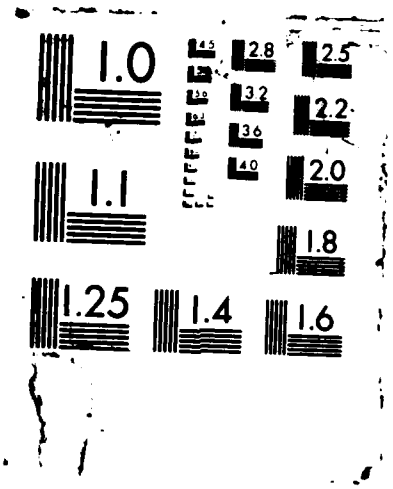
AD-A190 278 ON HOTELLING'S FORMULA FOR THE VOLUME OF TUBES AND NAIMAN'S INEQUALITY(U) STANFORD UNIV CA DEPT OF STATISTICS I JOHNSTONE ET AL. DEC 87 TR-5 1/1

UNCLASSIFIED N00014-87-K-0078 F/G 12/1 NL


1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.5
2.8
3.2
3.6
4.0
4.5

ON HOTELLING'S FORMULA FOR THE VOLUME OF TUBES AND NAIMAN'S INEQUALITY

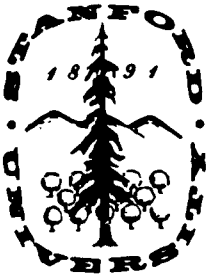
by

Iain Johnstone and David Siegmund
Stanford University

TECHNICAL REPORT NO. 5
DECEMBER 1987

PREPARED UNDER CONTRACT
N00014-87-K-0078 (NR-042-373)
FOR THE OFFICE OF NAVAL RESEARCH

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA



ON HOTELLING'S FORMULA FOR THE VOLUME OF TUBES AND NAIMAN'S INEQUALITY

by

Iain Johnstone and David Siegmund
Stanford University

TECHNICAL REPORT NO. 5
DECEMBER 1987

PREPARED UNDER CONTRACT
N00014-87-K-0078 (NR-042-373)
FOR THE OFFICE OF NAVAL RESEARCH

Also prepared under National Science Foundation grant DMS86-002335 and issued as Technical Report No. 284, Department of Statistics, Stanford University.

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

On Hotelling's Formula for the Volume of Tubes and Naiman's Inequality

by

Iain Johnstone and David Siegmund
Stanford University

December 1987

Abstract.

Two new derivations of the Hotelling–Naiman results on volume of tubes about curves in spheres are presented. The first involves simple differential inequalities. The second is probabilistic, using the concept of upcrossing borrowed from the theory of Gaussian processes. The upcrossings method is extended to a harmonic regression problem not covered by the Hotelling–Naiman formulation.

AMS 1980 Subject Classification: Primary: 62E15; Secondary: 53A04

Key Words and Phrases: differential inequalities, upcrossings

Research supported in part by NSF Grants DMS8600235, DMS8451750 (IJ) and ONR Contract N00014-87-K-0078 (DS).

1. Introduction.

Motivated by the question of testing for a nonlinear parameter in a regression model with independent, homoscedastic normal residuals, Hotelling (1939) was led to consider the geometric problem of computing the volume of a tube of given radius around a curve in the unit sphere in $\mathbb{R}^n$. The answer involves only the arc length of the curve and not its curvature, providing the radius of the tube is sufficiently small that there is no self overlap in the tube. Starting from a somewhat different statistical setting Naiman (1986) arrived at the same geometric problem and showed that Hotelling's result (properly interpreted) is an upper bound for the volume of a tube of arbitrary radius.

The purpose of this note is to give two new derivations of the Hotelling-Naiman results. The first involves differential inequalities. The second is probabilistic, using the concept of upcrossing borrowed from the theory of Gaussian processes. In the context of Gaussian processes Knowles (1987) has observed that approximations obtained from Hotelling's result and bounds derived via upcrossings are related.

Hotelling's statistical motivation and geometric problem are reviewed briefly in Section 2, which also establishes our basic notation. Sections 3 and 4 contain our derivations.

For a more extensive discussion of applications and several numerical examples, see Johansen and Johnstone (1985) and Knowles and Siegmund (1988).

2. The Problem.

Assume $y_i = \beta f_i(\theta) + \epsilon_i \quad (i = 1, 2, \ldots, n)$, where the f_i are known functions depending on an unknown parameter θ and the ϵ_i are independent $N(0, \sigma^2)$ errors. In principle one can also consider the more general model $y_i = \langle \beta, x_i \rangle + \beta_{p+1} f_i(\theta) + \epsilon_i$, where β and the x_i are p–dimensional vectors, but for our purposes the simpler one suffices.

The primary example given by Hotelling is $f_i(\theta) = \cos(\mu t_i + \omega)$, where the t_i are known constants and $\theta = (\mu, \omega)$. A second example is the broken line regression $f_i(\theta) = (t_i - \theta)^+$. See Davies (1987) for an interesting discussion of both these examples.

