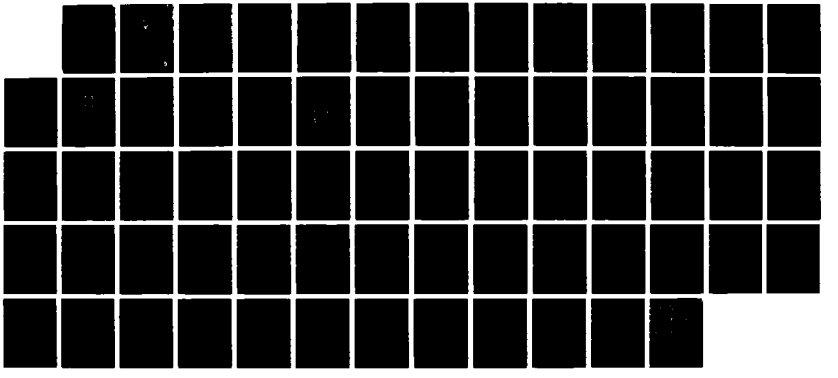
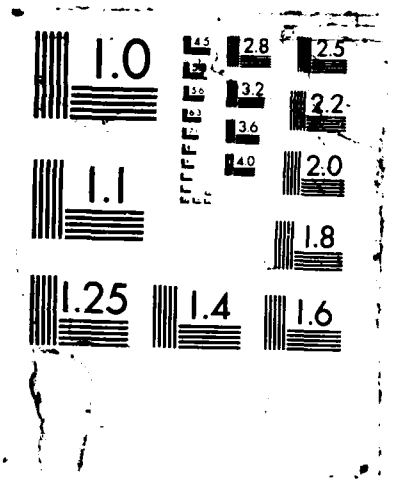
AD-A190 863 BAYESIAN SOLUTIONS TO A 2 X 2 DECISION MATRIX USING INTERVAL SCALED PAYOF..(U) NAVAL POSTGRADUATE SCHOOL MONTEREY CA A R KING SEP 87 1/1

UNCLASSIFIED F/G 12/3 NL




1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.5
2.8
3.2
3.6
4.0

DTIC FILE COPY

AD-A190 863

NAVAL POSTGRADUATE SCHOOL

Monterey, California

THESIS

BAYESIAN SOLUTIONS TO A 2 X 2 DECISION MATRIX USING INTERVAL SCALED PAYOFFS WITH AN APPLICATION TO FOREIGN POLICY DECISIONS

by

Alan R. King

September 1987

Thesis Advisor: Glenn F. Lindsay

Approved for public release; distribution is unlimited

DTIC
ELECTE
MAR 3 1 1988
S D
E

88 3 30 044

A190 863

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a REPORT SECURITY CLASSIFICATION: UNCLASSIFIED	1b RESTRICTIVE MARKINGS
2a SECURITY CLASSIFICATION AUTHORITY	3 DISTRIBUTION/AVAILABILITY OF REPORT: Approved for public release; distribution is unlimited.
2b DECLASSIFICATION/DOWNGRADING SCHEDULE	
4 PERFORMING ORGANIZATION REPORT NUMBER(S)	5 MONITORING ORGANIZATION REPORT NUMBER(S)
6a NAME OF PERFORMING ORGANIZATION: Naval Postgraduate School; 6b OFFICE SYMBOL (If applicable): 55	7a NAME OF MONITORING ORGANIZATION: Naval Postgraduate School
6c ADDRESS (City, State, and ZIP Code): Monterey, California 93943-5000	7b ADDRESS (City, State, and ZIP Code): Monterey, California 93943-5000
8a NAME OF FUNDING/SPONSORING ORGANIZATION; 8b OFFICE SYMBOL (If applicable)	9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
8c ADDRESS (City, State, and ZIP Code)	10 SOURCE OF FUNDING NUMBERS: PROGRAM ELEMENT NO; PROJECT NO; TASK NO; WORK UNIT ACCESSION NO

11 TITLE (Include Security Classification) BAYESIAN SOLUTIONS TO A 2 X 2 DECISION MATRIX USING INTERVAL SCALED PAYOFFS WITH AN APPLICATION TO FOREIGN POLICY DECISIONS

12 PERSONAL AUTHOR(S) KING, Alan R.

13a TYPE OF REPORT	13b TIME COVERED FROM ____ TO ____	14 DATE OF REPORT (Year, Month, Day)	15 PAGE COUNT
Master's Thesis		1987 SEPTEMBER	65

16 SUPPLEMENTARY NOTATION

17 COSATI CODES: FIELD	GROUP	SUB GROUP	18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
			DECISION THEORY, BAYES THEOREM

19 ABSTRACT (Continue on reverse if necessary and identify by block number)

Statistical Decision Theory is applied to assessing objectively the uncertainty involved in foreign policy decisions. In an international conflict of interest situation, a protagonist is presented with the problem of estimating what course of action an antagonist is going to pursue. This problem is addressed by taking observations of the antagonist's behavior. These observations are interpreted as being associated with a specific course of action. Bayes formula is then used to update a conjectured a priori probability function to estimate the course of action being pursued by the antagonist. This updated (a posteriori) conditional probability function is then used to develop a decision rule to select an appropriate response. The decision rule is based on an ordering of possible outcomes, the values assigned to those outcomes, and greatest expected value of return.

20 DISTRIBUTION/AVAILABILITY OF ABSTRACT	21 ABSTRACT SECURITY CLASSIFICATION
☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	UNCLASSIFIED
22a NAME OF RESPONSIBLE INDIVIDUAL: Prof. Glenn F. Lindsay	22b TELEPHONE (Include Area Code): 408-646-2688; 22c OFFICE SYMBOL: 55Ls

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete. SECURITY CLASSIFICATION OF THIS PAGE

Approved for public release; distribution is unlimited.

Bayesian Solutions to a 2 x 2 Decision Matrix Using Interval Scaled Payoffs with an Application to Foreign Policy Decisions

by

Alan R. King
Lieutenant, United States Navy
B.A., University of Hawaii, 1979

Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN OPERATIONS RESEARCH

from the

NAVAL POSTGRADUATE SCHOOL
September 1987

Author: _____
Alan R. King

Approved by: _____
Glenn F. Lindsay, Thesis Advisor

_____
Bard Manstger, Second Reader

_____
Peter Purdue, Chairman,
Department of Operations Research

_____
James M. Fremgen,
Acting Dean of Information and Policy Sciences

ABSTRACT

Statistical Decision Theory is applied to assessing objectively the uncertainty involved in foreign policy decisions. In an international conflict of interest situation, a protagonist is presented with the problem of estimating what course of action an antagonist is going to pursue. This problem is addressed by taking observations of the antagonist's behavior. These observations are interpreted as being associated with a specific course of action. Bayes formula is then used to update a conjectured *a priori* probability function to estimate the course of action being pursued by the antagonist. This updated (*a posteriori*) conditional probability function is then used to develop a decision rule to select an appropriate response. The decision rule is based on an ordering of possible outcomes, the values assigned to those outcomes, and greatest expected value of return. (Th2585),

Accession For	
NTIS GRA&I	☒
DTIC TAB	☐
Unannounced	☐
Justification	
By	
Distribution/	
Availability Codes	
Dist	Avail and/or Special
A-1	

QUALITY INSPECTED

QUALITY INSPECTED 4

TABLE OF CONTENTS

LIST OF FIGURES

I. INTRODUCTION

This thesis models foreign policy decisions using a 2 x 2 decision under risk matrix with interval scaled payoffs. The general application concerns two nations with a conflict of interest situation existing between them. Conflict of interest situations will continue to arise as a result of the ideological differences that exist in the world today. These are part of the overall East vs. West *protracted conflict* as characterized by Strause-Hupe, Kintner, Dougherty and Cottrell in their book *Protracted Conflict*. "The United States in this atmosphere will be repeatedly confronted with the enigma of making a decision in the face of uncertainty." [Ref. 1] Foreign policy decisions involve deciding what course of action is the most appropriate, given the evidence at hand. The courses of action available could be considered either peaceful or aggressive actions. The *evidence at hand* could be the behaviors the antagonist is presenting exhibiting. Are his actions peaceful in the present situation or are they aggressive?

A conflict of interest situation involving the United States and the Soviet Union could be considered as a simple 2 x 2 decision under risk matrix, where the United States, considered as the decision maker, would choose one of two alternatives (peaceful or aggressive action). The Soviet Union could be considered to have these same two courses of action open to it. For the purposes of this thesis, all actions are considered to be one of these two alternatives.

A. CATEGORIZING PEACEFUL OR AGGRESSIVE ACTIONS

Grouping the alternative courses of action into the appropriate category of peaceful or aggressive should be a fairly easy task. For example, during the Berlin crisis, troop and combat aircraft buildups in the vicinity of Berlin by the Soviet Union might be interpreted by the United States as the U.S.S.R. pursuing a *hot-war* (aggressive) course of action. On the other hand, the U.S.S.R.'s permitting American aircraft the use of prescribed air corridors while in-flight to Berlin might be interpreted to mean that the U.S.S.R. is pursuing a *cold-war* (peaceful) course of action.

Historical examples of exercising a *hot-war* course of action by the United States might include:

- the landing and liberation of Grenada by U.S. Armed Forces,
- the sending of Marines into Lebanon,
- and the Libyan bombings.

Examples of *Cold-war* alternatives might include:

- the banning of trade with China after the Communist takeover,
- the SALT Talks,
- and any passive act including the withdrawal of military forces or military aid.

B. A DECISION UNDER RISK

Two elements in the structure of a decision are the set of alternatives available to the decision maker and a set of factors (over which he has no control) influencing the decision outcome. These latter elements are called futures or future states of nature. A decision under *certainty* is one in which we assume that one and only one future state will occur. A decision under *uncertainty* is one in which we are unable to estimate the probabilities of these events. A decision under *risk* is one in which we *can* estimate the probabilities of these future states and more than one of them is greater than zero. [Ref. 2] The relative payoffs for different outcomes in this 2 x 2 decision under risk matrix are not the same for the United States and the Soviet Union. Therefore, this situation cannot be viewed as a two-person zero-sum game. However, the conflict of interest situation could be viewed as a statistical decision problem if the frequency of employment of peaceful (or aggressive) actions by the Soviet Union was regarded as a random state of nature. If the proportion of actions that are peaceful by the U.S.S.R. could be described probabilistically and the advisors in the State Department and the Joint Chiefs of Staff could estimate that distribution, then a good or reasonable estimate could provide the President with a statistical foundation for making a decision. The decision maker, in order to reduce the risks of selecting alternatives, would like to know what proportion of actions by the Soviet Union are peaceful.

The decision chosen for a course of action by the United States could be determined using a decision rule that depended in part the probability distribution that was used to describe the Soviet Union's (or any antagonist's) past behavior. This probability distribution might initially be a subjective one, with which the United States (or any protagonist) conjectured as to best describing the antagonist's behavior. This probability distribution is known as an *a priori* probability distribution.

C. A PRIORI TO A POSTERIORI PROBABILITY DISTRIBUTION

After the establishment of the *a priori* probability distribution, the advisors could observe the conduct of the antagonist and interpret specific events as associated with one of the two possible courses of action. These observations may then be used to

update the previously conjectured probability distribution. Bayes Theorem provides a method for updating the *a priori* probability distribution. In general terms, Bayes Theorem uses an *a priori* probability function and derives an *a posteriori* probability function given the sample values of a random variable. After the sample values are known, the *a posteriori* probability function summarizes the *a priori* and sample information about the proportion of actions, by the antagonist, that are peaceful. Bayes Theorem, thus, improves the assessment of the uncertainty as to which conjectured course of action the antagonist is pursuing. The changed probabilities are known as *a posteriori* probabilities. These become the present estimates, by the protagonist, of the probabilities that the antagonist is using one or the other specific course of action. These *a posteriori* probabilities then become the *a priori* probabilities in further estimates of the antagonist's behavior.

D. DECISION RULES FOR THE PROTAGONIST

Once an *a posteriori* probability distribution has been established, statistical decision theory may then be used to choose objectively an appropriate alternative through the use of a decision rule that is based on the matrix payoffs and greatest expected value of return. This decision rule will assist the decision maker as to which strategy to pursue given any past history of the antagonist's behavior.

