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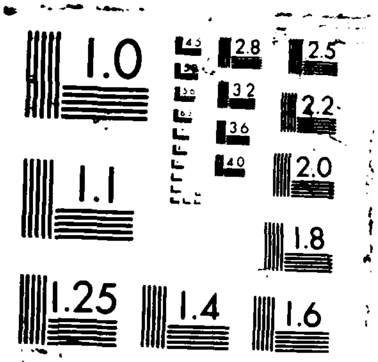
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19. ABSTRACT (Continue on reverse if necessary and identify by block number)

Efforts to develop computational methods for the identification and optimal control of linear and nonlinear systems governed by distributed parameter systems are reported on. Specifically, approximation methods for determining Optimal LOG compensators (feedback control and estimator gains) and functional parameters in linear and nonlinear partial differential equations and hereditary systems were developed, analyzed and tested. The study included theoretical, experimental and numerical components. Convergence theories for spline-based and modal finite element schemes were established and extensive numerical studies on both conventional (serial) and vector supercomputers were carried out.

A parameter estimation scheme was tested using experimental data taken from the RPL structure, a laboratory experiment designed to test control algorithms for the large angle slewing of spacecraft with flexible appendages, and other projects involving the identification of flexible structures based upon experimental data were initiated.

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Approximation Methods for the Identification and Control of Distributed Parameter Systems

Air Force Office of Scientific Research Contract No. AFOSR-84-0393

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We have developed computational methods for the identification and optimal control of distributed parameter systems. Our work has consisted of a theoretical, an experimental, and a numerical component. Using functional analytic techniques we demonstrate and study the convergence properties of our schemes. Extensive numerical studies are then carried out in order to fully assess their feasibility, performance, and limitations.

Central to our general approach is the notion of approximating infinite dimensional optimization problems by sequences of finite dimensional ones. Typically this involves the approximation of infinite dimensional state dynamics (in the form of distributed parameter systems such as partial differential equations or hereditary systems) by sequences of finite dimensional dynamical systems (such as ordinary differential or difference equations). In the case of the parameter estimation problems, when it is functional (i.e. spatially or temporally varying) parameters which are to be identified, the infinite dimensional admissible parameter spaces must also be discretized.

In general, we have used polynomial and Hermite spline function, as well as modal function based finite element methods to accomplish these tasks. We note that since we are not simply solving the so-called forward problem, i.e. the straight forward integration of the underlying dynamical equations, but rather are seeking to solve problems whose solutions depend in a highly nonlinear fashion on the system's infinite dimensional state transition, input / output or parameter space / output maps, the theoretical components of our investigations become especially important. Indeed, methods which have been known to do a good job integrating differential equations often perform less than satisfactorily when coupled with a scheme to solve a parameter estimation or control problem.

On the other hand, we have found our numerical studies to be extremely useful in allowing us to observe the limitations and short comings of our methods and to identify important directions for future research.

Below, we summarize our results and our progress in as yet incomplete but on-going projects. In the body of the report, we simply provide a brief outline and broad summary of our findings. The actual results are discussed in detail in the research papers which have been provided to AFOSR, and a sampling of which have been included in the appendix to this report.

1. Control

a. We have developed an abstract approximation framework for the discrete-time linear quadratic Gaussian (LQG) control, estimator, and compensator problems for systems whose state dynamics are described by linear semigroups of operators on infinite dimensional Hilbert spaces. The computational schemes included in the framework yield finite dimensional approximations to the optimal feedback control laws, estimator gains and LQG compensators. A convergence theory has been established and numerical studies involving parabolic (heat / diffusion), hereditary, and elastic systems have been carried out. Initially, our theory applied only to control systems whose continuous-time input and output operators were bounded. However, we have been able to extend these results to apply to discrete-time systems whose underlying continuous-time formulations involve unbounded input and/or output maps. An unbounded input operator is one that maps the control space into a space larger than the standard state space in which the problem is usually formulated. An unbounded output operator has domain smaller than the standard state space. We have been able to successfully apply our abstract theory to distributed parameter systems with boundary control, to hereditary systems with control delays, to boundary control systems with control delays, heat/diffusion equations with pointwise measurement of temperature, beam equations with pointwise measurement of strain or acceleration and distributed systems involving output delays. The computational schemes that we implemented and tested were either polynomial spline, Hermite spline, or modal function based and were able to handle reasonably complex problems when run on a micro-computer (an IBM PC - AT). This was joint work with Professor J.S. Gibson of the Department of Mechanical, Aerospace and Nuclear Engineering at the University of California, Los Angeles.

b. We have developed an α-shift technique which can be used in conjunction with schemes for the optimal LQ stablization of hereditary systems. This leads to control laws which yield a prescribed degree of stability, i.e. all system poles to the left of the line $\mathrm{Re}\, z = \alpha$ in the complex plane. Both the continuous and discrete time cases were considered. This was also joint work with Professor Gibson.

c. We have started to investigate and develop a finite dimensional approximation theory for the design of optimal fixed finite order compensators for distributed parameter systems. The approach we are taking is based upon and uses the Hyland-Bernstein optimal projection equations; a set of necessary conditions for optimality which, in an infinite dimensional setting, take the form of a coupled system of operator Riccati and Lyapunov equations. We replace the infinite dimensional system operators (i. e. A, B, and C) by finite element approximations. The resulting finite dimensional system of coupled matrix Riccati and Lyapunov equations are then solved using effective and efficient finite dimensional optimal projection algorithms and software developed by Hyland and Bernstein. At present, results from numerical studies carried out on examples involving heat and beam equations and hereditary systems are promising. Further computational studies along with theoretical analyses (i. e. convergence arguments, etc.) are currently underway and continuing. This work is joint with Dr. D. S. Bernstein of the Harris Corporation in Melbourne, Florida.

2. Parameter Identification

a. We have developed computational methods for the estimation of spatially varying material parameters (specifically flexural stiffness and Voigt-Kelvin viscoelastic damping coefficients) in Euler-Bernoulli models for the vibration of flexible beams with and without tip appendages. Our schemes involve spline-based finite element discretizations of the second order in time, fourth order in space system of partial differential or hybrid system of ordinary and parital differential equations and the function space admissible parameter set. A convergence theory was established and extensive numerical studies using simulation data was carried out on both conventional (serial) and vector computers. The schemes performed satisfactorily. This was joint work with Professor H.T. Banks of the Division of Applied Mathematics, Brown University and Dr. J.M. Crowley of the United States Air Force Academy.

b. We have tested our general approach for the estimation of unknown parameters in models for the vibration of flexible structures on actual expermental data taken from the RPL experiment. The RPL structure consists of four flexible beams cantilevered to a rotating hub. The structure was designed and built (and currenty resides) at the Charles Stark Draper Laboratory (CSDL) in Cambrigdge, MA with support from the Air Force Rocket Propulsion Laboratory (RPL) (now the Air Force Astronautical Laboratory (AFAL)) at Edwards Air Force Base in California. Using accelerometer data we were able to successfully identify parameters in a distributed parameter model for the structure. We hope to continue to test our theories and methods on this structure in the future. This was joint work with Professor Banks, Mr. S.S. Gates of CSDL and a former student, Ms. Y. Wang who is currently a research associate in the Division of Applied Mathematics at Brown University.