The likelihood ratio statistic for testing $H_0 : \beta = 0$ against $H_1 : \beta \neq 0$ is easily seen to be

equivalent to

$$\max_{\theta} \{ [\Sigma f_i(\theta) y_i]^2 / [\Sigma f_i^2(\theta) \Sigma y_i^2] \}.$$

Letting $f(\theta) = (f_1(\theta), \ldots, f_n(\theta))$ and $y = (y_1, \ldots, y_n)$, we can write this as

$$\max_{\theta} \{ \langle f(\theta), y \rangle^2 / [\, \| f(\theta) \|^2 \| y \|^2 \,] \}.$$

Putting $\gamma(\theta) = f(\theta) / \| f(\theta) \|$ and $U = y / \| y \|$, we see that the rejection region of the likelihood ratio test,

$$\max_{\theta} \langle \gamma(\theta), U \rangle^2 \geq w^2,$$

is the union of the two tubes, one about $\gamma(\theta)$, the other about $-\gamma(\theta)$, of geodesic radius $\cos^{-1} w$. Here the tube about $\gamma(\theta)$ of geodesic radius φ is the set of all points $U \in S^{n-1}$, the unit sphere in $\mathbb{R}^n$, within geodesic distance φ of the curve $\gamma(\theta)$. Under H_0, U is distributed uniformly on S^{n-1}, and hence the significance level of the likelihood ratio test is the normalized surface area on S^{n-1} of the union of the two tubes.

If $\gamma, u \in S^{n-1}$, then $\langle \gamma, u \rangle = 1 - 2^{-1} \| \gamma - U \|^2$. Hence the tube about $\gamma(\theta)$ of geodesic radius $\varphi = \cos^{-1} w$ can also be defined as the set of all points in S^{n-1} within Euclidean distance $[2(1-w)]^{1/2}$ of the curve. See Figure 1.

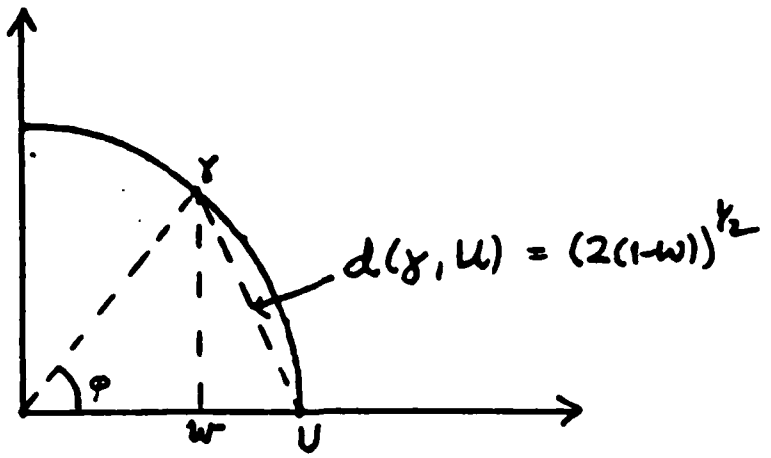


Figure 1. Illustration of the relation between ρ, w and $d(\gamma, U)$.

A simpler geometric problem is to compute the volume of a tube about a curve in Euclidean space. In this context Hotelling's result is very easy to state. If the curve is smooth, closed, and there is no self overlap in the tube (precise definitions are given below), the volume of the tube is the product of the arc length of the curve and the cross sectional area of the tube. If the curve is not closed, the volume of two hemispherical caps must be added to account for parts of the tube associated with endpoints of the curve.

We shall use the following notation. Given a closed interval I of real numbers, $\alpha : I \to \mathbb{R}^n$ is a piecewise regular (continuous and piecewise continuously differentiable with non-vanishing derivative) curve of arc length $|\alpha|$. The Euclidean distance between two points is $d(x,y) = \| x - y \|$, between a point and a set is $d(x,B) = \inf\{\| x - y \| : y \in B\}$, and between two sets is $d(A,B) = \sup\{d(x,B) : x \in A\}$. The tube (in $\mathbb{R}^n$) of radius R about α is $\alpha^R = \{x : d(x,\alpha(I)) < R\}$. For any (measurable) $A \subset S^{n-1}$ or $A \subset \mathbb{R}^n$, $V(A)$ = volume of A. Also let Ω_n denote the volume of the n dimensional unit ball in $\mathbb{R}^n$ and ω_{n-1} the volume (surface area) of S^{n-1}, the unit sphere in $\mathbb{R}^n (\Omega_n = \pi^{n/2}/\Gamma(n/2+1), \omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2))$. Finally, let $\dot{\alpha}(t) = d\alpha(t)/dt$.

Hotelling's result in $\mathbb{R}^n$ says that if α is twice continuously differentiable and there is no self overlap in the tube then

$$V(\alpha^R) = |\alpha|\Omega_{n-1}R^{n-1} + \Omega_n R^n \tag{2.1}$$

if the curve is not closed and

$$V(\alpha^R) = |\alpha|\Omega_{n-1}R^{n-1} \tag{2.2}$$

if the curve is closed.

If α does not actually intersect itself, the condition of no self overlap is essentially the condition that R be sufficiently small. Naiman (1986) has proved the elegant result that if α is only assumed piecewise regular,

$$V(\alpha^R) \leq |\alpha|\Omega_{n-1}R^{n-1} + \Omega_n R^n \tag{2.3}$$

for all $R \geq 0$.

