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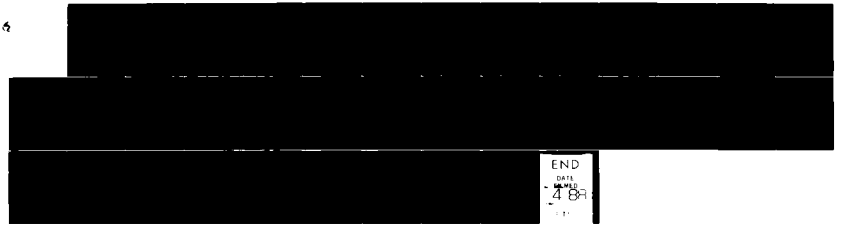

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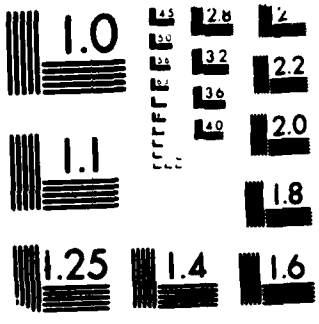

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NORMED BELLMAN EQUATION WITH DEGENERATE DIFFUSION COEFFICIENTS AND ITS APPLICATION TO DIFFERENTIAL EQUATIONS

by

Masatoshi Fujisaki

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NORMED BELLMAN EQUATION WITH DEGENERATE DIFFUSION COEFFICIENTS AND ITS APPLICATION TO DIFFERENTIAL EQUATIONS

by

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Abstract

The purpose of this paper is to prove the existence and uniqueness of generalized solution of the normed Bellman equation with degenerate diffusion coefficients and also to prove that this unique solution is the cost function of a stochastic control problem associated with the normed Bellman equation.

We can apply these results to some interesting degenerate and nonlinear differential equations.

Key words: Normed Bellman equation; degenerate diffusion coefficients; stochastic control problem; generalized solution; normalizing multiplier; cost function.

0. INTRODUCTION

Consider the following Bellman equation with degenerate diffusion coefficients:

$$
(0.1)\qquad \begin{cases} \inf_{\alpha\in A}\{\partial_s v + \frac{1}{2}\sum_{i,j=1}^{\nu} a_{ij}(\alpha,s,x)\partial_i\partial_i v + \sum_{i=1}^{d} b_i(\alpha,s,x)\partial_i v - c(\alpha,s,x)v \\ + f(\alpha,s,x)\} = 0 \qquad (s,x)\in(0,T)\times R^d. \\ v(T,x) = g(x), \quad x\in R^d \end{cases}
$$

where $d \geq 2$, $1 \leq \nu < d$. $\bar{a} = (a_{ij})$, $1 \leq i,j \leq \nu$, is a positive definite matrix, which is written as $\bar{a} = \bar{\sigma}\bar{\sigma}^*$ ($\bar{\sigma}^*$ denotes the transposed matrix of $\bar{\sigma}$) and $\bar{\sigma}$ is a (ν,ν)-matrix, and A is a separable metric space.

If the coefficients a, b, c, and f are bounded with respect to α, then it is known ([3]) that under suitable conditions about regularities and growth there exists a unique generalized solution of Equation (0.1) and furthermore, it is the cost function of a stochastic control problem associated with Equation (0.1).

In the case where the coefficients are not bounded with respect to α, it is necessary to consider a modified form of Equation (0.1), so called, the normed Bellman equation, which was originally considered by N. V. Krylov ([5]). This case is very important from the point of view of application. There are also several results in this case ([3], [4], [5], etc), though they are rather restrictive.

In this paper, we shall extend these results to more general cases, in which are included many interesting examples. We shall also discuss some applications to degenerate nonlinear differential equations.

1. FORMULATION AND PRELIMINARIES

Here we shall consider a stochastic control problem associated with Equation (0.1) and summarize some properties of the cost function. All of the notations follow [3].

Let T be a finite positive number which is fixed. Let A be a separable metric space which is a countable union of non-empty increasing sets A_n: $A = \bigcup_{n\geq 1} A_n$, $A_{n+1} \supset A_n$ (possibly $A_1 = A_2 = \ldots = A$), and we fix this representation. Put $Q_T = (0,T) \times R^d$ and also $\bar{Q}_T = [0,T] \times R^d$. For each $(s,x) \in Q_T$, consider the following stochastic control problem for a system described by stochastic differential equations of the type:

$$(1.1) \qquad \begin{cases} dX_t = b(\alpha_t, s+t, X_t)dt + \sigma(\alpha_t, s+t, X_t)d\beta_t, & 0 < t \leq T-s \\ X_0 = x, \end{cases}$$

where (β_t), $0 \leq t \leq T$, is a d-dimensional Brownian motion process. Assume that the coefficients σ and b satisfy the following conditions:

$$(A.1) \qquad \begin{cases} \sigma(\alpha,t,x): A \times \bar{Q}_T \rightsquigarrow R^d \otimes R^d & (d\times d \text{ - matrix}) \\ b(\alpha,t,x): A \times \bar{Q}_T \rightsquigarrow R^d & (d \text{ - vector}) \end{cases} \quad *$$

We assume that they are continuous with repsect to (α,t,x). Moreover, let there exist a sequence of nonnegative constants $\{k_n\}$, $n=1,2,\ldots$, such that for each n,

$$(1.2) \qquad \| \sigma(\alpha,t,x) - \sigma(\alpha,t,y) \| + |b(\alpha,t,x) - b(\alpha,t,y)| \leq k_n |x-y|,$$

and

$$(1.3) \qquad \| \sigma(\alpha,t,x) \| + |b(\alpha,t,x)| \leq k_n(1 + |x|)$$

for all $\alpha \in A_n$, $0 \leq t \leq T$, $x \in R^d$ and $y \in R^d$ ($\| \cdot \|$ denotes the matrix norm).

---

*We also write $\sigma^\alpha(t,x)$ and $b^\alpha(t,x)$, and so on.

Assume further that for each $\alpha \in A$, σ^α and $b^\alpha \in C^{1,2}(\bar{Q}_T)$, and, in addition, all of their derivatives $\partial_t\sigma^\alpha$, $\partial_i\sigma^\alpha$, $\partial_i\partial_j\sigma^\alpha$, $\partial_t b^\alpha$, $\partial_i b^\alpha$ and $\partial_i\partial_j b^\alpha$, $1 \le i,j \le d$, satisfy the following inequalities:

There exists a nonnegative constant $m \ge 0$ such that for all $n \ge 1$ and $(\alpha,t,x) \in A_n \times \bar{Q}_T$,

$$(1.4)\qquad \sum_{1\le i\le d} \| \partial_i\sigma^\alpha(t,x)\| + \sum_{1\le i,j\le d} \| \partial_i\partial_j\sigma^\alpha(t,x)\| + \sum_{1\le i\le d} \| \partial_i b^\alpha(t,x)\|$$

$$+ \sum_{1\le i,j\le d} \| \partial_i\partial_j b^\alpha(t,x)\| + |\partial_t\sigma^\alpha(t,x)| + |\partial_t b^\alpha(t,x)| \le k_n(1+|x|)^m.$$

Let $1 \le \nu < d$ and let us assume that (d,d) matrix σ in Equation (1.1) is written as follows:

$$(1.5)\qquad \sigma^\alpha(t,x) = \begin{pmatrix} \bar{\sigma}^\alpha(t,x) & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} \} \nu \\ \} d-\nu \end{matrix}$$

$$\qquad\qquad\qquad\qquad\quad \underbrace{\qquad}_{\nu} \quad \underbrace{\qquad}_{d-\nu}$$

where $\bar{\sigma}$ is a (ν,ν) matrix such that for all $(t,x) \in \bar{Q}_T$ and $\xi \in R^\nu$ such that $\|\xi\| = 1$,

$$(1.6)\qquad \sup_{\alpha\in A_1} n^\alpha(t,x)(\xi,\bar{\sigma}^\alpha(t,x)\bar{\sigma}^\alpha(t,x)^*\xi) > 0, \quad \text{where}$$

$$n^\alpha(t,x) = (1 + \mathrm{tr}.a^\alpha(t,x) + |b^\alpha(t,x)| + c^\alpha(t,x) + |f^\alpha(t,x)|)^{-1} \qquad \square$$

The definition of strategy is the same as one given in [3].

Definition 1.1 *Let $n \ge 1$. We write $\alpha \in \mathfrak{A}_n$ if the process $\alpha = (\alpha_t(\omega))$, $0 \le t \le T$, is defined on a probability space $(\Omega,F,P;F_t)$ satisfying the usual conditions, which are progressively measurable with respect to $\{F_t\}$, having values in A_n. Let $\mathfrak{A} = \bigcup_{n\ge1} \mathfrak{A}_n$. The elements of a set $\mathfrak{A}$ are called strategies.*

From the assumption (A.1), it is well-known that for each strategy $\alpha \in \mathfrak{A}$ and $(s,x) \in \bar{Q}_T$, there exists a unique solution of Equation (1.1) and we denote it by $(X_t^{\alpha,s,x})$. Remark that in this case it holds that for each $\alpha \in \mathfrak{A}$, $(s,x) \in [0,T]$,

$$E[\sup_{0\le t\le T-s} |X_t^{\alpha,s,x}|^p] < \infty \tag{1.7}$$

for all $p \ge 1$.

