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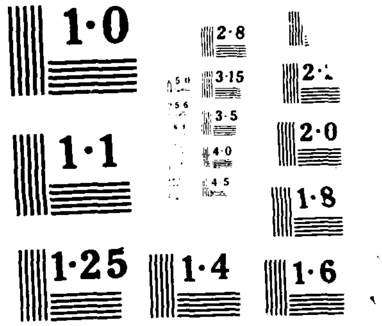
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A Note on Phase-Type, Almost Phase-Type, Generalized Hyperexponential and Coxian Distributions

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Report No. GMU/22461/102

October 1987

(revised January 1988)

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Technical Report

Office of Naval Research
Contract No. N00014-86-K-0029

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REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A Note on Phase-Type, Almost Phase-Type, Generalized Hyperexponential and Coxian Distributions		5. TYPE OF REPORT & PERIOD COVERED Technical Report 6. PERFORMING ORG. REPORT NUMBER GMU/22461/102
7. AUTHOR(s) Carl M. Harris Robert F. Botta		8. CONTRACT OR GRANT NUMBER(s) N00014-86-K-0029
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Operations Research and Applied Statistics George Mason University, Fairfax, Va. 22030		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Project 4118150
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research 800 North Quincy Street Arlington, Va. 22217		12. REPORT DATE October 1987 13. NUMBER OF PAGES 9
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) 15a. DECLASSIFICATION DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report)		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) applied probability computational probability queueing systems communication systems probability random processes stochastic modeling telecommunications		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number)		

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Abstract

In this note, we update earlier results on how various types of Coxian distributions relate to each other. Some more recent uses of such distributions are discussed, with an emphasis on their connection to generalized hyperexponential distributions. We pay particular attention to the class of densities having Laplace transforms with only real negative zeros and poles and their relationship with the GH class.

1. Background

A number of recent papers (e.g., Ott, 1987, and Shanthikumar, 1985) have used distribution functions related to the classical exponential staging formulation introduced by Erlang (see Brockmeyer et al., 1948), with primary modifications over the years by Jensen (1954) and Cox (1955). Much of the current popularity of such distributions is due to the work of Neuts and colleagues on the so-called phase-type distributions (see, e.g., Neuts, 1975a,b and 1981), exploiting relationships to the theory of Markov chains and putting the theory to effective computational use in a large variety of stochastic models. Some recent work by these authors, together with a number of different coauthors, has focused on the generalized mixed exponential form (called GH) of Coxian distribution (see Botta & Harris, 1986, Botta, Harris & Marchal, 1987, Harris & Sykes, 1987). These are linear, but not necessarily convex combinations of negative exponential densities. Botta and Harris (1986) showed, critically, that the GH class is dense in the set of all CDFs relative to an appropriate metric. Denseness is likewise a property of both the PH and Coxian classes (the latter henceforth called R_n).

For purposes of clarification, we note the following relationships between the major classes of cumulative distribution functions (CDFs). For the most part, all of this stems from Erlang's early idea of modeling a duration or lifetime as a sum of (k) independent and identical exponential stages. Much later, Jensen (1954) generalized Erlang's device to allow the stages to have non-identical CDFs, and indeed recognized the natural connection between Erlang's method of stages and absorption-time distributions for finite Markov chains. (Note that when there is possible exit from any stage, an Erlang becomes a random sum of exponentials.) An important milestone in this development came very shortly thereafter in work by Cox (1955), who generalized to cover all distributions with rational Laplace transforms, by involving a more complicated stage-to-stage movement possibly using complex transition probabilities and even an infinite number of steps. More precisely, the sequential flow corresponding to a Coxian distribution can have negative branching probabilities and complex scale parameters (with negative real parts). While such stages may not have a physical reality, the resulting CDF can well be legitimate. The class R_n contains the class of all phase-type distributions, which in turn contains all generalized Erlangs.

Recently, Shanthikumar (1985) worked with two classes of functions, which he called generalized and bilateral phase types. The CDF of the generalized phase type (GPH) is created from an infinite mixing on the number of convolutions. The bilateral phase type is defined over the entire real line in an analogous fashion using Erlang mixing. It follows that a GPH distribution is an ordinary PH whenever the mixing distribution has finite support or has an infinite phase-type representation.