Precise analogues of (2.1) and (2.3) for tubes about curves in S^{n-1} are given in Section 3.

REMARK. It is illuminating to consider the case where α is the unit circle in $\mathbb{R}^2$. One easily verifies (2.2) for $R \leq 1$ and (2.3) for all $R \geq 0$. Also (2.3) is asymptotically sharp as $R \to \infty$. We do not have a simple geometric explanation why the second term on the right hand side of (2.3), which is obviously necessary when α is not closed, works efficiently and in complete generality.

Anticipating applications to the examples presented above, we note that the broken line regression requires Naiman's formulation because the curve $\gamma(\theta)$ is only piecewise smooth. Hotelling's problem of testing for a periodic component in a regression model poses other difficulties because the parameter $\theta = (\mu, \omega)$ is two–dimensional and hence the "curve" $\gamma(\theta)$ is a surface in S^{n-1}. Weyl (1939) in a companion paper to Hotelling's calculates the volume of a tube of small radius about an arbitrary closed differentiable manifold imbedded in $\mathbb{R}^n$ or in S^{n-1}. However, his results must be modified for manifolds with boundary before they can be applied to the problem at hand. See Knowles and Siegmund (1988) for an appropriate modification and numerical examples. In Section 4 we show that the special structure of Hotelling's problem allows one to use an upcrossing argument to give an upper bound for the significance level of the likelihood ratio test.

3. The Hotelling–Naiman Theorem.

We begin with a technical lemma which summarizes several well known facts about the uniform distribution on S^{n-1}.

LEMMA 3.1. *Suppose $U = (U_1, \ldots, U_n)$ is uniformly distributed on S^{n-1}.*

(i) The distribution of U_1^2 is Beta$(1/2, (n-1)/2)$; the probability density function of U_1 is

$$f_{n-1}(x) = \frac{\Gamma(n/2)}{\pi^{1/2}\Gamma\big[(n-1)/2\big]}(1-x^2)^{(n-3)/2} \qquad (|x| < 1).$$

(ii)

$$E(U_1^+) = \Gamma(n/2)/\{2\pi^{1/2}\Gamma[(n+1)/2]\}.$$

(iii) For $k < n$, given $U_1, \ldots, U_k$, the conditional distribution of $(U_{k+1}, \ldots, U_n)$ is uniform on a sphere of dimension $n-k-1$ and radius $(1 - \Sigma_1^k U_i^2)^{1/2}$.

(iv) The random variable $U_1^2 + U_2^2$ has a Beta$(1, (n-2)/2)$ distribution and is independent of U_1^2/U_2^2.

PROOF. All these results can be proved by means of the representation $U_i = Z_i/(Z_1^2 + \ldots + Z_n^2)^{1/2}$, where $Z_1, \ldots, Z_n$ are independent, standard normal random variables, and Basu's theorem (cf. Lehmann, 1986, p. 191). ∎

A simple picture underlies the differential inequalities approach to Naiman's inequality. A sphere centered at one end of the curve is sliced along the plane perpendicular to the curve at that endpoint. The hemisphere intersecting the curve is moved along the curve at unit speed. The volume swept out is greatest when the curve is a geodesic, which leads to the Hotelling–Naiman result.

It is easiest to begin with the version of Naiman's bound for tubes in Euclidean space.

THEOREM 3.1. *Let $\alpha : I \to \mathbb{R}^n$ be a piecewise regular curve of length $|\alpha|$, and for $R \geq 0$ let $\alpha^R = \{x \in \mathbb{R}^n : d(x, \alpha(I)) \leq R\}$. Then $V(\alpha^R)$ satisfies (2.3) for all $R \geq 0$.*

PROOF. Without loss of generality we can assume α is parameterized by arc length, so $I = [0, |\alpha|]$. Let the image of α on a subinterval $[a, b]$ of I be denoted by $\alpha_{[a,b]}$. Let $v(s) = V(\alpha_{[0,s]}^R)$. Clearly $v(0) = \Omega_n R^n$, and to complete the proof it suffices to show that v is absolutely continuous and $\dot{v}(s) \leq \Omega_{n-1} R^{n-1}$ for a.e. s.

We say a point x is R–close to $\alpha_{[a,b]}$ if $x \in \alpha_{[a,b]}^R$. Let $\delta > 0$. If a point is R–close to $\alpha_{[0,s+\delta]}$ but not R–close to $\alpha_{[0,s]}$, it must be R–close to $\alpha_{[s,s+\delta]}$ but not R–close to $\alpha_{[s,s]}$. Hence

$$\begin{aligned} v(s+\delta) - v(s) &= V(\alpha_{[0,s+\delta]}^R) - V(\alpha_{[0,s]}^R) \\ &\leq V(\alpha_{[s,s+\delta]}^R) - V(\alpha_{[s,s]}^R). \end{aligned} \tag{3.1}$$