Next, for each $\alpha \in \mathfrak{A}$ and $(s,x) \in \bar{Q}_T$, define a function v^α by the formula:

$$v^\alpha(s,x) = E[\int_0^{T-s} e^{-\phi_t^{\alpha,s,x}} f(\alpha_t, s+t, X_t^{\alpha,s,x})dt + e^{-\phi_{T-s}^{\alpha,s,x}} g(X_{T-s}^{\alpha,s,x})], \tag{1.8}$$

where

$$\phi_t^{\alpha,s,x} = \int_0^t c(\alpha_r, s+r, X_r^{\alpha,s,x})dr, \tag{1.9}$$

and $(X_t^{\alpha,s,x})$ is the solution of Equation (1.1) associated with (α,s,x). Here we assume that the coefficients c, f and g satisfy the following conditions.

$$\text{(A.2)} \quad \begin{cases} c \text{ and } f\colon A \times Q_T \rightsquigarrow R, & c \ge 0, \\ g \quad\quad : R^d \rightsquigarrow R. \end{cases}$$

We assume that all of these functions are bounded from below and also continuous with respect to (α,t,x). Assume further that for each α, c^α, $f^\alpha \in c^{1,2}(\bar{Q}_T)$, and $g \in c^2(R^d)$ and that the foregoing functions and all their derivatives satisfy the following conditions: $\partial_t c$, $\partial_i c (1 \le i \le d)$, $\partial_i \partial_j c (1 \le i,j \le d)$ g, $\partial_i g (1 \le i \le d)$ and $\partial_i \partial_j g$ $(1 \le i,j \le d)$ are uniformly bounded with respect to $(\alpha,t,x) \in A \times \bar{Q}_T$. Moreover, assume that for all $n \ge 1$, $(\alpha,t,x) \in A_n \times \bar{Q}_T$,

$$|f(\alpha,t,x)| + |\partial_t f(\alpha,t,x)| + |f_{(\ell)}(\alpha,t,x)| + |f_{(\ell)(\ell)}(\alpha,t,x)| + c^\alpha(t,x) \le k_n(1+|x|)^m, \tag{1.10}$$

where $f_{(\ell)}$ and $f_{(\ell)(\ell)}$ mean the first and the second derivatives of $f^{\alpha}(t,x)$ along spatial direction $\ell \in R^d$ respectively (see [5] p. 46). □

For each n, define a function v_n by the formula:

$$(1.11) \qquad v_n(s,x) = \inf_{\alpha \in \mathfrak{A}_n} v^{\alpha}(s,x)$$

and also v is defined by

$$(1.12) \qquad v(s,x) = \inf_{\alpha \in \mathfrak{A}} v^{\alpha}(s,x)$$

It is easy to show by the assumptions (A.1) and (A.2) that the functions v_n and v are locally bounded over $\bar{Q}_T$. In fact, it holds that there exist a constant N and a sequence of nonnegative constants k_n', $n \geq 1$, such that for each n

$$(1.13) \qquad N \leq v_n(s,x) \leq k_n'(1 + |x|)^m,$$

and

$$(1.14) \qquad N \leq v(s,x) \leq k_1'(1 + |x|)^m$$

for all $(s,x) \in \bar{Q}_T$, because $v(s,x) \leq v_1(s,x)$ for $\forall\ (s,x) \in \bar{Q}_T$.

For further discussions, let us assume the following conditions about the coefficients:

(A.3) (1.15) $$c^{\alpha}(t,x) \geq 28\| \nabla_x \sigma^{\alpha}(t,x) \|^2 + 8|\nabla_x b^{\alpha}(t,x)| + |\nabla_x f^{\alpha}(t,x)|^2$$

for all $(\alpha,t,x) \in A \times \bar{Q}_T$, where $\nabla_x \sigma^{\alpha} = (\partial_1 \sigma^{\alpha}, \ldots, \partial_d \sigma^{\alpha})$, etc. Moreover, assume that there is a nonnegative function u, belonging to $c^{1,2}(\bar{Q}_T)$, such that

$$(1.16) \qquad |f^{\alpha}(t,x)|^2 + \sum_{i=1}^{d} |\partial_i f^{\alpha}(t,x)|^2 + |\partial_t f^{\alpha}(t,x)|^2$$

$$+ |\partial_t b^\alpha(t,x)|^2 + |\partial_t \sigma^\alpha(t,x)|^2 + \sum_{i,j=1}^{d} |\partial_i \partial_j f^\alpha(t,x)|^2$$

$$+ \sum_{i,j=1}^{d} |\partial_i \partial_j b^\alpha(t,x)|^4 + \sum_{i,j=1}^{d} |\partial_i \partial_j \sigma^\alpha(t,x)|^4 \le -L^{\alpha,\varepsilon} u(t,x),$$

for all $(\alpha,t,x) \in A \times \bar{Q}_T$, $0 \le \varepsilon \le 1$, where $L^{\alpha,\varepsilon}$ is a second order differential operator given by

$$(1.17) \quad L^{\alpha,\varepsilon} u(t,x) = \partial_t u + \frac{1}{2} \sum_{i,j=1}^{\nu} a_{ij}^\alpha(t,x) \partial_i \partial_j u + \varepsilon^2/2 \sum_{i=\nu+1}^{d} \partial_i^2 u$$

$$+ \sum_{i=1}^{d} b_i^\alpha(t,x) \partial_i u - c^\alpha(t,x) u,$$

where $a = \sigma\sigma^*$.

Finally, for technical reasons we assume that there exists a constant η such that for all $\alpha \in A$, $x \in R^d$ and $\varepsilon \in [0,1]$,

$$(1.18) \qquad f^\alpha(T,x) + L^{\alpha,\varepsilon} g(T,x) \ge \eta. \qquad \square$$

Some examples in which the assumptions (A.1), (A.2) and (A.3) can be easily verified will be given in Section 3 later. In Section 3, we shall also discuss another assumption different from the preceding one.

Under the assumptions (A.1), (A.2) and (A.3) in which $\varepsilon = 0$, we have the following result about v_n and v.

Proposition 1.1 *(a)* v_n *is locally bounded in* $\bar{Q}_T$ *uniformly with respect to* n. *(b)* $\{v_n, n \ge 1\}$ *is equicontinuous in each cylinder* $\bar{Q}_{T,R} \equiv [0,T] \times \{x \in R^d; |x| \le R\}$. *(c)* $\lim_{n\to\infty} v_n(s,x) = v(s,x)$ *uniformly in each cylinder* $\bar{Q}_{T,R}$. *(d)* v *is absolutely continuous in* (s,x); *hence there exist first order generalized derivatives with respect to* (s,x), $\partial v/\partial s$ *and* $\partial v/\partial x_i$, $1 \le i \le d$, *and, furthermore, the foregoing derivatives are bounded in each*

$\bar{Q}_{T,R}$. *(e) There exist second order generalized derivatives:* $\partial^2 v/\partial x_i \partial x_j$, $1 \le i,j \le \nu$, *which are also bounded in each* $\bar{Q}_{T,R}$. *(f) It holds that*

$$\inf_{\alpha \in A} F^{\alpha}[v](s,x) \ge 0 \qquad \text{a.e. } (s,x) \in Q_T, \tag{1.19}$$

where $F^{\alpha}[v]$ *is given by the formula:*

$$F^{\alpha}[v](s,x) = \partial_s v + \frac{1}{2} \sum_{i,j=1}^{\nu} a_{ij}^{\alpha}(s,x)\partial_i \partial_j v + \sum_{i=1}^{d} b_i^{\alpha}(s,x)\partial_i v - c^{\alpha}(s,x)v + f^{\alpha}(s,x). \tag{1.20}$$

Proof The method is the same as [3] (Proposition 5.1). (a) is clear from the definition of v_n. Let us prove (b). Since for each $\alpha \in \mathfrak{A}$, $v^{\alpha} \in c^{1,2}(Q_T) \cap c(\bar{Q}_T)$ under the assumptions (A.1) ~ (A.3), we have the following:

$$\begin{aligned} \partial v^{\alpha}(s,x)/\partial x_i = E\Big[& \int_0^{T-s} e^{-\phi_t^{\alpha,s,x}} \sum_{j=1}^{d} \partial_j f(\alpha_t, s+t, X_t^{\alpha,s,x})\, \partial_i X_{t,j}^{\alpha,s,x} dt \\ & + \int_0^{T-s} e^{-\phi_t^{\alpha,s,x}} f(\alpha_t, s+t, X_t^{\alpha,s,x}) \{-\int_0^t \sum_j \partial_j c(\alpha_r, s+r, X_r^{\alpha,s,x}) \partial_i X_{r,j}^{\alpha,s,x} dr\} dt \\ & + \sum_j \partial_j g(X_{T-s}^{\alpha,s,x}) \partial_i X_{T-s,j}^{\alpha,s,x} \times e^{-\phi_{T-s}^{\alpha,s,x}} + \{-g(X_{T-s}^{\alpha,s,x}) e^{-\phi_{T-s}^{\alpha,s,x}} \\ & \times \int_0^{T-s} \sum_j \partial_j c(\alpha_t, s+t, X_t^{\alpha,s,x}) dt\Big] \equiv \sum_{k=1}^{4} I_k^{\alpha,s,x}, \text{ where } \phi_t^{\alpha,s,x} \text{ is given by} \end{aligned} \tag{1.21}$$

$$\phi_t^{\alpha,s,x} = \int_0^t c(\alpha_r, s+r, X_r^{\alpha,s,x}) dr, \quad \text{and, in addition,} \tag{1.22}$$

$$\partial_j f(\alpha,t,x) = \partial f(\alpha,t,x)/\partial x_j, \quad \partial_i X_{t,j}^{\alpha,s,x} = \partial X_{t,j}^{\alpha,s,x}/\partial x_i, \text{ etc.}$$