The particular ordering of outcomes in terms of preferences selected and the values that the protagonist assigns to these outcomes is an important step in the use of this model. As will be seen later in the general development of the decision rule, this process can be the sole determining factor by which the model proposed here will indicate an aggressive or a peaceful action be taken by the protagonist.

The model in this thesis asks a decision maker to choose an ordering, from the most desirable to the least desirable, of four possible outcomes which show how each course of action for the protagonist (the matrix rows) combines with a course of action for the antagonist (the matrix columns) into four possible outcomes. Then it asks him to quantify, on an interval scale, the desirability of two of these outcomes relative to two standards. Next, a judgement is required stating which of many conjectured *a priori* probability distributions he believes is the best description of the antagonist's behavior. Based on the *a priori* distribution and observation of actual antagonist behavior an updated *a posteriori* probability distribution is established. Given the *a posteriori* probability distribution, which will not be the same for different occurrences of antagonist behavior, a decision rule is derived based on greatest expected value of return.

E. THESIS OUTLINE

Theory and general model development will occur over the next five chapters. Chapter II will discuss the specific categorizations of actions. It will show how these actions are combined into outcomes (or states) in what will be called a conflict situation matrix. Chapter II will also include an explanation of the ranking and quantification of these outcomes. Chapter III will discuss the optimal solutions to the decision problem represented by the conflict situation matrix. Chapter IV will show how to estimate the antagonist's behavior using a conjectured *a priori* probability distribution and Bayes Theorem. Chapters V and VI will show the difference between using a discrete or a continuous *a priori* probability distribution and the decision rules that result. Chapter VII contains specific examples. Chapter VIII contains concluding remarks. We turn now to a discussion of the general two alternative, two states of nature conflict situation matrix that is the basis for the model in this thesis.

II. CONFLICT SITUATION MATRIX

This chapter begins with a definition of the two alternative courses of action available to the protagonist and the antagonist. We will see how these strategies result in four possible outcomes or states, in a conflict situation matrix, to which the decision maker assigns utility values. This chapter will conclude with a discussion of how these assigned utility values relate to each other on an interval scale.

A. ALTERNATIVE COURSES OF ACTION

Imagine a situation involving a conflict of interests between two nations designated as the protagonist (P) and the antagonist (A). The protagonist will be the nation making the decision in the face of uncertainty. The protagonist has a set of two possible courses of action that it can pursue. The set of possible courses of action for the antagonist also consists of two different actions. These courses of action are denoted as P_c or P_h for the protagonist, and A_c or A_h for the antagonist. Let the courses of action for the nations involved have the following meaning:

P_c - cold war, non-aggressive or peaceful action,

P_h - hot war, aggressive or violent action,

A_c - cold war, non-aggressive or peaceful action,

A_h - hot war, aggressive or violent action.

The four possible situations which can then occur as the nations involved exercise their options may be denoted as:

$$(P_c,A_c),\ (P_c,A_h),\ (P_h,A_c) \text{ and } (P_h,A_h).$$

The pair (P_c,A_c) describes the condition of both nations pursuing peaceful actions towards each other. The pair (P_c,A_h) denotes the protagonist displaying either peaceful or non-violent actions towards the aggressor while the antagonist is displaying aggressive or *hot-war* like actions. The pair (P_h,A_c) is the reverse of the previous situation and (P_h,A_h) is both nations acting *hot-war* like or displaying aggressive

actions towards each other. These two alternative courses of action for both the protagonist and the antagonist could be represented as a 2 x 2 matrix. The two columns are the two possible behaviors the antagonist can exhibit, and the two rows would be the two alternative courses of action available to the protagonist. The four cells of this matrix are the four outcomes that can occur. A generalized representation of this conflict situation matrix is shown in Figure 2.1.

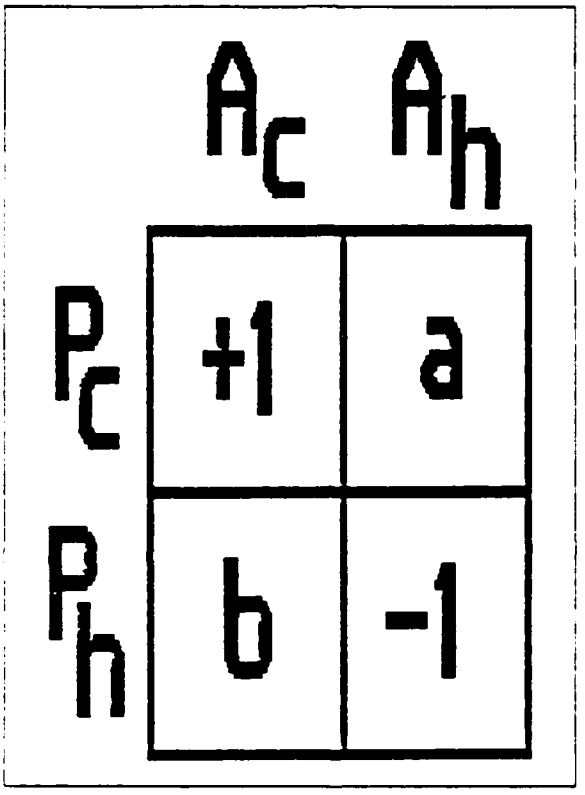


Figure 2.1 Conflict Situation Matrix.

B. RANKING OF OUTCOMES

A decision maker must select an alternative act, course of action or strategy from a recognized set of decision alternatives. The connection between decisions and rankings of outcomes (or preferences) in any decision scenario is made by assuming that preferences are the driving force behind decision choices. That is, it is assumed a decision maker would rather implement a course of action that led to a more preferred outcome over one that led to a less preferred outcome. [Ref. 3]

Suppose for a moment that the preferences chosen, by the protagonist, for the four possible outcomes are represented as

$$V(P_c,A_c) > V(P_h,A_c) > V(P_h,A_h) > V(P_c,A_h),$$

with the meaning that the situation to the left of the symbol " > " is preferred, by the protagonist, to the situation to the right of the symbol, and the notation, V( ), is the value of that outcome. These and all orderings of outcomes, by the protagonist, shall be subject to the general requirements of utility theory in that they are transitive. That is, if A > B and B > C then A > C or if the protagonist is indifferent between A and B and is indifferent between B and C then he is indifferent between A and C. This establishes the notion of coherence in decision theory or transitivity in the laws of mathematics. [Ref. 4: p. 74]

Notice that rankings, other than the one suggested above, could occur. There may be situations, in the eyes of the decision maker, where being at war while his antagonist is not, (P_h,A_c), is preferred to both being at peace, (P_c,A_c), so that $V(P_h,A_c)$ is greater than $V(P_c,A_c)$. Another possibility may arise in the case of a political decision maker looking for justification of aggressive maneuvers against a less capable antagonist. He might prefer the antagonist to take aggressive actions towards him first, (P_c,A_h), to all other outcomes, thus justifying a preference of (P_h,A_h), to all other outcomes later on.

C. UTILITY VALUES FOR THE OUTCOMES

Once an ordering of possible outcomes or preferences has been selected, a quantification of those outcomes needs to be determined by the protagonist in order to work within the framework of this model. As described in *Fundamentals of Decision Analysis* by Irving H. LaValle, [Ref. 4: p. 72] utility values assigned reflect the relative desirabilities of the possible outcomes. A generalized utility value assignment of the outcomes in this model might be indicated as follows:

$$V(P_c,A_c) = + 1.0$$

$$V(P_h,A_h) = - 1.0$$

$$V(P_c,A_h) = \mathbf{a}$$

$$V(P_h,A_c) = \mathbf{b}.$$

Note that this utility scale uses two specific outcomes as references, whereas the usual procedure assigns the value of zero to the least preferred outcome and a value of one to the most preferred outcome. The two representations are merely linear transformations of each other and are equivalent in information content. [Ref. 5] The meaning of the value of + 1.0 attached to $V(P_c,A_c)$ is a standard of preference for a desirable situation -- both nations at peace. The value of - 1.0 attached to $V(P_h,A_h)$ is a standard of preference for an undesirable situation -- both nations at war. In the example considered in the previous section **a** has a value less than - 1.0 and **b** has a value less than + 1.0 and greater than - 1.0. More succinctly, for this example of preferences:

$$V(P_c,A_c) > V(P_h,A_c) > V(P_h,A_h) > V(P_c,A_h),$$

or

$$+ 1.0 \quad > \quad \mathbf{b} \quad > \quad - 1.0 \quad > \quad \mathbf{a.}$$

The assignment of reference values to the outcomes (P_c,A_c) and (P_h,A_h) of + 1.0 and - 1.0 forces the $V(P_c,A_c)$ to *always* be greater than $V(P_h,A_h)$ in this model. This modeling restriction allows 12 possible orderings of outcomes. Which ordering of outcomes the protagonist selects will ultimately affect the alternative course of action he should pursue to gain his greatest expected value of return. The utility values ***a*** and ***b*** can be any arbitrary real numbers for the situations (P_c,A_h) and (P_h,A_c). Each numerical value is the *Utility* to the protagonist of each possible situation. A generalized matrix representation of the conflict situation with associated utility values assigned by the protagonist would appear as previously illustrated in Figure 2.1.

D. INTERVAL SCALING

The utility values of the conflict situation matrix are on an interval scale. Interval scales have no natural origin. There is a zero point on this scale but it is a *defined* point, while a natural origin would indicate the absence of the property being measured. [Ref. 6] If V_i is the utility value of i^{th} outcome, then transformations of the type $Y = m(V_i) - c$ (linear with a positive slope) are allowed. Due to the lack of a natural origin on the interval scale, one value of V_i cannot be said to be a proportion of another value on that scale. When using interval scales, one may only speak of relative *differences*. For example, the decision maker would, after quantification, want

to be able to say that he believes that one outcome, relative to the worst outcome is ρ times as good as another outcome relative to the worse. In other words, if V_i is the value for outcomes i = 1,2,3 or 4 and $V_1 > V_2 > V_3 > V_4$. It could then be stated that:

$$\rho = \frac{V_1 - V_4}{V_2 - V_4}.$$

This would imply that the difference between outcomes V_1 and V_4 is ρ times as good as the difference between outcomes V_2 and V_4. This is the basic information contained in an interval scale.

In this chapter we have introduced the conflict situation matrix (Figure 2.1) showing the possible courses of actions available, their resulting outcomes and the concept of utility values associated with these outcomes. The utility values assigned are rated on an interval scale relative to two standards of preference. In Chapter III we will discuss the optimal solutions to the conflict situation matrix using dominance and greatest expected value of return when the decision is made with knowledge about the proportion of antagonist actions that are peaceful.

III. OPTIMAL SOLUTIONS WITH CERTAINTY ABOUT THE PROPORTION OF ANTAGONIST ACTIONS THAT ARE PEACEFUL

The uncertainty in any decision making process may usually be expressed by saying that we do not know what *event* will occur on some future occasion, where events are defined as subsets of a sample space. [Ref. 7]

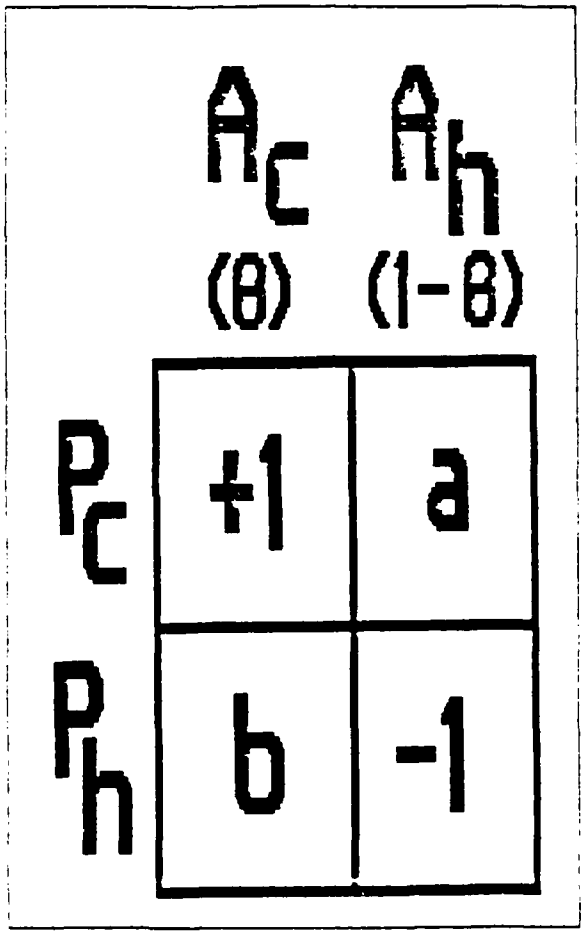


Figure 3.1 Conflict Situation Matrix Under Risk.

