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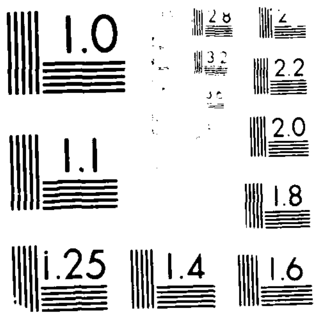
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BREAKING WAVE SPECTRUM IN WATER OF FINITE DEPTH IN THE PRESENCE OF CURRENT

by

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This report presents an approximate method to compute the mean value, the mean-square value, and the spectrum of waves in water of finite depth taking into account the effect of wave breaking in the presence of current. It is assumed that there exists a linear and Gaussian ideal wave train whose spectrum is first obtained using the wave energy flux balance equation without considering wave breaking. The Miche wave breaking criterion for waves in finite water depth is used to limit the wave elevation and establish an expression for the breaking wave elevation in terms of the elevation and its second time derivative of the ideal waves. Simple expressions for the mean value, the mean-square value and the spectrum are obtained. These results are applied to the case in which a deepwater unidirectional wave train, propagating normally toward a straight shoreline over gently varying sea bottom of parallel and straight contours, encounters an adverse steady current whose velocity is assumed to be uniformly distributed with depth. Numerical results are obtained and presented in graphic form.

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PREFACE

This report presents results of the development of an approximate method to compute the spectrum of breaking waves in water of finite depth taking into account the presence of current. The research in this report was authorized by the Office, Chief of Engineers (OCE), US Army Corps of Engineers, under the Harbor Entrances and Coastal Channels Program of Civil Works Research and Development, through "Waves at Entrances" Work Unit 31673, at the Coastal Engineering Research Center (CERC) of the US Army Engineer Waterways Experiment Station (WES). Messrs. John H. Lockhart, Jr., and John G. Housley of OCE were the Technical Monitors. Dr. Charles L. Vincent of CERC is the Program Manager.

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The CERC contract monitor for this study was Dr. H. S. Chen, Coastal Oceanography Branch (CR-O), CERC, under direct supervision of Dr. Edward F. Thompson, Chief, CR-O, and Mr. H. Lee Butler, Chief, Research Division; and under general supervision of Mr. Charles C. Calhoun, Jr., and Dr. James R. Houston, Assistant Chief and Chief, CERC, respectively. This report was edited by Ms. Shirley A. J. Hanshaw, Information Products Division, Information Technology Laboratory, WES.

Commander and Director of WES is COL Dwayne G. Lee, CE, and Technical Director is Dr. Robert W. Whalin.

CONTENTS

BREAKING WAVE SPECTRUM IN WATER OF FINITE DEPTH IN THE PRESENCE OF CURRENT

PART I: INTRODUCTION

1. There are many forms of wave energy spectrum. All of these spectra, however, are for specific conditions. For example, the Pierson-Moskowitz spectrum is for a fully developed sea, the Joint North Sea Wave Project spectrum is for a fetch-limited developing sea, and the Wallops spectrum (Huang et al. 1981) is derived based on wave dynamics but without considering wave breaking.

2. When conditions differ from those for which these spectra are intended or, as the waves move into regions where the conditions are changed, these spectra undergo corresponding changes. For example, as the steepness of the wave increases, wave breaking occurs; the Wallops spectrum, which does not consider wave breaking, must be modified. This problem was treated recently by Yuan, Tung, and Huang (1986) and by Tung and Huang (1987) for deepwater waves.

3. As the waves propagate from deep to shallow water, wave breaking takes place when they reach the surf zone. There have been a number of publications on the subject of wave breaking in shallow water such as those by Battjes and Janssen (1978) and Thornton and Guza (1983). These authors used the energy flux balance equation including energy dissipation, and the results are in good agreement with measurements. The equation, however, must be integrated numerically, and the methods do not give the breaking wave spectrum directly.

4. Similarly, when a wave train encounters an adverse current, wave breaking results. The method usually employed to obtain the spectrum of the waves interacting with current is to first resort to the classical energy flux balance equation without considering wave breaking (Huang et al. 1972 and Hedges, Burrows, and Mason 1979). To account for the effect of wave breaking on the wave spectrum, Hedges, Burrows, and Mason (1979) applied the equilibrium range spectrum to limit the spectral ordinates. The equilibrium range spectrum, however, contains a numerical constant whose value is difficult to specify. Furthermore, the equilibrium range spectrum only applies to

frequencies much higher than those corresponding to the peak of the spectrum and therefore cannot be extended to cover the range of frequencies where most of the wave energy resides.

5. In this study, we extend the method introduced earlier (Yuan, Tung, and Huang 1986, Tung and Huang 1987) for a deepwater breaking wave spectrum to waves in water of finite depth and apply the results to the situation where the waves encounter a current. The method consists essentially of first assuming that there exists an original ideal wave train at the locale under consideration, the spectrum of which is obtained from the equation of energy flux balance without considering wave breaking. By imposing the Miche wave breaking criterion (Battjes 1974), an expression for the elevation of the breaking waves is established in terms of the original ideal wave elevation and its second derivative which are assumed to be jointly Gaussian. Based on this breaking wave model, the expressions for the mean value, the mean-square value, and the spectrum of the breaking waves are derived. These results are then applied to the case in which a unidirectional deepwater wave train, propagating normally toward a straight shoreline over a gently varying sea bottom with straight and parallel contours, meets a steady current whose flow velocity is uniformly distributed in the vertical direction. Numerical results are obtained and given in graphical form. The simpler breaking wave model for deepwater waves is first presented and modified for waves in water of finite depth.

6. It is emphasized here that the studies carried out in this report are based on heuristic wave breaking models and simplified current and coast configurations. A number of approximations are introduced in the derivations, but the results have not yet been checked against either field or laboratory experiments. It is clear that the models have yet to be modified and that more detailed studies should be performed to examine the effect of utilizing various spectral forms for the original ideal waves.

PART II: BREAKING WAVE MODELS

7. Stokes (1880) showed that in deep water, when the vertical downward acceleration at the crest of the wave reaches a value of 0.5g* (g being gravitational acceleration), the wave breaks and its amplitude is reduced according to the ratio of 0.5g and the magnitude of the acceleration of the original ideal wave at the crest. The following equation expresses this relationship:

$$a_b = a \frac{0.5g}{a\omega^2} = \frac{0.5g}{\omega^2} \qquad (1)$$

where

a_b = amplitude of the breaking wave

a = amplitude of the ideal wave

ω = frequency of this ideal wave

8. Longuet-Higgins (1969) applied this criterion to a narrow-band wave train in which the amplitude of the breaking wave is given by

$$a_b = \frac{0.5g}{\bar{\omega}^2} \qquad (2)$$

where

$$\bar{\omega} = \left[\frac{\int \omega^2 S(\omega) d\omega}{S(\omega) d\omega} \right]^{1/2} \qquad (3)$$

is the characteristic wave frequency and $S(\omega)$ is the energy spectrum of the ideal waves.

9. To obtain the spectrum of the breaking waves, we assume (Phillips 1980) that the wave breaks whenever the local vertical downward acceleration at any point on the surface reaches a fraction of the gravitational acceleration. Referring to Figure 1, let $\zeta(t)$ and $\zeta_b(t)$ represent, respectively, the elevations of the ideal and breaking waves at a fixed point in space where

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* For convenience, symbols and abbreviations are listed in the Notation (Appendix B).

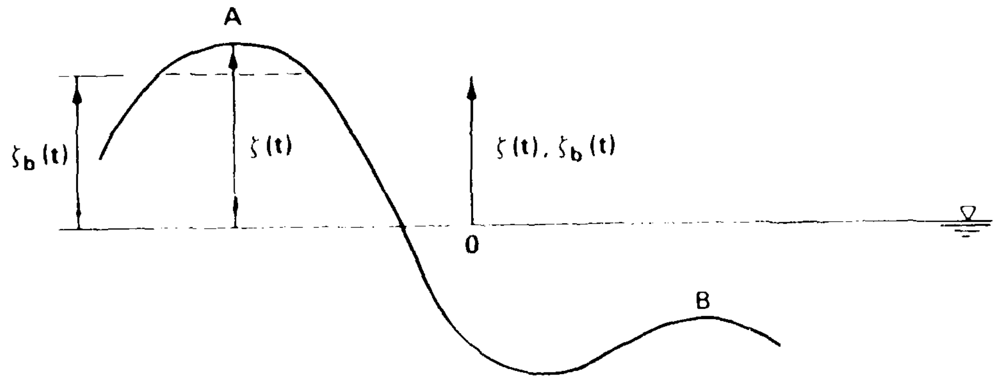

Figure 1. Wave profile

t is time. Wave breaking may take place at points such as A and B where $\ddot{\zeta}(t) < 0$. (Here, and hereafter, overdot denotes differentiation with respect to time.) At points such as A where $\zeta(t) > 0$ and when $\ddot{\zeta}(t) < -Kg$ (K is shown to range between 0.4 and 0.5 (Ochi and Tsai 1983)), the breaking wave elevation is given by

$$\zeta_b(t) = \zeta(t)\,\frac{-Kg}{\ddot{\zeta}(t)} \tag{4}$$

This expression is a restatement of Equation 1 or Equation 2; that is, when the wave breaks, the local wave elevation is reduced according to the ratio of Kg and the magnitude of the local acceleration of the ideal wave.