c. We have initiated a collaboration with Dr. Alok Das and his associates at the Air Force Astronautical Laboratory (AFAL) at Edwards Air Force Base in California for the purpose of collecting experimental data for use in the testing of our theory and computational schemes for the identification of distributed parameter systems. Specifically, we have made two visits to AFAL and taken data from an experimental flexible 5 ' x 5 ' alluminum grid which has been constructed by Dr. Das and his group. We are planning to develop appropriate theory and computational methods which can be used to fit a two dimensional distributed parameter model to the structure. In addition, we are also currently planning a series of flexible structure experiments to be carried out in the spring of 1988 in the 30ft thermal vaccuum chamber at AFAL. The purpose of these experiments is the collection of data for use in the study of thermal effects on internal damping mechanisms of composite materals. Our general approach will involve the identification of appropriate distributed thermoelastic or thermoviscoelastic models. Appropriate models, theory and computational schemes are being developed as the planning of the experiments and the preparation of the vaccuum chamber and experimental structure continues. The primary motivation for this investigation is the solar heating of orbiting large flexible spacecraft. This effort is joint with Dr. H. T. Banks of the Division of Applied Mathematics at Brown University and Dr. D. J. Inman of the Department of Mechanical and Aerospace Engineering at The State University of New York at Buffalo.

d. We have developed an abstract approximation framework for the identification of parameters in nonlinear distributed parameter systems. Using the theory of monotone operators and nonlinear evolution systems, we establish convergence results for Galerkin finite element methods for inverse problems involving broad classes of autonomous and nonautonomous nonlinear partial differential equations. This new nonlinear theory completely subsumes the existing linear theory and serves to generalize many of our earlier results. In addition, it can be applied to parameter estimation problems for a frequently cited model for nonlinear heat conduction. In addition to the theoretical results, we have carried out some preliminary numerical studies on a vector machine with the aid of a grant (of computer time) from the San Diego Supercomputer Center. This work is joint with Dr. H. T. Banks of the Division of Applied Mathematics at Brown University and Dr. S. Reich of the Department of Mathematics at the University of Southern California.

3. Publications Carrying AFOSR Grant Number AFOSR-84-0393

1. A Numerical Scheme for the Identification of Hybrid Systems Describing the Vibration of Flexible Beams with Tip Bodies, Journal of Math Analysis and Applications, 116 (1986), 262-288.

2. Spline-Based Rayleigh-Ritz Methods for the Approximation of the Natural Modes of Vibration for Flexible Beams with Tip Bodies, Quarterly of Appl. Math., Volume XLIV (1986) 169 - 185.

3. Approximation Methods for Inverse Problems Involving theVibration of Beams with Tip Bodies, Proceedings, 23rd IEEE Conference on Decision and Control, Las Vegas, Nevada, December, 1984.

4. A Galerkin Method for the Estimation of Parameters in Hybrid Systems Governing the Vibration of Flexible Beams with Tip Bodies, (with H.T. Banks), ICASE Report No. 85-7, Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, VA, February, 1985.

5. Approximation Methods for the Solution of Inverse Problems in Lake and Sea Sediment Core Analysis, (with H.T. Banks), Proceedings, 24th IEEE Conference on Decision and Control, Ft. Lauderdale, Florida, December, 1985.

6. Numerical Schemes for the Estimation of Functional Parameters in Distributed Models for Mixing Mechanisms in Lake and Sea Sediment Cores, (with H.T. Banks), Inverse Problems, 3(1987), 1-23.

7. Numerical Approximation for the Infinite-Dimensional Discrete-Time Optimal Linear-Quadratic Regulator Problem, (withJ.S. Gibson), SIAM J. Control and Optimization, 26(1988), to appear.

8. Shifting the Closed-Loop Spectrum in the Optimal Linear Quadratic Regulator Problem for Hereditary Systems, (with J.S. Gibson), IEEE Transactions on Automatic Control, AC-32(1987), 831-836.

9. Estimation of Stiffness and Damping in Cantilevered Euler-Bernoulli Beams with Tip Bodies, (with H.T. Banks and C. Wang), Proceedings, Fourth IFAC Symposium on Control of Distributed Parameter Systems, Los Angeles, CA, June, 1986.

10. Computational Methods for the Identification of Spatially Varying Stiffness and Damping in Beams (with H.T. Banks), Control - Theory and Advanced Technology, 3(1987), 1-32.

11. Methods for the Identification of Material Parameters in Distributed Models for Flexible Structures, (with H.T. Banks and J.M. Crowley), Mathematica Aplicada e Computacional, 5 (1986), 139-168.

12. The Identification of a Distributed Parameter Model for a Flexible Structure, (with H.T. Banks, S.S. Gates and Y. Wang), SIAM J. Control and Optimization, to appear.

13. Computational Methods for Optimal Linear - Quadratic Compensators for Infinite Dimensional Discrete-Time Systems, (with J.S. Gibson), Proceedings of International Conference on Control and Identification of Distributed Systems, Springer-Verlag Lecture Notes in Control and Information Sciences, to appear.

14. Inverse Problems in the Modeling of Vibrations of Flexible Beams, (with H. T. Banks and R. K. Powers), Proceedings of the International Conference on Control and Identification of Distributed Parameter Systems, Springer-Verlag Lecture Notes in Control and Information Sciences, to appear.

15. Approximation of Discrete-Time LQG Compensators for Distributed Systems with Boundary Input and Unbounded Measurement, (with J. S. Gibson), Automatica, to appear.

16. Approximation in Discrete-Time Boundary Control of Flexible Structures, (with J. S. Gibson), Proceedings of the 26th IEEE Conference on Decision and Control, Dec. 9-11, 1987, Los Angeles, CA, to appear.

17. Computational Methods for the Solution of Infinite Dimensional Discrete-Time Regulator Problems with Unbounded Input (with M. A. Lie) Proceedings of IMACS/IFAC International Symposium on Modeling and Simulation of Distributed Parameter Systems, Oct. 6-9, 1987, Hiroshima, Japan.

18. Approximation of Discrete-Time LQR Problems for Boundary Control Systems with Control Delays, Proceedings of IFIP Conference on Optimal Control of Systems Governed by Patial Differential Equations, July 6-9, 1987, Santiago de Compostela, Spain, Springer-Verlag, to appear.

19. An Approximation Framework for the Identification of Nonlinear Distributed Parameter Systems, (with H. T. Banks and S. Reich), in preparation.

4. Meetings Attended, Talks Given, and Papers Presented

Invited Participant, Workshop on Control Systems Governed by Partial Differential Equations with Application to Large Flexible Structures, The Pennsylvania State University, Clearwater, FL, March 4 - 8, 1985.

Invited Speaker, Applied Mathematics Seminar, Department of Mathematics, Harvey Mudd College, Claremont, CA., January 31, 1986.

Invited Speaker, Control Systems Seminar, Departments of Mathematics and Electrical Engineering, The Institute of Technology, University of Minnesota, Minneapolis, MN, June 5, 1986.