In this model, the sample space is the set of values allowed, for the proportion of antagonist actions that are peaceful. An event about which we are informed, as to whether it will occur or not, is called a *certain* event. Otherwise, it is referred to as an *uncertain* event. [Ref. 8: p. 8]

In this chapter we look first at the case where the decision maker is certain about the proportion of actions by the antagonist that are peaceful. In particular we define θ as the proportion of actions by the antagonist that are peaceful. Since the decision maker is certain as to the value of θ, this implies $Pr(\Theta = \theta) = 1.0$, where θ is a realization of the random variable Θ. In the notation used here, A_c denotes a peaceful (*cold-war*) act and A_h (its complement) denotes an aggressive (*hot-war*) act. The proportion of actions that are not peaceful is $(1 - \theta)$, since A_h is the only other act that can occur. The conflict situation matrix under risk is shown in Figure 3.1.

A. DOMINANCE

From the decision matrix it is clear that if **a** is greater than or equal to - 1.0, *and* **b** is less than or equal to + 1.0, then the decision maker will find P_c optimal by dominance regardless of the value of θ. Similarly, when **b** is greater than or equal to + 1.0, *and* **a** is less than or equal to - 1.0, P_h will be optimal, regardless of the value of θ. In each case he will be indifferent when the equalities hold, again, regardless of the value of θ. Regions when the values of **a** and **b** lead to dominance are seen in the shaded regions in Figure 3.2.

Considering that the conflict situation matrix in this model sets the utility values of outcomes (P_h,A_h) and (P_c,A_c) equal to - 1.0 and + 1.0 respectively, we find that there are 12 possible ordering of preferences. Five of these orderings are

$$1)\ V(P_c,A_c) \geq V(P_h,A_c) \geq V(P_c,A_h) \geq V(P_h,A_h)$$

or

$$+ 1.0 \geq \mathbf{b} \geq \mathbf{a} \geq - 1.0,$$

$$2)\ V(P_c,A_h) \geq V(P_c,A_c) > V(P_h,A_h) \geq V(P_h,A_c)$$

or

$$\mathbf{a} \geq + 1.0 > - 1.0 \geq \mathbf{b},$$

$$3)\ V(P_c,A_c) \geq V(P_c,A_h) \geq V(P_h,A_c) \geq V(P_h,A_h)$$

or

$$+ 1.0 \geq \mathbf{a} \geq \mathbf{b} \geq - 1.0,$$

$$4)\ V(P_c,A_c) \geq V(P_c,A_h) \geq V(P_h,A_h) \geq V(P_h,A_c)$$

or

$$+ 1.0 \geq \mathbf{a} \geq - 1.0 \geq \mathbf{b},$$

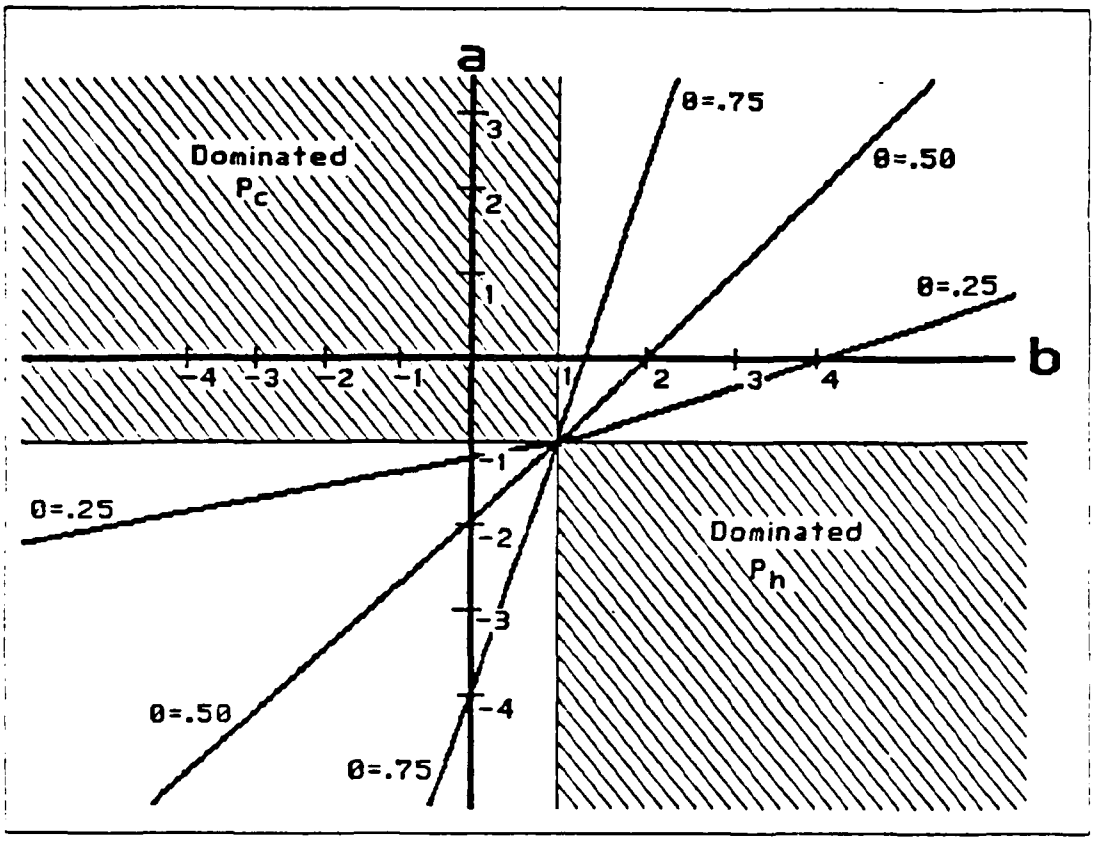


Figure 3.2 Graph of Decision Rule Equation with Dominance Regions.

$$5)\ V(P_c,A_h) \geq V(P_c,A_c) \geq V(P_h,A_c) \geq V(P_h,A_h)$$

or

$$a \geq +1.0 \geq b \geq -1.0.$$

These five preference patterns all have

$$b \leq +1.0 \text{ and } a \geq -1.0.$$

They all will find alternative P_c optimal by dominance.

Only one of the remaining seven preference selections also exhibits the characteristic of dominance. This particular ordering of outcomes is dominated by an

aggressive response, alternative P_h for the protagonist. This preference pattern is

$$6)\ V(P_h,A_c) \geq V(P_c,A_c) > V(P_h,A_h) \geq V(P_c,A_h)$$

or

$$\mathbf{b} \geq +1 > -1 \geq \mathbf{a},$$

and always has

$$\mathbf{b} \geq +1.0 \text{ and } \mathbf{a} \leq -1.0.$$

This preference choice is where the decision maker would prefer 1) acting *hot-war* like while his antagonist was not, to 2) peaceful conditions which would be preferred, to 3) total war which would be preferred, to 4) being peaceful while the antagonist acted aggressively. In this case the decision maker would always find P_h optimal by dominance.

How is the protagonist to decide, when there is no dominance, what course of action is most advantageous to himself, given that the value of θ is a *certainty*? One basis of choice he might use is to select the alternative that gives him the greatest expected value of return.

B. EXPECTED VALUE AS A BASIS FOR CHOICE

The expected value from the conflict situation matrix for alternatives P_c is

$$\theta + \mathbf{a}(1 - \theta)$$

and for alternative P_h, is

$$\mathbf{b}\theta - (1 - \theta).$$

The decision maker would need only select the alternative with the highest expected value, that is, whichever of the two quantities above is the maximum. Since the values from the conflict situation are utility values, the protagonist would be selecting the alternative with the highest expected utility.

A decision rule, for the options presented in this matrix, can be defined as the assignment of a particular course of action for any value of θ, remembering that θ is

the proportion of actions by the antagonist that are peaceful. We can create such a decision rule by setting the two possible decisions' expected value quantities equal to each other. We find, after simplifying and rearranging, that we have the following linear relationship between **a** and **b**,

$$\mathbf{a} = \mathbf{b}(\theta \;\; 1 - \theta) - (1, \; 1 - \theta), \qquad \text{(eqn 3.1)}$$

where $(\theta \;\; 1 - \theta)$ is the slope of the line and $(1 \;\; 1 - \theta)$ is its intercept. This relationship is a family of lines that passes through the point $\mathbf{a} = -1.0$ and $\mathbf{b} = +1.0$ for all values of θ, $0 < \theta < +1.0$. Equation 3.1 is plotted in Figure 3.2 for three values of θ.

C. MAXIMIZING EXPECTED UTILITY IN THE NON-DOMINATED CASES

We now turn to a discussion of the remaining six possible orderings that do not exhibit the characteristic of dominance. This is where the decision maker always rates

$$\mathbf{a} > -1.0 \text{ and } \mathbf{b} > +1.0$$

or

$$\mathbf{a} < -1.0 \text{ and } \mathbf{b} < +1.0.$$

Two examples of these preference patterns are

$$7) \;\; V(P_h,A_c) > V(P_c,A_c) > V(P_c,A_h) > V(P_h,A_h)$$

or

$$\mathbf{b} > +1 > \mathbf{a} > -1,$$

and

$$8) \;\; V(P_c,A_c) > V(P_h,A_c) > V(P_h,A_c) > V(P_c,A_h)$$

or

$$+1 > \mathbf{b} > -1 > \mathbf{a}.$$

The decision maker will find in the non-dominated cases that expected utility will be maximized by choosing P_c if

$$a > b(\theta \cdot 1 - \theta) - (1 \quad 1 - \theta), \qquad \text{(eqn 3.2)}$$

or by choosing P_h if

$$a < b(\theta \cdot 1 - \theta) - (1 \cdot 1 - \theta). \qquad \text{(eqn 3.3)}$$

He will be indifferent when the equality holds (Equation 3.1). Equations 3.2 and 3.3 gives the protagonist a well defined, unambiguous, decision rule that determines what action he should take, given a known value of θ. In Chapter V we will look at how this type of decision rule can be used when the value of θ must be estimated. However, first we look at how to estimate the value of θ using conjectured *a priori* probability distributions and a likelihood function in Bayes Theorem to provide reasonable estimates of θ.

IV. ESTIMATING THE UNCERTAINTY ABOUT THE PROPORTION OF ANTAGONIST ACTIONS THAT ARE PEACEFUL.

Ultimately, the protagonist must choose an action based on his estimate as to which course of action the antagonist is pursuing. In the last chapter we discussed the case where the protagonist's choice would be that course of action which gives him the greatest expected value of return. We also described the case where the proportion of actions by the antagonist that were peaceful was known, or that the value of θ was a *certainty*. What if protagonist has no information or at best limited information on the course of action the antagonist is pursuing? In this chapter we will see how to estimate the antagonist's future behavior using Bayes Theorem.