In order to get the assertion (b), it is sufficient to show that for any $n \ge 1$, for all $(\alpha,s,x) \in \mathfrak{A}_n \times Q_T$, $\partial v^{\alpha}/\partial x_i$ is locally bounded uniformly with respect to (α,n). Using Schwartz's inequality to $I_1^{\alpha,s,x}$, the first term of

the right hand side of (1.21), we have the following inequality:

$$(1.23)\qquad |I_1^\alpha|^2 \le E\Big[\int_0^{T-s} e^{-\phi_t^\alpha} \sum_j |\partial_j f(\alpha_t, s+t, X_t^\alpha)|^2 dt\Big|$$

$$\times E\Big[\int_0^{T-s} e^{-\phi_t^\alpha} \sum_j |\partial_i X_{t,j}^\alpha|^2 dt\Big] \equiv I_5^\alpha \times I_6^\alpha$$

(we omit superindices (s,x) so long as they are fixed).

Now, by means of (A.3), there exists a nonnegative function $u \in c^{1,2}(\bar{Q}_T)$, satisfying (1.16). Put $\tilde{u}(t,x) = u(s+t,x)$, $0 \le t \le T-s$, $(s,x) \in \bar{Q}_T$, then from Ito's formula, we have the following:

$$(1.24)\qquad e^{-\phi_{T-s}^\alpha}\tilde{u}(T-s, X_{T-s}^\alpha) - \tilde{u}(0,x) = \int_0^{T-s} e^{-\phi_t^\alpha}\{\partial_t \tilde{u}(t, X_t^\alpha) +$$

$$\frac{1}{2}\sum_{i,j=1}^{\nu} a_{ij}^\alpha(s+t, X_t^\alpha)\partial_i\partial_j\tilde{u}(t, X_t^\alpha) + \sum_{i=1}^{d} b_i^\alpha(s+t, X_t^\alpha)\partial_i\tilde{u}(t, X_t^\alpha) - c^\alpha(s+t, X_t^\alpha)$$

$$\times \tilde{u}(t, X_t^\alpha)\}dt + M_t, \quad \text{where } M_t \text{ is a square integrable martingale.}$$

Note that the right side of (1.24) is equal to $\int_0^{T-s} e^{-\phi_t^\alpha} L^{\alpha t} u(s+t, X_t^\alpha) \times dt + M_t$, because of the definition of $\tilde{u}$. If we add the quantity $\int_0^{T-s} e^{-\phi_t^\alpha} \cdot \sum_j |\partial_j f^\alpha(s+t, X_t^\alpha)|^2 dt$ to both terms of (1.24), and if we take the mathematical expectation, then, taking account of the inequality (1.16), we have the inequality:

$$(1.25)\qquad I_5^{\alpha,s,x} \le E[-e^{-\phi_{T-s}^{\alpha,s,x}} u(T, X_{T-s}^{\alpha,s,x}) + u(s,x)] \le u(s,x),$$

for all $\alpha \in \mathfrak{A}$, $(s,t) \in Q_T$, since u is nonnegative.

On the other hand, as to I_6, since X_t^α is (LB -) differentiable with respect to x, we have the formula:

$$(1.26)\qquad \partial_i X_{t,j}^\alpha = \delta_{ij} + \int_0^t \sum_k \partial_k b_j^\alpha(s+r, X_r^\alpha)\partial_i X_{r,k}^\alpha dr$$

$$+ \int_0^t \sum_{k,\ell} \partial_\ell \sigma_{jk}^\alpha(s+r, X_r^\alpha)\partial_i X_{r,\ell}^\alpha d\beta_k(r)$$

(see [5], Chapter 2, Section 8). By using again Ito's formula (and also taking the mathematical expectation), we get the following:

$$(1.27)\quad \sum_{i,j} E[e^{-\phi^{\alpha}_t}\{\partial_i X^{\alpha}_{t,j}\}^2] = \delta_{ij} + \sum_{i,j} E[\int_0^t - e^{-\phi^{\alpha}_r}\{\partial_i X^{\alpha}_{r,j}\}^2 c^{\alpha}_r dr$$

$$+ 2\int_0^t e^{-\phi^{\alpha}_r}\partial_i X^{\alpha}_{r,j} \sum_k \partial_k b^{\alpha}_j \cdot \partial_i X^{\alpha}_{r,k} dr + \int_0^t e^{-\phi^{\alpha}_r} \sum_{\ell,k,m} \partial_\ell \sigma^{\alpha}_{jk} \partial_m \sigma^{\alpha}_{jk} \partial_i X^{\alpha}_{r,\ell} \partial_i X^{\alpha}_{r,m} dr]$$

$$\le \tfrac{1}{4} N(1 + E[\int_0^t e^{-\phi^{\alpha}_r}\{-c^{\alpha}_r + 2|\nabla b^{\alpha}(r)| + \|\nabla\sigma^{\alpha}(r)\|^2\}|\nabla X^{\alpha}_r|^2 dr]),$$

where $|\nabla b^{\alpha}(r)|^2 = \sum_{i,j} |\partial_i b^{\alpha}_j(r)|^2$, $\|\nabla\sigma^{\alpha}(r)\|^2 = \sum_{i,j,k} |\partial_i \sigma^{\alpha}_{jk}(r)|^2$

and $|\nabla X^{\alpha}_r|^2 = \sum_{i,j} |\partial_i X^{\alpha}_{r,j}|^2$.

Then, on account of (A.3) (1.15), we have the inequality:

$$(1.28)\quad \sum_{i,j} E[e^{-\phi^{\alpha,s,x}_t}\{\partial_i X^{\alpha,s,x}_{t,j}\}^2] \le \tfrac{1}{4} N, \quad \text{for all } (\alpha,s,x,t) \in \mathfrak{A} \times \bar{Q}_T \times [0,T-s],$$

where N is a nonnegative constant independent of (α,s,x,t). Therefore, from (1.25) and (1.28), we can conclude that there exists a nonnegative function $\tilde{u}$ over $\bar{Q}_T$ such that $\tilde{u} \in c^{1,2}(\bar{Q}_T)$ and for all $n, (\alpha,s,x) \in \mathfrak{A}_n \times Q_T$,

$$(1.29)\quad |I_1^{\alpha,s,x}| \le \tilde{u}(s,x).$$

For the other I_k's (k=2,3 or 4), we can also obtain the same kind of estimates (1.29) for I_1, by using the assumption (A.3). Thus it is shown that there exists a nonnegative function $u' \in c^{1,2}(\bar{Q}_T)$ such that for all $(\alpha,s,x) \in \mathfrak{A} \times Q_T$,

$$(1.30)\quad \sum_{i=1}^{d} |\partial v^{\alpha}(s,x)/\partial x_i| \le u'(s,x).$$

Similarly, it is not hard to see that $\partial v^{\alpha}/\partial s$ is also locally bounded uni-

formly with respect to α by using (A.3) and (1.28). The assertions (c) ~ (f) can be obtained by the same way as [3] (Proposition 5.1). □

2. NORMED BELLMAN EQUATION

As it is well-known (see Section 3 and also [5], 6.3.14, p. 273), generally the inverse relation of (1.19) does not hold if the coefficients are not bounded with respect to α. Therefore, in fact, we need to introduce some auxiliary notations. Let $m^\alpha(t,x)$ be a nonnegative function with respect to $(\alpha,t,x) \in A \times \bar{Q}_T$, and define G^m by the formula;

$$(2.1)\quad G^m(u_0,u_{ij},u_i,u,t,x) = \inf_{\alpha\in A} m^\alpha(t,x)\{u_0 + \frac{1}{2}\sum_{1\le i,j\le\nu} a^\alpha_{ij}(t,x)u_{ij} + \sum_{1\le i\le d} b^\alpha_i(t,x)u_i - c^\alpha(t,x)u + f^\alpha(t,x)\} \qquad (a = \sigma\sigma^*)$$

Definition 2.1 *A nonnegative function $m^\alpha(t,x)$ over $A \times \bar{Q}_T$ is said to be a normalizing multiplier if for all u_0, u_{ij}, u_i, u, t, x,*

$$(2.2)\quad G^m(u_0,u_{ij},u_i,u,t,x) > -\infty.$$

Moreover, the normalizing multiplier $m^\alpha(t,x)$ is called regular if there exists a function $N(t,x) < \infty$ such that for all $(\alpha,t,x) \in A \times \bar{Q}_T$,

$$(2.3)\quad m_0^\alpha(t,x) \le N(t,x)m^\alpha(t,x),$$

where the function m_0^α is given by the formula:

$$(2.4)\quad m_0^\alpha(t,x) = \{1 + \frac{1}{2}\sum_{1\le i,j\le\nu} |a^\alpha_{ij}(t,x)|^2 + \sum_{i=1}^{d} |b^\alpha_i(t,x)|^2 + |c^\alpha(t,x)|^2 + |f^\alpha(t,x)|^2\}^{-\frac{1}{2}}. \qquad \square$$

Let us assume the conditions (A.1) ~ (A.3). Then we have the following main result.