Invited Speaker, Conference on Control and Indentification of Distributed Systems, The Institute of Mathematics of the University of Graz, Vorau, Austria, July 6 - 12, 1986.

Invited Speaker and Session Chairman, Meeting of the Society for Engineering Science, State University of New York at Buffalo, Buffalo, NY, August 25-27, 1986.

Invited Participant, Second Workshop on the Control of Systems Governed by Partial Differential Equations sponsored by AFOSR, NSF and the University of Montreal, Val David, Quebec, Canada, October 5 - 9, 1986.

Invited Speaker, Control Systems Seminar, Department of Electrical and Computer Engineering, University of California, Santa Barbara, Santa Barbara, CA, October 27, 1986.

Paper Presented, 1984 IEEE Conference on Decision and Control, Las Vegas, Nevada, December, 1984.

Paper Presented, 1985 SIAM Fall Meeting, Arizona State University, Tempe, Arizona, October, 1985.

Paper Presented, 1985 IEEE Conference on Decision and Control, Ft. Lauderdale, Florida, December, 1985.

Invited Speaker, IFIP Conference on Optimal Control of Systems Governed by Partial Differential Equations, July, 6-9, 1987, Santiago de Compostela, Spain.

Speaker and Invited Session Chairman, IMACS/IFAC International Symposium on Modeling and Simulation of Distributed Parameter Systems, Oct. 6-9, 1987, Hiroshima, Japan.

Attendee, ICIAM '87, First International Conference on Industrial and Applied Mathematics, June 29 - July 3, 1987, Paris, France.

5. Students Supported

1. Ms. Y. Wang, MSEE, University of Southern California, 1984, MS Applied Mathematics, University of Southern California, 1986. Carried out computations for identification of RPL structure. Thesis: An Inverse Problem for a Flexible Structure. Supported: 1 June, 1986 - 31 July, 1986.

2. Mr. M. Lie, BSEE, University of Southern California, Carried out computations for optimal discrete-time LQG compensators for infinite dimensional systems. Supported: 1 June, 1986 - 31, May, 1987

3. Mr. P. Feehan, MSEE, University of Southern California, Carried out computations for preliminary studies on the identification of material parameters in distributed parameter models for flexible structures using modal or spectral data. Supported: 1 June, 1986 - 31 August, 1986.

4. Mr. C. Lo, MSCE, University of Southern California, BSEE, George Washington University, carried out supercomputer calculations for studies on the identification of nonlinear distributed parameter systems. Attended San Diego Supercomputer Center Summer Institute, Summer, 1987. Supported 1, June, 1987 - Present.

5. Mr. C. Mao, Sc. D. Mathematics, Wuhan University, carried out preliminary theoretical studies on thermomechanical models in flexible structures. Supported 1, June, 1987 - 31, August, 1987.

6. Equipment Purchased

1. IBM PC AT and peripherals. Used to carry out many of the computations reported on above.

2. AST Premier/286 and peripherals. Used by P. I. and students to carry out computations reported on above.

7. Abstracts

1. A Numerical Scheme for the Identification of Hybrid Systems Describing the Vibration of Flexible Beams with Tip Bodies, I. G. Rosen.

   A cubic spline-based Galerkin-like method is developed for the identification of a class of hybrid systems which decribe the transverse vibration of flexible beams with attached tip bodies. The identification problem is formulated as a least-squares fit to data subject to the system dynamics given by a coupled system of ordinary and partial differential equations recast as an abstract evolution equation (AEE) in an appropriate infinite-dimensional Hilbert space. Projecting the AEE into spline-based subspaces leads naturally to a sequence of approximating finite-dimensional identification problems. The solutions to these problems are shown to exist, are relatively easily computed, and are shown to, in some sense, converge to solutions to the original identification problem. Numerical results for a variety of examples are discussed.

2. Spline-Based Rayleigh-Ritz Methods for the Approximation of the Natural Modes of Vibration for Flexible Beams with Tip Bodies, I. G. Rosen.

   Rayleigh-Ritz methods for the approximation of the natural modes for a class of vibration problems involving flexible beams with tip bodies using subspaces of piecewise polynomial spline functions are devloped. An abstract operator-theoretic formulation of the eigenvalue problem is derived and spectral properties investigated. The existing theory for spline-based Rayleigh-Ritz methods applied to elliptic differential operators and the approximation properties of interpolatory splines are used to argue convergence and establish rates of convergence. An example and numerical results are discussed.

3. Approximation Methods for Inverse Problems Involving the Vibration of Beams with Tip Bodies, I. G. Rosen.

   In this short paper we briefly outline two cubic spline based approximation schemes for the solution of inverse problems involving the vibration of flexible beams with attached tip bodies. The identification problem is formulated as the least squares fit to data of a hybrid system of coupled partial and ordinary differential equations describing the dynamics of the beam and tip bodies. The resulting optimization problem is infinite dimensional and as such, necessitates the use of some form of approximation. The schemes we have developed are based upon the construction of a sequence of approximating identification problems in which the underlying constraining state equations are semi-discrete finite dimensional approximations to the infinite dimensional distributed system which governs the original identification problem. Our study includes both theoretical convergence results and numerical testing.

4. A Galerkin Method for the Estimation of Parameters in Hybrid Systems Governing the Vibration of Flexible Beams with Tip Bodies, H Thomas Banks and I. G. Rosen.

   In this report we develop an approximation scheme for the identification of hybrid systems describing the transverse vibrations of flexible beams with attached tip bodies. In particular, problems involving the estimation of functional parameters (spatially varying stiffness and/or linear mass density, temporally and/or spatially varying loads, etc.) are considered. The identification problem is formulated as a least squares fit to data subject to the coupled system of partial and ordinary differential equations describing the transverse displacement of the beam and the motion of the tip boodies respectively. A cubic spline-based Galerkin method applied to the state equations in weak form and the discretization of the admissible parameter space yield a sequence of approximting finite dimensional identification problems. We demonstrate that each of the approximating problems admits a solution and that from the resulting sequence of optimal solutions a convergent subsequence can be extracted, the limit of which is a solution to the original identification problem. The approximating identification problems can be solved using standard techniques and readily available software. Numerical results for a variety of examples are provided.

5. Numerical Schemes for the Estimation of Functional Parameters in Distributed Models for Mixing Mechanisms in Lake and Sea Sediment Cores, H. T. Banks, and I. G. Rosen.

We consider distributed parameter models for vertical mixing in lake and sea sediment cores. Finite dimensional approximation schemes are developed for the solution of associated inverse problems. The schemes permit one to estimate temporally and spatially varying functional parameters which appear in the parabolic partial differential equations and boundary conditions constituting the models. Thecretical convergence results are established. Numerical findings are presented which demonstrate the potential of the methods. An example involving the identification of a depth-dependent mixing parameter based upon volcanic ash data from the North Atlantic is included.

6. Numerical Approximation for the Infinite-Dimensional Discrete-Time Optimal Linear-Quadratic Regulator Problem, J. S. Gibson, and I. G. Rosen.