10. At point B where $\zeta(t) < 0$ and when $\ddot{\zeta}(t) < -Kg$, the breaking wave elevation is given by

$$\zeta_b(t) = \zeta(t)\,\frac{-\ddot{\zeta}(t)}{Kg} \tag{5}$$

11. Based on the above considerations and noting that no wave breaking takes place when $\ddot{\zeta}(t) > -Kg$, in which case $\zeta(t)$ remains unchanged, $\zeta_b(t)$ may be written as

$$\zeta_b = \zeta\left(-\frac{Kg}{\ddot{\zeta}}\right)H(-\ddot{\zeta} - Kg)H(\zeta) + \zeta\left(-\frac{\ddot{\zeta}}{Kg}\right)H(-\ddot{\zeta} - Kg)H(-\zeta) + \zeta H(\ddot{\zeta} + Kg) \tag{6}$$

where $H(\cdot)$ is the Heaviside unit step function and, for brevity, the argument t in $\zeta_b(t)$, $\zeta(t)$, and $\ddot{\zeta}(t)$ is omitted. In Equation 6, the first and the second terms correspond to the points such as A and B in Figure 1 when wave breaking occurs, and the third term merely states that ζ remains unchanged as long as $\ddot{\zeta}(t) > -Kg$ regardless of the point under consideration.

12. In Equation 6, the breaking wave elevation ζ_b is a nonlinear function of ζ and $\ddot{\zeta}$, the elevation and its second derivative of the original ideal waves which are assumed to be stationary and jointly Gaussian with zero mean values. The determination of the mean value, mean-square value, and the spectrum of ζ_b may therefore be achieved in a straightforward manner (Papoulis 1965).

13. In water of finite depth, for a single wave, the breaking wave amplitude is (Battjes 1974)

$$a_b = 0.44d \frac{\tanh kd}{kd} \qquad (7)$$

where d is the local water depth and k is the wave number. The above may be expressed approximately in terms of k_o , the wave number in deep water; that is, using

$$k_o = k \tanh kd \qquad (8)$$

and

$$k_o \doteq k \sqrt{\tanh k_o d} \qquad (9)$$

we have

$$a_b \doteq 0.44 \frac{\tanh k_o d}{k_o} \qquad (10)$$

The deepwater wave number is

$$k_o = \frac{\omega^2}{g} \qquad (11)$$

where ω is the wave frequency which remains constant independent of water depth and is related to the local wave elevation ζ and the surface acceleration $\ddot{\zeta}$ at the point under consideration by

$$\omega^2 = -\frac{\ddot{\zeta}}{\zeta} \qquad (12)$$

14. For random waves in water of finite depth, for reasons purely of mathematical convenience, we replace a_b in Equation 10 by ζ_b and k_o in the denominator by $k_o = -\ddot{\zeta}/\zeta g$ but substitute the same in the numerator by $\bar{k}_o = \bar{\omega}^2/g$, the characteristic wave number in deep water, so that, for points such as A in Figure 1, the breaking wave elevation ζ_b takes the same form as that shown in Equation 4 where

$$K = 0.44 \tanh \bar{k}_o d \qquad (13)$$

which reduces to $K = 0.44$ in deep water.

15. Following the same argument leading to Equation 6 for deepwater waves, we see that the breaking wave elevation in water of finite depth is also given by Equation 6 with K replaced by Equation 13. In arriving at this expression we have ignored the situation where the magnitude of the negative wave elevation may exceed the water depth. We chose not to consider such possibility and restrict the application of the model to regions that are not unduly shallow where other factors such as bottom friction may come into play, the linear and Gaussian assumptions of the original waves are no longer valid, and the model may be overly strained.

16. In subsequent derivations of the expressions for the mean value, the mean-square value, and the spectrum of ζ_b, the second term in Equation 6 is ignored based on the consideration that the probability of occurrence of negative peaks such as point B in Figure 1 is usually small, especially when the spectrum of the waves under consideration is reasonably narrow. In this way, the derivation is much shortened, and our computation shows that the error incurred by ignoring the second term in Equation 6 is indeed imperceptibly small.

PART III: MEAN VALUE, MEAN-SQUARE VALUE, AND SPECTRUM OF ζ_b

17. Although the original ideal waves are a zero mean process, from Equation 6 it is obvious that the elevation of the breaking wave is not. From Equation 6 (with the second term deleted), it is not difficult to show that

$$E[\zeta_b] = \sqrt{r}\left\{\frac{\beta E_1\left[\frac{\left(\frac{\beta}{\varepsilon}\right)^2}{2}\right]\varepsilon}{4\pi} - \sqrt{1-\varepsilon^2}\,Z(\beta) + \frac{\beta\sqrt{1-\varepsilon^2}}{\sqrt{2\pi}}\,L\left(\beta,\ o,\ -\sqrt{1-\varepsilon^2}\right)\right\} \tag{14}$$

where

$E[\cdot]$ = expected value of the quantity enclosed in brackets

β = wave breaking parameter (defined in equation 19)

ε = spectral bandwidth parameter (defined in Equation 18)

$Z(\cdot)$ = probability function (defined in Equation 20)

Let

$$r = \int S(\omega)d\omega \tag{15}$$

$$r^{(2)} = -\int \omega^2 S(\omega)d\omega \tag{16}$$

$$r^{(4)} = \int \omega^4 S(\omega)d\omega \tag{17}$$

where $S(\omega)$ is the spectrum of the original ideal waves. The quantity

$$\varepsilon = 1 - \frac{\left[r^{(2)}\right]^2}{r\,r^{(4)}} \tag{18}$$

lies between zero and unity and is known as the bandwidth parameter of $S(\omega)$ (Cartwright and Longuet-Higgins 1956) and

$$\beta = \frac{Kg}{\sqrt{r^{(4)}}} \tag{19}$$

is a measure of the extent of wave breaking as will be shown later. The functions

$$Z(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{20}$$

$$L(c_1, c_2, p) = \int_{c_1}^{\infty} Z(x)Q(w)dx \ , \ w = \frac{c_2 - px}{\sqrt{1 - p^2}} \tag{21}$$

where

$L(\cdot, \cdot, \cdot)$ = probability function

c_1, c_2 = parameters

p = parameter

Q = probability function

and

$$Q(x) = \int_{x}^{\infty} Z(y)dy \tag{22}$$

are probability functions (Abramowitz and Stegun 1968), and

$$E_1(x) = \int_{x}^{\infty} \frac{e^{-y}}{y} dy \tag{23}$$

is the exponential integral (Abramowitz and Stegun 1968), and y is a dummy variable. It is seen that the mean value of ζ_b is a nonzero constant which depends on the value of water depth d through K in β in Equation 19 and the zeroth, second, and fourth spectral moments of the ideal waves.

18. Similarly, it may be shown that the mean-square value of ζ_b is (with the second term in Equation 6 deleted)

$$E\left[\zeta_b^2\right] = r[Q(-\beta) - (1 - \epsilon^2)\beta Z(\beta)] + r\,\beta^2 \left\{ \frac{\epsilon\sqrt{1 - \epsilon^2}\;\; E_1\left[\dfrac{\left(\frac{\beta}{\epsilon^2}\right)}{2}\right]}{4\pi} + (1 - \epsilon^2)\; L\left(\beta, 0, -\sqrt{1 - \epsilon^2}\right) + \epsilon^2 N \right\} \tag{24}$$

where

$$N = \int_{\beta}^{\infty} \frac{Z(x)}{x^2}\, Q\left(- \frac{\sqrt{1 - \epsilon^2}}{\epsilon}\, x\right) dx \tag{25}$$

19. To obtain the spectrum of ζ_b, we first form its autocorrelation function. For convenience, let subscripts 1 and 2 refer to quantities evaluated at time instants $t_1 = t + \tau$ (where τ is time lag) and $t_2 = t$, respectively. Furthermore, let $H = H(\zeta)$, $H''_+ = H(\ddot{\zeta} + Kg)$, and $H''_- = H(-\ddot{\zeta} - Kg)$. By anticipating that ζ_b is stationary, the autocorrelation function of ζ_b, denoted $R_b(\tau)$, is, from Equation 6 (with the second term deleted)

$$R_b(\tau) = E[\zeta_{b1}\zeta_{b2}] = (Kg)^2\, E\left[\frac{\zeta_1\zeta_2}{\ddot{\zeta}_1\ddot{\zeta}_2}\, H''_{1-}H''_{2-}H_1H_2\right] - 2Kg\, E\left[\frac{\zeta_1\zeta_2}{\ddot{\zeta}_1}\, H''_{1-}H''_{2+}H_1\right] + E[\zeta_1\zeta_2 H''_{1+}H''_{2+}] \tag{26}$$

The expected values in Equation 26 involve the random variables ζ_1, ζ_2, $\ddot{\zeta}_1$, and $\ddot{\zeta}_2$ which are jointly Gaussian with zero mean values. These expected values may all be obtained, although the task is tedious. In Appendix A, the last expected value of Equation 26 is evaluated to illustrate the techniques employed in obtaining these expected values.