Recall that for two sets A and B, $d(A,B) = \inf_{a\in A}\sup_{b\in B} \| a-b \|$, and observe that $d(A,B) < \eta$ implies $A^R \subset B^{R+\eta}$. Since α is right differentiable by assumption, the linear approximation to $\alpha(s+t)$ for $t \geq 0$, namely $\beta(t) = \alpha(s) + t\dot{\alpha}(s+)$ satisfies

$$d\big(\alpha_{[s,s+\delta]}, \beta_{[0,\delta]}\big) = \eta(\delta) = o(\delta) \tag{3.2}$$

uniformly in s provided we exclude left neighborhoods of length δ of discontinuity points of $\dot{\alpha}$. An explicit evaluation starting from (3.1) yields

$$\begin{aligned} 0 \leq v(s+\delta) - v(s) &\leq V\big(\beta^{R+\eta}_{[0,\delta]}\big) - V\big(\alpha^R_{[s,s]}\big) \\ &= \delta\Omega_{n-1}(R+\eta)^{n-1} + \Omega_n(R+\eta)^n - \Omega_n R^n \\ &= \delta\Omega_{n-1}R^{n-1} + O\big(\eta(\delta)\big). \end{aligned} \tag{3.3}$$

It follows that v is increasing and absolutely continuous, hence that $\dot{v}(s)$ exists a.e. and satisfies the required bound. ∎

REMARKS. (i) One can avoid the appeal to Lebesgue theory by showing directly that $v(s)$ is Lipschitz continuous with Lipschitz constant $M = \Omega_{n-1}R^{n-1}$. Fix $\varepsilon > 0$. Formula (3.3) and the analogous inequality for $\delta < 0$ imply that about each $s \in I$ there is a neighborhood N_s, open relative to I, of points s' satisfying $|v(s')-v(s)| \leq (M+\varepsilon)|s'-s|$. The compactness of I provides a finite subcover $\{N_{s_i}\}$ from which it follows by chaining that $v(s)$ is Lipschitz $(M+\varepsilon)$ on I. Since $\varepsilon > 0$ is arbitrary, we recover (2.3) from the decomposition $v(s) = v(0) + \big[v(s) - v(0)\big]$.

(ii) Lalley and Robbins (1987) have also exploited the idea of volume swept out by a moving sphere in a differential games setting.

To see that equality holds in (2.3) when α is continuously differentiable, not closed, and R is sufficiently small, we define the cross section $C\big[\alpha(s)\big]$ of the tube α^R at the point $\alpha(s)$ as the set of all $x \in \alpha^R$ such that $\langle x-\alpha(s), \dot{\alpha}(s)\rangle \leq 0$, $= 0$, or ≥ 0 according as $s = 0, s \in (0,|\alpha|)$, or $s = |\alpha|$. Clearly

$$\alpha^R = \cup_{s\in[0,|\alpha|]} C[\alpha(s)],$$

and we say that no self overlap occurs in the tube if this union is disjoint. For a closed curve, we require only that the union over $s \in (0,|\alpha|)$ be disjoint. The critical radius is

$R_c = \inf\{R \geq 0 : \text{self overlap occurs}\}$. If the curve does not actually intersect itself, $R_c > 0$, but its exact value may be difficult to determine analytically. Johansen and Johnstone (1985) give an easily computed bivariate function whose minimum is R_c.

THEOREM 3.2. *If $\alpha : I \longrightarrow \mathbb{R}^n$ is regular and closed (resp. not closed) then the equality (2.2) (resp. (2.1)) holds for all $R \leq R_c$.*

PROOF. Assume first that α is not closed. Again assume α is parameterized by arc length and that $R < R_c$. The key is to show that equality holds in (3.1): indeed if a point is R–close to $\alpha_{[s,s+\delta]}$ but not to $\alpha_{[s,s]}$, then it lies in $\cup_{t\in(s,|\alpha|]}C[\alpha(t)]$ but not $\alpha^R_{[s,s]}$ and hence not in $\alpha^R_{[0,s]}$.

If we combine equality in (3.1) with the inequality $V\big(\alpha^R_{[s,s+\delta]}\big) \geq V\big(\beta^{R-\eta}_{[0,\delta]}\big)$ and calculate as in (3.3) we obtain in addition to (3.3) the inequality

$$v(s+\delta) - v(s) \geq \delta\Omega_{n-1}R^{n-1} + O\big(\eta(\delta)\big).$$

The two inequalities show that $\dot{v}(s) = \Omega_{n-1}R^{n-1}$ for all $s \in [0, |\alpha|]$. Combined with the initial value $v(0) = \Omega_n R^n$, this implies (2.1).

Finally, suppose that α is closed. The idea is to split α into two non–closed curves and apply what we have just proved. Choose a pair (s_0, t_0) maximizing the distance function $(s,t) \rightarrow \|\alpha(s)-\alpha(t)\|^2$. By reparameterization, we may assume that $s_0 = 0$. Since $\alpha(0)-\alpha(t_0)$ is orthogonal to both $\dot{\alpha}(0)$ and $\dot{\alpha}(t_0)$ it follows that $\|\alpha(0)-\alpha(t_0)\| \geq 2R$. In turn, this implies that neither of the tubes $\alpha^R_{[0,t_0]}$ and $\alpha^R_{[t_0,|\alpha|]}$ suffers any self–overlap. Equality (2.1) applies to each of these non–closed curves. Since the two tubes intersect in precisely two disjoint balls, equality (2.2) is established by subtracting the double counted portions. ∎

REMARK. It is interesting to note that the curvature of α plays no role in the preceding argument. This contrasts with Hotelling's argument where the curvature appears and then is found after an integration to have a coefficient of zero.