Theorem 2.1 *Let $m^\alpha(t,x)$ be a regular normalizing multiplier. Then it holds that*

(2.5) $\quad G^m[v](t,x) = 0 \quad$ a.e. (Q_T),

where $G^m[v](t,x) = G^m(\partial_t v, \partial_i\partial_j v, \partial_i v, v, t, x)$. □

We call (2.5) the *normed Bellman equation.* For the same reason as [3], Section 5 (see also [5], pp. 269~271), in order to prove (2.5), it is sufficient to show the following:

Lemma 2.2 *If* $m^\alpha(t,x) = m_0^\alpha(t,x)$, *then* (2.5) *is correct.* □

This can be shown by the same way as [3], Section 5, the so-called perturbation method. Since the proof is almost the same as [3], it is sufficient to describe different points from it. Let us start introducing several notations as usual. Let ε be an arbitrary number between 0 and 1, and for each ε, $(\alpha,s,x) \in A \times \overline{Q}_T$, define σ^ε by the formula:

(2.6)
$$\sigma^\varepsilon(\alpha,t,x) = \begin{pmatrix} \overbrace{\bar{\sigma}^\alpha(t,x)}^{\nu} & \overbrace{0}^{d-\nu} \\ 0 & \varepsilon \cdot \\ & \cdot \\ & \cdot \\ & \varepsilon \end{pmatrix} \begin{matrix} \} \nu \\ \} d-\nu \end{matrix}$$

For each $\alpha \in \mathfrak{A}$, $(s,x) \in \overline{Q}_T$, $0 < \varepsilon < 1$, let $(X_t^{\alpha,s,x,\varepsilon})$ be a solution of Equation (1.1) in which σ is replaced by σ^ε, and also let $v^{\alpha,\varepsilon}$ be given by (1.8) in which $(X_t^{\alpha,s,x})$ is replaced by $(X_t^{\alpha,s,x,\varepsilon})$. For each $0 < \varepsilon < 1$ and $n \geq 1$, define v_n^ε and v^ε by the formulas:

(2.7)
$$\begin{cases} v_n^\varepsilon(s,x) = \inf\limits_{\alpha \in \mathfrak{A}_n} v^{\alpha,\varepsilon}(s,x), \quad \text{and} \\ v^\varepsilon(s,x) = \inf\limits_{n \geq 1} v_n^\varepsilon(s,x). \end{cases}$$

Then we have the following:

Proposition 2.3 *(a)* v_n^ε *is uniformly (in* (ε,n)*) bounded and also equicontinuous in* (s,x) *uniformly with respect to* ε *in each cylinder* $\bar{Q}_{T,R}$. *(b) For each* $\varepsilon > 0$, $\lim_{n\to\infty} v_n^\varepsilon(s,x) = v^\varepsilon(s,x)$ *uniformly in each* $\bar{Q}_{T,R}$, *and* v^ε *is continuous in* $(s,x) \in \bar{Q}_T$. *(c) For each* $\varepsilon > 0$, $n \geq 1$, $v_n^\varepsilon \in W_p^{1,2}(Q)$ *and* $v^\varepsilon \in W_p^{1,2}(Q)$ (cf. Section 4) *for any bounded subregion* $Q \subset Q_T$, $p \geq 1$. *Moreover, all their first order generalized derivatives with respect to* s *and* x_i, $1 \leq i \leq d$, *and second order generalized derivatives with respect to* $x_i x_j$, $1 \leq i,j \leq \nu$ *are locally bounded in* Q_T *uniformly with respect to* (ε,n). *(d)* $\lim_{\varepsilon\to 0} v^\varepsilon(s,x) = v(s,x)$, *uniformly in each cylinder* $\bar{Q}_{T,R}$. □

Proof. Since the assertions (a) ~ (e) are the same as [3] (Proposition 5.4) it is sufficient to show (d). Consider the following equality:

$$(2.8)\qquad v^\varepsilon(s,x) - v(s,x) = \{v^\varepsilon(s,x) - v_n^\varepsilon(s,x)\} + \{v_n^\varepsilon(s,x) - v_n(s,x)\}$$
$$+ \{v_n(s,x) - v(s,x)\} \equiv I_1^{n,\varepsilon} + I_2^{n,\varepsilon} + I_3^n.$$

Note that for each $\varepsilon > 0$, $\lim_{n\to\infty} I_1^{n,\varepsilon} = 0$ by means of (b) above and also that $\lim_{n\to\infty} I_3^n = 0$ from (c) of Proposition 1.1. Note also that the convergence is uniform in each cylinder $\bar{Q}_{T,R}$ in both cases. Let us show that $I_2^{n,\varepsilon} \to 0$ as $\varepsilon \to 0$ uniformly with respect to n.

For each $\alpha \in \mathfrak{A}$, $0 < \varepsilon < 1$, $(s,x) \in \bar{Q}_T$, $v^{\alpha,\varepsilon}(s,x)$ is written as follows:

$$(2.9)\qquad v^{\alpha,\varepsilon}(s,x) = E\Big[\int_0^{T-s} e^{-\phi_t^{\alpha,s,x,\varepsilon}} f(\alpha_t, s+t, X_t^{\alpha,s,x,\varepsilon})dt + e^{-\phi_{T-s}^{\alpha,s,x,\varepsilon}} g(X_{T-s}^{\alpha,s,x,\varepsilon})\Big].$$

It is easily shown that for each (α,s,x), $v^{\alpha,\varepsilon}$ is continuously differentiable with respect to ε, and, moreover, $\partial v^{\alpha,\varepsilon}/\partial\varepsilon$ is given by the following: (we omit also the superindices (s,x)) (see [5], Section 2.8).

$$(2.10)\qquad \partial v^{\alpha,\varepsilon}/\partial\varepsilon = E\Big[\int_0^{T-s}\Big\{-\int_0^t \sum_{i=1}^d \partial_i c^{\alpha,\varepsilon}\, \partial X_{r,i}^{\alpha,\varepsilon}/\partial\varepsilon \cdot dr\Big\} e^{-\phi_t^{\alpha,\varepsilon}} f^{\alpha,\varepsilon} dt$$

$$+\int_0^{T-s} e^{-\phi_t^{\alpha,\varepsilon}} \sum_i \partial_i f^{\alpha,\varepsilon} \cdot \partial X_{t,i}^{\alpha,\varepsilon}/\partial\varepsilon \, dt + \{-\int_0^{T-s} \sum_i \partial_i c^{\alpha,\varepsilon} \cdot \partial X_{t,i}^{\alpha,\varepsilon}/\partial\varepsilon \cdot dt\}$$

$$\times e^{-\phi_{T-s}^{\alpha,\varepsilon}} g(X_{T-s}^{\alpha,\varepsilon}) + e^{-\phi_{T-s}^{\alpha,\varepsilon}} \sum_i \partial_i g(X_{T-s}^{\alpha,\varepsilon}) \, \partial X_{T-s,i}^{\alpha,\varepsilon}/\partial\varepsilon]$$

Here, note that for each $\alpha \in \mathfrak{A}$, $(s,x) \in \bar{Q}_T$, $\partial X_i^{\alpha,\varepsilon}(t)/\partial\varepsilon$ denotes the "derivative" (in the sense of N. V. Krylov) with respect to ε and that it satisfies the following formula:

$$(2.11) \quad \partial X_{t,i}^{\alpha,\varepsilon}/\partial\varepsilon = \int_0^t \sum_{1\le j\le d} \partial_j b_i^{\alpha_r}(s+r, X_r^{\alpha,\varepsilon}) \times \partial X_{r,j}^{\alpha,\varepsilon}/\partial\varepsilon \, dr$$

$$+ \int_0^t \sum_{\substack{1\le k\le d \\ 1\le j\le \nu}} \partial_k \bar{\sigma}_{ij}^{\alpha_r}(s+r, X_r^{\alpha,\varepsilon}) \times \partial X_{r,k}^{\alpha,\varepsilon}/\partial\varepsilon \, d\beta_{r,j}$$

$$+ \sum_{i=\nu+1}^{d} \beta_{t,i}.$$

Then, by using the Ito formula, taking account of the condition (A.2), we have the following: there exists a nonnegative constant N, independent of $(\alpha,s,x\ \varepsilon)$, such that

$$(2.12) \quad E[e^{-\phi_t^{\alpha,s,x,\varepsilon}} \sum_{i=1}^{d} |\partial X_{t,i}^{\alpha,s,x,\varepsilon}/\partial\varepsilon|^2] \le N,$$

(see also (1.26) ~ (1.28)).