20. The resulting autocorrelation function is a nonlinear function of the correlation functions $r_{12}(\tau) = E[\zeta_1\zeta_2]$, $r_{12}^{(2)}(\tau) = E[\zeta_1\ddot{\zeta}_2]$, and $r_{12}^{(4)}(\tau) = E[\ddot{\zeta}_1\ddot{\zeta}_2]$ of the original wave elevation ζ and its second derivative $\ddot{\zeta}$ evaluated at time instants t_1 and t_2 . The autocorrelation function $R_b(\tau)$, viewed as a function of the correlation coefficient functions $r_{12}(\tau)/r$, $r_{12}^{(2)}(\tau)/r^{(2)}$, and $r_{12}^{(4)}(\tau)/r^{(4)}$ may be expanded by Taylor's series (Borgman 1965). By retaining only the zeroth and the first-order terms of the series, it may be verified that the zeroth order term is equal to the square of the expected value $E[\zeta_b]$ of ζ_b . The first-order approximate autocovariance function

$$K_b(\tau) = R_b(\tau) - E^2[\zeta_b] \tag{27}$$

is therefore a linear function of $r_{12}(\tau)$, $r_{12}^{(2)}(\tau)$, and $r_{12}^{(4)}(\tau)$ the Fourier transforms of which are, respectively, $S(\omega)$, $-\omega^2 S(\omega)$, and $\omega^4 S(\omega)$. Thus, by taking the Fourier transform of Equation 27, we have the approximate spectrum of the breaking wave simply related to $S(\omega)$ as

$$S_b(\omega) = F(\omega)\ S(\omega) \tag{28}$$

in which

$$F(\omega) = A_1^2 \left(\frac{\omega^2}{\omega_1^2} - 1 \right)^2 \tag{29}$$

is a fourth order polynominal function of ω and may be looked upon as a filter function which accounts for the effects of wave breaking on the spectrum $S(\omega)$ of the ideal waves.

21. In Equation 29,

$$\omega_1^2 = \left| \frac{A_1}{A_2} \right| \left| \frac{r^{(4)}}{r^{(2)}} \right| \quad . \tag{30}$$

$$A_1 = \beta\bar{N} + Q(-\beta) > 0 \tag{31}$$

and

$$A_2 = \beta\bar{N} - \beta Z(\beta)Q\left(\frac{\beta\sqrt{1-\varepsilon^2}}{\varepsilon}\right) - \frac{\beta Q\left(\frac{\beta}{\varepsilon}\right)}{\sqrt{2\pi(1-\varepsilon^2)}} + \beta Z(\beta) \qquad (32)$$

where

$$\bar{N} = \int_{\beta}^{\infty} \frac{Z(x)}{x} Q\left(-\frac{\sqrt{1-\varepsilon^2}}{\varepsilon}x\right)dx \qquad (33)$$

22. To examine the properties of the filter function $F(\omega)$ and the breaking wave spectrum $S_b(\omega)$, it is first noted that the value of β , which is the ratio of Kg and the standard deviation $\sqrt{r^{(4)}}$ of the surface acceleration of the ideal waves, may be given a rough estimate. By referring to Figure 1, let us assume that the acceleration in those portions of the surface, where $|\ddot{\zeta}|$ reaches or exceeds Kg , remains at the value of Kg ; but in the remaining portion of the surface the acceleration vanishes. The standard deviation $\sqrt{r^{(4)}}$ of $\ddot{\zeta}$ is therefore equal to $Kg\sqrt{A_B/A}$, and $\beta = 1/\sqrt{A_B/A}$ where A_B and A are, respectively, the area of wave surface with $|\ddot{\zeta}| > Kg$ and the total area. The ratio A_B/A is normally a small quantity so that β may be expected to be larger than unity. For example, in stormy situations, the ratio A_B/A may be as high as 1/4 giving $\beta = 2$; whereas in calmer situations, if the ratio A_B/A is equal to 1/9, β is approximately equal to 3.

23. Having established that β is larger than unity, by employing variously the series representation and the asymptotic behavior of $Q(\cdot)$ for large values of its argument (Abramowitz and Stegun 1968), it may be verified that $A_2 > 0$ and $A_1/A_2 >> 1$. Since $\left(\left|r^{(4)}/r^{(2)}\right|^{1/2}\right) > \left(\left|r^{(2)}/r\right|^{1/2}\right)$ and the latter quantity is in fact the characteristic wave frequency $\bar{\omega}$ (see Equation 3), it is seen that $\omega_1 >> \bar{\omega}$ in view of Equation 30. The filter function $F(\omega)$ is a monotonically decreasing function of ω for $0 \leq \omega < \omega_1$ decreasing from $F(o) = A_1^2$ to $F(\omega_1) = 0$. Beyond $\omega = \omega_1$, $F(\omega)$ increases indefinitely. The range of frequency of wind waves of

practical interest, however, is usually limited to within $0 < \omega < \omega_1$ $(\gg \bar{\omega})$ as the numerical results to be presented later will show. The manner in which $F(\omega)$ varies with β, a measure of the sea state, may be seen by taking the derivative of A_1 with respect to β. It may be verified that A_1 is a monotonically increasing function of β, and A_1 approaches unity as β approaches infinity which means that in mild seas β and ω_1 are both rather large so that $F(\omega) = 1$ for $0 < \omega < \omega_1$ and $S_b(\omega) = S(\omega)$. No wave breaking takes place, and the original ideal wave spectrum remains unchanged. In high seas, on the other hand, $A_1 < 1$ and so is $F(\omega)$ for $0 < \omega < \omega_1$. Thus, the original wave spectrum is reduced as a consequence of wave breaking, as expected.

PART IV: WAVE-CURRENT INTERACTIONS

24. Consider a unidirectional linear wave train entering a region of current. Let the current be steady in time and the flow velocity U, considered positive in the direction of the waves, be uniformly distributed in the vertical direction. For each wave component, the apparent frequency ω_a, in a stationary frame of reference, is related to the relative or intrinsic frequency ω_r in the frame of reference moving with the current as

$$\omega_a = \omega_r + kU \tag{34}$$

where ω_r and the wave number k are related as

$$\omega_r^2 = gk \tanh kd \tag{35}$$

25. Ignoring wave breaking and using the energy flux balance (Huang et al. 1972) or the conservation of wave action (Hedges, Burrows, and Mason 1979), it was shown that the wave spectrum $S(\omega_a)$, under the influence of current, is related to $S_o(\omega_a)$, the spectrum in quiescent deep water, as

$$S(\omega_a) = \frac{c_{go}}{U + c_{gr}} \frac{\omega_r}{\omega_a} S_o(\omega_a) \tag{36}$$

where

$$c_{go} = \frac{g}{2\omega_a} \tag{37}$$

and

$$c_{gr} = \frac{1}{2}\left(1 + \frac{2kd}{\sinh 2kd}\right)\frac{\omega_r}{k} \tag{38}$$

The subscript "o" is used to refer to quantities evaluated in deep water in zero current condition.

26. In the relative frame of reference, the wave spectrum $\bar{S}(\omega_r)$ may be obtained from $S(\omega_a)$ in Equation 36 by changing the frame of reference (Hedges, Burrows, and Mason 1979) as

$$\bar{S}(\omega_r) = S(\omega_a) \frac{d\omega_a}{d\omega_r} \tag{39}$$

27. The above is but a brief exposition of the basic equations for the determination of the ideal wave spectra $S(\omega_a)$ and $\bar{S}(\omega_r)$ for waves in water of finite depth in the presence of current. Details of many of the considerations and operations involved are well explained in Hedges, Burrows, and Mason (1979). For example, the report gives an account of the solutions of Equations 34 and 35 given the values of ω_a, d, and U, discusses the cutoff frequency of ω_r (and ω_a) in a negative current ($c_{gr} = |U|$), and shows a numerical scheme by which the transformation of the spectrum from the stationary to the relative frame of reference and vice versa may be achieved.

28. To account for wave breaking, the spectrum $\bar{S}(\omega_r)$ given by Equation 39 may be used as the original ideal wave spectrum in place of $S(\omega)$ in Equations 15, 16, and 17 for the calculation of r, $r^{(2)}$ and $r^{(4)}$ from which the mean value $E[\zeta_b]$ and the mean-square value $E\left[\zeta_b^2\right]$ of the breaking waves ζ_b are obtained from Equations 14 and 24, respectively. Similarly, the breaking wave spectrum in the relative frame of reference, denoted by $\bar{S}_b(\omega_r)$ may be obtained from Equation 28 with the ideal wave spectrum $S(\omega)$ replaced by $\bar{S}(\omega_r)$ in Equation 39. Finally, the breaking wave spectrum in the stationary frame of reference is determined from

$$S_b(\omega_a) = \bar{S}_b(\omega_r) \frac{d\omega_r}{d\omega_a} \tag{40}$$

by changing the frame of reference.

PART V: NUMERICAL RESULTS

29. The preceding development enables a consideration of the effect of wave breaking on the mean value, the mean-square value, and the spectrum of a unidirectional deepwater wave train that is free of current, propagating over a gently varying sea bottom with straight and parallel contours normally incident toward a straight shoreline, where it meets an adverse horizontal variable current steady in time and uniformly distributed with depth. The following computation, though not entirely realistic, treats the current speed as a constant.

30. Let the deepwater wave spectrum be the Wallops spectrum (Huang et al. 1981) which takes the form

$$S_o(\omega) = \frac{\alpha g^2}{\omega^m \omega_o^{5-m}} \exp\left[-\frac{m}{4}\left(\frac{\omega_o}{\omega}\right)^4\right] \qquad (41)$$

where

α = coefficient defined in Equation 44

ω_o = parameter of Wallops wave spectrum

The quantity m gives the magnitude of the slope of the spectrum (on log-log scale) in high frequency range and is given by

$$m = \left|\frac{\log(2\pi^2 \S^2)}{\log 2}\right| \qquad (42)$$

where

$$\S = \frac{\sqrt{r}}{\lambda_o} \qquad (43)$$

is the significant slope of the waves, $\lambda_o = 2\pi/\bar{k}_o$ being the characteristic wave length. The quantity α is given by

$$\alpha = \frac{m^{\frac{m-1}{4}}}{4^{\frac{m-5}{4}}} \frac{(2\pi\S)^2}{\Gamma\left[\frac{(m-1)}{4}\right]} \tag{44}$$

where $\Gamma(\cdot)$ is the Gamma function (Abramowitz and Stegun 1968). The Wallops spectrum, therefore, is seen to depend on two parameters, $\S$ and ω_o, the frequency corresponding to the peak of the "single-peak" Wallops spectrum.