A. ANTAGONIST ACTIONS AS BERNOULLI TRIALS

One way for the protagonist to gather information in order to estimate θ might be to observe the antagonist's behavior and from these observations estimate what course of action he is pursuing. The two types of actions, (A_c or A_h), will be viewed as two mutually exclusive and exhaustive events, in the sense that every action (or behavior) observed can be categorized as peaceful or aggressive. [Ref. 9] A random variable $X_n = i$ could then be defined, where:

$$i = \begin{cases} 0 \text{ if an aggressive behavior is observed} \\ 1.0 \text{ if a peaceful behavior is observed} \end{cases}$$

and n is the n^{th} behavior observed. These observations of the behaviors can be viewed as the occurrence of a Bernoulli random variable with parameter θ. We would then have:

$$Pr(X_n = i) = (\theta)^i \ (1 - \theta)^{1-i}. \qquad \text{(eqn 4.1)}$$

Our problem considers the case where there is also uncertainty as to the value of the parameter θ. The protagonist doesn't know what course of action or mixture of courses of action will occur. He may, however, be willing to make some subjective

guesses or conjectures as to what values of θ best describe the proportion of actions, by the antagonist, that are peaceful. This would make our Bernoulli trial, the $Pr(X_n = i)$, conditional on that proportion, θ. Therefore, we define the probability that the antagonist displays either peaceful or aggressive behavior, given that the proportion of actions that are peaceful (*cold-war* like) is θ, as:

$$Pr(X_n = i | \theta) = (\theta)^i (1 - \theta)^{1-i}. \qquad \text{(eqn 4.2)}$$

B. A PRIORI PROBABILITY DISTRIBUTIONS ESTIMATE UNCERTAINTY

The initial estimates about the proportion of antagonist's actions that are peaceful, θ, maybe subjective on the part of the protagonist or may come from historic data. In the discrete case, we can generalize these initial estimates as:

$$Pr(\Theta = \theta_r).$$

For example, the protagonist might conjecture that the antagonist acts peacefully only a small proportion of the time and might represent his uncertainy about the value of θ as

$$Pr(\Theta = .99) = .05$$

and

$$Pr(\Theta = .10) = .95.$$

The probability distribution function of Θ has the properties, that the

$$Pr(\Theta = \theta_r) \geq 0 \text{ and } \sum_r Pr(\Theta = \theta_r) = 1.0$$

for r = 1,2,3,...., where θ_r are the possible values for the parameter. Both the discrete and the continuous cases will be considered in this thesis. In the continuous case, the *a priori* probability distribution will be denoted as $f(\theta)$.

For the discrete case only certain values for θ_r in the interval (0,1) are allowed. The interval (0,1) is intentionally displayed as an open interval because attaching probabilities of 0 or 1.0 to uncertain events can lead to intractable conclusions suggested by Cromwell's rule. This will be discussed in detail in the next section on Bayes Theorem.

These probability functions represent the protagonist's *a priori* estimates of θ. Before or without the introduction of more information, these *a priori* estimates of the probability distributions are the only estimates the protagonist has available in this model. However, after the protagonist observes the antagonist's behavior, defined above as $X_n = i$ where $i = 0$ or 1.0, a resulting *a posteriori* probability distribution conditioned on the value of θ is given by Bayes Theorem.

C. BAYES THEOREM

Ultimately we want to show that the conditional probability distribution being considered, the probability that the proportion of actions that are peaceful, or $Pr(\Theta = \theta)$, given his most recently observed behavior is $X_n = i$, where

$X_n = 0$ - occurrence of an aggressive behavior (*hot-war*)
$X_n = 1$ - occurrence of a peaceful behavior (*cold-war*),

is given in this thesis as:

$$Pr(\Theta = \theta_r | X_n = i) = \frac{Pr(\Theta = \theta_r)\, Pr(X_n = i | \theta_r)}{\sum_s Pr(\Theta = \theta_s)\, Pr(X_n = i | \theta_s)}, \quad r = 1,2,3,... \qquad \text{(eqn 4.3)}$$

[Ref. 10] We begin with the definition of conditional probability, which is given as:

$$P(\Theta = \theta_r | X_n = i) = \frac{Pr(\Theta = \theta_r \cap X_n = i)}{P(X_n = i)}.$$

In order to obtain the unconditional distribution of $X_n = i$, (the denominator) we invoke a widely used theory in the study of stochastic processes, that is, the Theorem

of Total Probability, [Ref. 11] which states, that if $Pr(X_n = i \mid \theta_r)$ for $r = 1,2,3,\ldots$ is defined, then we have:

$$Pr(Xn = i) = \sum_r Pr(\Theta = \theta_r) Pr(X_n = i \mid \theta_r).$$

The conditional probability, or in Bayesian terms the likelihood of occurrence $X_n = i$, was defined in the section on Bernoulli trials (Equation 4.2) as

$$Pr(X_n = i \mid \theta_r) = (\theta_r)^i (1 - \theta_r)^{1-i}. \qquad \text{(eqn 4.2)}$$

We can also shown from the definition of conditional probability that:

$$Pr(\Theta = \theta_r \cap X_n = i) = Pr(\Theta = \theta_r) Pr(X_n = i \mid \theta_r).$$

Substituting these into our definition of conditional probability, we have our original expression:

$$Pr(\Theta = \theta_r \mid X_n = i) = \frac{Pr(\Theta = \theta_r) Pr(X_n = i \mid \theta_r)}{\sum_r Pr(\Theta = \theta_r) Pr(X_n = i \mid \theta_r)}, \quad r = 1,2,3,\ldots \qquad \text{(eqn 4.3)}$$

After substituting in our likelihood function, Equation 4.2 we have the general form of the Bayesian conditional *a posteriori* probability function used throughout this thesis:

$$Pr(\Theta = \theta_r \mid X_n = i) = \frac{Pr(\Theta = \theta_r) (\theta_r)^i (1 - \theta_r)^{1-i}}{\sum_r Pr(\Theta = \theta_r) (\theta_r)^i (1 - \theta_r)^{1-i}}, \quad r = 1,2,3,\ldots \qquad \text{(eqn 4.4)}$$

This equation and its continuous case analogue, update the *a priori* probability function into an *a posteriori* probability function that is conditioned on the proportion

of actions that are peaceful by the antagonist. Equation 4.4 is the conditional probability that the antagonist's proportion of peaceful actions is θ_r in the present conflict situation given that he has just recently been observed acting aggressively (for $X_n = 0$). This is the result of the *a priori* probability of the proportion of peaceful actions times the conditional probability that the antagonist acted aggressively in the past, given that his proportion of peaceful actions is θ_r, divided by the sum of all the ways these events can occur.

The random variable X_n will always be considered as a discrete random variable, but the *a priori* probability distributions may be discrete, $Pr(\Theta = \theta_r)$, or *continuous*, $f(\theta)$. For the case where it is continuous, Bayes Theorem is:

$$f(\theta \mid X_n = i) = \frac{f(\theta)\,(\theta)^i\,(1-\theta)^{1-i}}{\int f(\theta)\,(\theta)^i\,(1-\theta)^{1-i}\,d\theta}. \qquad \text{(eqn 4.5)}$$

The role of Bayes Theorem is to establish a connection between the protagonist's *a priori* probability estimates, an expression of his uncertainty that the antagonist will act peacefully, $Pr(\Theta = \theta)$, and the effect that additional information or evidence will be to revise these probabilities. If the protagonist were to obtain perfect information, one of the probabilities would go to + 1.0 so that the $Pr(\Theta = \theta) = +1.0$. This would be the one corresponding to the true distribution of Θ. The remaining probabilities would go to zero. Partial or less than perfect information, like that obtained through the model proposed here, produce a less marked effect on the *a posteriori* probabilities through the use of Bayes Theorem.

A result that follows from Bayes Theorem is that it is inadvisable to attach probabilities of zero to uncertain events. Uncertain events of zero probability remain so, no matter what additional information is provided. Therefore, the decision maker must be aware of the possibility that he may be mistaken, for if he thinks that something cannot be true (e.g., peaceful intentions on the part of his antagonist) and interprets this as having zero probability, his decisions will never be influenced by any contradictory information which is surely absurd. This result is known in the study of Bayesian techniques as Cromwell's Rule. [Ref. 8: p. 104] This same effect holds true, when a probability of 1.0 is assigned, since the assignment of a probability of 1.0 to an uncertain event implies the probability of its complement is zero.

Assuming probabilistic independence of the successive observations, the *a posteriori* probability after any set of observations becomes the *a priori* estimate for the next observation(s). It can also be shown that for any number of observations, the conditional likelihood function in Bayes Theorem is

$$\Pr(X_n = i \mid \theta_r) = (\theta_r)^{(\sum X_n)} (1 - \theta_r)^{(n - \sum X_n)} \qquad \text{(eqn 4.6)}$$

where ***n*** is the total number of behaviors occurring, and $\sum X_n$ is the summation of the random variable $X_n = i$ for i = 0 or 1.0. [Ref. 10] In this thesis we will deal with the behaviors one at a time, with specific examples worked out in the Chapter VII.

In this chapter we introduced the concept of the antagonist's exhibition of peaceful or not peaceful behavior as the realization of a Bernoulli random variable conditioned on the proportion of antagonist actions that are peaceful. This Bernoulli trial, combined with an expert's conjectured *a priori* probability distribution, are used in Bayes Theorem to update this probability distribution into a current estimate of the antagonist's intentions. In the next chapters we will take these *a posteriori* probability distributions for the discrete and the continuous cases and see how they can be used to create decision rules based on maximizing expected utility.

V. DISCRETE CASE DECISION RULES WITH BAYES THEOREM

In this chapter we will consider the discrete case for the *a priori* probability distribution of the proportion of antagonist actions that are peaceful, θ. We will begin by showing a general decision rule for the decision problem represented by the conflict situation matrix. This will be followed by a discussion of how to use *critical values* and *most likely values* as a means of selecting θ. This chapter concludes with a section on using maximum expected utility values considering *uncertain* values of θ. Chapter VI will consider the continuous cases as the *a priori* probability distributions.

A. A GENERALIZED DECISION RULE

The decision rule from Chapter III for maximizing expected utility value, with θ certain, is to choose P_c if

$$a > b(\theta \quad 1 - \theta) - (1 \quad 1 - \theta), \qquad \text{(eqn 3.2)}$$

and P_h if

$$a < b(\theta \quad 1 - \theta) - (1 \quad 1 - \theta). \qquad \text{(eqn 3.3)}$$

Returning to our original definition of expected value for the two alternatives, (P_c or P_h), in the conflict situation matrix we define

$$\theta_r + a(1 - \theta_r) = E(P_c, \theta_r),$$

and

$$b\theta_r - (1 - \theta_r) = E(P_h, \theta_r).$$

Here, $E(P_c, \theta_r)$ is the expected utility value for alternative P_c, given any particular value of θ. Also, the $E(P_h, \theta_r)$ is the expected utility value for alternative P_h, given

any particular value of θ. We can now generalize our decision rule to choose the alternative such that it is

$$\text{Max}_j \; E(P_j, \theta_r) \qquad \text{(eqn 5.1)}$$

where j = c or h for any value of θ_r.

B. CRITICAL AND MOST LIKELY VALUES

The question that still remains, given that the decision maker has selected a preference pattern (and utility values for **a** and **b**), is what value to use for θ_r? The protagonist might select a *critical value* that any estimates of θ_r should equal or exceed, and if he had the option, not consider a value of θ_r until its probability of occurrence equalled or exceeded this stated *critical value*. Suppose the decision maker selected .95 as his *critical value*, so that he would not select an alternative until any particular probability estimate equalled or exceeded this value.[1] After the $\Pr(\Theta = \theta_r)$ equals or exceeds .95 he could substitute that value for θ_r into Equations 3.2 and 3.3, and based on whichever equation holds, choose the corresponding alternative. This changing of probabilities is best demonstrated with an example.