Ott (1987) has renamed the GPH distributions as almost phase types (APH), and has developed a relationship between the Wiener-Hopf factorization for the G/G/1 queue and infinite matrix (possibly phase-type) representations of at least one of the interarrival and service CDFs. An iterative numerical procedure is formulated for solving the special cases of the GPH/G/1 and G/GPH/1 queues. In these cases, the algorithm works from a finite mixture of Erlangs, even though this might be just one element in an infinite sequence. Since the GPH class is dense in the set of all CDFs, one can theoretically always find such a close approximant.

In another recent paper, Swensen (1986) dealt with a variation on the GI/M/c model in which customers renege when their waiting times exceed a fixed value. To make the problem as general as possible, the CDF for interarrival times was assumed to belong to a general class, said to be Coxian. However, by restricting the mixing probabilities and means of the exponential stages to positive values, Swensen's class of CDFs is, in fact, identical to the class of finite mixtures of generalized Erlang distributions. While this restricted class is a subset of the PH class of distributions, the larger class of Coxian distributions actually *contains* the PH class, contrary to Swensen's statement that the Coxian distributions are a subset of the phase-type CDFs.

Although these observations do not affect any of the results in the paper, it is important to note that when the author speaks of Coxian distributions, he is actually considering only a very restricted subset of those distributions. Confusion may result if this limited definition of Coxian distributions is not borne in mind when interpreting Swensen's results.

The newest class added to the list has been offered by Sumita and Masuda (1987). This is the class (called Ω^+) of probability distribution functions which have Laplace-Stieltjes transforms with only real negative zeros and poles. These densities are the newest and therefore the least well known, and because their possible use raises some interesting questions, we devote the next section of the note to a discussion of the position of Ω^+ relative to the other types of phase classes.

2. The Density Class Ω^+

The formal definition of Ω^+ is the class of densities admitting Laplace transforms

$$f^*(s) = \frac{\prod_{i=1}^{m} (1 + s/\eta_i)}{\prod_{j=1}^{n} (1+s/\theta_j)}, \quad 0 \leqq m < n < \infty,$$

$$\eta_i \neq \theta_j \quad \eta_i, \theta_j > 0 \text{ for all } i,j.$$

[We assume that the numerator is 1 for $m = 0$.]

Clearly, all such transforms are rational and thus $\Omega \subset R_n$. Since all $\{\theta_j\}$ are positive, it appears at first glance that $f^*(s)$ should also be generalized hyperexponential whenever the $\{\theta_j\}$ are district. However, this is not the case.

Sumita and Masuda (1987) have actually given the skeleton of a counterexample pdf which is in GH but not Ω^+. Consider the transform

$$f^*(s) = \frac{6}{13} \; \frac{2s^2+10s+13}{(s+1)(s+2)(s+3)} .$$

This is easily inverted to be the GH density

$$f(t) = \frac{15}{13}(e^{-t}) - \frac{3}{13}(2e^{-2t}) + \frac{1}{13}(3e^{-3t}).$$

However, we see that f(t) is not in Ω^+ since the roots of the numerator of $f^*(s)$ are complex. This particular example arose as a mixture of two Ω^+ densities, thus showing that Ω^+ is not closed under mixing.

Clearly, this is not a pathological counterexample, for there are many examples of GH distributions which have similar transform properties. Another illustration is Example 2.2.2 (page 124) of Botta, Harris and Marchal (1987), namely,

$$f(t) = 4(e^{-t}) - 6(2e^{-2t}) + 3(3e^{-3t}).$$

The numerator of its transform is the polymomial

$$f_n(s) = s^2 - s + 6 \quad ,$$

which also has complex roots. Furthermore, similar complex arithmetic issues create examples of distributions in the Neuts phase-type class PH which are also not in Ω^+.

Some of the major results of Sumita and Masuda (1987) have interesting application to the theory of Coxian and generalized hyperexponential distributions. A prime example of these is their Theorem 1.2, which provides a simple sufficient condition for $f \in GH$ when f is also in Ω^+. We have been able to extend their argument to derive a sufficient condition for $f \in GH$ independent of whether it is in Ω^+.