31. For current speed of $U = -2$m/sec , $\S = 0.015$ (the value of $\S$ rarely exceeding 0.025 in the field), and $\omega_o = 0.6$ rad/sec , the quantities $E[\zeta_b]$, $E\left[\zeta_b^2\right]$, $\bar{S}_b(\omega_r)$, and $S_b(\omega_a)$ are computed for various values of water depth d . The solutions are carried out in an iterative manner; that is, upon obtaining $\bar{S}_b(\omega_r)$ in Equation 28, it is treated as the original ideal wave spectrum, and the solution process is repeated until convergence is reached. Based on the final values of $\bar{S}_b(\omega_r)$, the quantities $E[\zeta_b]$, $E\,\zeta_b^2$, and $S_b(\omega_a)$ are then determined. The results presented in the following are obtained after four cycles of iteration. It should be mentioned that the above iterative scheme, strictly speaking, is not valid since some of the assumptions underlying the derivation of these quantities are violated because it was originally assumed that the ideal waves must be zero mean and Gaussian. Our results show, however, that the mean value of ζ_b is insignificantly small, and preliminary investigation indicates that the breaking wave elevation ζ_b deviates but slightly from Gaussian.

32. In Figure 2, $E[\zeta_b]$ is plotted as a function of $\bar{k}_o d$ for $\bar{k}_o d$ ranging between 3 and 0.5 where $\bar{k}_o = \bar{\omega}^2/g$ is the characteristic deepwater wave number, $\bar{\omega}$ being given by Equation 3 with $S(\omega)$ replaced by $S_o(\omega)$, the Wallops spectrum. If we denote by $\bar{k} = \bar{k}_o/\tanh^{1/2} \bar{k}_o d$ according to Equation 8, these values of $\bar{k}_o d$ correspond to $\bar{k}d = 4.1$ and 1 (or $d = 81.3^m$ and 13.6^m). It is seen that $E[\zeta_b]$ is always negative, as expected, and indeed very small.

33. Figure 2 also gives the standard deviation $\sqrt{r}$ (see Equation 15) of the elevation of the ideal waves and $\sqrt{r_b} = \sqrt{E\left[\zeta_b^2\right] - E^2[\zeta_b]}$ that of the breaking waves. While $\sqrt{r_b}$ decreases monotonically from deep to shallow water, $\sqrt{r}$ first decreases slightly. Beyond $k\,\bar{d}_o = 0.6$ shoreward, however, it begins to rise because of shoaling. Owing to the relatively small value of

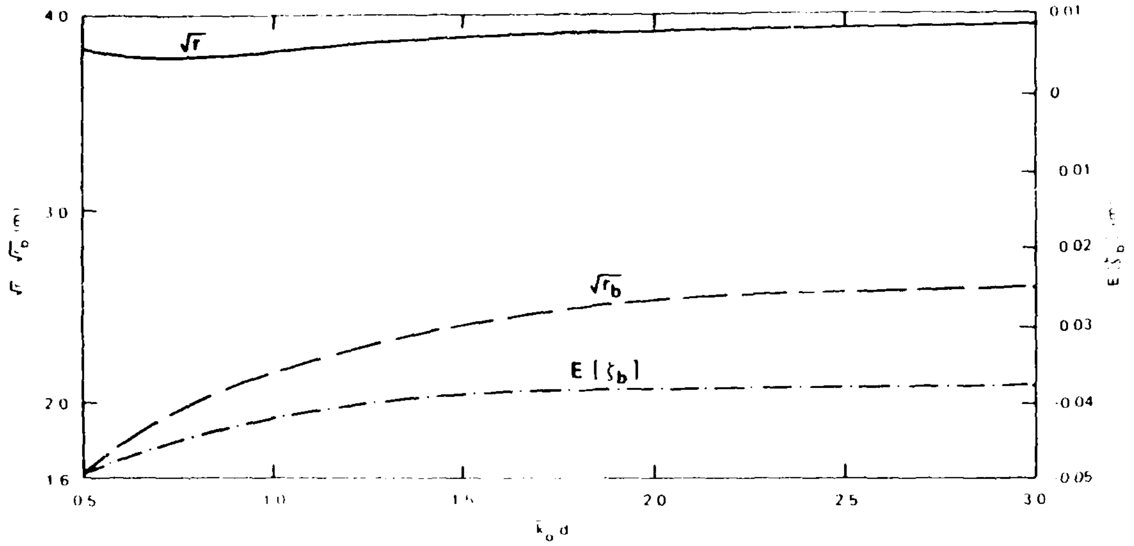

Figure 2. Mean value $E[\zeta_b]$ of breaking waves, standard deviations $\sqrt{r}$ of ideal waves, and $\sqrt{r_b}$ of breaking waves plotted against $\bar{k}_o d$ for $\S = 0.015$, $\omega_o = 0.6$ rad/sec, and $U = -2$ m/sec

§ and the strong negative current speed used, wave breaking is seen to occur everywhere but more so in shallower than in deeper water.

34. For $\bar{k}_o d = 3, 2, 1$ and 0.5 , the spectra $S_o(\omega_r)$, $\bar{S}(\omega_r)$, and $\bar{S}_b(\omega_r)$ are in Figures 3, 5, 7, and 9, respectively, and those in the stationary frame of reference $S_o(\omega_a)$, $S(\omega_a)$, and $S_b(\omega_a)$ are given in Figures 4, 6, 8, and 10, respectively. An adverse current feeds energy into the wave system so that the ideal wave spectra always exceed those in deep water where there is no current. Wave breaking, however, dissipates wave energy, and the breaking wave spectra are seen to fall below $S_o(\cdot)$. Close examination of these spectra also shows that this pattern of variation with water depth is consistent with that of the standard deviations shown in Figure 2.

35. As mentioned earlier, the quantity r gives an indication of the extent of wave breaking and is expected to be larger than unity, with the larger values corresponding to milder sea state. It was also shown that the quantity ω_1 , the cutoff frequency of the breaking wave spectrum given in Equation 30, is expected to be much larger than ω , the characteristic wave frequency. It is, therefore, of interest to examine the variation of these two quantities as the waves move toward the shore. In Figure 11 the

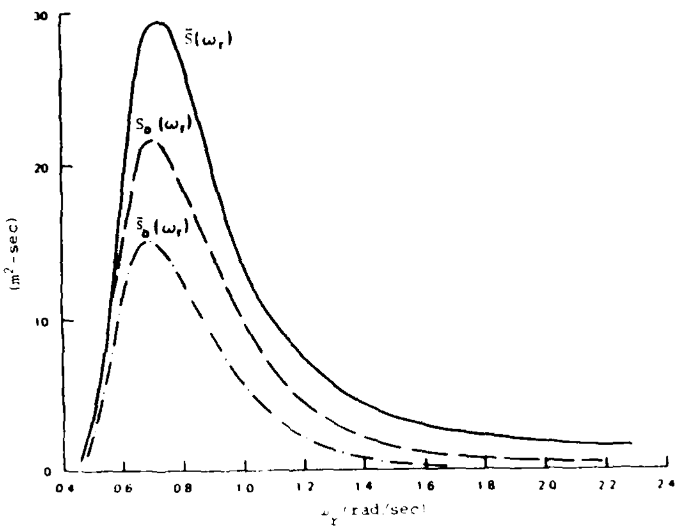

Figure 3. Deepwater wave spectrum $S_o(\omega_r)$, ideal wave spectrum $\bar{S}(\omega_r)$, and breaking wave spectrum $\bar{S}_b(\omega_r)$ in relative frame of reference for $\bar{k}_o d = 3.0$, $\delta = 0.015$, $\omega_o = 0.6$ rad/sec, and $U = -2$ m/sec

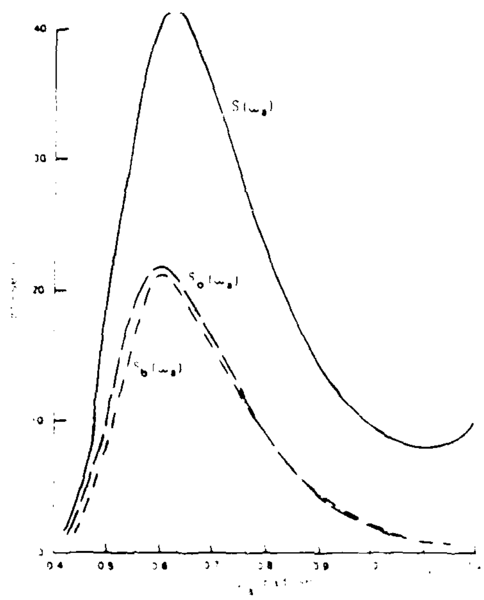

Figure 4. Deepwater wave spectrum $S_o(\omega_a)$, ideal wave spectrum $S(\omega_a)$, and breaking wave spectrum $S_b(\omega_a)$ in stationary frame of reference for $\bar{k}_o d = 3.0$, $\delta = 0.015$, $\omega_o = 0.6$ rad/sec, and $U = -2$ m/sec