To discuss tubes in spheres, let S^{n-1} be the unit sphere in $\mathbb{R}^n$ and let $\gamma : I \longrightarrow S^{n-1}$ be a piecewise regular curve parameterized by arc length.

Although it is appropriate to use geodesic distance to define tube radii, in order to adapt as directly as possible the preceding arguments we shall think of S^{n-1} imbedded in $\mathbb{R}^n$ and use Euclidean distance. The relation between geodesic distance $\varphi, w = \cos\varphi$, and Euclidean distance $R = \left[2(1-w)\right]^{1/2}$ is illustrated in Figure 1. The tube of radius R about γ in S^{n-1} is

$$\gamma^R = \{y \in S^{n-1} : \max_s \langle y, \gamma(s)\rangle > \cos\varphi\} = \{y \in S^{n-1} : d(y, \gamma(I)) < R\}.$$

Now the "linear continuation" of $\gamma(s)$ is continuation along a geodesic, defined by $\beta(t) = \gamma(s)\cos t + \dot\gamma(s+)\sin t$. Since Euclidean distances on S^{n-1} are inherited from $\mathbb{R}^n$, (3.1) and (3.2) remain valid with this new definition of β.

To complete the calculation analogous to (3.3), we must evaluate $v(0)$ and $V\left(\beta^{R+\eta}_{[0,\delta]}\right)$. Let $U = (U_1, \ldots, U_n)$ be uniformly distributed on S^{n-1} and assume without loss of generality that $\gamma(0) = (1, 0, \ldots, 0)$. Then

$$\begin{aligned} v(0) &= V\{(u_1, \ldots, u_n) \in S^{n-1} : u_1 \geq w\} \\ &= \omega_{n-1} P\{U_1 \geq w\} \\ &= \omega_{n-2} \int_w^1 (1-x^2)^{(n-3)/2} dx \end{aligned}$$

by Lemma 3.1. To compute $V\left(\beta^{R(w)}_{[0,\delta]}\right)$ observe that since $\langle \gamma(s), \dot\gamma(s+)\rangle = 0$ we can without loss of generality assume that $\gamma(s) = (1, 0, \ldots, 0)$ and $\dot\gamma(s+) = (0, 1, 0, \ldots, 0)$. Then $\beta_{[0,\delta]}$ is a portion of the equator subtending an angle δ at the origin. The portion of the tube which does not involve the two caps, $B = \{u \in S^{n-1} : \Sigma_{i\geq 3}\, u_i^2 \leq 1 - w^2, 0 \leq u_2/u_1 \leq \tan\delta, u_1 \geq 0\}$, has volume

$$V(B) = 4^{-1}\omega_{n-1} P\{U_1^2 + U_2^2 \geq w^2, U_2^2/U_1^2 \leq \tan^2\delta\},$$

which can be evaluated by Lemma 3.1. Hence

$$V\left(\beta^{R(w)}_{[0,\delta]}\right) = (2\pi)^{-1}\delta\omega_{n-1}(1-w^2)^{(n-2)/2} + \omega_{n-2}\int_w^1 (1-x^2)^{(n-3)/2} dx.$$

If $\tilde R = R + \eta$, the corresponding $\tilde w = 1 - \tilde R^2/2 = w - R\eta + \eta^2/2 = w + O(\eta)$. A calculation like (3.3) yields

$$0 \leq v(s+\delta) - v(s) \leq (2\pi)^{-1}\delta\omega_{n-1}(1-w^2)^{(n-2)/2} + O\left(\eta(\delta)\right),$$

from which follows $\dot{v}(s) \leq (2\pi)^{-1}\omega_{n-1}(1-w^2)^{(n-2)/2}$ a.e. s since $\eta(\delta) = o(\delta)$. From this inequality we obtain Naiman's bound for a tube in S^{n-1}.

An argument analogous to that of Theorem 3.2 shows that there is equality in the Hotelling–Naiman bound for tubes in S^{n-1} whenever the curve is smooth and the tube radius is less than R_c, the radius of first overlap. Details of the definition of R_c together with a computational method for evaluating R_c and some statistical examples are given in Johansen and Johnstone (1985).

The Naiman–Hotelling result we have just derived is summarized in Theorem 3.3, which is then proved by a different method.

THEOREM 3.3. *Let $\gamma : [0, t_0] \longrightarrow S^{n-1}$ be a regular curve. Let U be uniformly distributed on S^{n-1} and put $Z(t) = \langle \gamma(t), U \rangle$. Then for any $0 < w < 1$,*

$$P\Big\{ \max_{0 \leq t \leq t_o} Z(t) \geq w \Big\} \leq \frac{\Gamma(n/2)}{\pi^{\frac{1}{2}}\Gamma[(n-1)/2]} \int_w^1 (1-x^2)^{(n-3)/2} dx + (2\pi)^{-1}|\gamma|(1-w^2)^{(n-2)/2}. \quad (3.4)$$

If no self overlap occurs in the tube $\gamma^R_{[0,t_0]}$, where $R = \big[2(1-w)\big]^{1/2}$, there is equality in (3.4). The inequality (3.4) continues to hold if γ is only assumed to be continuous and piecewise regular.