As we saw in Section 1, since it is not difficult to estimate $\partial v^{\alpha,\varepsilon}/\partial\varepsilon$ by using (A.3) and also the inequality (2.12) just as we proved above, it holds that there exists a nonnegative function u' over $\bar{Q}_T$ such that u' is locally bounded, independent of (α,ε) and

$$(2.13) \quad |\partial v^{\alpha,\varepsilon}(s,x)/\partial\varepsilon| \le u'(s,x)$$

for all $(\alpha,s,x,\varepsilon) \in \mathfrak{A} \times \bar{Q}_T \times (0,1)$. Now it is clear from the above inequality that

$$(2.14)\qquad |v_n^{\varepsilon}(s,x) - v_n(s,x)| = \left|\inf_{\alpha\in\mathfrak{A}_n} v^{\alpha,\varepsilon}(s,x) - \inf_{\alpha\in\mathfrak{A}_n} v^{\alpha}(s,x)\right|$$

$$\le \sup_{\alpha\in\mathfrak{A}_n} |v^{\alpha,\varepsilon}(s,x) - v^{\alpha}(s,x)| \le \sup_{\alpha\in\mathfrak{A}} |v^{\alpha,\varepsilon}(s,x) - v^{\alpha}(s,x)|$$

$$= \sup_{\alpha\in\mathfrak{A}} \left|\int_0^1 \{\partial v^{\alpha,\lambda\varepsilon}(s,x)/\partial\varepsilon\}\varepsilon d\lambda\right| \le \varepsilon \times u'(s,x),$$

which implies that $I_2^{n,\varepsilon} \to 0$ as $\varepsilon \to 0$ uniformly with respect to n (we used Hadamard's theorem). □

Recall the definition (2.1) of G^m, and for each n=1,2,..., denote by G_n^m the right side of (2.1) if we replace A by A_n. In addition to these notations we further need the following one: for each function m and $\varepsilon \in (0,1)$, define $G^{m,\varepsilon}$ by the formula

$$(2.15)\qquad G^{m,\varepsilon}(u_0,u_{ij},u_i,u,s,x) = \inf_{\alpha\in A} m^{\alpha}(s,x)\{u_0 + (\tfrac{1}{2}) \sum_{1\le i,j\le\nu} a_{ij}^{\alpha}(s,x)u_{ij}$$

$$+ (\varepsilon^2/2) \sum_{i=\nu+1}^{d} u_{ii} + \sum_{i=1}^{d} b_i^{\alpha}(s,x)u_i - c^{\alpha}(s,x)u + f^{\alpha}(s,x)\}.$$

Then we can obtain the following, where proof is the same as [3] (Lemma 5.5).

Lemma 2.4 *For each $\varepsilon > 0$ it holds that*

$$(2.16)\qquad G^{m_0,\varepsilon}[v^{\varepsilon}](s,x) \equiv G^{m_0,\varepsilon}(\partial_t v^{\varepsilon},\ \partial_i\partial_j v^{\varepsilon},\ \partial_i v^{\varepsilon},\ v^{\varepsilon},s,x) = 0$$

a.e. (Q_T). □

Let $\varepsilon \to 0$ in (2.16). In order to show Lemma 2.2, we also need the following transformation of variables: let us fix an arbitrary pair (ξ,ε) such that $\xi = (\xi_{\nu+1},\ldots,\xi_d) \in R^{d-\nu}$ and $0 < \varepsilon < 1$, and define new variables $(s,y) \in \bar{Q}_T$ in the following way:

$$(2.17)\qquad \begin{cases} s = s \\ y_i = x_i, \ 1 \le i \le \nu \\ \varepsilon y_i = x_i - \xi_i, \ \nu+1 \le i \le d \end{cases}$$

For any $y \in R^d$, let $\bar{y}$ and $\hat{y}$ denote the first ν and the last $d-\nu$ components of y respectively (similar to $x = (\bar{x}, \hat{x})$). Then the last two expressions of (2.17) are written by $\bar{y} = \bar{x}$ and $\varepsilon\hat{y} = \hat{x} - \xi$ respectively. For each (ξ,ε), define a function $\psi^{\xi,\varepsilon}$ over $\bar{Q}_T$ by

$$(2.18)\qquad \psi^{\xi,\varepsilon}(s,y) = v^{\varepsilon}(s,x)\,(\equiv v^{\varepsilon}(s,\bar{y},\varepsilon\hat{y}+\xi)).$$

If we assume the conditions (A.1) ~ (A.3). it is not hard to show that $\psi^{\xi,\varepsilon}$ has the following properties.

Lemma 2.5 *(a) For each* (ξ,ε), $\psi^{\xi,\varepsilon} \in W^{1,2}_{p,\mathrm{loc}}(Q_T)$ *for any* $p \ge 1$. *(b) The function* $\psi^{\xi,\varepsilon}$ *itself and its generalized derivatives* $\psi^{\xi,\varepsilon}_s$, $\psi^{\xi,\varepsilon}_{y_i}$ $(1 \le i \le d)$, $\psi^{\xi,\varepsilon}_{y_i y_j}$ $(1 \le i,j \le d)$, *are locally bounded in each cylinder* $\bar{Q}_{T,R}$ *uniformly with repsect to* ε. *(c) For any* $(s,y) \in \bar{Q}_T$, $\lim_{\varepsilon\to 0} \psi^{\xi,\varepsilon}(s,y) = v(s,\bar{y},\xi)$ $(y = (\bar{y},\hat{y}) \in R^d)$, *whose convergence is uniform in each cylinder* $\bar{Q}_{T,R}$. □

Since the proof is the same as [3], we omit it here.

Proof of Lemma 2.2 Now, in order to prove Lemma 2.2, it is sufficient to show that $G^{m_0}[v](s,x) \le 0$ a.e., due to Proposition 1.1 (1.19). At first, it is easily seen that Equation (2.16) is equivalent to the following:

$$(2.19)\qquad 0 = \inf_{\alpha\in A} \tilde{m}_0^{\alpha,\xi,\varepsilon}(s,y)\{\psi^{\xi,\varepsilon}_s + \frac{1}{2}\sum_{1\le i,j\le\nu} \tilde{a}^{\alpha,\xi,\varepsilon}(s,y)\psi^{\xi,\varepsilon}_{y_i y_j}$$

$$+ \frac{1}{2}\sum_{i=\nu+1}^{d} \psi^{\xi,\varepsilon}_{y_i y_i} + \tilde{\chi}^{\alpha,\xi,\varepsilon}(s,y) - \tilde{c}^{\alpha,\xi,\varepsilon}(s,y)\psi^{\xi,\varepsilon}\} \quad \text{a.e. } (Q_T),$$

where $\tilde{m}_0^{\alpha,\xi,\varepsilon}$, $\tilde{a}^{\alpha,\xi,\varepsilon}$, $\tilde{c}^{\alpha,\xi,\varepsilon}$ and $\tilde{\chi}^{\alpha,\xi,\varepsilon}$ are given by the following

$(0 \le \varepsilon < 1)$ respectively:

$$(2.20)\qquad \begin{cases} \tilde{m}_0^{\alpha,\xi,\varepsilon}(s,y) = m_0^{\alpha}(s,\bar{y},\varepsilon\hat{y}+\xi), \quad y=(\bar{y},\hat{y}), \\ \tilde{a}^{\alpha,\xi,\varepsilon}(s,y) = a^{\alpha}(s,\bar{y},\varepsilon\hat{y}+\xi), \\ \tilde{c}^{\alpha,\xi,\varepsilon}(s,y) = c^{\alpha}(s,\bar{y},\varepsilon\hat{y}+\xi), \end{cases}$$

and

$$\tilde{\chi}^{\alpha,\xi,\varepsilon}(s,y) = \chi^{\alpha}(s,\bar{y},\varepsilon\hat{y}+\xi),$$

$$(\chi^{\alpha}(s,x) = \sum_i b_i^{\alpha}(s,x)v_{x_i}(s,x) + f^{\alpha}(s,x)).$$

From (2.19) we obtain the following inequality:

$$(2.21)\qquad 0 \ge \tilde{G}^{\tilde{m}_0^{\xi}}[\psi^{\xi,\varepsilon}](s,y) + \sum_{i=1}^{2} \tilde{f}_i^{\xi,\varepsilon}(s,y), \quad \text{a.e.},$$

where $\tilde{G}^{\tilde{m}_0^{\xi}}[\psi^{\xi,\varepsilon}]$ and $\tilde{f}_i^{\xi,\varepsilon}$ $(i=1,2)$ are given by the formulas:

$$(2.22)\qquad \tilde{G}^{\tilde{m}_0^{\xi}}[\psi^{\xi,\varepsilon}](s,y) \equiv \tilde{G}^{\tilde{m}_0^{\xi}}(\psi_s^{\xi,\varepsilon}, \psi_{y_iy_j}^{\xi,\varepsilon}, \psi^{\xi,\varepsilon}, s, y) = \inf_{\alpha\in A} \tilde{m}_0^{\alpha,\xi,0}(s,y)\{\psi_s^{\xi,\varepsilon}$$

$$+ (\tfrac{1}{2}) \sum_{i,j=1}^{\nu} \tilde{a}_{ij}^{\alpha,\xi,0}(s,y)\psi_{y_iy_j}^{\xi,\varepsilon} + (\tfrac{1}{2}) \sum_{i=\nu+1}^{d} \psi_{y_iy_i}^{\xi,\varepsilon} - \tilde{c}^{\alpha,\xi,0}(s,y)\psi^{\xi,\varepsilon} + \tilde{\chi}^{\alpha,\xi,0}(s,y)\},$$

$$(2.23)\qquad \tilde{f}_1^{\xi,\varepsilon}(s,y) = \inf_{\alpha\in A}[\tilde{m}_0^{\alpha,\xi,\varepsilon}(s,y)\{\psi_s^{\xi,\varepsilon} + \frac{1}{2} \sum_{i,j=1}^{\nu} \tilde{a}_{ij}^{\alpha,\xi,\varepsilon}(s,y)\psi_{y_iy_j}^{\xi,\varepsilon}$$

