tag{2.7}$$

where $\underset{\sim}{y}$ and $\underset{\sim}{\epsilon}$ are the vectors of observations and random errors, respectively, $\underset{\sim}{X}_i$ is a matrix of zeros and ones of order $N \times c_i$ $(i=0,1,\ldots,\nu)$ with $\underset{\sim}{X}_0$ being equal to $\underset{\sim}{1}_N$, the vector of ones of order $N \times 1$, and $\underset{\sim}{\beta}_i$ is a vector consisting of the c_i elements of $\gamma_{\theta_i(\bar{\theta}_i)}$. The integer c_i is given by

$$c_i = \prod_{k_j \in \psi_i} a_j, \quad i = 0, 1, \ldots, \nu, \tag{2.8}$$

where, if we recall, ψ_i is the set of subscripts associated with the i^{th} effect. Note that $c_i = 1$ for $i = 0$.

Let $\bar{y}_\tau$ denote the mean of y_θ averaged over the range of subscript k_s for a given $\tau = (k_1, k_2, \ldots, k_{s-1})$ in T, that is,

$$\bar{y}_\tau = \frac{1}{n_\tau} \sum_{k_s=1}^{n_\tau} y_\theta, \quad \tau \epsilon T. \tag{2.9}$$

From (2.1) we have

$$\bar{y}_\tau = \sum_{i=0}^{\nu} \gamma_{\theta_i(\bar{\theta}_i)} + \bar{\epsilon}_\tau, \tag{2.10}$$

where $\bar{\epsilon}_\tau = \left(\sum_{k_s=1}^{n_\tau} \epsilon_\theta \right) / n_\tau$. Formula (2.10) may be written in matrix form as

$$\underset{\sim}{\bar{y}} = \sum_{i=0}^{\nu} \underset{\sim}{H}_i \underset{\sim}{\beta}_i + \underset{\sim}{\bar{\epsilon}}, \tag{2.11}$$

where $\underset{\sim}{H}_i$ is a matrix of order $c \times c_i$. Since $\sum_{i=0}^{\nu} \gamma_{\theta_i(\bar{\theta}_i)}$ in (2.10) is of the same form as in a balanced model, the matrix $\underset{\sim}{H}_i$ can be expressed as a direct product of the form (see Khuri (1982), p. 2908)

$$\underset{\sim}{H}_i = \bigotimes_{\ell=1}^{s-1} \underset{\sim}{L}_{i\ell}, \quad i = 0, 1, \ldots, \nu, \tag{2.12}$$

where $\underset{\sim}{L}_{i\ell}$ is given by

$$\underset{\sim}{L}_{i\ell} = \begin{matrix} \underset{\sim}{I}_{a_\ell}, & k_\ell \in \psi_i \\ \underset{\sim}{1}_{a_\ell}, & k_\ell \in \psi_i^c \end{matrix} \quad \begin{matrix} i = 0, 1, \ldots, \nu; \\ \ell = 1, 2, \ldots, s\text{-}1. \end{matrix} \tag{2.13}$$

In (2.13), $\underset{\sim}{I}_{a_\ell}$ and $\underset{\sim}{1}_{a_\ell}$ denote, respectively, the identity matrix of order $a_\ell \times a_\ell$ and the vector of ones of order $a_\ell \times 1$. We recall that ψ_i^c in (2.13) is the complement of ψ_i with respect to τ. Let $\underset{\sim}{A}_i = \underset{\sim}{H}_i \underset{\sim}{H}_i'$, then

$$\underset{\sim}{A}_i = \bigotimes_{\ell=1}^{s-1} \underset{\sim}{M}_{i\ell}, \quad i = 0, 1, \ldots, \nu, \tag{2.14}$$

where

$$\underset{\sim}{M}_{i\ell} = \begin{matrix} \underset{\sim}{I}_{a_\ell}, & k_\ell \in \psi_i \\ \underset{\sim}{J}_{a_\ell}, & k_\ell \in \psi_i^c \end{matrix} \quad \begin{matrix} i = 0, 1, \ldots, \nu; \\ \ell = 1, 2, \ldots, s\text{-}1. \end{matrix} \tag{2.15}$$

In (2.15), $\underset{\sim}{J}_{a_\ell}$ denotes the matrix of ones of order $a_\ell \times a_\ell$.

The variance-covariance matrix of $\bar{\underset{\sim}{y}}$ in (2.11) can now be written as

$$\text{Var}(\bar{\underset{\sim}{y}}) = \sum_{i=1}^{\nu} \sigma_i^2 \underset{\sim}{A}_i + \sigma_\epsilon^2 \underset{\sim}{K}, \tag{2.16}$$

where

$$\underset{\sim}{K} = \text{diag}\left(\frac{1}{n_\tau}\right)_{\tau \epsilon T}. \tag{2.17}$$

The right side of (2.17) denotes a diagonal matrix of order $c \times c$. Its diagonal elements are the reciprocals of the n_τ's for $\tau \epsilon T$, the set of all values of τ in (2.3). It can be verified that $\underset{\sim}{A}_i \underset{\sim}{A}_{i'} = \underset{\sim}{A}_{i'} \underset{\sim}{A}_i$ for $i \neq i'$. It follows that there exists an orthogonal matrix, $\underset{\sim}{Q}$, of order $c \times c$ such that

$$\underset{\sim}{Q} \underset{\sim}{A}_i \underset{\sim}{Q}' = \underset{\sim}{\Lambda}_i, \quad i = 0, 1, \ldots, \nu, \tag{2.18}$$

where $\underset{\sim}{\Lambda}_i$ is a diagonal matrix. The construction of the matrix $\underset{\sim}{Q}$ will be described in Section 3.