An abstract approximation framework is developed for the finite and infinite horizon discrete-time linear-quadratic regulator problems for systems whose state dynamics are described by a linear semigroup of operators on an infinite-dimensional Hilbert space. The schemes included in the framework yield finite-dimensional approximations to the linear state feedback gains which determine the optimal control law. Convergence agruments are given. Examples involving hereditary and parabolic systems and the vibration of a flexible beam are considered. Spline-based finite element schemes for these classes of problems, together with numerical results, are presented and discussed.

7. Shifting the Closed-Loop Spectrum in the Optimal Linear Quadratic Regulator Problem for Hereditary Systems, J. S. Gibson and I. G. Rosen.

In the optimal linear quadratic regulator problem for finite dimensional systems, the method known as an α-shift can be used to produce a closed-loop system whose spectrum lies to the left of some specified vertical line; that is, a closed-loop system with a prescribed degree of stability. This paper treats the extension of the α-shift to hereditary systems. As in finite dimensions, the shift can be accomplihed by adding α times the identity to the open-loop semigroup generator and then solving an optimal regulator problem. However, this approach does not work with a new approximation scheme for hereditary control problems recently developed by Kappel and Salamon. Since this scheme is among the best to date for the numerical solution of the linear regulator problem for hereditary systems, an alternative method for shifting the closed-loop spectrum is needed. An α-shift technique that can be used with the Kappel-Salamon approximation scheme is developed. Both the continuous-time and discrete-time problems are considered. A numerical example which demonstrates the feasibility of the method is included.

8. Estimation of Stiffness and Damping in Cantilevered Euler-Bernoulli Beams with Tip Bodies, H. T. Banks and I. G. Rosen.

    We develop finite dimensional approximation schemes for the identification of spatially varying material parameters, i. e. flexural stiffness and viscous damping coefficients in hybrid models for flexible beams with tip bodies. Our schemes are derived via an application of spline-based Galerkin techniques to the conservative form state space representation for the coupled system of ordinary and partial differential equations and boundary conditions which describe the dynamics of the system. A convergence theory is briefly outlined and a discussion of our findings based upon extensive numerical studies carried out on both conventional and vector processors is included.

9. Computational Methods for the Identification of Spatially Varying Stiffness and Damping in Beams, H. T. Banks and I. G. Rosen.

    A numerical approximation scheme for the estimation of functional parameters in Euler-Bernoulli models for the transverse vibration of flexible beams with beams with tip bodies is developed. The method permits the identification of spatially varying flexural stiffness and Voigt-Kelvin viscoelastic damping coefficients which appear in the hybrid system of ordinary and partial differential equations and boundary conditions describing the dynamics of such structures. An inverse problem is formulated as a least squares fit to data subject to constraints in the form of a vector system of abstract first order evolution equations. Spline-based finite element approximations are used to finite dimensionalize the problem. Theoretical convergence results are given and numerical studies carried out on both conventional (serial) and vector computers are discussed.

10. Methods for the Identification of Material Parameters in Distributed Models for Flexible Structures, H. T. Banks, J. M. Crowley and I. G. Rosen.

    In this paper we present theoretical and numerical results for inverse problems involving estimation of spatially varying parameters such as stiffness and damping in distributed models for elastic structures such as Euler-Bernoulli beams. An outline of algorithms we have used and a summary of our computational experiences are presented.

11. The Identification of a Distributed Parameter Model for a Flexible Structure, H. T. Banks, S. S. Gates, I. G. Rosen, and Y. Wang.

    We develop a computational method for the estimation of parameters in a distributed model for a flexible structure. The structure we consider (part of the "RPL experiment") consists of a cantilevered beam with a thruster and linear accelerometer at the free end. The thruster is fed by a pressurized hose whose horizontal motion effects the transverse vibration of the beam. We use the Euler-Bernoulli theory to model the vibration of the beam and treat the hose-thruster assembly as a lumped or point mass-dashpot-spring system at the tip. Using measurements of linear accleration at the tip, we estimate the hose parameters (mass, stiffness, damping) and a Voigt-Kelvin viscoelastic structural damping parameter for the beam using a least squares fit to the data.

    We consider spline based approximations to the hybrid (coupled ordinary and partial differential equations) system; theoretical convergence results and numerical studies with both simulation and actual experimental data obtained from the structure are presented and discussed.

12. Computational Methods for Optimal Linear-Quadratic Compensators for Infinite Dimensional Discrete-Time Systems, J. S. Gibson and I. G. Rosen.

    An abstract approximation theory and computational methods are developed for the determination of optimal linear-quadratic feedback controls, observes and compensators for infinite dimensional discrete-time systems. Particular attention is paid to systems whose open-loop dynamics are described by semigroup of operators on Hilbert spaces. The approach taken is based upon the finite dimensional approximation of the infinite dimensional operator Riccati equations which characterize the optimal feedback control and observer gains. Theoretical convergence results are presented and discussed. Numerical results for an example involving a heat equation with boundary control are presented and used to demonstrate the feasibility of our methods.

13. Inverse Problems in the Modeling of Vibrations of Flexible Beams, H. T. Banks, R. K. Powers and I. G. Rosen.

    The formulation and solution of inverse problems for the estimation of parameters which describe damping and other dynamic properties in distributed models for the vibration of flexible structures is considered. Motivated by a slewing beam experiment, the identification of a nonlinear velocity dependent term which models air drag damping in the Euler-Bernoulli equation is investigated. Galarkin techniques are used to generate finite dimensional approximations. Convergence estimates and numerical results are given. The modeling of, and related inverse problems for the dynamics of a high pressure hose line feeding a gas thruster actuator at the tip of a cantilevered beam are then considered. Approximation and convergence are discussed and numerical results involving experimental data are presented.

14. Approximation of Discrete-Time LQG Compensators for Distributed Systems with Boundary Input and Unbounded Measurement, J. S. Gibson and I G. Rosen.

    We consider the approximation of optimal discrete-time linear quadratic Gaussian (LQG) compensators for distributed parameter control systems with boundary input and unbounded measurement. Our approach applies to a wide range of problems that can be formulated in a state space on which both the discrete-time input and output operators are continuous. Approximating compensators are obtained via application of the LQG theory and associated approximation results for infinite dimensional discrete-time control system with bounded input and output. Numerical results for spline and modal based approximation schemes used to compute optimal compensators for a one dimensional heat equation with either Neumann or Dirichlet boundary control and pointwise measurement of temperature are presented and discussed.

15. Approximation in Discrete-Time Boundary Control of Flexible Structures, J. S. Gibson and I. G. Rosen.

    This paper treats discrete-time LQG optimal control of flexible structures with boundary control and what normally are unbounded measurement operators.

    The application of recently developed approximation theory for infinite dimensional discrete-time LQG problems to the problem here is discussed, and numerical examples are presented.

16. Computational Methods for the Solution of Infinite Dimensional Discrete-Time Regulator Problems with Unbounded Input, I. G. Rosen and M. A. Lie.

An approximation framework for the closed-loop solution of discrete-time linear-quadratic regulator problems for infinite dimensional systems with unbounded control inputs is developed. Sufficient conditions for the convergence of approximations to Riccati operators and feedback gains which characterize the optimal control law are provided. General theories for abstract partial differential systems with boundary control and distributed systems with control delays are developed. Spline-based schemes and numerical results for heat and beam equations with boundary control and a hereditary system with delayed control are presented and discussed.