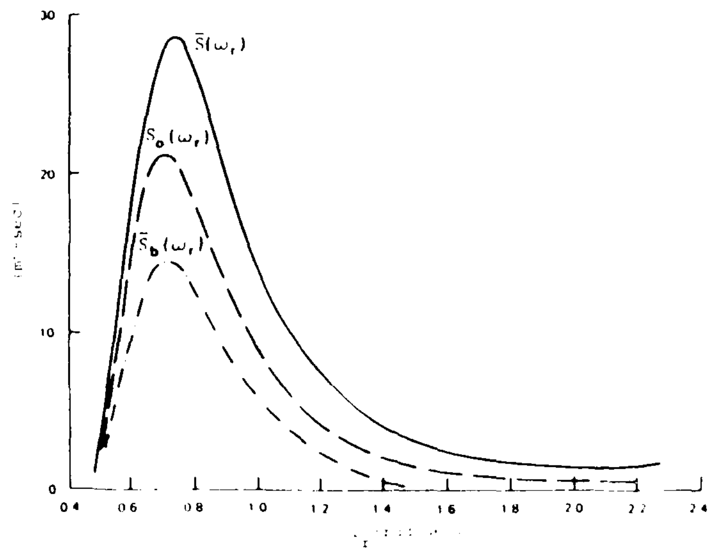

Figure 5. Deepwater wave spectrum $S_o(\omega_r)$, ideal wave spectrum $\bar{S}(\omega_r)$, and breaking wave spectrum $\bar{S}_b(\omega_r)$ in relative frame of reference for $\bar{k}_o d$ = 2.0 , § = 0.015 , ω_o = 0.6 rad/sec, and U = -2 m/sec

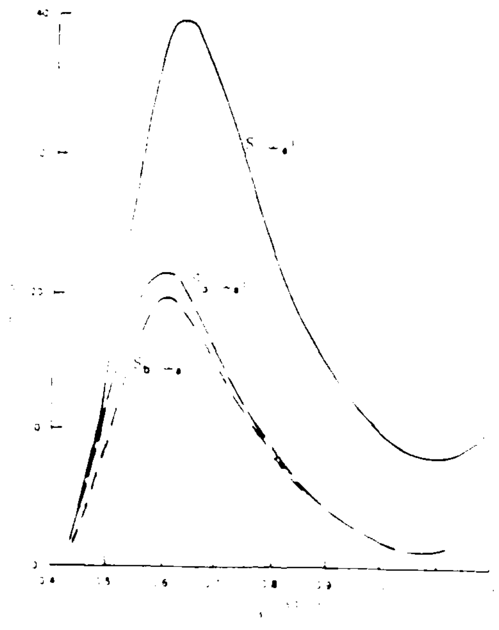

Figure 6. Deepwater wave spectrum $S_o(\omega_a)$, ideal wave spectrum $S(\omega_a)$, and breaking wave spectrum $S_b(\omega_a)$ in stationary frame of reference for $\bar{k}_o d$ = 2.0 , § = 0.015 , ω_o = 0.06 rad/sec, and U = -2 m/sec

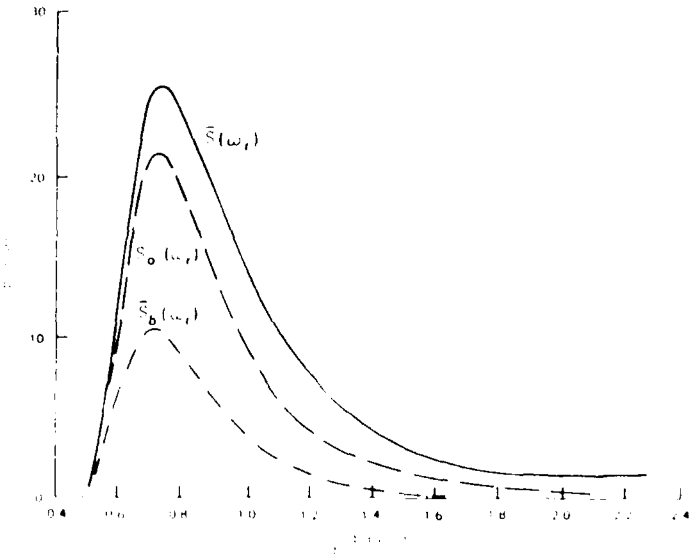

Figure 7. Deepwater wave spectrum $S_o(\omega_r)$, ideal wave spectrum $\bar{S}(\omega_r)$, and breaking wave spectrum $\bar{S}_b(\omega_r)$ in relative frame of reference for $\bar{k}_o d = 1.0$, $\delta = 0.015$, $\omega_o = 0.6$ rad/sec, and $U = -2$ m/sec

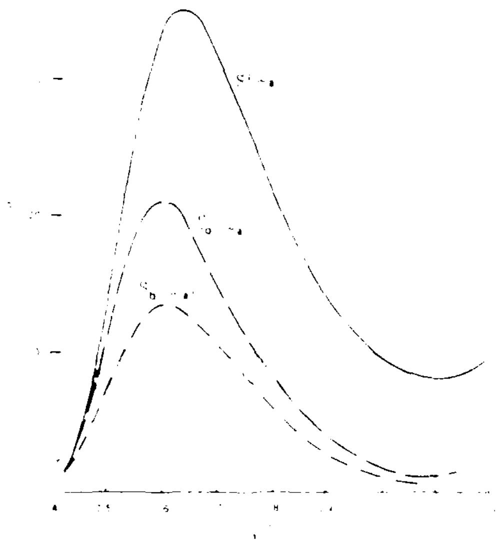

Figure 8. Deepwater wave spectrum $S_o(\omega_a)$, ideal wave spectrum $S(\omega_a)$, and breaking wave spectrum $S_b(\omega_a)$ in stationary frame of reference for $\bar{k}_o d = 1.0$, $\delta = 0.015$, $\omega_o = 0.6$ rad/sec, and $U = -2$ m/sec

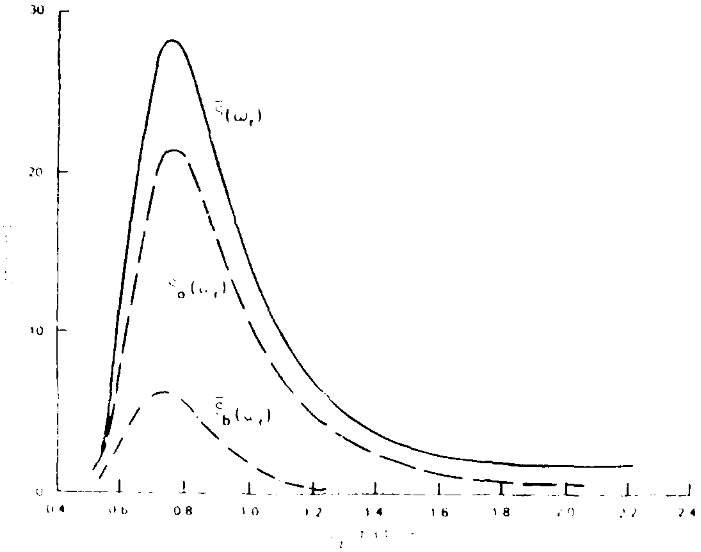

Figure 9. Deepwater wave spectrum $S_o(\omega_r)$, ideal wave spectrum $\bar{S}(\omega_r)$, and breaking wave spectrum $\bar{S}_b(\omega_r)$ in relative frame of reference for $\bar{k}_o d$ = 0.05 , δ = 0.015 , ω_o = 0.6 rad/sec, and U = -2 m/sec

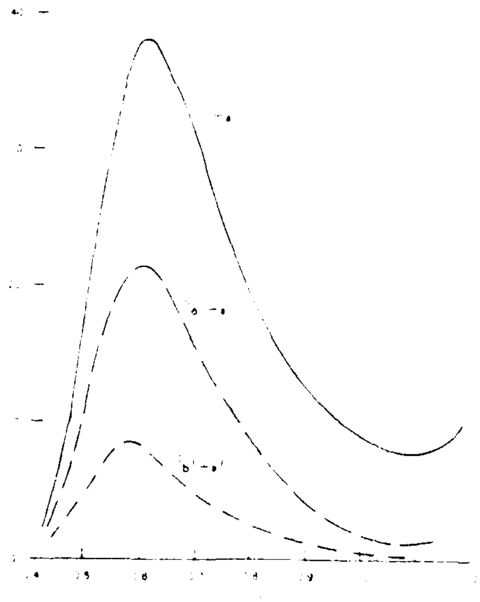

Figure 10. Deepwater wave spectrum $S_o(\omega_a)$, ideal wave spectrum $\bar{S}(\omega_a)$, and breaking wave spectrum $\bar{S}_b(\omega_a)$ in stationary frame of reference for $\bar{k}_o d$ = 0.5 , δ = 0.015 , ω_o = 0.06 rad/sec, and [illegible]

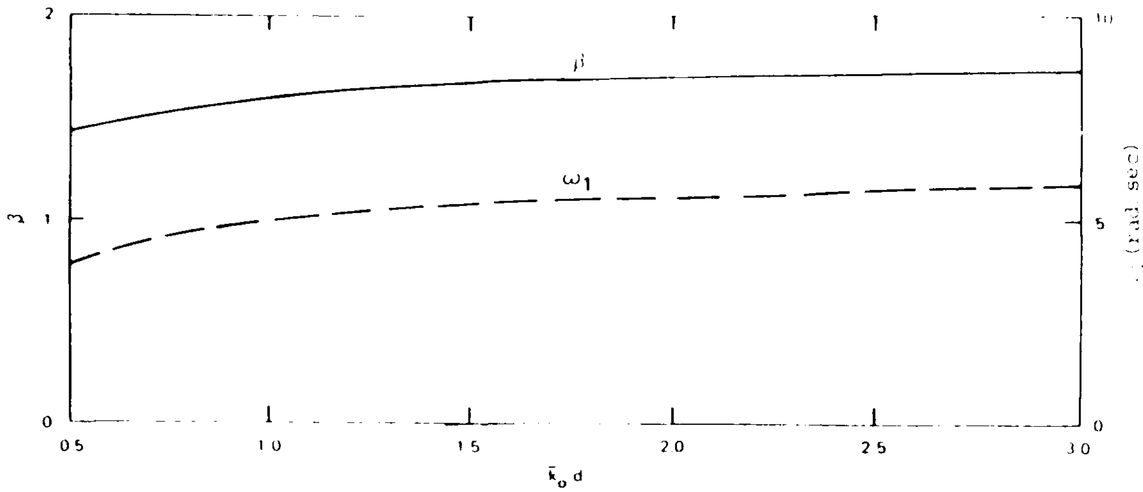

Figure 11. Wave breaking parameter β and cutoff frequency ω_1 of breaking wave spectrum plotted against $\bar{k}_o d$ for $\S = 0.015$, $\omega_o = 0.6$ rad/sec, and $U = -2$ m/sec

quantities β and ω_1 are plotted against $\bar{k}_o d$. The value of β decreases from 1.8 in deep water to about 1.4 at $\bar{k}_o d = 0.5$ and that of ω_1 from 6.0 to 4.0 rad/sec.