PROOF. The argument uses the notion of an upcrossing of the level w, which plays an important role in the theory of Gaussian processes. See Leadbetter, Lindgren, and Rootzen (1983, Chapter 7) for the definition and basic properties. Let $N_w(t_0)$ denote the number of upcrossings of the level w by $Z(t)$, $0 \leq t \leq t_0$. Then

$$\begin{aligned} P\Big\{ \max_{0 \leq t \leq t_0} Z(t) > w \Big\} &= P\big\{ Z(0) > w \big\} + P\big\{ Z(0) < w, N_w(t_0) \geq 1 \big\} \\ &\leq P\big\{ Z(0) > w \big\} + E\big\{ N_w(t_0) \big\} \end{aligned} \quad (3.5)$$

The proof of (3.4) is completed by the evaluation of the right hand side of (3.5) given in Lemma 3.1 and Lemma 3.2, which follows.

If there is no self overlap in S^{n-1} in the tube of geodesic radius $\cos^{-1}(w)$ about γ, then no sample path $Z(t)$, $0 \leq t \leq t_0$, can leave and subsequently re–enter the tube. It follows that $P\big\{ Z(0) > w, N_w(t_0) \geq 1 \big\} + P\big\{ N_w(t_0) \geq 2 \big\} = 0$, so equality holds in (3.5), hence also in (3.4).

Since a continuous, piecewise regular curve is the uniform limit of a sequence of regular curves, the arc lengths of which also converge to that of the given curve, one sees from Fatou's lemma that (3.4) continues to hold under the weaker condition of continuity and piecewise regularity. ∎

LEMMA 3.2. *If γ is a regular curve in S^{n-1}, the expected number of upcrossings $N_w(t_0)$ of the level w by the process $Z(t)$, $0 \le t \le t_0$, is given by*

$$EN_w(t_0) = (2\pi)^{-1}|\gamma|(1-w^2)^{(n-2)/2}.$$

PROOF. Assume without loss of generality that γ is parameterized by arc length $s = s(t)$, and let $s_0 = s(t_0)$. Then $\dot{\gamma}(s) = d\gamma/ds \in S^{n-1}$. A standard argument (cf. Leadbetter, Lindgren, and Rootzen, 1983, Chapter 7) shows that if as $h \to 0$ the joint density of $Z(s)$ and $\big[Z(s+h) - Z(s)\big]/h$ satisfies certain regularity conditions discussed below, then

$$EN_w(s_0) = f_{n-1}(w) \lim_{m\to\infty} 2^{-m} \sum_{k=1}^{2^m s_0} E\big[\dot{Z}^+(k/2^m) | Z(k/2^m) = w\big],$$

where f_{n-1} is the density function of $Z(s)$ given in Lemma 3.1. Since $\langle \gamma(s), \dot{\gamma}(s)\rangle = 0$, for the purpose of evaluating the conditional distribution of $\dot{Z}(s) = \langle \dot{\gamma}(s), U\rangle$ given $Z(s) = \langle \gamma(s), U\rangle$, we can let $U = (U_1, \ldots, U_n)$ and by a rotation of the coordinate axes assume that $\gamma(s) = (1, 0, \ldots, 0)$ and $\dot{\gamma}(s) = (0, 1, 0, \ldots, 0)$. This means that

$$EN_w(s_0) = s_0 f_{n-1}(w) E[U_2^+ | U_1 = w]. \tag{3.6}$$

The conditional expectation in (3.6) is evaluated with the help of Lemma 3.1 (ii) and (iii).

To justify the preceding calculation it suffices that as $h \to 0$ the joint density function of $Z(s)$ and $\big[Z(s+h) - Z(s)\big]/h$, say $p_{s,h}(x, y)$, converge uniformly in s and x, at least for x in a neighborhood of w, to the joint density of $Z(s)$ and $\dot{Z}(s)$ (cf. Leadbetter, Lindgren, and Rootzen, 1983, Theorem 7.2.4).

By a rotation of the coordinate axes so that $\gamma(s) = (1, 0, \ldots, 0)$ and $\dot{\gamma}(s) = (0, 1, 0, \ldots, 0)$, we see that the joint density function of $Z(s)$ and $\big[Z(s+h) - Z(s)\big]/h = \dot{Z}(s) + \big\{\big[Z(s+h) -$

$Z(s)\big]/h - \dot{Z}(s)\big\}$ has the form

$$P\big\{Z(s) \in dx, \big[Z(s+h) - Z(s)\big]/h \in dy\big\} = P\left\{U_1 \in dx, U_2 + \sum_{i\geq 1} \epsilon_i U_i \in dy\right\}, \qquad (3.7)$$

where the $\epsilon_i \to 0$ uniformly in s as $h \to 0$, because γ is regular.