17. Approximation of Discrete-Time LQR Problems for Boundary Control Systems with Control Delays, I. G. Rosen.

In this short note we consider the extension and application of the approximation theory for discrete-time linear-quadratic regulator problems with either bounded or unbounded inputs we developed earlier to boundary control systems with control delays. We synthesize our earlier, existing results for distributed systems with boundary controls and for systems with control delays into a theory which is applicable to systems that simultaneously exhibit both forms of unbounded input.

Appendix

Approximation of Discrete-Time LQG Compensators for Distributed Systems with Boundary Input and Unbounded Measurement†

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ABSTRACT

We consider the approximation of optimal discrete-time linear quadratic Gaussian (LQG) compensators for distributed parameter control systems with boundary input and unbounded measurement. Our approach applies to problems that can be formulated in a state space on which both the discrete-time input and output operators are continuous. Approximating compensators are obtained via application of the LQG theory and associated approximation results for infinite dimensional discrete-time control systems with bounded input and output. Numerical results for spline and modal approximation schemes used to compute optimal compensators for a one dimensional heat equation with either Neumann or Dirichlet boundary control and pointwise measurement of temperature are presented and discussed.

1. Introduction

In this paper we develop an approximation theory for the computation of optimal discrete-time linear quadratic Gaussian (LQG) compensators (combined feedback control law and state estimator) for distributed parameter systems with boundary input or control and unbounded output or measurement. In a continuous time setting, boundary input typically results in an unbounded input operator. That is, the system's input operator maps the control input into a space larger than the state space in which the open-loop system is usually formulated. In the discrete-time case, on the other hand, for a wide class of distributed systems, the resulting input operator is bounded on the usual underlying state space. An unbounded output, or measurement, operator has domain smaller than the usual open-loop state space.

For continuous time systems, Pritchard and Salamon (1987) have established an abstract semigroup theoretic framework for treating the linear quadratic regulator problem (control only) for infinite dimensional systems with unbounded input and output operators. Their approach is based upon a weak or distributional formulation of the Riccati equations which characterize the optimal feedback control laws in an appropriate dual space . Curtain (1984) provides a procedure for the design of finite dimensional compensators for parabolic systems with unbounded control and observation. In (Curtain and Salamon, 1986) a finite dimensional compensator design procedure for a wider class of infinite dimensional systems with unbounded input (but bounded output) including hereditary systems with control delays and partial differential systems with boundary control is developed. Lasiecka and Triggiani have looked at linear regulator problems for parabolic (1983a, 1987a) and hyperbolic (1983b, 1986) systems with boundary control and obtained, among other things, global and local regularity results for the optimal controls and state trajectories. In (Lasiecka and Triggiani, 1987b) Galerkin approximations and an associated convergence theory for closed-loop solutions to regulator problems for parabolic systems with Dirichlet boundary input are studied. A more complete survey of the boundary control literature including references to some of the poineering work in this area can be found in (Pritchard and Salamon, 1987).

In our treatment here, we consider the discrete-time problem (i.e. piecewise constant input and sampled output). Our interest in the discrete-time or digital formulation is motivated by 1) the fact that it represents a more accurate or realistic description of how the linear-quadratic theory for distributed systems would actually be applied in practice, and by 2) how the boundedness of the discrete-time input operator in the usual underlying state space facilitates the development of an approximation theory which can simultaneously handle both unbounded input and unbounded output. Our approach is based upon an application of the theory we developed earlier in (Gibson and Rosen, 1985 and 1986) for the approximation of optimal discrete-time LQG compensators for infinite dimensional systems with bounded input and output. Our results are applicable to boundary control systems in which a restriction of the state transition operator and the discrete-time input operator are bounded on a space on which the output operator is bounded as well. To illustrate our approach, in this paper we describe in detail the application of our theory to a one dimensional heat equation with either Neumann or Dirichlet boundary control and pointwise measurement of temperature. Elsewhere (see Gibson and Rosen, 1987) we have applied our results to develop approximation schemes for the computation of optimal LQG compensators for flexible structures (i.e. Euler-Bernoulli beams) with shear force input at the boundary and a pointwise measurement of strain.

An outline of the remainder of the paper is as follows. In Section 2 we describe an abstract framework for the study of boundary control systems and their discrete-time formulation. In Section 3 we review the LQG theory for infinite dimensional discrete-time systems and associated abstract approximation results. In the fourth section, we discuss spline and modal subspace based approximation schemes for the heat equation example. Section 5 contains some concluding remarks.

2. The Boundary Control System and its Discrete-Time Formulation

We employ a semigroup theoretic formulation that has been used previously for a class of abstract boundary control systems. See, for example, (Curtain and Salamon, 1986). Let W,V and H be Hilbert spaces with W and V densely and continuously embedded in H. We consider boundary control systems of the form

(2.1) $\dot{w}(t) = \Delta w(t), \quad t > 0$

(2.2) $w(0) = w_0$

(2.3) $\Gamma w(t) = v(t), \quad t \geq 0$

(2.4) $y(t) = Cw(t), \quad t \geq 0$

where $\Delta \varepsilon \mathcal{L}(W,H)$, the boundary input operator Γ is an element in $\mathcal{L}(W,R^m)$ and the output operator C is an element in $\mathcal{L}(V,R^p)$. Note that the operator Δ need not be the Laplacian. Our choice of Δ to denote a general, most often differential, operator satisfying the conditions set forth below is consistent with the notation used in earlier treatments of boundary control systems elsewhere in the literature.

We assume that 1) Γ is surjective and its null space, $\mathcal{N}(\Gamma) = \{\varphi \varepsilon W: \Gamma\varphi = 0\}$, is dense in H, 2) the operator $\mathcal{A}$, defined to be the restriction of the operator Δ to $\mathcal{N}(\Gamma)$, is a closed operator on H and has non-empty resolvent set and 3) for each $T > 0$, all $w_0 \varepsilon W$, and $v \varepsilon C^1(0,T; R^m)$ with $\Gamma w_0 = v(0)$, there exists a unique $w \varepsilon C([0,T]; W) \cap C^1([0,T]; H)$ which depends continuously on w_0 and v and which satisfies (2.1) - (2.3) for each $t \varepsilon [0,T]$. It then follows (see Hille and Phillips, 1957) that the operator $\mathcal{A} : \mathrm{Dom}(\mathcal{A}) \subset H \rightarrow H$ given by $\mathcal{A}\varphi = \Delta\varphi$ for $\varphi \varepsilon \mathrm{Dom}(\mathcal{A}) = \mathcal{N}(\Gamma)$ is the infinitesimal generator of a $\mathcal{C}_0$ semigroup, $\{\mathcal{T}(t) : t \geq 0\}$, of bounded linear operators on H.