36. Although our primary objective in this study is to devise a method for the calculation of the breaking wave spectrum under the influence of an adverse current, the method, as is obvious, may be applied to the special case in which there is no current. The results presented in the following are for $U = 0$, $\S = 0.015$, and $\omega_o = 0.6$ rad/sec for $\bar{k}_o d = 3$ to 0.5 corresponding to $\bar{k}d = 3$ to 0.74.

37. In Figure 12, $E[\zeta_b]$, $\sqrt{r}$, and $\sqrt{r_b}$ are given as functions of $\bar{k}_o d$. As expected, $E[\zeta_b]$ is always negative and even smaller than when $U = -2$ m/sec in Figure 2. Because of the small value of significant slope $\S = 0.015$ used, the two curves $\sqrt{r}$ and $\sqrt{r_b}$ are practically indistinguishable until $\bar{k}_o d < 1.5$ $\left(\bar{k}d < 1.6,\ d < 41.7^{m}\right)$ when wave breaking becomes noticeable and they begin to diverge from each other.

38. The variation of the quantities β and ω_1 with $\bar{k}_o d$ is shown in Figure 13. The value of β changes from 2.9 to 1.9, and that of ω_1 from 20 rad/sec to 3 rad/sec as $\bar{k}_o d$ goes from 3 to 0.5. Comparison of Figure 13

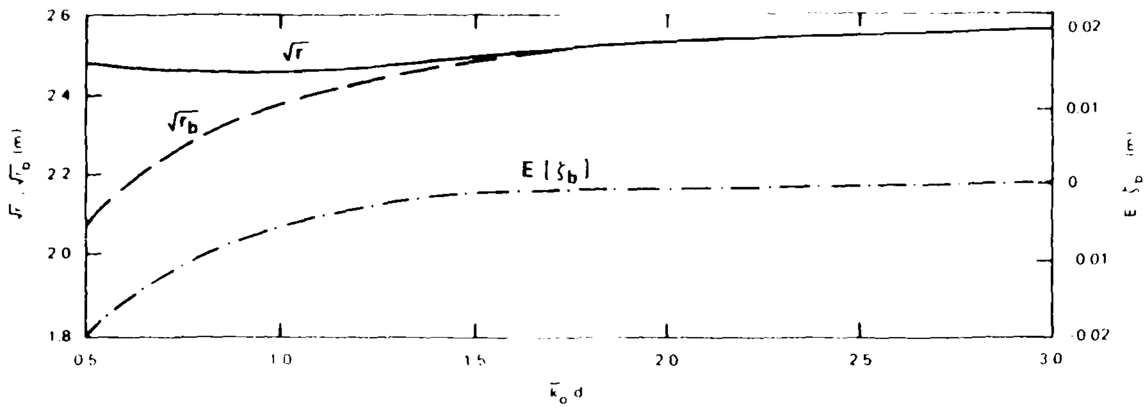

Figure 12. Mean value $E[\zeta_b]$ of breaking waves, standard deviations $\sqrt{r}$ of ideal waves, and $\sqrt{r_b}$ of breaking waves plotted against $\bar{k}_o d$ for § = 0.015 , ω_o = 0.6 rad/sec , and U = 0

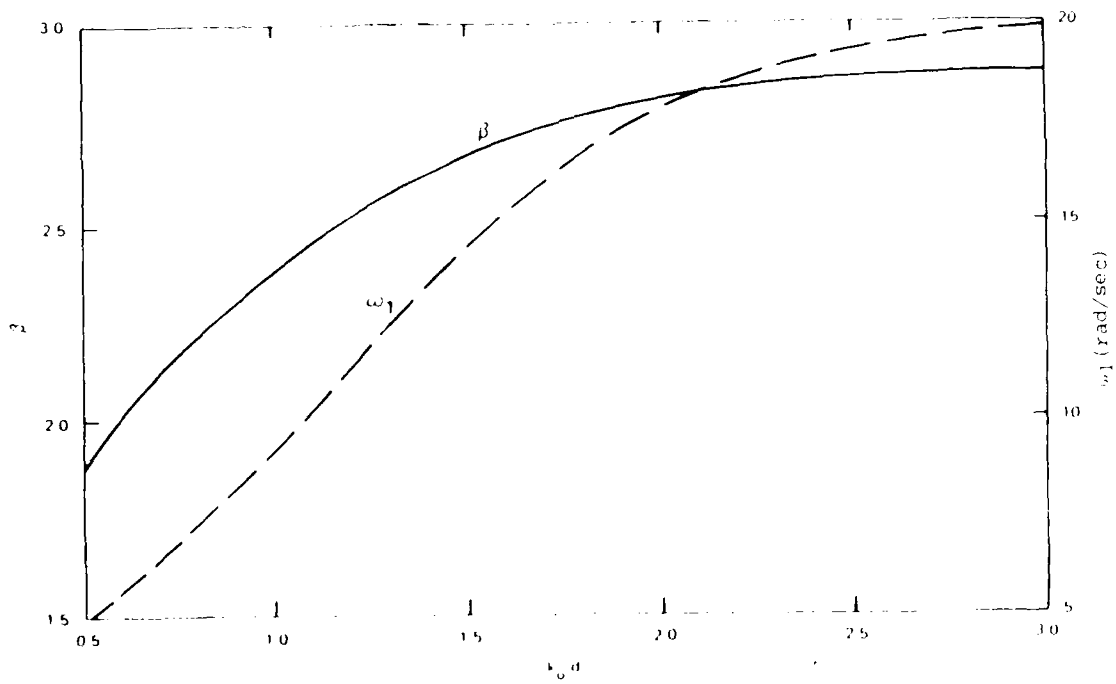

Figure 13. Wave breaking parameter β and cutoff frequency ω_1 of breaking wave spectrum plotted against $\bar{k}_o d$ for § = 0.015 , ω_o = 0.6 rad/sec , and U = 0

with Figure 11 for the case of $U = -2$ m/sec shows that the quantities F and ω_1 undergo more variation in the present case of $U = 0$ than in the case $U = -2$ m/sec because in the latter case the current dominates the flow field so that wave breaking is uniformly present regardless of the locale under consideration.

39. The spectra of the deepwater waves $S_o(\omega)$, the local ideal waves $S(\omega)$, and the breaking waves $S_b(\omega)$ will not be shown; but the peak values (in square metres per second) are recorded in the following tabulation:

$\bar{k}_o d$	$S_o(\omega)$	$S(\omega)$	$S_b(\omega)$
3	22.3	22.0	21.7
2	22.3	21.1	20.8
1	22.3	19.5	18.3
0.5	22.3	20.7	15.0

PART VI: CONCLUSION

40. We have given in this report a method to compute wave spectra in water of finite depth taking into account the effect of wave breaking and considering the presence of current. The breaking wave spectrum is simply related to that of the original ideal waves. The method, however, is approximate because the following is assumed: (a) there exists an ideal original wave train which is linear and Gaussian; (b) the wave breaking model is heuristic, and some approximations have been introduced; and (c) the higher order terms in the expression of the breaking wave spectrum are ignored. As such, the model should only be applied to the energy containing part of the spectrum and must not be used for points too close to the shore where all the assumptions underlying this model will be violated.

REFERENCES

Abramowitz, M., and Stegun, I. A. 1968. Handbook of Mathematical Functions, Dover Publications, Inc., New York, N.Y., pp 228-936.

Battjes, J. A. 1974. "Computation of Set-up, Long Shore Currents, Run-up and Overtopping due to Wind-generated Waves," Communications on Hydraulics No. 74-2, Department of Civil Engineering, Delft University of Technology, Delft, The Netherlands.

Battjes, J. A., and Janssen, J. P. F. M. 1978. "Energy Loss and Set-up due to Breaking of Random Waves," Proceedings of the 16th International Conference of Coastal Engineering, American Society of Civil Engineers, New York, N.Y., pp 569-587.

Borgman, L. E. 1965. "A Statistical Theory for Hydrodynamic Forces on Objects," Technical Report HEL-9-6, Hydraulics Engineering Laboratory, University of California, Berkeley, Calif.

Cartwright, D. E., and Longuet-Higgins, M. S. 1956. "The Statistical Distribution of the Maxima of a Random Function," Proceedings of the Royal Society of London, Ser A, Vol 237, pp 212-232.