3. The development of the exact tests

Let us again consider model (2.10). As was noted earlier in Section 2, the first part to the right of this model, namely, $\sum_{i=0}^{\nu} \gamma_{\theta_i(\bar{\theta}_i)}$, is not affected by the imbalance in the last stage of the design, that is, in the τ-cells, where τ is described in (2.3). Let us therefore consider the derived model

$$z_\tau = \sum_{i=0}^{\nu} \gamma_{\theta_i(\bar{\theta}_i)}. \tag{3.1}$$

Model (3.1) is balanced with one observation in each τ-cell. Let $\underset{\sim}{P}_i$ be a c×c matrix associated with the sum of squares for the i^{th} effect ($i=0,1,\ldots,\nu$). From Khuri (1982) we have the following lemmas:

Lemma 3.1.

(i) $\underset{\sim}{P}_i$ is idempotent ($i=0,1,\ldots,\nu$).

(ii) $\underset{\sim}{P}_i \underset{\sim}{P}_{i'} = \underset{\sim}{0}$ for $i \neq i'$.

(iii) $\sum_{i=0}^{\nu} \underset{\sim}{P}_i = \underset{\sim}{I}_c$.

Lemma 3.2. The matrix $\underset{\sim}{P}_i$ can be expressed in terms of the $\underset{\sim}{A}_i$'s in (2.14) as

$$\underset{\sim}{P}_i = \sum_{j=0}^{\nu} \left(\lambda_{ij}/b_j\right) \underset{\sim}{A}_j, \quad i = 0, 1, \ldots, \nu, \tag{3.2}$$

where λ_{ij} is a known constant equal to the coefficient of the j^{th} admissible mean in the i^{th} component for the balanced model in (3.1) (possible values of λ_{ij} are -1, 0, and 1), and b_j is given by

$$b_j = \prod_{k_\ell \in \psi_j^c} a_\ell, \quad j = 0, 1, \ldots, \nu, \tag{3.3}$$

where ψ_j^c is the complement of ψ_j with respect to τ.

Lemma 3.3. The $\underset{\sim}{A}_j$ and $\underset{\sim}{P}_i$ matrices in (3.2) are related by the following identity:

$$\underset{\sim}{A}_j \underset{\sim}{P}_i = \kappa_{ij} \underset{\sim}{P}_i, \quad \begin{array}{l} i = 0, 1, \ldots, \nu; \\ j = 0, 1, \ldots, \nu, \end{array} \tag{3.4}$$

where κ_{ij} is given by

$$\kappa_{ij} = \begin{array}{ll} 0, & \psi_i \not\subset \psi_j \\ b_j, & \psi_i \subset \psi_j, \end{array} \tag{3.5}$$

and ψ_i is the set of subscripts associated with the i^{th} effect in (3.1), $i = 0, 1, \ldots, \nu$.

Let m_i be the rank of $\underset{\sim}{P}_i$ $(i=0,1,\ldots,\nu)$, then by Lemma 3.1, $\sum_{i=0}^{\nu} m_i = c$. Let $\underset{\sim}{Q}_i$ be a matrix of order $m_i \times c$ and rank m_i whose rows are orthonormal and span the row space of $\underset{\sim}{P}_i$ $(i=0,1,\ldots,\nu)$. The matrix $\underset{\sim}{Q}_i$ can be easily obtained as the result of a Gram-Schmidt orthonormal factorization of the rows (or columns) of $\underset{\sim}{P}_i$. The proof of the following lemma is given in Appendix A:

Lemma 3.4. The matrices $\underset{\sim}{Q}_0, \underset{\sim}{Q}_1, \ldots, \underset{\sim}{Q}_\nu$ have the following properties:

(i) $\underset{\sim}{Q}_0 = \underset{\sim}{1}'_c/\sqrt{c}$.

(ii) $\begin{array}{ll} \underset{\sim}{Q}_i \underset{\sim}{Q}'_i = \underset{\sim}{I}_{m_i}, & i = 0, 1, \ldots, \nu \\ \underset{\sim}{Q}_i \underset{\sim}{Q}'_{i'} = \underset{\sim}{0}, & i \neq i'. \end{array}$

(iii) $\underset{\sim}{A}_j \underset{\sim}{Q}'_i = \begin{array}{ll} \underset{\sim}{0}, & \psi_i \not\subset \psi_j \\ b_j \underset{\sim}{Q}'_i, & \psi_i \subset \psi_j \end{array} \quad \begin{array}{l} i = 0, 1, \ldots, \nu; \\ j = 0, 1, \ldots, \nu. \end{array}$

Let us now define the matrix

$$\underset{\sim}{Q} = \left[\underset{\sim}{Q}'_0 : \underset{\sim}{Q}'_1 : \cdots : \underset{\sim}{Q}'_\nu\right]', \tag{3.6}$$

which is of order $c \times c$. By Lemma 3.4, $\underset{\sim}{Q}$ is an orthogonal matrix and diagonalizes $\underset{\sim}{A}_0, \underset{\sim}{A}_1, \ldots \underset{\sim}{A}_\nu$ simultaneously as in (2.18). From (2.16) we have

$$\mathrm{Var}\left(\underset{\sim}{Q}_i \, \bar{\underset{\sim}{y}}\right) = \underset{\sim}{Q}_i \left(\sum_{j=1}^{\nu} \sigma_j^2 \underset{\sim}{A}_j + \sigma_\epsilon^2 \underset{\sim}{K}\right) \underset{\sim}{Q}'_i, \quad i = 1, 2, \ldots, \nu,$$