3. Sufficient Conditions

The condition offered by Sumita and Masuda requires that there exists an indexing of the $\{\theta_i\}$ and the $\{\eta_j\}$ (assuming $\eta_1 < \eta_2 < \ldots < \eta_m$ and $\theta_1 < \theta_2 < \ldots < \theta_n$) in which $\theta_i < \eta_i$ for $1 \leq i \leq m$. An informal proof of this result follows from writing the transform as the product of quotients

$$\frac{1 + s/\eta_1}{1 + s/\theta_1} \; \frac{1 + s/\eta_2}{1 + s/\theta_2} \cdots \frac{1 + s/\eta_m}{1 + s/\theta_m} \; \frac{1}{1 + s/\theta_{m+1}} \cdots \frac{1}{1 + s/\theta_n} \, .$$

The inverse Stieltjes transform of each of the m factors is of the form

$$F(t) = \frac{\theta_i}{\eta_i} + (1 - \frac{\theta_i}{\eta_i})(1 - e^{-\theta_i t}) \, .$$

The requirement that $\theta_i < \eta_i$ yields a mixture of an atom at the origin and an exponential.

Clearly, each of the last n−m terms corresponds to an exponential. Since each of the n factors thus corresponds to a legitimate probability distribution, the convolution of the corresponding functions will yield a true distribution (without an atom since there is at least one purely exponential term in the convolution).

This suggests the following extension to the case where the numerator polynomial of the transform can have complex roots occurring in conjugate pairs.

Theorem Suppose a rational transform has the form

$$f^*(s) = \frac{\prod_{i=1}^{m} (1+s/\eta_i)}{\prod_{j=1}^{n} (1+s/\theta_j)} \qquad (0 < m < n),$$

where the $\{\theta_j\}$ are real, positive and arranged in ascending order, and the $\{\eta_i\}$ are either real and positive or occur in complex conjugate pairs with positive real parts. Suppose (without loss of generality) that

$$\mathrm{Re}\,(\eta_1) \leq \mathrm{Re}\,(\eta_2) \leq \ldots \leq \mathrm{Re}\,(\eta_m) \text{ [where equality holds only in the case of complex conjugates]}$$

and that for $i = 1, 2, \ldots, m$, $\theta_i < \eta_i$ when η_i is real and $(\theta_i + \theta_{i+1})/2 \leq \mathrm{Re}(n_i)$ when (η_i, η_{i+1}) are a complex conjugate pair. Then the inverse transform of $f^*(s)$ is a probability distribution.

Proof: The idea of the proof is the same as before, except that we now can have factors of the form

$$\frac{\left(1+\frac{s}{a+ib}\right)\left(1+\frac{s}{a-ib}\right)}{(1+s/r_1)(1+s/r_2)}\,.$$

Each of such factors can be expressed as

$$\frac{r_1 r_2}{a^2+b^2} + \frac{\frac{2a-r_1-r_2}{a^2+b^2}\,s + 1 - \frac{r_1 r_2}{a^2+b^2}}{(1+s/r_1)(1+s/r_2)}\,.$$

Expanding the second term in partial fractions yields

$$\frac{r_1 r_2}{a^2+b^2} + \frac{\frac{r_2}{r_1-r_2}\cdot\frac{2ar_1-r_1^2-a^2-b^2}{a^2+b^2}}{1+s/r_1} + \frac{\frac{r_1}{r_1-r_2}\cdot\frac{r_2^2-2ar_2+a^2+b^2}{a^2+b^2}}{1+s/r_2}\,.$$

The inverse Stieltjes transform $F(t)$ of this is

$$\frac{r_1 r_2}{a^2+b^2} + \frac{1}{(r_1-r_2)(a^2+b^2)}\left\{r_1[(r_2-a)^2+b^2][1-e^{-r_2 t}]-r_2[(r_1-a)^2+b^2][1-e^{-r_2 t}]\right\}$$

$$= \frac{r_1 r_2}{a^2+b^2} + \frac{1}{(r_2-r_1)(a^2+b^2)}\left\{r_2[(r_1-a)^2+b^2][1-e^{-r_1 t}]-r_1[(r_2-a)^2+b^2][1-e^{-r_2 t}]\right\}$$

Since $r_2 < r_1$, $e^{-r_1 t} > e^{-r_2 t}$ and thus the term in the braces will be nondecreasing if

$$(r_2-a)^2 \leq (r_1-a)^2$$

Now, if $a \leq r_2 > r_1$, this inequality is satisfied;
if $r_2 > r_1 \geq a$, the inequality is violated; and
if $r_2 > a > r_1$, the inequality is satisfied as long
as $a \geq (r_1 + r_2)/2$. Therefore $a \geq (r_1 + r_2)/2$ assures that F(t) is nondecreasing and, since $F(\infty) = 1$, it is a legitimate CDF.