Given $U_1 = x, (U_2, \ldots, U_n)$ are uniformly distributed on an $n-2$ dimensional sphere of radius $(1-x^2)^{1/2}$ (cf. Lemma 3.1), and consequently the right hand side of (3.7) equals

$$f_{n-1}(x)dx P\left\{\tilde{U}_2(1+\epsilon_2) + \sum_{i\geq 3} \epsilon_i \tilde{U}_i \in (dy - \epsilon_1 x)/(1-x^2)^{1/2}\right\},$$

where $(\tilde{U}_2, \ldots, \tilde{U}_n)$ is uniformly distributed on S^{n-2}. This last probability can be written as an integral with respect to the joint density of $\tilde{U}_2$ and $\sum_{i\geq 3} \epsilon_i \tilde{U}_i$, and by a similar conditioning argument it can be shown to converge uniformly in s and $|x|$ bounded away from 1 to $P\big\{\tilde{U}_2 \in dy/(1-x^2)^{1/2}\big\} = P\big\{\langle \dot{\gamma}(s), U\rangle \in dy | \langle \gamma(s), U\rangle = x\big\}$. The details are omitted.

REMARK. It is possible (although not particularly natural) to derive Theorems 3.1 and 3.2 by a (down) crossing argument. Since the tube cannot be defined by an inner product, the appropriate process is $Z(s) = \| U - \alpha(s) \|$, where U is uniformly distributed in a box large enough to contain α^R.

4. Testing for an Harmonic.

As indicated in Section 2, Hotelling's problem of testing for an harmonic of undetermined frequency and phase does not fall within the scope of the results of Section 3 because it involves two nonlinear parameters. However, by writing

$$\beta \cos(\mu t_i + \omega) = \beta_1 \cos \mu t_i + \beta_2 \sin \mu t_i,$$

where $\beta_1 = \beta \cos \omega, \beta_2 = -\beta \sin \omega$, and observing that $\beta = 0$ if and only if $\beta_1 = \beta_2 = 0$, we reduce the number of nonlinear parameters to one. The upshot is a likelihood ratio test with a rejection region of the form

$$\sup_\theta \left[\langle \gamma_1(\theta), U\rangle^2 + \langle \gamma_2(\theta), U\rangle^2\right]^{1/2} \geq w,$$

where $\gamma_i(\theta) \in S^{n-1}, \langle \gamma_1(\theta), \gamma_2(\theta) \rangle = 0$ for all θ, and under $H_0 : \beta_1 = \beta_2 = 0$ U is uniformly distributed on S^{n-1}. Although in this form the rejection region does not have a simple geometric interpretation, its one dimensional structure permits one to obtain an inequality based on an upcrossings argument. The following theorem is similar to a result of Davies (1987), who assumes that σ^2 is known and therefore can modify known results about upcrossings of χ^2 processes.

THEOREM 4.1. *For $i = 1, 2$ let $\gamma_i : [0, t_0] \to S^{n-1}$ be regular curves. Assume $\langle \gamma_1(t), \gamma_2(t) \rangle = 0$ for all t. Let U be uniformly distributed on S^{n-1} and put $Z(t) = \{\langle \gamma_1(t), U \rangle^2 + \langle \gamma_2(t), U \rangle^2\}^{1/2}$. Then for $0 < w < 1$*

$$P\left\{\max_{0 \le t \le t_0} Z(t) > w\right\} \le (1-w^2)^{(n-2)/2} \tag{4.1}$$
$$+ \frac{\Gamma(n/2) w (1-w^2)^{(n-3)/2}}{2\pi^{3/2}\Gamma[(n-1)/2]} \int_0^{t_0} \int_0^{2\pi} \left[\|\dot\gamma_1 \cos\omega + \dot\gamma_2 \sin\omega\|^2 - \langle \dot\gamma_1, \gamma_2 \rangle^2 \right]^{1/2} d\omega dt.$$

REMARKS. (i) Theorem 4.1 has been formulated with a view towards application to Hotelling's problem of testing for an harmonic of undetermined frequency and phase. There has been no attempt at generality.

(ii) A special case of Knowles and Siegmund's (1988) formula for the volume of a tube about a surface imbedded in S^{n-1} shows that equality holds in (4.1) for all w sufficiently close to 1. However, their method does not yield the inequality (4.1) for all w.

(iii) See Knowles and Siegmund (1988) for a numerical example related to Theorem 4.1.

PROOF OF THEOREM 4.1. The inequality (3.5) is again applicable. The probability density function of $Z(t)$ is (cf. Lemma 3.1)

$$f_{n-1}^{(2)}(x) = (n-2)x(1-x^2)^{(n-4)/2}, \tag{4.2}$$

and hence $P\{Z(t) \ge w\} = (1-w^2)^{(n-2)/2}$. The standard recipe for calculating $EN_w(t_0)$ yields

$$EN_w(t_0) = f_{n-1}^{(2)}(w) \int_0^{t_0} E\left[\dot Z^+(t) | Z(t) = w\right] dt. \tag{4.3}$$