Suppose, that the protagonist feels that the antagonist acts peacefully a small proportion of times and that he acts aggressively a large proportion of times, and that he suggests this *a priori* probability distribution, such that

$$\Pr(\Theta = .9 \mid .1) = .4$$

and

$$\Pr(\Theta = .1 \mid .1) = .6.$$

Now, suppose that the protagonist observes peaceful behavior on the part of the antagonist, or the value of $X_n = 1$. Recalling Bayes Theorem (Equation 4.4) from Chapter IV:

$$\Pr(\Theta = \theta_r \mid X_n = i) = \frac{\Pr(\Theta = \theta_r)(\theta_r)^i (1 - \theta_r)^{1-i}}{\sum_r \Pr(\Theta = \theta_r)(\theta_r)^i (1 - \theta_r)^{1-i}}, \quad r = 1,2,3,\ldots, \qquad \text{(eqn 4.4)}$$

[1] It is conceded here that a third alternative, that of *doing nothing* is *not* included in the conflict situation matrix. The critical value discussion is predicated on the existence of this third alternative.

and using the situation of our example we find

$$\Pr(\Theta = .9 \mid .10 \mid X_n = 1) = \frac{(1 \quad .4)(.9 \quad .10)}{(1 \quad .4)(.9 \quad .10) + (.3 \quad .4)(.1 \quad .4)} = (.18 \quad .33) = .55,$$

and

$$\Pr(\Theta = .1 \mid .4 \mid X_n = 1) = \frac{(.3 \quad .4)(.1 \quad .4)}{(.3 \quad .4)(.1 \quad .4) + (.1 \quad .4)(.9 \quad .10)} = (.15 \quad .33) = .45.$$

As can be seen, none of the values of the *a posteriori* probability distributions exceed our decision makers stated *critical value* of .95. So let's say he observes a second behavior, and this time he observes aggressive behavior by the antagonist, or $X_n = 0$. We again return to Bayes Theorem to update our probability distribution, remembering that the *a priori* distribution now was the *a posteriori* distribution above. We now have

$$\Pr(\Theta = .9 \mid .10 \mid X_n = 0) = \frac{(.18 \; .33)(.1 \quad .10)}{(.18 \; .33)(.1 \quad .10) + (.15 \; .33)(.3 \quad .4)} = .138,$$

and

$$\Pr(\Theta = .1 \mid .4 \mid X_n = 0) = \frac{(.15 \; .33)(.3 \quad .4)}{(.15 \; .33)(.3 \quad .4) + (.18 \; .33)(.1 \quad .10)} = .862.$$

The protagonist might continue taking observations of the antagonist's behavior until one of the probabilities does actually exceed his stated *critical value*. He could then use this value of θ_r in Equations 3.2 and 3.3 to choose his alternative. What if, after one or more observations, the protagonist finds that none of the proposed probability distributions show estimates that exceed the *critical value* (as in the example above) and a decision must be made ***now*** as to which course of action to pursue? This is the case when the conditional *a posteriori* probabilities have not achieved a *sufficient probability* to exceed any stated *critical value*. The protagonist might then use the *most likely value*, or the maximum $\Pr(\Theta = \theta_r \mid X_n = 0)$ and use this value of θ_r in Equations 3.2 and 3.3, to select the alternative with the maximum expected utility. Which, after the

first observation would mean that he would have used $\theta = .90$, since its probability was the maximum, whereas, after the second observation he would have used the value of $\theta = .25$ as its probability of occurrence was the maximum in that instance. The decision maker could also use this criterion to select a value for θ_r before the observation of any antagonist actions. In that case he would have selected $\theta = .25$.

C. MAXIMIZE EXPECTED UTILITY VALUE CONSIDERING ALL PROPOSED PARAMETER VALUES

The procedures just described for selecting θ have a weakness since they do not use all the information available in the problem. If the protagonist is interested in maximizing his expected utility value relevant to *all of his estimates* of an antagonist's behavior, then he would not want to lose any information available to him. This would be the case in selecting the *most likely* and *critical value* criteria. Therefore, he would want to consider all possible parameter values proposed and the relative frequency with which they can occur, and use the expected value of return.[2]

In the discrete case, the protagonist computes the expected value of return over θ_r if he uses course of action P_c, and he computes the expected value of return over these same proposed values of θ_r if he uses course of action P_h. The course of action he uses corresponds to whichever of these two values is the maximum. This maximum value is

$$\text{Max}_j \sum_r E(P_j,\theta_r)\, \Pr(\Theta = \theta_r \mid X_n = i) \qquad \text{(eqn 5.2)}$$

for j = c or h and r = 1,2... This is the Bayesian decision rule relative to the *a posteriori* probability estimates. [Ref. 12]

Returning to our discrete case example above, we have after two observations and assuming that the protagonist had selected the eighth listed preference pattern from Chapter III,

$$V(P_c,A_c) > V(P_h,A_c) > V(P_h,A_h) > V(P_c,A_h),$$

or

$$+1 > b > -1 > a,$$

---

[2]The term expected value of return is not to be interpreted as the value that will be obtained by the protagonist on any single occurrence but rather as the expected value of return due to the uncertainty of the value of θ.

with **b** = 0 and **a** = - 3.0. In accordance with 5.2, we seek the maximum of

$$\sum_r a(1 - \theta_r) \Pr(\Theta = \theta_r \mid X_n = i),$$

or

$$\sum_r (b\theta_r - 1) \Pr(\Theta = \theta_r \mid X_n = i)$$

for both values of θ_r. Thus we have

$$(-3.0)(1\ 10)(.138) + (-3)(3\ 4)(.862) = -1.98,$$

or

$$(0 - 1.0)(.138) + (0 - 1.0)(.862) = -1.0.$$

The protagonist would choose alternative P_h.

This procedure of selecting alternatives P_c or P_h is based on preference pattern selection, utility value assignment, Bayes Theorem and the use of the decision rule, 5.2. This decision rule can be used for any number of proposed values for the parameter θ_r thus allowing the use of all the information available in the problem. In the next chapter we will look at the Bayesian decision rules that result from three different continuous case *a priori* probability distributions.

VI. CONTINUOUS CASE DECISION RULES WITH BAYES THEOREM

Rather than just considering the *a priori* probability distribution to be finite, we would like to consider the case where there are an infinite number of values for θ_r possible. In this case, the general Bayesian decision rule for the continuous case *a priori* probability function is to choose P_j such that

$$\text{Max}_j \int E(P_j,\theta)\, f(\theta \mid X_n = i)\, d\theta \qquad \text{(eqn 6.1)}$$

for j = c or h. There are three particular cases of continuous probability functions considered in this chapter, the first case being that all values in the range of θ are equally likely.

A. DECISION RULES USING A UNIFORM A PRIORI DISTRIBUTION

The uniform case presents a good starting point for the protagonist if he has no information (or no idea) as to what proportion of actions, by the antagonist, are peaceful. The assumption that all possible values of θ are equally likely suggests an unbiased viewpoint, and when combined with Bayes Theorem, allows the distribution to weight future probabilities on observed behaviors alone. This can be seen by the form of the *a posteriori* probability distributions plotted in Figure 6.1. In this section we assume the protagonist uses a uniform *a priori* probability distribution, then we show the two *a posteriori* probability distributions that result after an observation of each type of behavior. From these two updated probability distributions a generalized decision rule, based on the conflict situation matrix, is derived.

The protagonist assumes that all values in the range of θ are equally likely and uses a uniform distribution as his *a priori* distribution. The general form of the uniform distribution is

$$f(\theta) = 1/(d - c), \quad c \le \theta \le d.$$

The limits for θ are 0 and 1.0, so, the general form becomes

$$f(\theta) = 1/(1 - 0),$$

or

$$f(\theta) = 1.0.$$

The initial input into Bayes Theorem is this conjectured *a priori* probability distribution. The protagonist, next, observes the anatagonist's behavior. There are two possibilities for this random variable; $X_n = 0$ or $+ 1.0$. If he observes aggressive behavior then $X_n = 0$ and the *a posteriori* distribution becomes

$$f(\theta \mid X_n = 0) = 2 - 2\theta . \qquad \text{(eqn 6.2)}$$

This result can be seen by beginning with the general form of Bayes theorem stated in Chapter IV as

$$f(\theta \mid X_n = i) = \frac{f(\theta)(\theta)^i (1 - \theta)^{1-i}}{\int_\theta f(\theta)(\theta)^i (1 - \theta)^{1-i} d\theta}, \qquad \text{(eqn 4.5)}$$

where for the uniform case with $i = 0$ becomes

$$f(\theta \mid X_n = 0) = \frac{(1.0)(1 - \theta)^1}{\int_\theta (1.0)(1 - \theta)^1 d\theta},$$

which simplifies to

$$f(\theta \mid X_n = 0) = 2 - 2\theta . \qquad \text{(eqn 6.2)}$$

If the protagonist observes peaceful behavior on the part of the antagonist, then the random variable, $X_n = + 1.0$, and the *a posterior* distribution becomes

$$f(\theta \mid X_n = 1.0) = 2\theta . \qquad \text{(eqn 6.3)}$$

This result can be seen by beginning with the general form of Bayes theorem which with i = + 1.0 becomes

$$f(\theta \mid X_n = 1.0) = \frac{(1.0)(\theta)}{\int_\theta (1.0)(\theta)\, d\theta},$$

which reduces to

$$f(\theta \mid X_n = 1.0) = 2\theta . \qquad \text{(eqn 6.3)}$$

The effect that Bayes Theorem has on the *a priori* distribution can be seen in Figure 6.1, where all three (the *a priori* and the two resulting *a posteriori*) distributions are plotted.

Both of these *a posteriori* probability distributions are legitimate probability distributions since

$$\int_\theta 2\theta = 1.0$$

and

$$\int_\theta 2 - 2\theta = 1.0.$$

1. **Uniform Case Decision Rule for $X_n = 0$**

The *a posteriori* distribution that results after $X_n = 0$ is $f(\theta \mid X_n = 0) = 2 - 2\theta$ (Equation 6.3). The decision rule is to choose P_j that maximizes $\int_\theta E(P_j,\theta) f(\theta)\, d\theta$, in accordance with 6.1, and we have the integral below to evaluate. The decision is to choose alternative P_c if the quantity on the left exceeds the quantity on the right:

$$\int_\theta (\theta - a\theta + a)(2 - 2\theta)\, d\theta > \int_\theta (\theta + b\theta - 1)(2 - 2\theta)\, d\theta.$$

After integration this decision rule, in the case for a uniform *a priori* and the observation of $X_n = 0$, becomes select P_c if

$$a > b/2 - 3/2. \qquad \text{(eqn 6.4)}$$

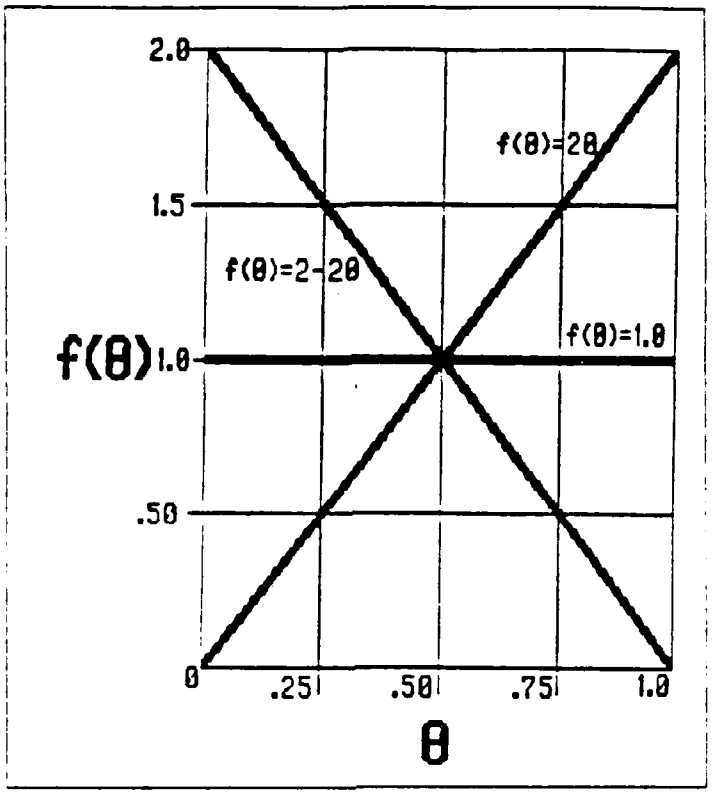


Figure 6.1 Uniform Case - A Priori and A Posteriori Distributions.

or P_h if

$$a < b\,2 - 3.2. \qquad \text{(eqn 6.5)}$$

This result can be seen after evaluation of the two integrals on either side of the inequality. The integral on the left is

$$\int_\theta (\theta - a\theta + a)(2 - 2\theta)\, d\theta ,$$

which equals

$$(-2\theta^2 + 2a\theta^2 - 4a\theta + 2\theta + 2a)\, d\theta ,$$

which reduces to

$$2a\ 3 - 1\ 3.$$

The integral on the right is

$$\int_\theta (\theta - b\theta - 1)(2 - 2\theta)\, d\theta ,$$

which equals

$$(-4\theta - 2b\theta - 2 - 2\theta^2 - 2b\theta^2)\, d\theta ,$$

which reduces to

$$b\ 3 - 2\ 3.$$

Use of these two values from our integrals brings us to our decision rules of choosing P_c if $a > b\ 2 - 3\ 2$ or P_h if $a < b\ 2 - 3\ 2$.