$$= \sum_{j=1}^{\nu} \sigma_j^2 \left(\underset{\sim}{Q}_i \underset{\sim}{A}_j \underset{\sim}{Q}_i' \right) + \sigma_\epsilon^2 \left(\underset{\sim}{Q}_i \underset{\sim}{K} \underset{\sim}{Q}_i' \right), \qquad i = 1, 2, \ldots, \nu, \tag{3.7}$$

where $\underset{\sim}{\bar{y}}$ is the vector of all τ-cell means defined in (2.11). But, by Lemma 3.4 $\left((ii), (iii)\right)$

$$\sum_{j=1}^{\nu} \sigma_j^2 \left(\underset{\sim}{Q}_i \underset{\sim}{A}_j \underset{\sim}{Q}_i' \right) = \delta_i \underset{\sim}{I}_{m_i}, \qquad i = 1, 2, \ldots, \nu, \tag{3.8}$$

where

$$\delta_i = \sum_{j \in W_i} b_j \sigma_j^2. \tag{3.9}$$

In (3.9), W_i is the set

$$W_i = \left\{ j : \psi_i \subset \psi_j, \quad 1 \leq j \leq \nu \right\}.$$

From (3.7) and (3.8) we obtain

$$\text{Var}\left(\underset{\sim}{Q}_i \underset{\sim}{\bar{y}} \right) = \delta_i \underset{\sim}{I}_{m_i} + \sigma_\epsilon^2 \left(\underset{\sim}{Q}_i \underset{\sim}{K} \underset{\sim}{Q}_i' \right), \qquad i = 1, 2, \ldots, \nu. \tag{3.10}$$

Furthermore, for $i \neq i'$,

$$\text{Cov}\left(\underset{\sim}{Q}_i \underset{\sim}{\bar{y}}, \underset{\sim}{\bar{y}}' \underset{\sim}{Q}_{i'}' \right) = \underset{\sim}{Q}_i \left(\sum_{j=1}^{\nu} \sigma_j^2 \underset{\sim}{A}_j + \sigma_\epsilon^2 \underset{\sim}{K} \right) \underset{\sim}{Q}_{i'}'$$

$$= \sigma_\epsilon^2 \left(\underset{\sim}{Q}_i \underset{\sim}{K} \underset{\sim}{Q}_{i'}' \right). \tag{3.11}$$

Let $\underset{\sim}{u}$ be a vector of order $(c-1) \times 1$ defined as

$$\underset{\sim}{u} = \left[\underset{\sim}{Q}_1' : \underset{\sim}{Q}_2' : \cdots : \underset{\sim}{Q}_\nu' \right]' \underset{\sim}{\bar{y}}. \tag{3.12}$$

By (3.10) and (3.11), the variance-covariance matrix of $\underset{\sim}{u}$ is of the form

$$\text{Var}(\underset{\sim}{u}) = \text{diag}\left(\delta_1 \underset{\sim}{I}_{m_1}, \delta_2 \underset{\sim}{I}_{m_2}, \ldots, \delta_\nu \underset{\sim}{I}_{m_\nu} \right) + \sigma_\epsilon^2 \underset{\sim}{G}, \tag{3.13}$$

where

$$\tilde{G} = \tilde{Q}\tilde{K}\tilde{Q}', \tag{3.14}$$

and $\tilde{Q} = \left[Q_1' : Q_2' : \cdots : Q_\nu'\right]'$.

Now, the vector $\bar{y}$ can be expressed as $\bar{y} = Dy$, where y is the vector of observations in (2.7) and D is the direct sum

$$D = \underset{\tau \epsilon T}{\oplus}\left(1_{n_\tau}'/n_\tau\right), \tag{3.15}$$

where τ is defined in (2.3) and T is the set of all values of τ. The matrix D is of order $c \times N$, where N is the total number of observations. Also, the residual sum of squares for the original unbalanced model in (2.1) can be written as

$$SS_E = \sum_\theta (y_\theta - \bar{y}_\tau)^2$$

$$= \sum_{\tau \epsilon T}\left[\sum_{k_s=1}^{n_\tau}(y_\theta - \bar{y}_\tau)^2\right], \tag{3.16}$$

where, if we recall, $\theta = (\tau, k_s)$. Formula (3.16) can be rewritten as

$$SS_E = y'Ry, \tag{3.17}$$

where

$$R = \underset{\tau \epsilon T}{\oplus}\left(I_{n_\tau} - J_{n_\tau}/n_\tau\right). \tag{3.18}$$

Lemma 3.5.

(i) R is idempotent of rank N-c.

(ii) $DR = 0$.

(iii) $RX_i = 0$, $i = 1, 2, \ldots, \nu$, where $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_\nu$ are the matrices given in (2.7).

Proof. See Appendix B.

From Lemma 3.5 we conclude that

$$\underset{\sim}{D}\underset{\sim}{\Sigma}\underset{\sim}{R} = \underset{\sim}{0}, \tag{3.19}$$

where $\underset{\sim}{\Sigma}$ is the variance-covariance matrix of $\underset{\sim}{y}$ in (2.7), which is equal to

$$\underset{\sim}{\Sigma} = \sum_{i=1}^{\nu} \sigma_i^2 \underset{\sim}{X}_i \underset{\sim}{X}_i' + \sigma_\epsilon^2 \underset{\sim}{I}_N. \tag{3.20}$$

Furthermore,

$$\underset{\sim}{R}\underset{\sim}{\Sigma}/\sigma_\epsilon^2 = \underset{\sim}{R}. \tag{3.21}$$

It follows from (3.19) and (3.21) that $\bar{\underset{\sim}{y}} = \underset{\sim}{D}\underset{\sim}{y}$ is independent of SS_E and that SS_E/σ_ϵ^2 has the chi-square distribution with N-c degrees of freedom, the rank of $\underset{\sim}{R}$.