$$+ \frac{1}{2} \sum_{i=\nu+1}^{d} \psi_{y_iy_i}^{\xi,\varepsilon} - \tilde{c}^{\alpha,\xi,\varepsilon}(s,y)\psi^{\xi,\varepsilon}\} - \tilde{m}_0^{\alpha,\xi,0}(s,y)\{\psi_s^{\xi,\varepsilon}$$

$$+ \frac{1}{2} \sum_{i,j=1}^{\nu} \tilde{a}_{ij}^{\alpha,\xi,0}(s,y)\psi_{y_iy_j}^{\xi,\varepsilon} + \frac{1}{2} \sum_{\nu+1}^{d} \psi_{y_iy_i}^{\xi,\varepsilon} - \tilde{c}^{\alpha,\xi,0}(s,y)\psi^{\xi,\varepsilon}\}],$$

and

$$(2.24)\qquad \tilde{f}_2^{\xi,\varepsilon}(s,y) = \inf_{\alpha\in A}\{\tilde{m}_0^{\alpha,\xi,\varepsilon}(s,y)\tilde{\chi}^{\alpha,\xi,\varepsilon}(s,y) - \tilde{m}_0^{\alpha,\xi,0}(s,y)\tilde{\chi}^{\alpha,\xi,0}(s,y)\}.$$

Let $\{\varepsilon_n\}$, n=1,2,..., be a sequence of arbitrary positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$, then by the same way as [3] (see the proof of Lemma 5.3) we can show that for a.e. $\xi \in R^{d-\nu}$, $\lim_{n\to\infty} \tilde{f}_i^{\xi,\varepsilon_n}(s,y) = 0$ (i=1,2) a.e. in each cylinder $\overline{Q}_{T,R}$. In fact, in this case it is sufficient to check, for example, the following inequality: there exists a constant $N_R(\xi)$ which depends upon (R,ξ) such that

$$(2.25)\quad |a_{ij}^{\alpha}(s,\bar{y},\varepsilon\hat{y}+\xi) - a_{ij}^{\alpha}(s,\bar{y},\xi)| \le \varepsilon N_R(\xi)\{c^{\alpha}(s,\bar{y},\varepsilon\hat{y}+\xi) + 1\}, \text{ for all}$$

$$\alpha \in A,\ (s,y) \in \overline{Q}_{T,R},\ \xi \in R^{d-\nu} \text{ and } 0 \le \varepsilon < 1,$$

because of the assumptions (A.2) and (A.3) (in fact, (1.15)). We can also make similar estimates for b, c and f as in (2.25). On account of the above estimates and the fact that the equality $\lim_{n\to\infty} |\nabla v^{\varepsilon_n}(s,\bar{y},\varepsilon_n\hat{y}+\xi) - \nabla v(s,\bar{y},\xi)| = 0$ still holds in this case, it is shown that $\lim_{n\to\infty} \tilde{f}_i^{\xi,\varepsilon_n}(s,y) = 0$ (i=1,2), whose proof is routine and thus omitted here.

Finally, for the same reason as [3], we can obtain the following inequality: for a.e. ξ,

$$(2.26)\quad 0 \ge \overline{\lim_{n\to\infty}}\, \tilde{G}^{\tilde{m}_0^{\xi,\varepsilon_n}}[\psi^{\xi,\varepsilon}](s,y) \ge \tilde{G}^{\tilde{m}_0^{\xi}}[\psi^{\xi}](s,y) \quad \text{a.e. } (s,y),$$

where $\psi^{\xi}(s,y) \equiv v(s,\bar{y},\xi)$.

But, since $\tilde{G}^{\tilde{m}_0^{\xi}}[\psi^{\xi}](s,y) = G^{m_0}[v](s,\bar{y},\xi)$, (2.26) implies the desired relation:

$$(2.27)\qquad 0 \ge G^{m_0}[v](s,x) \quad \text{a.e. } (Q_T). \qquad \square$$

3. EXAMPLES AND APPLICATIONS

3.1. Here we shall discuss another assumption different from (A.1), (A.2) and (A.3) in Section 1. Roughly speaking, the latter is the most general one in the case in which all of the coefficients are unbounded relative to the parameter α. We can, however, relax the assumptions with respect to the coefficients f and g if the bounds of σ and b are independent of α, which is often important from the point of view of applications. Let us consider the following example.

Let us assume that σ and b satisfy the following conditions:

(A.1)' . σ: $A \rightsquigarrow R^d \otimes R^d$

. b: $A \times \bar{Q}_T \rightsquigarrow R^d$

. σ is a continuous ft. in A and b is also continuous in (α,t,x). b *is uniform Lipshitz* continuous with respect to x, i.e. there exists a nonnegative constant k such that

$$|b(\alpha,t,x) - b(\alpha,t,x')| \leq k|x - x'| \tag{3.1}$$

for all $(t,x,x',\alpha) \in [0,T] \times R^{2d} \times A$, and also

$$\| \sigma(\alpha) \| + |b(\alpha,t,x)| \leq k(1 + |x|) \tag{3.2}$$

for all $\alpha \in A$, $(t,x) \in \bar{Q}_T$.

Furthermore, for each $\alpha \in A$, $b^\alpha \in c^{1,2}(\bar{Q}_T)$ and its first order derivatives with respect to (t,x), $\partial_t b$, $\partial_i b (1 \leq i \leq d)$ and second order one $\partial_i \partial_j b (1 \leq i,j \leq d)$ are uniformly bounded. σ^α is still assumed to satisfy the relations (1.5) and (1.6). □

As for c, f and g, assume the following conditions:

(A.2)' $c: A \rightsquigarrow R$,

$f: A \times \bar{Q}_T \rightsquigarrow R$, $g: R^d \rightsquigarrow R$.

c, f and g are continuous functions over A, $A \times \bar{Q}_T$ and R^d respectively. c is nonnegative and f is also uniformly bounded from below. Moreover, for each $\alpha \in A$, $f^\alpha \in c^{1,2}(\bar{Q}_T)$ and there exists a nonnegative sequence $\{k_n\}$ $n \geq 0$ such that for each $n = 1,2,\ldots,$

$$(3.3)\quad |f^\alpha(t,x)| + |\partial_t f^\alpha(t,x)| + \sum_{i=1}^{d} |\partial_i f^\alpha(t,x)| + \sum_{i,j=1}^{d} |\partial_i \partial_j f^\alpha(t,x)| + c(\alpha) \leq k_n(1+|x|)^m,$$

for all $(\alpha,t,x) \in A_n \times \bar{Q}_T$. As to g, $g \in c^2(R^d)$ and it satisfies the following condition: for all $x \in R^d$,

$$(3.4)\quad |g(x)| + \sum_{i=1}^{d} |\partial_i g(x)| + \sum_{i,j=1}^{d} |\partial_i \partial_j g(x)| \leq K(1 + |x|)^m,$$

where m is a nonnegative constant. □

In this case, we assume the following condition instead of (A.3):

(A.3)' There exists a nonnegtaive function u over $\bar{Q}_T$ such that $u \in c^{1,2}(\bar{Q}_T)$ and, moreover,

$$(3.5)\quad |f(\alpha,t,x)| + |\partial_t f(\alpha,t,x) + \sum_{i=1}^{d} |\partial_i f(\alpha,t,x)| + \sum_{i,j=1}^{d} |\partial_i \partial_j f(\alpha,t,x)| + L^{\alpha,\varepsilon} u(t,x) \leq 0$$

for all $(\alpha,t,x) \in A \times \bar{Q}_T$, $0 \leq \varepsilon < 1$, where $L^{\alpha,\varepsilon}$ is given in (1.17). Assume further the following condition between f and c:

$$(3.6)\quad \sum_{i=1}^{d} |\partial_i f(\alpha,t,x)| \leq kc(\alpha),$$

for all $(\alpha,t,x) \in A \times \bar{Q}_T$. Finally, we also assume the same condition as (1.8), i.e.

(3.7) $\quad f^{\alpha}(T,x) + L^{\alpha,\varepsilon}g(T,x) \geq \eta$

for all $(\alpha,t,x) \in A \times \bar{Q}_T$ and all $\varepsilon \in [0,1)$. □

Then we have the following result whose proof is the same as Theorem 2.1.

Theorem 3.1 *Assume the conditions* (A.1)' ~ (A.3)'. *Then the assertion of Theorem 2.1 is correct.* □

Remark 3.1 Thus we can consider other assumptions besides (A.1) ~ (A.3) or (A.1)' ~ (A.3)' under which Theorem 2.1 holds (e.g. in (A.1)' ~ (A.3)', if g has bounded derivatives then the conditions with respect to b can be relaxed, etc.) (cf. [3] examples 5.1, 5.2). □

Before we consider an example for which the assumptions (A.1)' ~ (A.3)' hold, notice the following fact.