2. Uniform Case Decision Rule with $X_n = 1.0$

The *a posteriori* distribution that results after $X_n = +1.0$ is $f(\theta \mid X_n = 1.0) = 2\theta$ (Equation 6.2). Following from the decision rule, 6.1, we have the integrals below to evaluate. Notice that only the *a posteriori* density function has changed. The decision maker would choose P_c if

$$\int_\theta (\theta - a\theta + a)(2\theta)\, d\theta > \int_\theta (\theta + b\theta - 1)(2\theta)\, d\theta ,$$

which reduces to choosing P_c if

$$a > 2b - 3. \qquad \text{(eqn 6.6)}$$

or P_h *if*

$$a < 2b - 3. \qquad \text{(eqn 6.7)}$$

This result can be seen by comparing the value of the integral on the left to the value of the integral on the right. The integral on the left becomes

$$2\int_\theta (\theta)^2 - a(\theta)^2 + a(\theta)\ d\theta\ ,$$

which equals

$$a\ 3 + 2\ 3.$$

The integral on the right becomes

$$2\int_\theta (\theta)^2 + b(\theta)^2 - (\theta)\ d\theta\ ,$$

which equals

$$2b\ 3 - 1\ 3.$$

These two values reduce to the decision rules of Equations 6.6 and 6.7 to choose *alternative* P_c *if* $a > 2b - 3$ *or* P_h *if* $a < 2b - 3$.

B. DECISION RULES USING A BETA ($\alpha = 2$, $\beta = 2$) A PRIORI DISTRIBUTION

Another conceivable distribution that the decision maker might consider as best describing the antagonist's behavior is a symmetric dome-shaped curve. This curve would have the mean equal to the mode, and be centered at $\theta = .50$. Such a curve would mean that the protagonist feels that most of the probability for θ is at the value .50. This would mean he feels it is most likely that half the actions of the antagonist are aggressive and half of them are peaceful, sloping off towards zero probability at the extreme values of θ. This kind of subjective feel for the *a priori* probability distribution can be captured by a Normal distribution, such that it is $N \sim (.5, \sigma^2)$. However, the range of the Normal distribution is $-\infty$ to $+\infty$ and the range of the parameter, θ, is 0 to + 1.0. A Beta distribution with parameters $\alpha = \beta > 1$ captures the essence of a symmetric dome-shaped curve centered at .50, and it limits itself to the range of our

random variable. The probability density function of a Beta distribution has the form:

$$f(\theta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} (\theta)^{\alpha-1}(1-\theta)^{\beta-1}.$$

For example, when $\alpha = \beta = 2.0$,

$$f(\theta) = \frac{\Gamma(4)}{\Gamma(2)\Gamma(2)} (\theta)(1-\theta),$$

or

$$f(\theta) = 6(\theta)(1 - \theta).$$

When this conjectured *a priori* probability distribution is used in Bayes theorem we get

$$f(\theta \mid X_n = i) = \frac{6(\theta)(1 - \theta)(\theta)^i (1 - \theta)^{1-i}}{6\int_\theta (\theta)(1 - \theta)(\theta)^i (1 - \theta)^{i-1} d\theta},$$

and when the observation of the antagonist's behavior yields $X_n = 0$, this equals

$$\frac{6(\theta)(1 - \theta)^2}{6\int_\theta (\theta)(1 - \theta)^2 d\theta}.$$

The denominator is (6)(Beta(2,3)) density function which equals 6(1/12). The entire expression for the *a posteriori* probability distribution, with a conjectured *a priori* distribution of a Beta (2,2), and a subsequent observation of $X_n = 0$, reduces to

$$f(\theta \mid X_n = 0) = 12(\theta)(1 - \theta)^2. \qquad \text{(eqn 6.8)}$$

If the subsequent observation of antagonist's behavior is peaceful, or $X_n = 1.0$, then our formula from Bayes theorem becomes

$$f(\theta \mid X_n = 1.0) = \frac{6(\theta)^2(1-\theta)}{6\int_\theta (\theta)^2(1-\theta)\, d\theta},$$

where the denominator is (6)(Beta(3,2)) density function which also equals 6(1/12), and we have

$$f(\theta \mid X_n = 1.0) = 12(\theta)^2(1-\theta). \qquad \text{(eqn 6.9)}$$

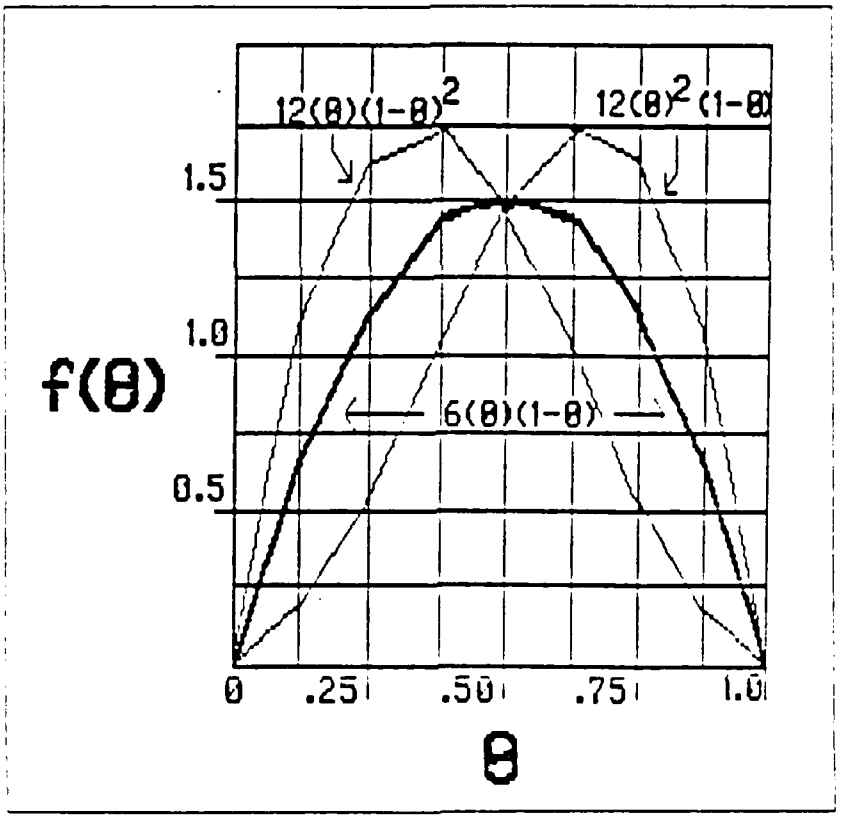


Figure 6.2 Beta(2,2) Case - A Priori and A Posteriori Distributions.

The effect that Bayes Theorem has on the *a priori* distribution can be seen in Figure 6.2 where all three (one *a priori* and two *a posteriori*) distributions for the Beta(2,2) case are plotted.

1. **Beta (2,2) Case Decision Rule with $X_n = 0$**

   The decision, after observation of $X_n = 0$, is to choose alternative P_c if

$$\int_\theta (\theta - a\theta + a)12(\theta)(1-\theta)^2 \, d\theta > \int_\theta (\theta + b\theta - 1)12(\theta)(1-\theta)^2 \, d\theta ,$$

and the decision rule reduces to selecting P_c if

$$a > 2b/3 - 5/3 , \qquad \text{(eqn 6.10)}$$

or P_h if

$$a < 2b/3 - 5/3 . \qquad \text{(eqn 6.11)}$$

This result can be seen by comparing the value of the integral on the left to the one on the right, where the integral on the left is

$$12(1-a)\int_\theta (\theta)^2(1-\theta)^2 \, d\theta + 12a\int_\theta (\theta)(1-\theta)^2 \, d\theta ,$$

and the two terms reduce to

$$12(1-a)\{\Gamma(3)\Gamma(3)/\Gamma(6)\} + 12a\{\Gamma(2)\Gamma(3)/\Gamma(5)\},$$

which equals

$$3a/5 + 2/5.$$

The integral on the right becomes

$$12(1+b)\int_\theta (\theta)^2(1-\theta)^2 \, d\theta - 12\int_\theta (\theta)(1-\theta)^2 \, d\theta,$$

yielding

$$12(1+b)\{\Gamma(3)\Gamma(3)\ \Gamma(6)\} \ - \ 12\{\Gamma(2)\Gamma(3)\ \Gamma(5)\},$$

which equals

$$2b\ 5 - 3\ 5,$$

and the two values reduce to the decision rules of Equations 6.10 and 6.11 to choose P_c if $a > 2b\ 3 - 5\ 3$ or P_h if $a < 2b\ 3 - 5\ 3$.

2. **Beta (2,2) Case Decision Rule with $X_n = 1.0$**

The decision after observation of $X_n = 1.0$ is to choose alternative P_c if

$$\int_\theta (\theta(1-a)+a)12(\theta)^2(1-\theta)\ d\theta \ > \int_\theta (\theta(1+b) - 1)12(\theta)^2(1-\theta)\ d\theta\ ,$$

and the decision rule reduces to selecting P_c if

$$a > 3b\ 2 - 5\ 2\ , \qquad \text{(eqn 6.12)}$$

or P_h if

$$a < 3b\ 2 - 5\ 2\ . \qquad \text{(eqn 6.13)}$$

This result can be seen by comparing the value of the integral on the left to the one on the right, where the integral on the left becomes

$$12(1-a)\int_\theta (\theta)^3(1-\theta)\ d\theta \ + \ 12a\int_\theta (\theta)^2(1-\theta)\ d\theta,$$

which reduces to

$$12(1-a)[\Gamma(4)\Gamma(2)\ \Gamma(6)] \ + \ 12a[\Gamma(3)\Gamma(2)\ \Gamma(5)],$$

which equals

$$2\mathbf{a}\ 5 + 3\ 5.$$

The integral on the right becomes

$$12(1+b)\int_{\theta}(\theta)^3(1-\theta)\,d\theta - 12\int_{\theta}(\theta)^2(1-\theta)\,d\theta.$$

giving

$$12(1+b)[\Gamma(4)\Gamma(2)\ \Gamma(6)] - 12[\Gamma(3)\Gamma(2)\ \Gamma(5)],$$

which equals

$$3\mathbf{b}\ 5 - 2\ 5.$$

These two values reduce to the decision rules, for the Beta(2,2) case and $X_n = 1$, of Equations 6.12 and 6.13 to choose P_c if $\mathbf{a} > 3\mathbf{b}\ 2 - 5\ 2$ or P_h if $\mathbf{a} < 3\mathbf{b}\ 2 - 5\ 2$.

C. DECISION RULES USING A BETA ($\alpha = 1/2$, $\beta = 1/2$) A PRIORI DISTRIBUTION

So far, in the continuous cases, an equally likely *a priori* probability distribution (uniform) and a symmetric dome-shaped *a priori* probability distribution centered at $\theta = .50$ (Beta 2,2) have been considered. What if the protagonist feels that the antagonist is an extremist type of thinker, in the sense that he has just as high a probability of acting peacefully as he does of acting aggressively and a low probability of an even split between the two? This would be the case if most of the probability were weighted at the values of $\theta = 0$ and $\theta = +1.0$ with a very low probability of $\theta = .5$. The essence of this type of thinking could be captured with another Beta distribution with $\alpha = \beta < 1$. This is a *U*-shaped probability density function centered again at $\theta = .50$, but with most of the probability weighted at the extreme values of θ.

Here we will use a Beta function with $\alpha = \beta = 1\ 2$. The probability density function of a Beta (1 2,1 2) distribution has the form:

$$f(\theta) = \frac{\Gamma(1)}{\Gamma(1\ 2)\Gamma(1\ 2)} (\theta)^{-1\ 2} (1-\theta)^{-1\ 2},$$

or

$$f(\theta) = (1\ \pi)(\theta)^{-1\ 2}(1 - \theta)^{-1\ 2},$$

where $\Gamma(1)\ \Gamma(1\ 2)\Gamma(1\ 2) = 1\ (\sqrt{\pi}\sqrt{\pi})$.