Define the space Z as the dual of $\mathrm{Dom}(\mathcal{A}^*)$ where the norm on $\mathrm{Dom}(\mathcal{A}^*)$ is taken to be the graph Hilbert space norm associated with the operator $\mathcal{A}^*$. Then H is densely and continuously embedded in Z and $\{\mathcal{T}(t) : t \geq 0\}$ can be uniquely extended to a $\mathcal{C}_0$ semigroup of bounded linear operators on Z. Its generator is the extension of the operator $\mathcal{A}$ to an the operator $\hat{\mathcal{A}}$ in $\mathcal{L}(H,Z)$ given by $(\hat{\mathcal{A}}\varphi)(\psi) = \langle\varphi, \mathcal{A}^*\psi\rangle_H$ for $\varphi \varepsilon H$ and $\psi \varepsilon \mathrm{Dom}(\mathcal{A}^*)$.

Since Γ was assumed to be a surjection, it has a right inverse. Let $\Gamma^+ : R^m \rightarrow W$ be any right inverse of Γ. Since $\mathrm{Dom}(\Gamma^+) = R^m$, we have $\Gamma^+ \varepsilon \mathcal{L}(R^m, W)$. For $v \varepsilon R^m$, we define $\mathcal{B} \varepsilon \mathcal{L}(R^m, Z)$ by $\mathcal{B}v = (\Delta - \hat{\mathcal{A}})\Gamma^+ v$. If Γ^+_1 and Γ^+_2 are two distinct right inverses of Γ then $\mathcal{R}(\Gamma^+_1 - \Gamma^+_2) \subset \mathcal{N}(\Gamma)$. Since $\hat{\mathcal{A}}$ coincides with Δ on $\mathcal{N}(\Gamma)$, it follows that the operator $\mathcal{B}$ is well defined. It can be shown (see Curtain and Salamon, 1986) that for each $w_0 \varepsilon H$ and $v \varepsilon L_2(0,T; R^m)$ there exists a unique $w \varepsilon C([0,T]; H) \cap H^1(0,T; Z)$ which depends continuously on w_0 and v and which satisfies

$$\dot{w}(t) = \mathcal{A} w(t) + \mathcal{B} v(t), \qquad t > 0$$
$$w(0) = w_0$$

in Z. The function w is given by

$$(2.5) \quad w(t) = \mathcal{T}(t)w_0 + \int_0^t \mathcal{T}(t-s)\,\mathcal{B} v(s)ds, \quad t \geq 0$$

and is referred to as a weak solution to the boundary control system (2.1) - (2.3).

The discrete-time formulation of (2.1) - (2.4) is found by considering piecewise-constant controls of the form

$$(2.6) \quad v(t) = u_k, \quad t \,\varepsilon\, [k\tau, (k+1)\tau), \quad k = 0,1,2,\ldots$$

where τ denotes the length of the sampling interval. Let $w_k = w(k\tau)$, $k = 0,1,2,\ldots$ where $w(\cdot)$ is the unique weak solution to (2.1) - (2.3) given by (2.5) corresponding to $w_0 \varepsilon H$ and input v given by (2.6). (We note that with piecewise constant input of the form (2.6), the solution w is in fact a strong solution on each subinterval $[k\tau, (k+1)\tau]$.) For each $k = 0,1,2,\ldots$ define $z_k \;\varepsilon\; C([k\tau, (k+1)\tau]; H)$ by $z_k(t) = w(t) - \Gamma^+ u_k$, $t \,\varepsilon\, [k\tau, (k+1)\tau]$. Then

$$\begin{aligned} \dot{z}_k(t) = \dot{w}(t) &= \mathcal{A} w(t) + \mathcal{B} u_k \\ &= \mathcal{A} z_k(t) + (\mathcal{A} + \mathcal{B}\Gamma)\Gamma^+ u_k \\ &= \mathcal{A} z_k(t) + \Delta\,\Gamma^+ u_k, \qquad t \,\varepsilon\, (k\tau, (k+1)\tau], \\ z_k(k\tau) &= w_k - \Gamma^+ u_k . \end{aligned}$$

Therefore

$$
\begin{aligned}
w_{k+1} &= z_k((k+1)\tau) + \Gamma^+u_k \\
&= \mathcal{T}(\tau)(w_k - \Gamma^+u_k) + \int_0^\tau \mathcal{T}(s)\,\Delta\,\Gamma^+u_k\,ds + \Gamma^+u_k \\
&= \mathcal{T}(\tau)w_k + (I - \mathcal{T}(\tau))\,\Gamma^+u_k + \int_0^\tau \mathcal{T}(s)\Delta\Gamma^+u_k ds,
\end{aligned}
$$

or

$$
w_{k+1} = Tw_k + Bu_k, \quad k = 0,1,2,...
$$

$$
w_0 \in H
$$

where $T \in \mathcal{L}(H)$ and $B \in \mathcal{L}(R^m, H)$ are given by $T = \mathcal{T}(\tau)$ and $B = (I - \mathcal{T}(\tau))\,\Gamma^+ + \int_0^\tau \mathcal{T}(s)\Delta\Gamma^+ds$ respectively.

We note that as in the case of the continuous-time input operator $\mathcal{B}$, the discrete-time input operator B is well defined and does not depend upon a particular choice for Γ^+. Indeed if B_1 and B_2 are the input operators which correspond to the choices Γ_1^+ and Γ_2^+ then for $u \in R^m$ we have

$$
(B_1 - B_2)u = (I - \mathcal{T}(\tau))(\Gamma_1^+ - \Gamma_2^+)u + \int_0^\tau \mathcal{T}(s)\Delta\,(\Gamma_1^+ - \Gamma_2^+)u ds.
$$

But $(\Gamma_1^+ - \Gamma_2^+)u \in \mathcal{N}(\Gamma) = \mathrm{Dom}(\mathcal{Q})$ and therefore

$$
\int_0^\tau \mathcal{T}(s)\Delta(\Gamma_1^+ - \Gamma_2^+)u ds = \int_0^\tau \mathcal{T}(s)\,\mathcal{Q}(\Gamma_1^+ - \Gamma_2^+)\,u ds
$$

$$
= \int_0^\tau \frac{d}{ds}\,\mathcal{T}(s)(\Gamma_1^+ - \Gamma_2^+)u ds = (\mathcal{T}(\tau) - I)(\Gamma_1^+ - \Gamma_2^+)\,u\,.
$$

In addition, if Γ^+ is chosen so that $\mathcal{R}(\Gamma^+) \subset \mathcal{N}(\Delta)$, B takes on the particularly simple form $B = (I - \mathcal{T}(\tau))\Gamma^+$. It is worth noting that a simple calculation reveals that

$$
B = \int_0^\tau \mathcal{T}(s)\,\mathcal{B} ds
$$

in agreement with the standard technique for obtaining the discrete or sampled time formulation of a continuous time system in either a finite dimensional or bounded input setting.

It is our intention here to apply the approximation theory we developed earlier in (Gibson and Rosen, 1986) for the design of optimal discrete-time LQG compensators for infinite dimensional systems with bounded input and output operators. We therefore require the additional assumptions that 4) $T = \mathcal{T}(\tau) \varepsilon \mathcal{L}(V)$ and 5) $\mathcal{R}(\Gamma^+) \subset V$. Although not all boundary control systems we might formulate would satisfy these conditions, there are many interesting and important systems which do (see, for example, Section 4 below and Gibson and Rosen, 1987). In this case, the control system (2.1) - (2.4) takes the form

(2.7) $\qquad w_{k+1} = Tw_k + Bu_k, \quad k = 0,1,2,\ldots$

(2.8) $\qquad w_0 \varepsilon V$

(2.9) $\qquad y_k = Cw_k, \quad k = 0,1,2,\ldots$.