Since $\underset{\sim}{R}$ is idempotent of rank N-c, then it can be written as

$$\underset{\sim}{R} = \underset{\sim}{C}\underset{\sim}{\Lambda}\underset{\sim}{C}', \tag{3.22}$$

where $\underset{\sim}{C}$ is an orthogonal matrix and $\underset{\sim}{\Lambda}$ is a diagonal matrix with N-c ones and c zeros. By the assumption in (2.6), $\underset{\sim}{\Lambda}$ can be partitioned as

$$\underset{\sim}{\Lambda} = \text{diag}\left(\underset{\sim}{I}_{\xi_1}, \underset{\sim}{I}_{\xi_2}, \underset{\sim}{0}\right), \tag{3.23}$$

where

$$\begin{aligned} \xi_1 &= c-1 \\ \xi_2 &= N - 2c+1>0, \end{aligned} \tag{3.24}$$

and $\underset{\sim}{0}$ in (3.23) is a zero matrix of order c×c. Accordingly, the matrix $\underset{\sim}{C}$ in (3.22) can be partitioned as $\underset{\sim}{C} = \left[\underset{\sim}{C}_1 : \underset{\sim}{C}_2 : \underset{\sim}{C}_3\right]$, where $\underset{\sim}{C}_1$, $\underset{\sim}{C}_2$, and $\underset{\sim}{C}_3$ are of orders $N\times\xi_1$, $N\times\xi_2$, and $N\times c$, respectively. We then have

$$\underset{\sim}{R} = \underset{\sim}{C}_1\underset{\sim}{C}_1' + \underset{\sim}{C}_2\underset{\sim}{C}_2'. \tag{3.25}$$

Note that

$$\begin{aligned} \underset{\sim}{C}_i'\underset{\sim}{C}_i &= \underset{\sim}{I}, \quad i = 1, 2, 3; \\ \underset{\sim}{C}_i'\underset{\sim}{C}_{i'} &= \underset{\sim}{0}, \quad i \neq i'. \end{aligned} \tag{3.26}$$

Let us now define the random vector $\underset{\sim}{\omega}$ as

$$\underset{\sim}{\omega} = \underset{\sim}{u} + \left(\lambda_{max}\underset{\sim}{I}_{\xi_1} - \underset{\sim}{G}\right)^{\frac{1}{2}}\underset{\sim}{C}_1'\underset{\sim}{y}, \tag{3.27}$$

where $\underset{\sim}{G}$ is the matrix in (3.14) and λ_{max} is its largest eigenvalue, $\left(\lambda_{max}\underset{\sim}{I}_{\xi_1} - \underset{\sim}{G}\right)^{\frac{1}{2}}$ is a symmetric matrix with eigenvalues equal to the square roots of the eigenvalues of $\lambda_{max}\underset{\sim}{I}_{\xi_1} - \underset{\sim}{G}$, which are nonnegative. Let $\underset{\sim}{\omega}$ be partitioned just like $\underset{\sim}{u}$ in (3.12) as

$$\underset{\sim}{\omega} = \left(\underset{\sim}{\omega}_1', \underset{\sim}{\omega}_2', \ldots, \underset{\sim}{\omega}_\nu'\right)', \tag{3.28}$$

where $\underset{\sim}{\omega}_i$ is of order $m_i\times 1$ and m_i is the number of rows $\underset{\sim}{Q}_i$ ($i=1,2,\ldots,\nu$) in (3.12). The distributional properties of the $\underset{\sim}{\omega}_i$'s are given in the following lemma:

Lemma 3.6.

(i) $E(\underset{\sim}{\omega}_i) = \underset{\sim}{0}, \quad i = 1,2,\ldots,\nu.$

(ii) $\underset{\sim}{\omega}_1, \underset{\sim}{\omega}_2, \ldots, \underset{\sim}{\omega}_\nu$ are independently distributed as normal random vectors with $\underset{\sim}{\omega}_i$ having the variance-covariance matrix

$$\mathrm{Var}(\underset{\sim}{\omega}_i) = \left(\delta_i + \lambda_{\max}\sigma_\epsilon^2\right)\underset{\sim}{I}_{m_i}, \quad i = 1,2,\ldots,\nu, \tag{3.29}$$

where δ_i is defined in (3.9).

(iii) $\underset{\sim}{\omega}_1, \underset{\sim}{\omega}_2, \ldots, \underset{\sim}{\omega}_\nu$ are independent of $SS_E^{(2)}$, where $SS_E^{(2)} = \underset{\sim}{y}'\underset{\sim}{C}_2\underset{\sim}{C}_2'\underset{\sim}{y}$ is the portion of the residual sum of squares, SS_E, in (3.17), which corresponds to the matrix $\underset{\sim}{C}_2$ in (3.25).

Proof.

(i) From (2.11) and Lemma 3.4, $E(\underset{\sim}{Q}_i\bar{\underset{\sim}{y}}) = \underset{\sim}{Q}_i\underset{\sim}{H}_0\beta_0 = \underset{\sim}{Q}_i\underset{\sim}{1}c\beta_0 = 0$ for $i = 1, 2, \ldots, \nu$. Hence, $E(\underset{\sim}{u}) = \underset{\sim}{0}$ by (3.12). From (3.18) we also have that $\underset{\sim}{R}\,\underset{\sim}{1}_N = \underset{\sim}{0}$, which by (3.25) can be rewritten as

$$\left(\underset{\sim}{C}_1\underset{\sim}{C}_1' + \underset{\sim}{C}_2\underset{\sim}{C}_2'\right)\underset{\sim}{1}_N = \underset{\sim}{0}. \tag{3.30}$$

Using (3.26) in (3.30) we get $\underset{\sim}{C}_1'\underset{\sim}{1}_N = \underset{\sim}{0}$. It follows from (2.7) that $E(\underset{\sim}{C}_1'\underset{\sim}{y}) = \underset{\sim}{C}_1'\underset{\sim}{1}_N\beta_0 = \underset{\sim}{0}$. The mean of $\underset{\sim}{\omega}$ in (3.27) is therefore equal to zero.