When this conjectured *a priori* probability distribution is used in Bayes theorem we get:

$$f(\theta | X_n = i) = \frac{(1\ \pi)(\theta)^{-1\ 2} (1-\theta)^{-1\ 2} (\theta)^{i} (1-\theta)^{i-1}}{\int_\theta (1\ \pi)(\theta)^{-1\ 2} (1-\theta)^{-1\ 2} (\theta)^{i} (1-\theta)^{i-1}\, d\theta}.$$

If the observation of $X_n = 0$, then, after cancelling the common factor of $1\ \pi$, the *a posteriori* distribution becomes

$$f(\theta | X_n = 0) = \frac{(\theta)^{-1\ 2} (1-\theta)^{1\ 2}}{\int_\theta (\theta)^{-1\ 2} (1-\theta)^{1\ 2}\, d\theta},$$

where the denominator is

$$\Gamma(1\ 2)\Gamma(3\ 2) \quad \Gamma(2),$$

which equals

$$\pi\ 2.$$

The *a posteriori* distribution for $X_n = 0$ becomes

$$f(\theta \mid X_n = 0) = (2/\pi)(\theta)^{-1/2}(1-\theta)^{1/2}. \qquad \text{(eqn 6.14)}$$

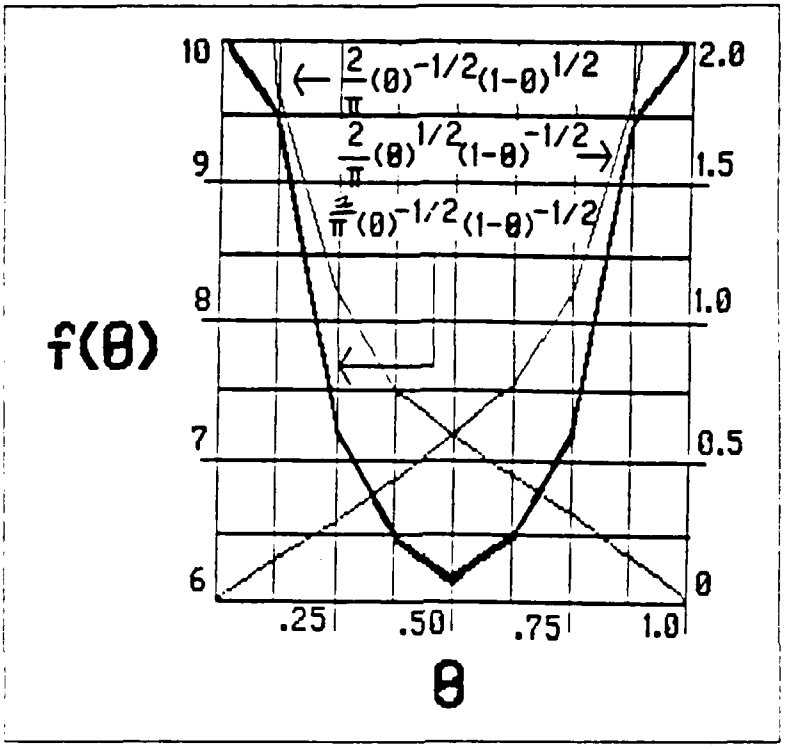


Figure 6.3 Beta (1/2, 1/2) Case - A Priori and A Posteriori Distributions.

In the instance where the observation yields $X_n = 1.0$, Bayes formula gives

$$f(\theta \mid X_n = 1.0) = \frac{(1/\pi)(\theta)^{1/2}(1-\theta)^{-1/2}}{\int_\theta (1/\pi)(\theta)^{1/2}(1-\theta)^{-1/2}\,d\theta}.$$

and after cancelling the common factor of $1/\pi$, the *a posteriori* distribution becomes

$$f(\theta \mid X_n = 1.0) = \frac{(\theta)^{1/2}(1-\theta)^{-1/2}}{\int_\theta (\theta)^{1/2}(1-\theta)^{-1/2}\,d\theta}.$$

where the denominator is

$$\frac{\Gamma(3/2)\Gamma(1/2)}{\Gamma(2)},$$

or

$$\pi/2.$$

The *a posteriori* distribution for $X_n = 1$ becomes

$$f(\theta \mid X_n = 1.0) = (2/\pi)(\theta)^{1/2}(1-\theta)^{-1/2}. \qquad \text{(eqn 6.15)}$$

The effect that Bayes Theorem has on the *a priori* distribution can be seen in Figure 6.3 where all three (one *a priori* and two *a posteriori*) distributions are plotted.

1. **Beta (1/2,1/2) Case Decision Rule with $X_n = 0$**

   The decision after the observation of $X_n = 0$ is to choose alternative P_c if

$$(2/\pi)\int_\theta\{\theta(1-a)+a\}\{(\theta)^{-1/2}(1-\theta)^{1/2}\}\,d\theta > (2/\pi)\int_\theta\{\theta(1+b)-1\}\{(\theta)^{-1/2}(1-\theta)^{1/2}\}\,d\theta,$$

which simplifies to choosing P_c if

$$a > b/3 - 4/3, \qquad \text{(eqn 6.16)}$$

or P_h if

$$a < b/3 - 4/3. \qquad \text{(eqn 6.17)}$$

This result can be seen by noting that the integral on the left is

$$2(1-a)/\pi\int_\theta \theta^{1/2}(1-\theta)^{1/2}d\theta - 2a/\pi\int_\theta \theta^{-1/2}(1-\theta)^{1/2}\,d\theta.$$

yielding

$$[2(1-a)\pi][\Gamma(3/2)\Gamma(3/2)/\Gamma(3)] + [2a\pi][\Gamma(1/2)\Gamma(3/2)/\Gamma(2)],$$

or

$$3a/4 + 1/4.$$

The integral on the right in our original expression is

$$2(1+b)\pi\int_\theta (\theta)^{1/2}(1-\theta)^{1/2}\,d\theta - (2/\pi)\int_\theta (\theta)^{-1/2}(1-\theta)^{1/2}\,d\theta,$$

giving

$$[2(1+b)\pi][\Gamma(3/2)\Gamma(3/2)/\Gamma(3)] - [2/\pi][\Gamma(1/2)\Gamma(3/2)/\Gamma(2)],$$

or

$$b/4 - 3/4.$$

These two values reduce to the decision rules of Equations 6.16 and 6.17 to choose alternative P_c if $a > b/3 - 4/3$ or P_h if $a < b/3 - 4/3$.

2. **Beta (1/2,1/2) Decision Rule with $X_n = 1.0$**

The decision after observation of $X_n = 1.0$ is to choose alternative P_c if

$$(2/\pi)\int_\theta \{\theta(1-a)+a\}\{(\theta)^{1/2}(1-\theta)^{-1/2}\}\,d\theta > (2/\pi)\int_\theta \{\theta(1+b)-1\}\{(\theta)^{1/2}(1-\theta)^{-1/2}\}\,d\theta,$$

which simplifies to choosing P_c if

$$a > 3b - 2, \qquad \text{(eqn 6.18)}$$

or P_h if

$$a < 3b - 2. \qquad \text{(eqn 6.19)}$$

This result can be seen by comparing the value of the integral on the left to the one on the right, where the integral on the left is

$$(2(1-a)\,\pi)\int_\theta (\theta)^{3/2}(1-\theta)^{-1/2}\,d\theta - (2a\,\pi)\int_\theta (\theta)^{1/2}(1-\theta)^{-1/2}\,d\theta ,$$

which evaluates to

$$\{2(1-a)\,\pi\}\{\Gamma(5/2)\Gamma(1/2)\,\Gamma(3)\} - \{2a\,\pi\}\{\Gamma(3/2)\Gamma(1/2)\,\Gamma(2)\},$$

yielding

$$a < 1/4.$$

The integral on the right in our original expression above is

$$(2(1-b)\,\pi)\int_\theta \theta^{3/2}(1-\theta)^{-1/2}\,d\theta - (2\,\pi)\int_\theta \theta^{1/2}(1-\theta)^{-1/2}\,d\theta ,$$

which reduces to

$$[2(1-b)\,\pi]\,[\Gamma(5/2)\Gamma(1/2)\,\Gamma(3)] - [2\,\pi]\,[\Gamma(3/2)\Gamma(1/2)\,\Gamma(2)],$$

yielding

$$3b > 1/4.$$

These two values result in the decision rules of Equations 6.18 and 6.19 to choose P_c if $a > 3b - 2$ or P_h if $a < 3b - 2$.

Depending on his subjective feel for the antagonists intentions, or any historical data that provide a reasonable fit, the decision maker could choose of one of the three continuous *a priori* probability distributions presented in this chapter. The *a*

priori distribution selected and the subsequent behavior observed will determine which decision rule he will follow. A summary of the results of this chapter are presented in Appendix A. In Chapter VII we will show some specific examples for the cases presented here and in Chapter V. Chapter VIII will contain concluding remarks.

VII. SPECIFIC EXAMPLES

In this chapter we will look at specific examples of the conjectured *a priori* probability distributions discussed in Chapters V and VI. There will be four cases for the *a priori* probability distributions (one discrete and three continuous) with eight examples. The discrete case will use four proposed values of θ_r. We will see how the behavior observed, through the use of Bayes Theorem, effects the alternative which maximizes the expected utility value. The three continuous case examples show how an observation of each type of behavior effects the decision rule used and ultimately the alternative (P_c or P_h) chosen. The examples used in this chapter will assume the following preference ordering of:

$$V(P_c,A_c) > V(P_h,A_c) > V(P_h,A_h) > V(P_c,A_h)$$

or

$$+1.0 > \mathbf{b} > -1.0 > \mathbf{a}.$$

We also assume in this chapter that the utility values chosen for $\mathbf{a} = -2.0$ and for $\mathbf{b} = -0.5$.

A. DISCRETE CASE WITH AGGRESSIVE BEHAVIOR OBSERVED

We consider the instance of the following *a priori* probability distribution:

$$\Pr(\Theta = 9/10) = (1/8) = .125$$
$$\Pr(\Theta = 3/4) = (1/4) = .25$$
$$\Pr(\Theta = 1/4) = (1/4) = .25$$
$$\Pr(\Theta = 1/10) = (3/8) = .375.$$

Now, we suppose that the protagonist observes aggressive behavior ($X_n = 0$) on the part of the antagonist. Bayes Theorem gives us for each of the possible values of θ_r that

$$\Pr(\Theta = 9/10 \mid X_n = 0)$$

equals

$$\{(1\ 10)(1\ 8)\} \quad \{(1\ 10)(1\ 8)+(1\ 4)(1\ 4)+(3\ 4)(1\ 4)+(9\ 10)(3\ 8)\} = (1\ 48) = .02.$$

Similarly,

$$\Pr(\Theta = 3\ 4 \mid X_n = 0) = \frac{(1\ 4)(1\ 4)}{(48\ 80)} = (5\ 48) = .10,$$

and

$$\Pr(\Theta = 1\ 4 \mid X_n = 0) = \frac{(3\ 4)(1\ 4)}{(48\ 80)} = (15\ 48) = .31,$$

and

$$\Pr(\Theta = 1\ 10 \mid X_n = 0) = \frac{(9\ 10)(3\ 8)}{(48\ 80)} = (27\ 48) = .56.$$

The course of action to choose in accordance with 5.2 is the maximum of

$$\sum_r a(1 - \theta_r) \Pr(\Theta = \theta_r \mid X_n = i),$$

or

$$\sum_r (b\theta_r - 1) \Pr(\Theta = \theta_r \mid X_n = i),$$

or the maximum of

$$(-2.0)\{(1\ 10)(1\ 48)+(1\ 4)(5\ 48)+(3\ 4)(15\ 48)+(9\ 10)(23\ 48)\} = -1.53$$

or

$$(-.5)(9\ 10)-1](1\ 48)+[(-.5)(3\ 4)-1](5\ 48)+[(-.5)(1\ 4)-1](15\ 48)+[(-.5)(1\ 10)-1](27\ 48)$$

which equals - 1.12. The protagonist in this case would choose P_h, since -1.53 is less than -1.12.