3. LQG Theory for Infinite Dimensional Discrete-Time Systems and Finite Dimensional Approximation

3.1. The Infinite Dimensional Probelm

The discrete-time linear-quadratic regulator problem for the boundary control system (2.1) - (2.3) is:

Find $u^* = \{u^*_k\}_{k=0}^{\infty} \varepsilon\ \ell_2(0, \infty; R^m)$ which minimizes the quadratic performance index

$$J(u) = \sum_{k=0}^{\infty} \langle Qw_k, w_k \rangle_V + u_k^T R u_k$$

where $Q \varepsilon \mathcal{L}(V)$ is self-adjoint and nonnegative, R is a symmetric positive definite m×m matrix and the state $w = \{w_k\}_{k=0}^{\infty}$ evolves according to the recurrence (2.7), (2.8).

An optimal control exists for each initial condition w_0 if and only if the operator algebraic Riccati equation

(3.1) $\qquad \Pi = T^*(\Pi - \Pi B(R + B^*\Pi B)^{-1}B^*\Pi)T + Q$.

has a bounded nonnegative, self-adjoint solution Π. In this case, the optimal control has the feedback form $u_k = -Fw_k$ where $F = (R + B^* \Pi B)^{-1} B^* \Pi T$. A control (sequence) u is admissible for the initial condition w_0 if the corresponding J(u) is finite. If there exists an admissible control for each initial condition, then (3.1) has a bounded nonnegative, self-adjoint solution. If each admissible control for each initial condition drives the state to zero asymptotically, then there exists at most one bounded nonnegative, self-adjoint solution to (3.1). The optimal trajectory $w^* = \{w^*_k\}_{k=0}^{\infty}$ evolves according to $w^*_k = S^k w_0$, $k = 0,1,2,\ldots$, where the closed loop state transition operator $S \in \mathcal{L}(V)$ is $S = T - BF$. If Q is coercive, then S has spectral radius less than one and is uniformly exponentially stable. From the finite dimensionality of the control space we obtain

$$(3.2) \qquad u^*_k = -\langle f, w^*_k \rangle_V, \quad k = 0,1,2,\ldots$$

where $f = (f_1, f_2, \ldots, f_m)^T \in \mathop{\times}\limits_{j=1}^{m} V$ is called the optimal functional feedback control gain. The results stated here for the optimal linear-quadratic regulator problem are summarized from (Gibson and Rosen, 1985).

When only a finite dimensional measurement $y = \{y_k\}_{k=0}^{\infty}$ of the infinite dimensional state w is available (recall (2.9)), a state estimator or observer is required. For a given input sequence u and corresponding output sequence y, the optimal LQG estimator is

$$(3.3) \qquad \hat{w}_{k+1} = T\hat{w}_k + Bu_k + \hat{F}\{y_k - C\hat{w}_k\}, \quad k = 0,1,2,\ldots$$

$$(3.4) \qquad \hat{w}_0 \in V$$

where the optimal estimator or observer gain $\hat{F} \in \mathcal{L}(R^p, V)$ is $\hat{F} = T\hat{\Pi}C^*(\hat{R} + C\hat{\Pi}C^*)^{-1}$ with $\hat{\Pi} \in \mathcal{L}(V)$ the minimal, self-adjoint, nonnegative solution (if one exists) to the operator algebraic Riccati equation

$$(3.5) \qquad \hat{\Pi} = T(\hat{\Pi} - \hat{\Pi}C^*(\hat{R} + C\hat{\Pi}C^*)^{-1}C\hat{\Pi})T^* + \hat{Q}.$$

Since $\hat{F} \in (R^p, V)$, it has the representation

$$\hat{F}y = \hat{f}^T y \ , \quad y \ \varepsilon \ R^p$$

where $\hat{f} = (f_1, f_2, ..., f_p)^T \ \varepsilon \ \times_{j=1}^{p} V$ is called the optimal functional observer gain. The operator $\hat{Q} \ \varepsilon \ \mathcal{L}(V)$ is self-adjoint, nonnegative and the p×p matrix $\hat{R}$ is symmetric, positive definite.

In a stochastic setting, the operator $\hat{Q}$ and the matrix $\hat{R}$ are, respectively, the covariance operator and covariance matrix for uncorrelated, zero-mean, stationary, Gaussian white noise processes that force the state and corrupt the measurement. In this case, if $\hat{Q}$ is trace class, (3.3), (3.4) is the infinite dimensional analog of the discrete-time Kalman-Bucy filter. In a strictly deterministic setting, $\hat{Q}$ and $\hat{R}$ are assumed to be determined via engineering design criteria such as stability margins, robustness of the closed-loop system, etc.

Replacing operators in the control problem with the adjoints of the appropriate operators in the estimator problem yields the usual duality between the LQG optimal control and estimator problems. Hence sufficient conditions for existence and uniqueness of solutions of (3.5) and the closed-loop estimator stability properties are analogous to the results for the control problem. In particular, if $e_k = \hat{w}_k^* - w_k$, then $e_k = \hat{S}^k e_0$, k = 0, 1,2, ..., where $\hat{S} = T - \hat{F}C$, and a sufficient condition for $\hat{S}$ to be uniformly exponentially stable is that $\hat{Q}$ be coercive.

The optimal LQG compensator consists of the state estimator in (3.3) and (3.4) and the control law

$$\text{(3.6)} \qquad \hat{u}_k^* = -F \ \hat{w}_k^* \ , \ k = 0,1,2,...$$

The resulting closed-loop system is given by

$$\mathcal{W}_k = \mathcal{A}^k \mathcal{W}_0, \quad k = 0,1,2,...$$

where $\mathcal{W}_k = (w_k, \hat{w}^*_k)^T$ with $\{w_k\}^*_{k=0}$ the state trajectory that results from the input (3.6) and $\mathcal{A} \ \varepsilon \ \mathcal{L}(V \times V)$ is

$$\mathcal{S} = \begin{bmatrix} T & -BF \\ \\ \hat{F}C & T-BF-\hat{F}C \end{bmatrix}.$$

It is easy to show that the spectrum of $\mathcal{S}$ is given by $\sigma(\mathcal{S}) = \sigma(S) \cup \sigma(\hat{S})$, so that the stability of the closed-loop plant-compensator system is determined by the stability of the plant with full state feedback and the stability of the estimator error.