(ii) It is clear that $\underset{\sim}{\omega}$ is normally distributed. Now, the vector $\underset{\sim}{u}$ in (3.27) is independent of $\underset{\sim}{C}_1'\underset{\sim}{y}$. To show this, we note from (3.19), (3.25), and (3.26) that

$$\underset{\sim}{D}\underset{\sim}{\Sigma}\underset{\sim}{C}_1 = \underset{\sim}{0}. \tag{3.31}$$

Hence,

$$\mathrm{Cov}\left(\bar{\underset{\sim}{y}}, \underset{\sim}{y}'\underset{\sim}{C}_1\right) = \underset{\sim}{D}\underset{\sim}{\Sigma}\underset{\sim}{C}_1 = \underset{\sim}{0},$$

since $\bar{\underset{\sim}{y}} = \underset{\sim}{D}\underset{\sim}{y}$. It follows that $\bar{\underset{\sim}{y}}$, and hence $\underset{\sim}{u}$ in (3.12), is independent of $\underset{\sim}{C}_1'\underset{\sim}{y}$.

The variance-covariance matrix of $\underset{\sim}{\omega}$ can therefore be expressed as

$$\mathrm{Var}(\underset{\sim}{\omega}) = \mathrm{Var}(\underset{\sim}{u}) + \left(\lambda_{\max}\underset{\sim}{I}_{\xi_1} - \underset{\sim}{G}\right)^{\frac{1}{2}}\underset{\sim}{C}_1'\underset{\sim}{\Sigma}\underset{\sim}{C}_1\left(\lambda_{\max}\underset{\sim}{I}_{\xi_1} - \underset{\sim}{G}\right)^{\frac{1}{2}}. \tag{3.32}$$

But, from (3.21) we have

$$\underset{\sim}{R}\underset{\sim}{\Sigma}\underset{\sim}{R} = \sigma_\epsilon^2\underset{\sim}{R}. \tag{3.33}$$

Also, from (3.25), (3.26), and (3.33) it can be verified that

$$\underset{\sim}{C}_1'\underset{\sim}{\Sigma}\underset{\sim}{C}_1 = \sigma_\epsilon^2\underset{\sim}{C}_1'\underset{\sim}{C}_1 = \sigma_\epsilon^2\,\underset{\sim}{I}_{\xi_1}. \tag{3.34}$$

From (3.32) and (3.34) we then have

$$\mathrm{Var}(\underset{\sim}{\omega}) = \mathrm{Var}(\underset{\sim}{u}) + \sigma_\epsilon^2\left(\lambda_{\max}\underset{\sim}{I}_{\xi_1} - \underset{\sim}{G}\right). \tag{3.35}$$

By using (3.13) in (3.35) we get

$$\mathrm{Var}(\underset{\sim}{\omega}) = \mathrm{diag}\left(\delta_1\underset{\sim}{I}_{m_1}, \delta_2\underset{\sim}{I}_{m_2}, \ldots, \delta_\nu\underset{\sim}{I}_{m_\nu}\right) + \lambda_{\max}\sigma_\epsilon^2\underset{\sim}{I}_{\xi_1}. \tag{3.36}$$

From (3.36) we conclude that $\underset{\sim}{\omega}_1, \underset{\sim}{\omega}_2, \ldots, \underset{\sim}{\omega}_\nu$ are independent and that $\mathrm{Var}(\underset{\sim}{\omega}_i)$ has the form described in (3.29).

(iii) $SS_E^{(2)}$ is independent of $\underset{\sim}{u}$ $\left(\text{since } SS_E \text{ is independent of } \bar{\underset{\sim}{y}} \text{ by (3.19)}\right)$ and is also independent of $\underset{\sim}{C}_1'\underset{\sim}{y}$. This is true because

$$\underset{\sim}{C}_1'\underset{\sim}{\Sigma}\underset{\sim}{C}_2 = \underset{\sim}{0}, \tag{3.37}$$

which follows from (3.25), (3.26), and (3.33). Consequently, $SS_E^{(2)}$ and $\underset{\sim}{\omega}$ in (3.27) are independent.

From Lemma 3.6 we conclude that if $SS_i = \underset{\sim}{\omega}_i'\underset{\sim}{\omega}_i$ $(i=1,2,\ldots,\nu)$, then the sums of squares, SS_1, $SS_2, \ldots, SS_\nu$ are independent and $SS_i/\left(\delta_i+\lambda_{\max}\sigma_\epsilon^2\right)$ is distributed as a central chi-square variate with m_i degrees of freedom $(i=1,2,\ldots,\nu)$. Furthermore, the SS_i's are independent of $SS_E^{(2)}/\sigma_\epsilon^2$, which has the chi-square distribution with ξ_2 degrees of freedom, where ξ_2 is defined in (3.24).

It follows that $SS_1, SS_2, \ldots, SS_\nu$ and $SS_E^{(2)}$ act like sums of squares in an ANOVA table for a balanced random model. In other words, the analysis concerning the variance components, $\sigma_1^2, \sigma_2^2, \ldots, \sigma_\nu^2$, can proceed using these sums of squares just like in a balanced data situation.