Remark 3.2 In order that $m^{\alpha}(t,x) \equiv 1$ (constant function) is a *normalizing multiplier*, it is sufficent and necessary that for any $r \geq 0$, $(t,x) \in \bar{Q}_T$,

(3.8) $\quad \inf_{\alpha \in A}\{-r[\frac{1}{2}\,\mathrm{tr}.a^{\alpha}(t,x) + |b^{\alpha}(t,x)| + c^{\alpha}(t,x)] + f^{\alpha}(t,x)\} > -\infty$

This is a small modification of the result due to N. V. Krylov ([5], Exercise 6.3.10). □

3.2 Consider the following simple example considered by N. V. Krylov in the case of d = 1.

(3.9) $\quad \inf_{0 \leq \alpha < \infty}\{\partial_t v + \frac{1}{2} v_{xx} - \alpha v + \alpha \tilde{f}(t,x,y)\} = 0,$

where $\tilde{f}$ is a bounded and $c^{1,2}(\bar{Q}_T)$ function with bounded derivatives. It is easily seen that the coefficients of Equation (3.9) satisfy the assumptions (A.1)' ~ (A.3)' (d = 2, ν = 1, A = $[0,\infty)$, $\sigma^{\alpha}(t,x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $b^{\alpha}(t,x) \equiv 0$, $c^{\alpha}(t,x) = \alpha$,

$(x,y) \in R^2$, $f^\alpha(t,x,y) = \alpha\tilde{f}(t,x,y)$, g is arbitrary, but also note that the constant function 1 is not normalizing multiplier for any such $\tilde{f}$ by means of (3.8). Therefore, we could not know whether the cost v, given by (1.12), satisfies Equation (0.1) and, in fact, there exists a counter example (cf. [5], Example 6.3.14).

For $0 \le \alpha < \infty$, $(t,x) \in \bar{Q}_T$, put

$$m^\alpha(t,x) = \frac{1}{1+\alpha}, \tag{3.10}$$

then it is easy to verify that the function m^α is a normalizing multiplier of Equation (3.9). It follows from Theorem 3.1 that v of (1.12) is a generalized solution of the following normed Bellman equation:

$$\inf_{0\le\alpha<\infty} m^\alpha(t,x)\{\partial_t v + \frac{1}{2} v_{xx} - \alpha v + \alpha\tilde{f}(t,x,y)\} = 0, \quad v(T,x) \equiv g(x). \tag{3.11}$$

If we put $\beta = \frac{\alpha}{1+\alpha}$, then Equation (3.11) is equal to the following one:

$$\inf_{0\le\beta<1} \{(1-\beta)(\partial_t v + \frac{1}{2} v_{xx}) + \beta(\tilde{f}(t,x,y) - v)\} = 0. \tag{3.12}$$

But it is equivalent to the following inequalities:

$$\begin{cases} \text{(a)} \quad \tilde{f}(t,x,y) = v(t,x,y), & \partial_t v + \frac{1}{2} v_{xx} \ge 0 \\ \text{(b)} \quad \tilde{f}(t,x,y) > v(t,x,y), & \partial_t v + \frac{1}{2} v_{xx} = 0. \end{cases} \tag{3.13}$$

3.3 Linear case (separate form of variables)

If all of the coefficients are of separate form in the variables (t,x) and α, then the assumptions (A.1) ~ (A.3) can be verified easily in the following way.

Let $A = R^e$ $(e \ge 1)$, and assume that $\bar{\sigma}$, b, f, c and g are of the following form: $\bar{\sigma}^\alpha(t,x) = K(t,x)$,

- $b^{\alpha}(t,x) = I(t,x) + J(\alpha), \qquad c^{\alpha}(t,x) \equiv 0,$
- $f^{\alpha}(t,x) = M(t,x) + N(\alpha),$

where K, I, J, M and N satisfy the following conditions:

(A.1)'' $K: \bar{Q}_T \rightsquigarrow R^{\nu} \otimes R^{\nu}$, $I: \bar{Q}_T \rightsquigarrow R^d$

- K and I are Lipshitz continuous, i.e. there exists a constant $k \geq 0$ such that for all $t \in [0,T]$, $x,x' \in R^d$,

$$\| K(t,x) - K(t,x') \| + |I(t,x) - I(t,x')| \leq k|x - x'|.$$

- For all $(t,x) \in \bar{Q}_T$,

$$\| K(t,x) \| + |I(t,x)| \leq k(1 + |x|).$$

- Assume that K and I belong to $c^{1,2}(\bar{Q}_T)$ and, further, that their derivatives $\partial_t K$, $\partial_t I$, $\partial_i K$, $\partial_i I$, $\partial_i \partial_j K$ and $\partial_i \partial_j I$ $(1 \leq i,j \leq d)$ are all bounded.
- σ is assumed to satisfy the relations (1.5) and (1.6), i.e. $\sigma^{\alpha}(t,x) = \begin{pmatrix} K(t,x) & 0 \\ 0 & 0 \end{pmatrix}$
- Let J be a continuous function over A with its values in R^d, and, furthermore, for each n, $|J(\alpha)| \leq k_n$ for all $\alpha \in A_n$. □

(A.2)'' $M \in c^{1,2}(\bar{Q}_T)$ and $g \in c^2(R^d)$. Moreover, all of their derivatives, M, $\partial_t M$, ∇M, $\nabla^2 M$, g, ∇g and $\nabla^2 g$ are bounded.

- $N(\alpha)$ is assumed to be continuous in A and bounded from below, and also satisfy that for each n, $|N(\alpha)| \leq k_n$ for $\forall\, \alpha \in A_n$. □

In this case, instead of (A.3) it is sufficient to assume the following:

(A.3)'' $M(T,x) + N(\alpha) + L^{\alpha,\varepsilon} g(T,x) \geq 0$ for all $x \in R^d$, $\alpha \in A$, $0 \leq \varepsilon < 1$. □

Corollary 3.2 *Assume* (A.1)''~(A.3)'', *then the assertion of Theorem 2.1 still holds.* □

Remark 3.3 If the function J is null, then the assumptions will respect to M and g are relaxed (see Theorem 3.1). We can also consider the case where σ depends upon α under which the assertion of Corollary 3.2 is correct. □

Remark 3.4 Linear Regulator Problem: $A = R^e$.

$$\sigma^\alpha(t) \equiv \sigma(t), \quad b^\alpha(t) = A(t)x + B(t)\alpha,$$

$$f^\alpha(t,x) = xM(t)x^* + \alpha N(t)\alpha^*, \quad g(x) = xDx^*$$

(M, N and D are symmetric matrices such that $M > 0$ and $N, D \geq 0$) is not included in (A.1)''~(A.3)'', but it is well-known that this problem can be solved completely by a particular method (see, for example, [2], p. 165). □

3.4 Consider the following 1-dim. (d = 1) Bellman equation:

$$\inf_{-\infty<\alpha<\infty} \{\partial_t v + \lambda v'' + \sqrt{2}\,\alpha v' + \alpha^2 + d(t,x)\} = 0, \tag{3.14}$$

where λ is a positive constant and d is a nonnegative, bounded and continuous function of (t,x). It is easily shown that this equation is equivalent to the following:

$$\partial_t v + \lambda v'' - \frac{(v')^2}{2} + d(t,x) = 0, \tag{3.15}$$

and, moreover, Equation (3.15) is known as "equation of burning of gas in a rocket" ([1], p. 23). Since the coefficients of Equation (3.14) satisfy the assumptions (A.1)''~(A.3)'', the cost v, given by the formula

$$v(s,x) = \inf_{-\infty<\alpha_t<\infty} E\Big[\int_0^{T-s} \{|\alpha_t|^2 + d(s+t, X_t^{\alpha,x})\}dt + g(X_{T-s}^{\alpha,x})\Big], \tag{3.16}$$

is a generalized (classical in this case) solution of Equation (3.14) (and

of Equation (3.15)) by means of Corollary 3.2. Here, $(X_t^{\alpha,x})$ is given by

$$X_t^{\alpha,x} = x + \int_0^t \sqrt{2}\,\alpha_r dr + \sqrt{2\lambda}\,\beta t, \tag{3.17}$$

and the function g may be taken appropriately so as to satisfy (A.2)'' and (A.3)''.

By using Corollary 3.2, we can easily extend the above result to a multi-dimensional and also degenerate one. For example, assume that $\alpha = 2$, $\nu = 1$ and $A = R^2$. Let us consider the following degenerate Bellman equation:

$$\inf_{\alpha\in R^2}\{\partial_t v + \frac{\lambda}{2} v_{xx} + \sqrt{2}\,(\alpha,\nabla v) + |\alpha|^2 + d(t,x,y)\} = 0, \tag{3.18}$$

then it is easily shown that this is equal to the following:

$$\partial_t v + \frac{\lambda}{2} v_{xx} - \frac{(v_x)^2}{2} - \frac{(v_y)^2}{2} + d(t,x,y) = 0. \tag{3.19}$$

Thus, by using Corollary 3.2, it is also shown that the cost v (see (3.16)) is a generalized solution of Equation (3.19), which is a nonlinear and degenerate differential equation. We can obtain a partial result if d is not bounded ([4]).

Remark 3.5 It is well-known (e.g. [1]) that certain kinds of equations, such as Burger's equation, are equal to Equation (3.15) by simple transformations. Therefore, by means of the above discussions, the cost v is an explicit representation of a solution for those (nonlinear, degenerate) partial differential equations. Moreover, by choosing the coefficients in Equations (0.1) or Equation (2.5) appropriately, we can consider many other differential equations than Equation (3.19). Conversely, it is well-known that the following equation,

$$\partial_t u + \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t,x)\partial_i\partial_i u + \sum_{i=1}^{d} b_i(t,x)\partial_i u - c(t,x)u = 0 \tag{3.20}$$

can be formally transferred to the Bellman equation by simple transformations (for the details, see [4]). □

4. UNIQUENESS

There arises naturally a problem whether the cost v, given by (1.12), is only one solution of Equation (0.1) or Equation (2.5), i.e. the uniqueness problem.