Since

$$\dot{Z} = \frac{\langle \gamma_1, U\rangle\langle \dot{\gamma}_1, U\rangle + \langle \gamma_2, U\rangle\langle \dot{\gamma}_2, U\rangle}{[\langle \gamma_1, U\rangle^2 + \langle \gamma_2, U\rangle^2]^{1/2}},$$

after rotation of axes so that $\gamma_1 = (1, 0, \ldots, 0)$ and $\gamma_2 = (0, 1, 0, \ldots, 0)$, the conditional expectation in (4.3) equals

$$w^{-1} E\big(E\{[U_1\langle \dot{\gamma}_1, U\rangle + U_2\langle \dot{\gamma}_2, U\rangle]^+ | U_1, U_2\} | U_1^2 + U_2^2 = w^2\big). \tag{4.4}$$

Defining α_1 and α_2 by $\dot{\gamma}_1 = \langle \dot{\gamma}_1, \gamma_2\rangle\gamma_2 + \alpha_1$ and $\dot{\gamma}_2 = \langle \dot{\gamma}_2, \gamma_1\rangle\gamma_1 + \alpha_2$, we see that α_1 and α_2 are orthogonal to both γ_1 and γ_2, and satisfy

$$\|\alpha_1\|^2 = \|\dot{\gamma}_1\|^2 - \langle \dot{\gamma}_1, \gamma_2\rangle^2, \|\alpha_2\|^2 = \|\dot{\gamma}_2\|^2 - \langle \gamma_1, \dot{\gamma}_2\rangle^2, \langle \alpha_1, \alpha_2\rangle = \langle \dot{\gamma}_1, \dot{\gamma}_2\rangle. \tag{4.5}$$

Since $\langle \dot{\gamma}_1, \gamma_2\rangle = -\langle \gamma_1, \dot{\gamma}_2\rangle$, we easily obtain $U_1\langle \dot{\gamma}_1, U\rangle + U_2\langle \dot{\gamma}_2, U\rangle = U_1\langle \alpha_1, U\rangle + U_2\langle \alpha_2, U\rangle$, and hence the inner conditional expectation in (4.4) equals

$$E\{[U_1\langle \alpha_1, U\rangle + U_2\langle \alpha_2, U\rangle]^+ | U_1, U_2\}.$$

By decomposing α_2 into a component along α_1 and a component orthogonal to α_1 and then rotating the coordinate axes while leaving the first two coordinate directions fixed, we see that this conditional expectation equals

$$\|\alpha_1 U_1 + \alpha_2 U_2\| \, E(U_3^+ | U_1, U_2) \tag{4.6}$$

Using (4.5) and Lemma 3.1 in (4.6) and substituting the result into (4.4), one can easily complete the proof of the theorem.

References

Davies, Robert B. (1987). Hypothesis testing when a nuisance parameter is present only under the alternative, *Biometrika* **74**, 33-43.

Hotelling, Harold (1939). Tubes and spheres in n–spaces and a class of statistical problems, *Amer. J. Math.* **61**, 440-460.

Johansen, Søren and Johnstone, Iain (1985). Some uses sof spherical geometry in simultaneous inference and data analysis, Stanford University Technical Report.

Knowles, Mark (1987). Simultaneous confidence bands for random functions, Stanford University Ph.D. thesis.

Knowles, Mark and Siegumnd, D. (1988). On Hotelling's geometric approach to testing for a nonlinear parameter in regression, Stanford University Technical Report.

Lalley, S. and Robbins, H. (1987). Stochastic search in a convex region, unpublished manuscript.

Leadbetter, M., Lindgren, G., and Rootzen, H. (1983). *Extremes and Related Properties of Random Sequences and Processes*, Springer–Verlag, New Yorkk–Heidelberg–Berlin.

Lehmann, E. L. (1986). *Testing Statistical Hypotheses*, 2nd Edition, John Wiley and Sons, New York.

Naiman, D. Q. (1986). Conservative confidence bands in curvilinear regression, *Ann. Statist.* **14**, 896-906.

Weyl, H. (1939). On the volume of tubes, *Amer. J. Math.* **61**, 461-472.

UNCLASSIFIED
SECURITY CLASSIFICATION OF TH

REPORT DOC

AD-A190 278

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: 5

CIPIENT'S CATALOG NUMBER: N/A

4. TITLE (and Subtitle)

ON HOTELLING'S FORMULA FOR THE VOLUME OF TUBES AND NAIMAN'S INEQUALITY

5. TYPE OF REPORT & PERIOD COVERED: Technical Report

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s): Iain Johnstone and David Siegmund

8. CONTRACT OR GRANT NUMBER(s): N00014-87-K-0078

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Department of Statistics - Sequoia Hall
Stanford University
Stanford, CA 94305-4065

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: NR-042-373

11. CONTROLLING OFFICE NAME AND ADDRESS
Statistics & Probability Program Code (411(SP))
Office of Naval Research
Arlington, Virginia 22217

12. REPORT DATE: December 1987

13. NUMBER OF PAGES: 15pp

14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)

15. SECURITY CLASS. (of this report): Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

NA

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

differential inequalities, upcrossings

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Two new derivations of the Hotelling–Naiman results on volume of tubes about curves in spheres are presented. The first involves simple differential inequalities. The second is probabilistic, using the concept of upcrossing borrowed from the theory of Gaussian processes. The upcrossings method is extended to a harmonic regression problem not covered by the Hotelling–Naiman formulation.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

END

DATE

FILMED

4-88

DTIC