In particular, if the data set is balanced, that is, if $n_\tau = n$, then $\underset{\sim}{K} = \underset{\sim}{I}_c/n$ $\left(\text{see } (2.17)\right)$ and $\underset{\sim}{G} = \underset{\sim}{I}_{\xi_1}/n$, where $\underset{\sim}{G}$ and ξ_1 are defined in (3.14) and (3.24), respectively. Consequently, λ_{max}, the largest eigenvalue of $\underset{\sim}{G}$, is equal to $1/n$. The vectors $\underset{\sim}{\omega}$ and $\underset{\sim}{u}$ in (3.27) are therefore identical. In this case,

$$\underset{\sim}{\omega}'\underset{\sim}{\omega} = \underset{\sim}{u}'\underset{\sim}{u} = \bar{\underset{\sim}{y}}'\Big(\sum_{i=1}^{\nu} \underset{\sim}{Q}_i'\underset{\sim}{Q}_i\Big)\bar{\underset{\sim}{y}} \quad \big(\text{by } (3.12)\big)$$

$$= \bar{\underset{\sim}{y}}'\Big(\underset{\sim}{I}_c - \underset{\sim}{J}_c/c\Big)\bar{\underset{\sim}{y}} \quad (\text{by Lemma 3.4})$$

$$= \bar{\underset{\sim}{y}}'\Big(\sum_{i=1}^{\nu} \underset{\sim}{P}_i\Big)\bar{\underset{\sim}{y}} \quad \Big(\text{by Lemma 3.1(iii) and the fact that } \underset{\sim}{P}_0 = \underset{\sim}{J}_c/c \text{ by formula } (3.2)\Big).$$

Thus,

$$\sum_{i=1}^{\nu} SS_i = \sum_{i=1}^{\nu} \bar{\underset{\sim}{y}}'\underset{\sim}{P}_i\bar{\underset{\sim}{y}}.$$

Now, from (3.2) we can write

$$\bar{\underset{\sim}{y}}'\underset{\sim}{P}_i\bar{\underset{\sim}{y}} = \bar{\underset{\sim}{y}}'\left[\sum_{j=0}^{\nu}\Big(\lambda_{ij}/b_j\Big)\underset{\sim}{A}_j\right]\bar{\underset{\sim}{y}}$$

$$= \bar{\underset{\sim}{y}}'\left[\sum_{j=0}^{\nu}\Big(\lambda_{ij}/b_j\Big)\Big(\bigotimes_{\ell=1}^{s-1} \underset{\sim}{M}_{j\ell}\Big)\right]\bar{\underset{\sim}{y}}. \qquad \big(\text{by } (2.14)\big) \tag{3.38}$$

Since $\bar{\underset{\sim}{y}} = \Big(\underset{\sim}{I}_c \otimes \underset{\sim}{1}_n'\Big)\underset{\sim}{y}/n$, formula (3.38) can be expressed as

$$\underset{\sim}{\bar{y}}'\underset{\sim}{P}_i\underset{\sim}{\bar{y}} = \frac{1}{n}\underset{\sim}{y}'\left[\sum_{j=0}^{\nu}\frac{\lambda_{ij}}{nb_j}\left\{\left(\bigotimes_{\ell=1}^{s-1}M_{j\ell}\right)\otimes J_n\right\}\right]\underset{\sim}{y}. \tag{3.39}$$

Formula (3.39) shows that $n\underset{\sim}{\bar{y}}'\underset{\sim}{P}_i\underset{\sim}{\bar{y}}$ is the usual sum of squares for the i^{th} effect in a balanced model of the form given in (2.1). In other words, nSS_1, nSS_2, ..., and nSS_ν reduce to the usual balanced ANOVA sums of squares associated with the corresponding ν effects whenever the data set is balanced.

4. Power of the exact tests

Power values for the exact tests in Section 3 can be easily obtained just like in a balanced model situation. As in Khuri and Littell (1987), it is easy to show that such power values are monotone decreasing with respect to λ_{max}, the largest eigenvalue of the matrix $\underset{\sim}{G}$ in (3.14). Upper and lower bounds on λ_{max} are given by the double inequality

$$\frac{1}{c}\sum_{\tau\epsilon T}\frac{1}{n_\tau} \leq \lambda_{max} \leq \frac{1}{\min\limits_{\tau\epsilon T}(n_\tau)}, \tag{4.1}$$

where n_τ is the frequency of the τ-cell (see (2.3)), and T is the set of all values of τ. The proof of (4.1) is similar to the one given in Lemma 2 in Khuri and Littell (1987) and will, therefore, be omitted. We note that the lower bound in (4.1) is the reciprocal of the harmonic mean of the τ-cell frequencies.

5. A numerical example

An example is given in Milliken and Johnson (1984, p. 264) of a study concerning the efficiency of workers in assembly lines at several plants. Three plants were randomly selected. Four assembly sites and three workers were randomly selected in each plant. For convenience, the efficiency scores are reproduced in Table 1.

The model for this experiment is

$$y_{ijk\ell} = \mu + \alpha_i + \beta_{i(j)} + \gamma_{i(k)} + (\beta\gamma)_{i(jk)} + \epsilon_{ijk\ell},$$

where α_i is the effect of the i^{th} plant (i=1,2,3), $\beta_{i(j)}$ is the effect of the j^{th} site within the i^{th} plant (j=1,2,3,4), $\gamma_{i(k)}$ is the effect of the k^{th} worker within the i^{th} plant (k=1,2,3), $(\beta\gamma)_{i(jk)}$ is the interaction effect of sites and workers within plant i, and $\epsilon_{ijk\ell}$ is the error term. The variance components are σ^2_α, σ^2_β, σ^2_γ, $\sigma^2_{\beta\gamma}$ and σ^2_ϵ, respectively.