Erdely, A., Magnum, W., Oberhettinger, F., and Tricomi, F. G. 1953. Higher Transcendental Functions, Vol 2, McGraw-Hill, New York, N.Y.

Hedges, T. S., Burrows, R., and Mason, W. G. 1979. "Wave-current Interaction and the Effects on Fluid Loading," Report No. MCE/3/79, Department of Civil Engineering, University of Liverpool, England.

Huang, N. E., Chen, D. T., Tung, C. C., and Smith, J. 1972. "Interactions between Steady Non-uniform Current and Gravity Waves with Applications for Current Measurements," Journal of Physical Oceanography, Vol 2, pp 420-431.

Huang, N. E., Long, S. R., Tung, C. C., Yuen, Y., and Bliven, L. F. 1981. "A Unified Two-parameter Wave Spectral Model for a General Sea State, Journal of Fluid Mechanics, Vol 112, pp 203-224.

Longuet-Higgins, M. S. 1969. "On Wave Breaking and the Equilibrium Spectrum of Wind-generated Waves," Proceedings of the Royal Society of London, Ser A, Vol 310, pp 151-159.

Ochi, M. K., and Tsai, C. H. 1983. "Prediction of Occurrences of Breaking Waves in Deep Water," Journal of Physical Oceanography, Vol 13, pp 2008-2019.

Papoulis, A. 1965. Probability, Random Variables, and Stochastic Processes, McGraw-Hill Book Co., Inc., New York, N.Y., pp 206-221, 239-258.

Phillips, O. M. 1980. The Dynamics of the Upper Ocean, Cambridge University Press New York, N.Y., p 142.

Stokes, G. G. 1880. "Supplement to a Paper on the Theory of Oscillatory Waves," Mathematical Physical Paper, Vol I, pp 225-228.

Thornton, E. B., and Guza, R. T. 1983. "Transformation of Wave Height Distribution," Journal of Geophysical Research, Vol 88, pp 5925-5938.

Tung, C. C., and Huang, N. E. 1987. "Spectrum of Deep Water Breaking Waves," Journal of Engineering Mechanics, American Society of Civil Engineers, Vol 113, pp 293-302.

Yuan, Y., Tung, C. C., and Huang, N. E. 1986. "Statistical Characteristics of Breaking Waves," Wave Dynamics and Radio Probing of the Ocean Surface, (eds. O. M. Phillips and K. L. Hasselmann), Plenum Press, New York, N.Y., pp 265-272.

APPENDIX A: DERIVATION OF $E[\zeta_1\zeta_2 H''_{1+} H''_{2+}]$

1. To obtain $E[\zeta_1\zeta_2 H''_{1+} H''_{2+}]$, the concept of conditional probability and conditional expectation to reduce the number of random variables is used, thus (Papoulis 1965),*

$$E[\zeta_1\zeta_2 H''_{1+} H''_{2+}] = E\left[H''_{1+} H''_{2+} E\left[\zeta_1\zeta_2 \middle| \ddot{\zeta}_1 , \ddot{\zeta}_2\right]\right] \quad \text{(A1)}$$

where

$$E\left[\zeta_1\zeta_2 \middle| \ddot{\zeta}_1, \ddot{\zeta}_2\right] = \iint_{-\infty}^{\infty} \zeta_1\zeta_2 f_{\zeta_1,\zeta_2|\ddot{\zeta}_1,\ddot{\zeta}_2}(\zeta_1,\zeta_2)\, d\zeta_1 d\zeta_2 \quad \text{(A2)}$$

is the conditional expected value of $\zeta_1\zeta_2$ given $\ddot{\zeta}_1$ and $\ddot{\zeta}_2$, and

$$f_{\zeta_1,\zeta_2|\ddot{\zeta}_1,\ddot{\zeta}_2}(\zeta_1,\zeta_2) = \frac{1}{2\pi\left(1-\rho_{12}^2\right)^{1/2}\sigma_1\sigma_2}$$

$$\exp\left\{-\frac{1}{2\left(1-\rho_{12}^2\right)}\left[\left(\frac{\zeta_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{\zeta_2-\mu_2}{\sigma_2}\right)^2 - 2\rho_{12}\left(\frac{\zeta_1-\mu_1}{\sigma_1}\right)\left(\frac{\zeta_2-\mu_2}{\sigma_2}\right)\right]\right\} \quad \text{(A3)}$$

is the jointly Gaussian conditional probability density function of ζ_1 and ζ_2 given $\ddot{\zeta}_1$ and $\ddot{\zeta}_2$, where

ρ_{12} = conditional covariance coefficient function of ζ_1 and ζ_2

σ_1^2, σ_2^2 = conditional variance functions of ζ_1 and ζ_2

μ_1, μ_2 = conditional mean value functions of ζ_1 and ζ_2

2. The five parameters, ρ_{12}, σ_1^2, σ_2^2, μ_1, and μ_2, may all be determined using the linear mean-square estimation technique (Papoulis 1965); that is,

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* References cited in the Appendix can be found in the References at the end of the main text.

$$\mu_1 = a_1\ddot{\xi}_1 + b_1\ddot{\xi}_2 \tag{A4}$$

and

$$\mu_2 = a_2\ddot{\xi}_1 + b_2\ddot{\xi}_2 \tag{A5}$$

where a_1 , b_1 , a_2 , and b_2 are determined based on the condition that $(\xi_1 - \mu_1)$ and $(\xi_2 - \mu_2)$ are orthogonal to, and hence independent of, $\ddot{\xi}_1$ and $\ddot{\xi}_2$ giving

$$a_1 = \left[\frac{r^{(2)}r^{(4)} - r_{12}^{(2)}r_{12}^{(4)}}{\Delta}\right] = b_2 \tag{A6}$$

$$b_1 = \left[\frac{r_{12}^{(2)}r^{(4)} - r^{(2)}r_{12}^{(4)}}{\Delta}\right] = a_2 \tag{A7}$$

$$\Delta = \left[r^{(4)}\right]^2 - \left[r_{12}^{(4)}\right]^2 \tag{A8}$$

The utilization of the same orthogonal properties leads to

$$\sigma_1^2 = E\left[\left(\xi_1 - \mu_1\right)^2 \mid \ddot{\xi}_1 , \ddot{\xi}_2\right] = E\left[\left(\xi_1 - \mu\right)^2\right] = r - a_1 r^{(2)} - b_1 r_{12}^{(2)} \tag{A9}$$

which may be shown to be the same as σ_2^2 and

$$\rho_{12} = \frac{E\left[(\xi_1 - \mu_1)(\xi_2 - \mu_2) \mid \ddot{\xi}_1 , \ddot{\xi}_2\right]}{\sigma_1\sigma_2} = \frac{E[(\xi_1 - \mu_1)(\xi_2 - \mu_2)]}{\sigma_1\sigma_2}$$

$$= \frac{\left[r_{12} - a_1 r_{12}^{(2)}\right] - b_1 r^{(2)}}{\sigma_1\sigma_2} \tag{A10}$$

3. In the above, $r = r_{12}(o)$, $r^{(2)} = r_{12}^{(2)}(o)$, and $r^{(4)} = r_{12}^{(4)}(o)$ are given in terms of $S(\omega)$ as indicated in Equations 15, 16, and 17, respectively. The argument τ (the time lag in $r_{12}(\tau)$, $r_{12}^{(2)}(\tau)$, and $r_{12}^{(4)}(\tau)$) is omitted for brevity. The quantities μ_1 , μ_2 , σ_1, σ_2 , and ρ_{12} are

all functions of $r_{12}(\tau)$, $r_{12}^{(2)}(\tau)$, and $r_{12}^{(4)}(\tau)$ and hence are functions of τ .

4. The conditional expected value $E[\zeta_1\zeta_2|\ddot{\zeta}_1,\ddot{\zeta}_2]$ is seen to be the conditional correlation function of ζ_1 and ζ_2 and is therefore by definition given by

$$E[\zeta_1\zeta_2|\ddot{\zeta}_1,\ddot{\zeta}_2] = \mu_1\mu_2 + \rho_{12}\sigma_1\sigma_2$$
$$= \left(a_1^2 + b_1^2\right)\ddot{\zeta}_1\ddot{\zeta}_2 + a_1b_1\left(\ddot{\zeta}_1^2 + \ddot{\zeta}_2^2\right) + \rho_{12}\sigma_1\sigma_2 \qquad \text{(A11)}$$

The expected value sought is, therefore, from Equation A1,

$$E[\zeta_1\zeta_2H''_{1+}H''_{2+}] = \iint_{-Kg}^{\infty} \left[\left(a_1^2 + b_1^2\right)\ddot{\zeta}_1\ddot{\zeta}_2 + a_1b_1\left(\ddot{\zeta}_1^2 + \ddot{\zeta}_2^2\right) + \rho_{12}\sigma_1\sigma_2\right] f_{\ddot{\zeta}_1,\ddot{\zeta}_2}(\ddot{\zeta}_1,\ddot{\zeta}_2)\,d\ddot{\zeta}_1 d\ddot{\zeta}_2 \qquad \text{(A12)}$$

where $f_{\ddot{\zeta}_1,\ddot{\zeta}_2}(\ddot{\zeta}_1,\ddot{\zeta}_2)$ is the jointly Gaussian probability density function of the zero-mean random variables $\ddot{\zeta}_1$ and $\ddot{\zeta}_2$ whose variances are $E\left[\ddot{\zeta}_1^2\right] = E\left[\ddot{\zeta}_2^2\right] = r^{(4)}$, and whose correlation coefficient function is $\rho_{12}^{(4)}(\tau) = E[\ddot{\zeta}_1\ddot{\zeta}_2]\big/r^{(4)} = r_{12}^{(4)}\big/r^{(4)}$.