Real terms are tested as before, so that every factor corresponds to a probability function, and the resulting convolution is also probability function (again with no atom at the origin). Consider the following example. First, let

$$f^{*}(s) = \frac{\left(1+\dfrac{s}{2+2i}\right)\left(1+\dfrac{s}{2+2i}\right)}{(1+s/1)(1+s/2)(1+s/3)}.$$

The inverse of

$$\frac{\left(1+\dfrac{s}{2+i2}\right)\left(1+\dfrac{s}{2-i2}\right)}{\left(1+s/1\right)\left(1+s/2\right)}$$

is

$$F_1(t) = \frac{1}{4} + \frac{5}{4}\,(1-e^{-t}) - \frac{1}{2}\,(1-e^{-2t}).$$

This is indeed a CDF since

$$\frac{\theta_1+\theta_2}{2} = \frac{3}{2} < \mathrm{Re}(\eta_1) = 2.$$

We also see that the inverse of $1/(1+s/3)$ is $F_2(t) = 1 - e^{-3t}$, and it follows that F(t) is a CDF.

As a second illustration, let $F_2(t) = 1 - e^{-t} + e^{-2t} - e^{-3t}$,
with

$$f^*(s) = \frac{1}{1+s} - \frac{2}{2+s} + \frac{3}{3+s}$$

$$= \frac{6 + 6s + 2s^2}{(1+s)(2+s)(3+s)}$$

$$= \frac{\left(1 + \frac{s}{(3+\sqrt{3}\,i)/2}\right)\left(1 + \frac{s}{(3-\sqrt{3}\,i)/2}\right)}{(1+s)(2+s)(3+s)} .$$

Taking

$$\eta_1 = \frac{3+\sqrt{3}\ i}{2}, \quad \eta_2 = \overline{\eta_1}, \quad r_1 = 1, \quad r_2 = 2,$$

we have

$$\frac{r_1+r_2}{2} = \frac{3}{2} = \mathrm{Re}(\eta_1)$$

so that the condition is satisfied and F(t) is therefore a CDF.

References

1. Robert F. Botta and Carl M. Harris (1986). "Approximation with Generalized Hyperexponential Distributions: Weak Convergence Results." *Queueing Systems* 2, 169–190.
2. Robert F. Botta, Carl M. Harris, and William G. Marchal (1987). "Characterizations of Generalized Hyperexponential Distribution Functions." *Comm. Stat. Stoch. Models* 3, 115–148.
3. F. Brockmeyer, H.L. Halstrom and Arne Jensen (1948). "The Life and Works of A.K. Erlang." *Trans. Dan. Acad. Techn. Sci.*, No. 2.
4. David R. Cox (1955). "A Use of Complex Probabilities in the Theory of Stochastic Processes." *Proc. Comb. Phil. Sor.* 51, 313–319.
5. Carl M. Harris and Edward A. Sykes (1987). "Likelihood Estimation for Generalized Mixed Exponential Distributions." *Naval Res. Logistics,* 34, 251–279.
6. Arne Jensen (1954). *A Distribution Model.* Munksgaard, Copenhagen
7. Marcel F. Neuts (1975a). "Probability Distributions of Phase Type" in *Liber Amicorum Prof. Emeritus H. Florin.* Dept. of Mathematics, Univ. of Louvain, Belgium, 173–206.
8. Marcel F. Neuts (1975b). "Computational Uses of the Method of Phases in the Theory of Queues." *Computers and Math. with Appl.*, 1, 151–166.
9. Marcel F. Neuts (1981). *Matrix–Geometric Solutions in Stochastic Models.* Johns Hopkins, Baltimore
10. Teunis J. Ott (1987). "On the Stationary Waiting–Time Distribution in the G/G/1 Queue, I: Transform Methods and Almost–Phase–Type Distributions." *Adv. Appl. Prob.* 19, 240–265.
11. J. George Shanthikumar (1985). "Bilateral Phase–Type Distributions." *Naval Res. Logis. Quart.*, 32, 119–136.
12. Ushio Sumita and Yasushi Masuda (1987). "Classes of Probability Density Functions Having Laplace Transforms with Negative Zeros and Poles. *Adv. Appl. Prob.*, 19, 632–651.
13. A.R. Swensen (1986). "On a GI/M/c Queue with Bounded Waiting Times." *Opus. Res.*, 34, 895–908.

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