To facilitate the understanding of the application of the exact testing procedure to this example, the reader is referred to Table 2 which lists the values of some key quantities used in the development of the exact tests. The expected mean square values of $MS_i = SS_i/m_i$ (i=1,2,3,4) and $MS_E^{(2)} = SS_E^{(2)}/\xi_2$ are given in Table 3.

From Tables 2 and 3 it can be seen that the value of the exact F-statistic for testing the hypothesis H_0: $\sigma^2_{\beta\gamma} = 0$ versus H_a: $\sigma^2_{\beta\gamma} > 0$ is $F = MS_4/MS_E^{(2)} = 7.090$ with 18 and 47 degrees of freedom. The significance level is 3.4×10^{-8}. The second hypothesis to be tested is H_0: $\sigma^2_\beta = 0$ versus H_a: $\sigma^2_\beta > 0$. The corresponding value of the F-statistic is $F = MS_2/MS_4 = .994$ with 9 and 18 degrees of freedom. This is a nonsignificant test. Next, the value of the F-statistic for the hypothesis H_0: $\sigma^2_\gamma = 0$ is $F = MS_3/MS_4 = 3.293$ with 6 and 18 degrees of freedom. The level of significance in this case is .023.

Finally, the testing of the hypothesis H_0: $\sigma^2_\alpha = 0$ versus H_a: $\sigma^2_\alpha > 0$ requires the use of Satterthwaite's procedure since no mean square exists in Table 3 with an expected value equal to that of MS_1 under H_0. The test statistic in this case is given by

$$F = \frac{MS_1}{MS_2 + MS_3 - MS_4} = 5.184,$$

which is approximately distributed as an F random variable with 2 and η degrees of freedom, where

$$\eta = \frac{\left(MS_2 + MS_3 - MS_4\right)^2}{(MS_2)^2/9 + (MS_3)^2/6 + (MS_4)^2/18} = 5.477.$$

The level of significance is .055.

Appendix A

This appendix gives the proof of Lemma 3.4.

Proof.

(i) From (3.2), $\underset{\sim}{P}_0 = \underset{\sim}{A}_0/b_0$ and $b_0 = c$ by (3.3) since ψ_0 is the empty set, hence $\psi_0^c = \tau$. But, by (2.14), $\underset{\sim}{A}_0 = \underset{\sim}{J}_c$. It follows that $\underset{\sim}{P}_0 = \underset{\sim}{J}_c/c$. Consequently, $\underset{\sim}{Q}_0 = \underset{\sim}{1}_c'/\sqrt{c}$.

(ii) $\underset{\sim}{Q}_i\underset{\sim}{Q}_i' = \underset{\sim}{I}_{m_i}$. This follows by the definition of $\underset{\sim}{Q}_i$. Furthermore, $\underset{\sim}{Q}_i = \underset{\sim}{V}_i\underset{\sim}{P}_i$ for some matrix $\underset{\sim}{V}_i$ of order $m_i \times c$ $(i=0,1,\ldots,\nu)$. Since $\underset{\sim}{P}_i\underset{\sim}{P}_{i'} = \underset{\sim}{0}$ for $i \neq i'$, then $\underset{\sim}{Q}_i\underset{\sim}{Q}_{i'}' = \underset{\sim}{V}_i\underset{\sim}{P}_i\underset{\sim}{P}_{i'}\underset{\sim}{V}_{i'}' = \underset{\sim}{0}$.

(iii) From (3.4) and (3.5) it can be seen that $\underset{\sim}{A}_j\underset{\sim}{Q}_i' = \underset{\sim}{A}_j\underset{\sim}{P}_i\underset{\sim}{V}_i' = \underset{\sim}{0}$, if $\psi_i \not\subset \psi_j$, and $\underset{\sim}{A}_j\underset{\sim}{Q}_i' = \underset{\sim}{A}_j\underset{\sim}{P}_i\underset{\sim}{V}_i' = b_j\underset{\sim}{P}_i\underset{\sim}{V}_i' = b_j\underset{\sim}{Q}_i'$, if $\psi_i \subset \psi_j$.

Appendix B

This appendix gives the proof of Lemma 3.5.

Proof.

(i) This is straightforward.

(ii) $$\underset{\sim}{D}\underset{\sim}{R} = \left[\underset{\tau\epsilon T}{\oplus}\left(\underset{\sim}{1}_{n_\tau}'/n_\tau\right)\right]\left[\underset{\sim}{I}_N - \underset{\tau\epsilon T}{\oplus}\left(\underset{\sim}{J}_{n_\tau}/n_\tau\right)\right] \qquad \text{(by (3.15) and (3.18))}$$

$$= \underset{\tau\epsilon T}{\oplus}\left(\underset{\sim}{1}_{n_\tau}'/n_\tau\right) - \underset{\tau\epsilon T}{\oplus}\left(\underset{\sim}{1}_{n_\tau}'/n_\tau\right) = \underset{\sim}{0}.$$

(iii) The matrix $\underset{\sim}{X}_i$ can be partitioned into c submatrices that correspond to the values of τ in (2.3). The submatrix corresponding to a particular τ is of order $n_\tau \times c_i$, where c_i is the number of columns of $\underset{\sim}{X}_i$ $(i=1,2,\ldots,\nu)$. Let us denote such a submatrix by $\underset{\sim}{U}_\tau$. Each column of $\underset{\sim}{U}_\tau$ consists of either n_τ zeros or n_τ ones. Therefore, $\underset{\sim}{R}\underset{\sim}{X}_i$ can be partitioned into c submatrices of orders $n_\tau \times c_i$ for the different values of τ. For a particular τ, the corresponding submatrix is of the form

$$\left(\underset{\sim}{I}_{n_\tau} - \underset{\sim}{J}_{n_\tau}/n_\tau\right)\underset{\sim}{U}_\tau = \underset{\sim}{U}_\tau - \underset{\sim}{U}_\tau = \underset{\sim}{0},$$

by the property of $\underset{\sim}{U}_\tau$ described earlier. It follows that $\underset{\sim}{R}\underset{\sim}{X}_i = \underset{\sim}{0}$ for $i=1,2,\ldots,\nu$.