B. DISCRETE CASE WITH PEACEFUL BEHAVIOR OBSERVED

This section uses the same conjectured probability distribution and utility values shown in the beginning of this chapter except with an observation of peaceful behavior, (X_n = 1.0). Bayes Theorem gives us for each of the possible values of θ_r that

$$\Pr(\Theta = 9\ 10 \mid X_n = 1.0)$$

equals

$$[(1\ 8)(9\ 10)] / [(1\ 8)(9\ 10)+(1\ 4)(3\ 4)+(1\ 4)(1\ 4)+(3\ 8)(1\ 10)] = (9\ 32) = .28.$$

Similarly,

$$\Pr(\Theta = 3\ 4 \mid X_n = 1.0) = \frac{(1\ 4)(3\ 4)}{(32\ 80)} = (15\ 32) = .47,$$

and

$$\Pr(\Theta = 1\ 4 \mid X_n = 1.0) = \frac{(1\ 4)(1\ 4)}{(32\ 80)} = (5\ 32) = .16,$$

and

$$\Pr(\Theta = 1\ 10 \mid X_n = 1.0) = \frac{(1\ 10)(1\ 8)}{(32\ 80)} = (3\ 32) = .09.$$

We look for the maximum of

$$(-2.0)\{(1.10)(9.32)+(1.4)(15.32)+(3.4)(5.32)+(9.10)(3.32)\} = -.70$$

or

$$\{(-.5)(9.10)-1\}(9.32)+\{(-.5)(3.4)-1\}(15.32)+\{(-.5)(1.4)-1\}(5.32)+\{(-.5)(1.10)-1\}(3.32) = -1.33.$$

Given the above proposed *a priori* probability distribution and an observation of $X_n = +1.0$, the protagonist would choose alternative P_c, since -.70 is greater than -1.33.

C. UNIFORM CASE WITH AGGRESSIVE BEHAVIOR OBSERVED

In the uniform case if the protagonist observes aggressive behavior then the decision rule (Equation 6.4) is to choose P_c if

$$\mathbf{a} > (1.2)\mathbf{b} - (3.2).$$

Since $a = -2.0$ and $b = -.50$, the the protagonist would choose P_h.

D. UNIFORM CASE WITH PEACEFUL BEHAVIOR OBSERVED

The uniform case with an observation of peaceful behavior ($X_n = 1.0$) uses the decision rule derived in Chapter VI (Equation 6.6). This equation states to choose P_c if

$$\mathbf{a} > 2\mathbf{b} - 3.$$

The protagonist would choose P_c as his choice of action.

E. BETA (2,2) CASE WITH AGGRESSIVE BEHAVIOR OBSERVED

In the case of a Beta (2,2) as the proposed *a priori* probability distribution and an observation of aggressive behavior ($X_n = 0$) the decision rule (Equation 6.10) is to choose P_c if

$$\mathbf{a} > (2/3)\mathbf{b} - (5/3).$$

The protagonist would find that he is indifferent between alternatives P_c and P_h.

F. BETA (2,2) CASE WITH PEACEFUL BEHAVIOR OBSERVED

In the Beta (2,2) case with an observation of peaceful behavior the decision rule (Equation 6.12) is to choose P_c if

$$\mathbf{a} > (3/2)\mathbf{b} - (5/2),$$

In this case, substituting in the values for *a* and *b* will lead the protagonist to select P_c.

G. BETA (1/2,1/2) CASE WITH AGGRESSIVE BEHAVIOR OBSERVED

A Beta (1/2,1/2) with $X_n = 0$ as the observation yields Equation 6.16 as the decision rule. This decision rule states to choose P_c if

$$\mathbf{a} > (1/3)\mathbf{b} - (4/3),$$

leading the protagonist to select P_h as his course of action.

H. BETA (1/2,1/2) CASE WITH PEACEFUL BEHAVIOR OBSERVED

Using a Beta (1/2,1/2) as the proposed *a priori* probability distribution, with an observation of $X_n = 1.0$, gives us the decision rule of Equation 6.18. The values of *a* and *b*, and the rule,

$$\mathbf{a} > 3\mathbf{b} - 2.$$

lead the protagonist to select alternative P_c.

The reader is reminded that while these specific examples appear to be definitive in their conclusions as to which course of action a protagonist should select, they are based on *subjective* assertions or historic data as to which probability distribution best describes the proportion of actions, by the antagonist, that are peaceful. These decision rules are only as good as the distributions they are based on.

VIII. SUMMARY AND SUGGESTIONS FOR FURTHER STUDY

This thesis has proposed a simple 2 x 2 decision under risk matrix as a model for decision making in an uncertain environment. This matrix is referred to as a conflict situation matrix. The 2 x 2 matrix design uses the two columns as two unknown states of nature and the two rows as the decision alternatives for a protagonist. The random states of nature are interpreted as being either the occurrence or non-occurrence of peaceful behavior on the part of an antagonist. The probability of occurrence of either state of nature is estimated through the use of Bayes Theorem.

Utility values, rated on an interval scale, are assigned to the four cells of the matrix. These utility values are used as payoffs in the conflict situation matrix. The estimated probabilities of occurrence of peaceful behavior of the antagonist are used to develop a Bayesian decision rule to maximize expected utility value for the protagonist. The Bayesian decision rules used are relative to the *a posteriori* probability distributions whereas the initial *a priori* probability distribution estimates are subjective or based on historic data. The Bayesian decision rules developed use discrete and continuous *a priori* probability distributions. The decision rule for the discrete case is to choose the alternative P_c or P_h that maximizes

$$\sum_r E(P_j,\theta_r) \Pr(\Theta = \theta_r \mid X_n = i),$$

where j = c or h and r = 1,2,... This decision rule and its continuous case analogue says to select the alternative that maximizes expected utility relative to the *a posteriori* Bayesian estimates. The three continuous case *a priori* probability distributions considered are the uniform, the Beta (2,2) and the Beta (1/2,1/2). The general form of the decision rule for these cases is to select the alternative P_c or P_h that maximizes

$$\int E(P_j,\theta) f(\theta \mid X_n = i)\, d\theta,$$

where j = c or h and θ_r is the continuous *a priori* probability distribution being considered.

The model proposed here provides reasonable decision rules that are *not* counter-intuitive. The idea of categorizing all protagonist and antagonist behaviors (or actions) as either peaceful or aggressive when incorporated with the Bayesian decision rules work well, and make for a usable, easily understood model. This model and its decision rules may also provide a building block for larger models. An area that could provide room for expansion might include a larger conflict situation matrix. If we allowed the protagonist and the antagonist additional alternatives, (e.g., no action taken, as suggested in in the fifth chapter), then we would open up the possibility for an n x m conflict situation matrix.

Another additional feature that wasn't considered here would be to let the decision maker rate all the outcomes in the matrix on an interval scale. This would surely affect the decision rules with the addition of two more variables and would allow all combinations of preference patterns to be considered.

Variance arguments for the *a priori* Beta probability functions, with α equal to β, could also be incorporated. The mean of these density functions is .50 as $\mu = \alpha/\alpha+\beta$. The variance of a Beta distribution is given by $\sigma^2 = \alpha\beta/(\alpha + \beta)^2(\alpha + \beta + 1)$. This fact could be used for the $\alpha = \beta > 1.0$ case, that increasing the values of α and β, while maintaining their equality, decreases the variance of the parameter and leaves the mean unchanged. This causes the distribution to weight more and more probability about the mean and less and less at the extremes. This characteristic of the Beta function could be exploited to reflect a degree of confidence by the decision maker in his *a priori* assumption of a Beta distribution with $\alpha = \beta > 1$. In the $\alpha = \beta < 1.0$ case we would have, as $\alpha = \beta \rightarrow +1.0$, the $\sigma^2 \rightarrow 0$, and this could also be used to reflect the decision makers confidence in his *a priori* assumption of a Beta distribution with $\alpha = \beta < 1.0$. This particular characteristic of Beta distributions with $\alpha = \beta$ could provide an area for further study. Other continuous *a priori* probability distributions, and particularly non-symmetric Beta distributions ($\alpha \neq \beta$), should also provide interesting further research.

The purpose of this model is to analyze the probable future consequences of decision choices. The decision criteria used in the model proposed is highest expected utility value. It is conceded that a simple model, such as this, and its suggested decision criteria cannot capture all the ramifications and consequences of political decisions that befall a decision maker at the national level. While it may be true that models cannot substitute for reality and analysts cannot substitute for decision

makers", [Ref. 13] it is hoped that this parsimonious, quantitative model will be able to provide an unbiased, objective input into the decision making process.

APPENDIX A
SUMMARY OF RESULTS FOR CONTINUOUS CASES

RESULTS WITH UNIFORM A PRIORI

A PRIORI PROBABILITY DISTRIBUTION

$$f(\theta) = 1$$

A POSTERIORI for action $X_n = i$

$X_n = 0$ (*aggressive*)	$X_n = 1$ (*peaceful*)
$2 - 2\theta$	2θ

Choose P_c if

$a > 1.2b - 3.2$	$a > 2b - 3$

Choose P_h if

$a < 1.2b - 3.2$	$a < 2b - 3$

RESULTS WITH BETA (2,2) A PRIORI

A PRIORI PROBABILITY DISTRIBUTION

$$f(\theta) = [\Gamma(4) \quad \Gamma(2)\,\Gamma(2)]\,(\theta)(1-\theta)$$

A POSTERIORI for action $X_n = i$

$X_n = 0$ (*aggressive*)	$X_n = 1$ (*peaceful*)
$12(\theta)(1-\theta)^2$	$12(\theta)^2(1-\theta)$

Choose P_c if

$a > 2\ 3b - 5\ 3$	$a > 3\ 2b - 5\ 2$

Choose P_h if

$a < 2\ 3b - 5\ 3$	$a < 3\ 2b - 5\ 2$

RESULTS WITH BETA (1/2,1/2) A PRIORI

A PRIORI PROBABILITY DISTRIBUTION

$$f(\theta) = [\Gamma(1) \quad \Gamma(1\ 2)\ \Gamma(1\ 2)]\ (\theta)^{-1\ 2}(1-\theta)^{-1\ 2}$$

A POSTERIORI for action $X_n = i$

$X_n = 0$ *(aggressive)*	$X_n = 1$ *(peaceful)*
$2\ \pi(\theta)^{-1\ 2}(1-\theta)^{1\ 2}$	$2\ \pi(\theta)^{1\ 2}(1-\theta)^{-1\ 2}$

Choose P_c if

$a > 1\ 3b - 4\ 3$	$a > 3b - 2$

Choose P_h if

$a < 1\ 3b - 4\ 3$	$a < 3b - 2$

LIST OF REFERENCES

1. Strausz-Hupe, R., Kintner, W.R., Dougherty, J.E., Cottrell, A.J., *Protracted Conflict*, Harper & Brothers, Publishers, 1959.

2. Morris, W.T., *The Analysis of Management Decisions*, Irwin, 1967.

3. Fishburn, P.C., *Utility Theory for Decision Making*, John Wiley & Sons, 1970.

4. LaValle, I.H., *Fundamentals of Decision Analysis*, Holt, Rinehart and Winston, 1978.

5. Savage, L.T., *The Foundations Of Statistics*, Dover Publications Inc., 1972.

6. Torgerson, W.S., *Theory and Methods of Scaling*, John Wiley & Sons, 1958.

7. Miller, I., Freund, J.E., *Probability and Statistics for Engineers*, Prentice-Hall Inc., 1985.

8. Lindley, D.V., *Making Decisions*, John Wiley & Sons, 1985.

9. Aitchison, John, *Choice Against Chance, An Introduction to Statistical Decision Theory*, Addison-Wesley Publishing Co. Inc., 1970.

10. Larson, H.J., *Introduction to Probability Theory and Statistical Inference*, John Wiley & Sons, 1982.

11. Ledermann, W., Chief Editor, *Handbook of Applicable Mathematics Vol. II Probability*, John Wiley & Sons, 1980.

12. Weiss, Lionel, *Statistical Decision Problems*, McGraw-Hill Book Co., 1961.

13. Hughes, W.P., Jr., Editor, *Military Modeling*, The Military Operations Research Society, Inc., 1984.

INITIAL DISTRIBUTION LIST

		No. Copies
1.	Defense Technical Information Center Cameron Station Alexandria, VA 22304-6145	2
2.	Library, Code 0142 Naval Postgraduate School Monterey, CA 93943-5002	2
3.	Professor G.F. Lindsay, Code 55Ls Naval Postgraduate School Monterey, CA 93943	1
4.	Major Bard Mansinger, USA, Code 55Ma Naval Postgraduate School Monterey, CA 93943	1
5.	Chairman Dept. of National Security Affairs, Code 56 Naval Postgraduate School Monterey, CA 93943	1
6.	Captain John A. King, USCGRet 1058 Mokapu Blvd. Kailua, HI 96734	1
7.	Lt. Alan R. King, USN c/o ComFairWestPac F.P.O. Seattle, WA 98767-2700	5

END

DATE

FILMED

5-88

DTIC