5. The above integrals may all be carried out giving

$$E[\zeta_1\zeta_2H''_{1+}H''_{2+}] = \left(a_1^2 + b_1^2\right)F_1 + 2a_1b_1F_2 + \rho_{12}\sigma_1\sigma_2F_3 \qquad \text{(A13)}$$

where

$$F_1 = r^{(4)}\left\{\left[1 - \left(\rho_{12}^{(4)}\right)^2\right]^{1/2} Z(\beta)Z(\alpha) - 2\beta\rho_{12}^{(4)}Z(\beta)Q(\alpha) + \rho_{12}^{(4)}F_3\right\} \qquad \text{(A14)}$$

$$F_2 = r^{(4)}\left\{\left[1 - \left(\rho_{12}^{(4)}\right)^2\right]^{1/2} \rho_{12}^{(4)} Z(\beta)Z(\Omega) - \beta\left[1 + \left(\rho_{12}^{(4)}\right)^2\right] Z(\beta)Q(\Omega) + F_3\right\} \tag{A15}$$

and

$$F_3 = L\left(-\beta,\ -\beta,\ \rho_{12}^{(4)}\right) \tag{A16}$$

(see Abramowitz and Stegun 1968). Here, the argument τ in $\rho_{12}^{(4)}(\tau)$ is omitted

$$\Omega = -\beta\left[\frac{1 - \rho_{12}^{(4)}}{1 + \rho_{12}^{(4)}}\right]^{1/2} \tag{A17}$$

and β , $Z(\cdot)$, and $Q(\cdot)$ are defined in Equations 19, 20, and 22, respectively.

6. The expected value in Equation A1 is a nonlinear function of $r_{12}(\tau)$, $r_{12}^{(2)}(\tau)$, and $r_{12}^{(4)}(\tau)$ and may be expanded by the Taylor series. By retaining only the zeroth and the first order terms, it is given approximately by

$$E[\epsilon_1\epsilon_2 H''_{1+}H''_{2+}] = a_1^2\, r^{(4)} Z^2(\beta) + a_1^2\, r^{(4)}\rho_{12}^{(4)}[-\beta Z(\beta) + Q(-\beta)]^2 + 2a_1 b_1 r^{(4)}\, Q(-\beta)[-\beta Z(\beta) + Q(-\beta)] + \rho_{12}\sigma_1\sigma_2 Q^2(-\beta) \tag{A18}$$

where

$$a_1 \doteq \left[\frac{r^{(2)}}{r^{(4)}}\right]^{1/2} \tag{A19}$$

$$h_1 \doteq \frac{\left[r_{12}^{(2)}\ r^{(4)} - r_{12}^{(4)}\right] r^{(2)}}{\left[r^{(4)}\right]^2} \tag{A20}$$

and

$$[illegible]_{12}[illegible]_1[illegible]_2 \doteq r_{12} - 2r_{12}^{(2)}\ \frac{r^{(2)}}{r^{(4)}} + r_{12}^{(4)} \left[\frac{r^{(2)}}{r^{(4)}}\right]^2 \tag{A21}$$

7. The integration in Equation A12 may also be facilitated by employing the Hermite polvrominal series representation (Erdely et al. 1953) as follows:

$$\frac{1}{\sqrt{1 - p^2}} \exp - \left[\frac{1}{2(1 - p^2)} (x^2 + y^2 - 2pxy)\right]$$

$$= 2\pi \sum_{n=0}^{\infty} p^n h_n(x) h_n(y) Z(x) Z(y) \tag{A22}$$

where

$$h_n(x) = \frac{(-1)^n}{(n!)^{1/2}} \frac{\frac{d^n Z(x)}{dx^n}}{Z(x)} \tag{A23}$$

is the Hermite polvnominal function. Upon expanding the jointly Gaussian probability density function $f_{\tilde{z}_1, \tilde{z}_2}(\tilde{z}_1, \tilde{z}_2)$ into the Hermite series representation, it is seen that the integrals in Equation A12 may be carried out easily. By retaining only the terms involving $n = 0$ and 1 in the series, Equation A18 may be obtained.

A, A_B	Total wave surface area and wave surface area with $\|\ddot{\zeta}\| > Kg$
A_1, A_2	Quantities defined in Equations 31 and 32, respectively
a, a_b	Amplitudes of ideal and breaking waves, respectively
a_1, a_2	Quantities defined in Equations A6 and A7 whose approximate values are given in Equations A19 and A20
b_1, b_2	Quantities defined in Equations A7 and A6 whose approximate values are given in Equations A20 and A19
c_{go}, c_{gr}	Group velocities defined in Equations 37 and 38, respectively
c_1, c_2	Parameters used in Equation 21
d	Water depth
$E[\cdot]$	Expected value of the quantity enclosed in the brackets
$E[\cdot\|\cdot]$	Conditional expected value
$E_1(\cdot)$	Exponential integral defined in Equation 23
F_1, F_2, F_3	Quantities defined in Equations A14, A15, and A16
$F(\cdot)$	Filter function defined in Equation 29
$f_{\ddot{\zeta}_1,\ddot{\zeta}_2}(\cdot,\cdot)$	Joint probability density function of the Gaussian zero-mean random variables $\ddot{\zeta}_1$ and $\ddot{\zeta}_2$
$f_{\zeta_1,\zeta_2\|\ddot{\zeta}_1,\ddot{\zeta}_2}(\cdot,\cdot)$	Conditional joint probability density function of ζ_1 and ζ_2 given $\ddot{\zeta}_1$ and $\ddot{\zeta}_2$
g	Gravitational acceleration
$H(\cdot)$	Heaviside unit step function
$H, H''_+ H''_-$	Abbreviations for $H(\zeta)$, $H(\ddot{\zeta} + Kg)$, and $H(-\ddot{\zeta} - Kg)$, respectively
$h_n(\cdot)$	Hermite polynominal function defined in Equation A23
K	Coefficient defined in Equation 13
$K_b(\tau)$	Autocovariance function of breaking wave elevation $\zeta_b(t)$
k, $\bar{k}$	Wave number and characteristic wave number, respectively, in water of finite depth
k_o, $\bar{k}_o$	Wave number and characteristic wave number, respectively, in deep water
$L(\cdot,\cdot,\cdot)$	Probability function defined in Equation 21

m	Magnitude of the slope of the Wallops spectrum [illegible] deepwater waves in the high frequency range [illegible] log-log scale given in Equation 42
$N, \bar{N}$	Quantities defined in Equations [illegible] and [illegible], respectively
n	Dummy index
p	Parameters used in Equations 21 and A[illegible]
$Q(\cdot)$	Probability function defined in Equation [illegible]
$R_b(\tau)$	Autocorrelation function of breaking wave elevation $\zeta_b(t)$
$r, r^{(2)}, r^{(4)}$	Quantities defined in Equations 15, 16, and 1[illegible]
r_b	Variance of breaking wave elevation $\zeta_b(t)$
$r_{12}(\cdot), r_{12}^{(2)}(\cdot)$, $r_{12}^{(4)}(\cdot)$	Correlation functions $E[\zeta_1\zeta_2]$, $E[\zeta_1\dot{\zeta}_2]$, and $E[\ddot{\zeta}_1\ddot{\zeta}_2]$ of the ideal waves, respectively
$S(\cdot), S_b(\cdot)$	Ideal and breaking wave spectra, respectively
$S_0(\cdot)$	Wave spectrum in deep water
$\bar{S}(\cdot), \bar{S}_b(\cdot)$	Ideal and breaking wave spectra, respectively, [illegible] relative frame of reference in the presence of current
t	Time
t_1, t_2	Time instants $t + \tau$ and t, respectively
U	Current speed
w	Quantity defined in Equation [illegible]
x, y	Dummy variables
$Z(\cdot)$	Probability function defined in Equation [illegible]
[illegible]	Coefficient defined in Equation 44
[illegible]	Wave breaking parameter defined in Equation [illegible]
$\Gamma(\cdot)$	The Gamma function
[illegible]	Quantity defined in Equation A8
[illegible]	Spectral bandwidth parameter defined in Equation [illegible]
$\zeta(t), \zeta_b(t)$	Elevations of ideal and breaking waves, respectively
λ_0	Characteristic wave length
μ_1, μ_2	Conditional mean value functions of ζ_1 and ζ_2, respectively, given $\ddot{\zeta}_1$ and $\ddot{\zeta}_2$
ρ_{12}	Conditional covariance coefficient function of ζ_1 and ζ_2 given $\ddot{\zeta}_1$ and $\ddot{\zeta}_2$ in Equation A10

$\rho_{12}^{(4)}$	Correlation coefficient function of $\ddot{\zeta}_1$ and $\ddot{\zeta}_2$
σ_1, σ_2	Conditional standard deviation of ζ_1 and ζ_2, respectively, given $\ddot{\zeta}_1$ and $\ddot{\zeta}_2$
τ	Time lag
Ω	Quantity defined in Equation A17
ω	Wave frequency
$\bar{\omega}$	Characteristic wave frequency defined in Equation 3
ω_a, ω_r	Wave frequency in stationary and relative frames of reference, respectively
ω_o	Parameter of Wallops wave spectrum
ω_1	Cutoff frequency of breaking waves given in Equation 30

Subscripts

1, 2	Quantities evaluated at time instants t_1 and t_2, respectively
o	Quantities evaluated in deep water in zero current condition

Symbols

•	Differentiation with respect to time
§	Significant wave slope defined in Equation 43

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