It is known (e.g. [3], [5]) that if all of the coefficients in Equation (0.1) are bounded with respect to α, then there exists only one solution of Equation (0.1) under relatively general conditions.

On the other hand, if the coefficients are unbounded with respect to α, then it is difficult to get such general results from Equation (2.5) as the preceding one. In the following, we shall discuss only some particular cases which were treated in Section 3 (cf. [3], Remark 5.2).

Following [3], let us start to define additional notations. Let $c(\overline{Q}_T)$ be a space of real valued continuous functions defined over $\overline{Q}_T$. For each $\nu(1 \le \nu \le d)$, $p \ge 1$, we say that a real valued function u given on $\overline{Q}_T$ belongs to $W^{1,2,\nu}_{p,loc}(Q_T)$ if there exist generalized derivatives, $\partial_t u$, $\partial_i u$ $(1 \le i \le d)$ and $\partial_i \partial_j u$ $(1 \le i,j \le \nu)$ such that they are locally p^{th} integrable on Q_T. We write $W^{1,2}_{p,loc}(Q_T)$ if $\nu = d$ which is the well-known Sobolev space.

Let $m^{\alpha}(t,x)$ be a regular normalizing multiplier.

Definition 4.1 *A real valued function* u *over* $\overline{Q}_T$ *is called* m-*superharmonic if there exist constants* p, λ *and* $k \ge 0$ *such that* $u \in W^{1,2,\nu}_{p,loc}(Q_T) \cap c(\overline{Q}_T)$.

$$|u(t,x)| \le k(1 + |x|)^{\lambda} , \tag{4.1}$$

and, moreover,

$$G^m[u](t,x) \ge 0 \quad \text{a.e. } (Q_T), \quad u(T,x) \le g(x), \ x \in R^d. \tag{4.2}$$

□

Then we have the following.

Theorem 4.1 *Let* m^α *be a regular normalizing multiplier and let* u *be* m*-superharmonic. Moreover, suppose that* $\partial_i u (1 \le i \le d)$ *and* $\partial_i \partial_j u (1 \le i,j \le \nu)$ *are locally bounded. Then for all* $(s,x) \in \overline{Q}_T$, $u(s,x) \le v(s,x)$, □

Proof. By means of (4.2) and the definition of m^α it follows that

$$F^\alpha[u]\ (t,x) \ge 0 \quad \text{for a.e. } (Q_T), \tag{4.3}$$

where $F^\alpha[u](t,x) = F^\alpha(\partial_t u,\ \partial_i u,\ \partial_i \partial_j u,\ u, t, x)$ and $F^\alpha(\cdot)$ is given by the formula:

$$F^\alpha(u_0, u_{ij}, u_i, u, t, x) = u_0 + \frac{1}{2} \sum_{i,j=1}^{\nu} a_{ij}^\alpha(t,x) u_{ij} + \sum_{i=1}^{d} b_i^\alpha(t,x) u_i - c^\alpha(t,x) u + f^\alpha(t.x). \tag{4.4}$$

Let us fix $\alpha \in \mathfrak{A}$; then there exists a number n such that $\alpha \in \mathfrak{A}_n$, and for such α it is well-known ([3]), Theorem 3.1) that

$$v^\alpha(t,x) \ge u(t,x) \qquad \text{for all } (t,x) \in \overline{Q}_T. \tag{4.5}$$

By means of the definition of v, the assertion follows immediately from (4.5). □

In order to show the inverse relation, we need a further new notation and also some assumptions about the coefficients. For any real valued function h on $\overline{Q}_T$ and $\ell \in R^d$ such that $|\ell| = 1$, $0 < \delta < 1$, define the quadratic difference, $D^2_{\ell,\delta} h(t,x)$, by

$$D^2_{\ell,\delta} h(t,x) = \{h(t,x + \delta\ell) + h(t,x - \delta\ell) - 2h(t,x)\}/\delta^2 \tag{4.6}$$

Note that if $h(t,\cdot) \in c^2(Q_T)$ then $D^2_{\ell,\delta} h(t,x) \to h_{(\ell)(\ell)}(t,x)$ as $\delta \to 0$.

Let $m^\alpha(t,x)$ be a regular normalizing multiplier, Borel measurable with respect to (t,x) and continuous with respect to α. Then we have the following (cf. [3], Lemma 4.2).

Lemma 4.2 *For some* p, *let* $u \in W^{1,2,\nu}_{p,\mathrm{loc}}(Q_T) \cap c(\bar{Q}_T)$, *Also, let* $G^m[u](t,x) \le 0$ a.e. (Q_T). *Then for each* $\kappa = 1,2,\ldots$ *there exists a Borel function* α_κ *over* $\bar{Q}_T$ *taking values in* A *such that*

$$1/\kappa > m^{\alpha_\kappa}(t,x)\, F^{\alpha_\kappa}[u](t,x) \qquad \text{a.e. } (Q_T), \tag{4.7}$$

where

$$\begin{cases} m^{\alpha_\kappa}(t,x) = m(\alpha_\kappa(t,x),t,x) & \text{and} \\ F^{\alpha_\kappa}[u](t,x) = u_t + \frac{1}{2}\sum_{i,j=1}^{\nu} a_{ij}(\alpha_\kappa(t,x),t,x)\partial_i\partial_j u \\ \qquad + \sum_{i=1}^{d} b_i(\alpha_\kappa(t,x),t,x)\partial_i u - c(\alpha_\kappa(t,x),t,x)u + f(\alpha_\kappa(t,x),t,x). \end{cases} \tag{4.8}$$

□

Let us assume the following conditions relative to the sequence $\{\alpha_\kappa\}$ obtained in the above lemma.

(A.4) There exists a constant $k \ge 0$ such that

$$\sup_{1 \le \kappa < \infty} |\alpha_\kappa(t,x)| \le k \tag{4.9}$$

for almost all $(t,x) \in Q_T$. □

Remark 4.1 For example, in 3.4, (3.18), we can take as α_κ the following function (we may put $m^\alpha(t,x) \equiv 1$): for any $\kappa = 1,2,\ldots$,

$$\alpha_\kappa(t,x) = -\sqrt{2}\,\nabla v(t,x)/2. \tag{4.10}$$

Note further that in this case ∇v is a.a. bounded on Q_T (for the details, see [4]). Also in 3.2, (3.12), $\{\alpha_\kappa\}$ can be taken such that (4.9) holds under additional assumptions. □

Let us assume (A.4); then we have the following:

Theorem 4.3 *Let* $u \in W^{1,2,\nu}_{p,loc}(Q_T) \cap c(\bar{Q}_T)$ *for any* $p \geq d+1$, *and also let* (4.1) *be satisfied. Let* $m^{\alpha}(t,x)$ *be a regular normalizing multiplier satisfying the conditions of Lemma 4.2 and the relation*

$$m_0^{\alpha}(t,x) \leq N(t,x)m^{\alpha}(t,x) \tag{4.11}$$

for all (t,x), *where* N *is bounded from below. Assume that for such* m,

$$\begin{cases} G^m[u](t,x) \leq 0 & \text{a.e. } (Q_T), \\ u(T,x) \geq g(x), & x \in R^d. \end{cases} \tag{4.12}$$

Moreover, assume that there exist nonnegative constants k, m *such that for all* $(t,x) \in Q_T$, $0 < \delta < 1$, $\ell \in R^d$ *such that* $|\ell| = 1$,

$$D^2_{\ell,\delta}u(t,x) \leq k(1+|x|)^m. \tag{4.13}$$

Then it holds that

$$u \geq v \quad \text{on } \bar{Q}_T. \tag{4.14}$$ □

Since the proof is almost the same as the one for Theorem 4.1, Lemmas 4.2 ~4.4 in [3], we omit it here (although we have to modify it slightly). Finally, by combining Theorem 4.1 with Theorem 4.3, we have the following uniqueness result of Equation (2.5).

Corollary 4.4 *Let* $u \in W^{1,2,\nu}_{p,loc}(Q_T) \cap c(\bar{Q}_T)$ *for any* $p \geq d+1$ *and also let* (4.1) *be verified. Suppose that* u *satisfies the normed Bellmann equation* $G^m[u](t,x) = 0$ a.e. *and* $u(T,x) = g(x)$, $x \in R^d$, *for a regular normalizing multiplier* m *satisfying the preceding conditions made in Theorems* 4.1 and 4.3. *Furthermore, let us assume all of the conditions made in Theorem* 4.1 *and* 4.3. *Then* $u \equiv v$ on $\bar{Q}_T$. □

Remark 4.2 It is not hard to prove that for the examples in Section 3, such as (3.12), (3.18), etc. (under some additional conditions, if necessary), the uniqueness theorem holds. In fact, it is sufficient to note that (A.4) is true and that m_0^{α} itself has such properties as m^{α}. It is also shown that the cost v satisfies the same condition (except (A.4)) as u in Corollary 4.4 under the assumptions (A.1) ~ (A.3) or (A.1)' ~ (A.3)' or (A.1)'' ~ (A.3)''. (cf. [3]).

□

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