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Table 1

Data for the numerical example

Plant	Worker	Site 1	Site 2	Site 3	Site 4
Plant 1	1	100.6 106.8 100.6	110.0 105.8	100.0 102.5 97.6 98.7 98.7	98.2 99.5
	2	92.3 92.0 97.2 93.9 93.0	103.2 100.5 100.2 97.7	96.4	108.0 108.9 107.9
	3	96.9 96.1 100.8	92.5 85.9 85.2 89.4 88.7	86.8	94.4 93.0 91.0
Plant 2	1	82.6	96.5 100.1 101.9 97.9 95.9	87.9 93.5 88.9 92.8	83.6 82.7 87.7 88.0 82.5
	2	72.7	71.7 72.1 72.4 71.4	78.4 80.4 83.8 77.7 81.2	82.1 79.9 81.9 82.6 78.6
	3	82.5 82.1 82.0	80.9 84.0 82.2 83.4 81.5	96.3 92.4 92.0 95.8	77.7 78.6 77.2 78.8 80.5
Plant 3	1	107.6 108.8 107.2 104.2 105.4	96.1 98.5 97.3 93.5	101.1	109.1
	2	97.1 94.2 91.5 99.2	91.9	88.0 91.4 90.3 91.5 85.7	89.6 86.0 91.2 87.4
	3	87.1	97.8 95.9	95.9 89.7	101.4 100.1 102.1 98.4

Table 2

Some key quantities used in the development of the exact tests for the numerical example

Quantity	Formula cited	Corresponding value
ν	(2.1)	4
τ	(2.3)	(i,j,k)
c	(2.5)	36
N	(2.6)	118
b_0	(3.3)	36
b_1	(3.3)	12
b_2	(3.3)	3
b_3	(3.3)	4
b_4	(3.3)	1
$\underset{\sim}{P}_1$	(3.2)	$\left(\underset{\sim}{I}_3 \otimes \underset{\sim}{J}_{12}\right)/12 - \underset{\sim}{J}_{36}/36$
$\underset{\sim}{P}_2$	(3.2)	$\left(\underset{\sim}{I}_{12} \otimes \underset{\sim}{J}_3\right)/3 - \left(\underset{\sim}{I}_3 \otimes \underset{\sim}{J}_{12}\right)/12$
$\underset{\sim}{P}_3$	(3.2)	$\left(\underset{\sim}{I}_3 \otimes \underset{\sim}{J}_4 \otimes \underset{\sim}{I}_3\right)/4 - \left(\underset{\sim}{I}_3 \otimes \underset{\sim}{J}_{12}\right)/12$
$\underset{\sim}{P}_4$	(3.2)	$\underset{\sim}{I}_{36} - \left(\underset{\sim}{I}_{12} \otimes \underset{\sim}{J}_3\right)/3 - \left(\underset{\sim}{I}_3 \otimes \underset{\sim}{J}_4 \otimes \underset{\sim}{I}_3\right)/4 + \left(\underset{\sim}{I}_3 \otimes \underset{\sim}{J}_{12}\right)/12$
m_1	(3.29)	2
m_2	(3.29)	9
m_3	(3.29)	6
m_4	(3.29)	18
δ_1	(3.9)	$12\sigma_\alpha^2 + 3\sigma_\beta^2 + 4\sigma_\gamma^2 + \sigma_{\beta\gamma}^2$
δ_2	(3.9)	$3\sigma_\beta^2 + \sigma_{\beta\gamma}^2$
δ_3	(3.9)	$4\sigma_\gamma^2 + \sigma_{\beta\gamma}^2$
δ_4	(3.9)	$\sigma_{\beta\gamma}^2$
ξ_1	(3.24)	35
ξ_2	(3.24)	47
λ_{max}	(3.27)	1
$SS_1 = \underset{\sim}{\omega}_1'\underset{\sim}{\omega}_1$	Lemma 3.6	1265.96
$SS_2 = \underset{\sim}{\omega}_2'\underset{\sim}{\omega}_2$	Lemma 3.6	332.313
$SS_3 = \underset{\sim}{\omega}_3'\underset{\sim}{\omega}_3$	Lemma 3.6	733.949
$SS_4 = \underset{\sim}{\omega}_4'\underset{\sim}{\omega}_4$	Lemma 3.6	668.634
$SS_E^{(2)}$	Lemma 3.6	246.245

Table 3

Expected mean square values for the numerical example

Mean square	Expected value
$MS_1 = SS_1/2$	$12\sigma_\alpha^2 + 3\sigma_\beta^2 + 4\sigma_\gamma^2 + \sigma_{\beta\gamma}^2 + \sigma_\epsilon^2$
$MS_2 = SS_2/9$	$3\sigma_\beta^2 + \sigma_{\beta\gamma}^2 + \sigma_\epsilon^2$
$MS_3 = SS_3/6$	$4\sigma_\gamma^2 + \sigma_{\beta\gamma}^2 + \sigma_\epsilon^2$
$MS_4 = SS_4/18$	$\sigma_{\beta\gamma}^2 + \sigma_\epsilon^2$
$MS_E^{(2)} = SS_E^{(2)}/47$	σ_ϵ^2

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