3.2. Approximation

For each $N = 1,2,\ldots$, let V_N be a finite dimensional subspace of V and let P_N be a bounded linear mapping from V onto V_N (for example, the orthogonal projection with respect to either the V or H inner product). Let T_N, Q_N, $\hat{Q}_N \in \mathcal{L}(V_N)$, $B_N \in \mathcal{L}(R^m,V_N)$ and $C_N \in \mathcal{L}(V_N,R^p)$ and set

$$F_N = (R + B_N^* \Pi_N B_N)^{-1} B_N^* \Pi_N T_N$$

and

$$\hat{F}_N = T_N \hat{\Pi}_N C_N^* (\hat{R} + C_N \hat{\Pi}_N C_N^*)^{-1}$$

where Π_N and $\hat{\Pi}_N$ are the minimal, self-adjoint, nonnegative solutions (assuming that they exist) to the finite dimensional operator algebraic Riccati equations

$$\Pi_N = T_N^*(\Pi_N - \Pi_N B_N (R + B_N^* \Pi_N B_N)^{-1} B_N^* \Pi_N) T_N + Q_N \tag{3.7}$$

and

$$\hat{\Pi}_N = T_N(\hat{\Pi}_N - \hat{\Pi}_N C_N^* (\hat{R} + C_N \hat{\Pi}_N C_N^*)^{-1} C_N \hat{\Pi}_N) T_N^* + \hat{Q}_N \tag{3.8}$$

respectively. The approximating optimal compensator is given by

$$\hat{u}_{N,k}^* = -F_N \hat{w}_{N,k}^*, \; k = 0,1,2,\ldots$$

where $\hat{w}^*_N = \{\hat{w}^*_{N,k}\}_{k=0}^{\infty}$ is determined according to the approximating observer

$$\hat{w}^*_{N,k+1} = T_N \hat{w}^*_{N,k} + B_N \hat{u}^*_{N,k} + \hat{F}_N\{y^*_{N,k} - C_N \hat{w}^*_{N,k}\}, \quad k = 0,1,2\ldots$$
$$\hat{w}^*_{N,0} = P_N \hat{w}_0 \in V_N .$$

The measurements y^*_N are given by $y^*_{N,k} = Cw_{N,k}$, $k = 0,1,2,\ldots$ where

$$w_{N,k+1} = Tw_{N,k} - B\hat{u}^*_{N,k} , \qquad k = 0,1,2,\ldots$$
$$w_{N,0} = w_0 .$$

The resulting closed-loop system is given by $\mathcal{W}_{N,k} = \mathcal{S}^k_N \mathcal{W}_{N,0}$, $k = 0,1,2,\ldots$ where $\mathcal{W}_{N,k} = (w_{N,k}, \hat{w}^*_{N,k})^T$ and $\mathcal{S}_N \in \mathcal{L}(V \times V_N)$ is given by

$$(3.9) \qquad \mathcal{S}_N = \begin{bmatrix} T & -BF_N \\ \hat{F}_N C & T_N - B_N F_N - \hat{F}_N C \end{bmatrix} .$$

Let $S_N = T_N - B_N F_N$ and $\hat{S}_N = T_N - \hat{F}_N C_N$ and assume that $P_N \to I$ strongly on V as $N \to \infty$. Assume further that $T_N P_N \to T$, $T^*_N P_N \to T^*$, $Q_N P_N \to Q$ and $\hat{Q}_N P_N \to \hat{Q}$ strongly on V and that $B_N \to B$ and $C_N P_N \to C$ in norm as $N \to \infty$. If the pairs (T_N, B_N) and (T^*_N, C^*_N) are uniformly exponentially stabilizable and the pairs (T_N, Q_N) and $(T^*_N, \hat{Q}_N)$ are detectable (see Kwakernaak and Sivan, 1972) then there exist unique, self-adjoint, nonnegative solutions Π_N and $\hat{\Pi}_N$ to the algebraic Riccati equations (3.1) and (3.5). If Π_N and $\hat{\Pi}_N$ are bounded from above uniformly in N, then $\Pi_N P_N$ and $\hat{\Pi}_N P_N$ converge weakly to Π and $\hat{\Pi}$, respectively, as $N \to \infty$.

If, in addition, S_N and $\hat{S}_N$ are uniformly exponentially stable, uniformly with respect to N, then $\Pi_N P_N$ and $\hat{\Pi}_N P_N$ converge strongly. Weak convergence of $\Pi_N P_N$ to Π yields strong convergence of $F_N P_N$ to F and $S_N P_N$ to S. If $\Pi_N P_N$ converges strongly then $F_N P_N \to F$ in norm. Weak convergence of $\hat{\Pi}_N P_N$ to $\hat{\Pi}$ yields weak convergence of $\hat{F}_N$ to $\hat{F}$ and $\hat{S}_N P_N$ to $\hat{S}$. When $\hat{\Pi}_N P_N \to \hat{\Pi}$ strongly, then $\hat{F}_N \to \hat{F}$ in norm and $\hat{S}_N P_N \to \hat{S}$ strongly in V as $N \to \infty$. Finally, if $\mathcal{P}_N$ is the mapping of $V \times V$ onto $V \times V_N$ given by $\mathcal{P}_N(w_1, w_2) = (w_1, P_N w_2)$, then $\Pi_N P_N \to \Pi$ weakly or strongly is sufficient to conclude that $\mathcal{S}_N \mathcal{P}_N \to \mathcal{S}$ weakly or strongly depending only upon whether $\hat{\Pi}_N P_N \to \hat{\Pi}$ weakly or strongly as $N \to \infty$. Under appropriate additional hypotheses on the spectral properties of the open-loop system and on the approximation scheme, it is possible to show that $\mathcal{S}_N \mathcal{P}_N$ converges to $\mathcal{S}$ in norm. (We have been able to obtain such a result only for modal approximations.) Norm convergence of the closed-loop state transition operators is sufficient to conclude that uniform exponential stability of $\mathcal{S}$ implies uniform exponential stability of $\mathcal{S}_N$ for all N sufficiently large (see Gibson and Rosen 1986).

In practice, the finite dimensional approximating subspaces V_N are often constructed using any of a number of common finite element bases, e.g. polynomial and hermite spline functions, mode shapes, orthogonal polynomials, etc. For the discrete-time boundary control systems of interest to us here, the approximations to T and B, T_N and B_N, are obtained by approximating the continuous time semigroup, $\{\mathcal{T}(t) : t \geq 0\}$, by a semigroup of bounded linear operators on V_N, $\{\mathcal{T}_N(t): t \geq 0\}$. In fact it is the infinitesimal generator $\mathcal{Q}$ of the semigroup $\{\mathcal{T}(t): t \geq 0\}$ that is approximated by a bounded linear operator $\mathcal{Q}_N$ on V_N with $\{\mathcal{T}_N(t): t \geq 0\}$ then being defined by $\mathcal{T}_N(t) = \exp(\mathcal{Q}_N t), t \geq 0$. With $T_N = \mathcal{T}_N(\tau)$ and $B_N = (I - \mathcal{T}_N(\tau))P_N\Gamma^+ + \int_0^\tau \mathcal{T}_N(s)P_N\Delta\Gamma^+ ds$, the required convergence can usually be proved using the Trotter-Kato semigroup approximation result (see (Kato, 1966) and (Pazy, 1983)). The approximations to Q, $\hat{Q}$ and C, Q_N, $\hat{Q}_N$ and C_N, respectively, typically are taken to be $Q_N = P_N Q$, $\hat{Q}_N = P_N \hat{Q}$ and $C_N = CP_N$.

Let $\{\varphi_j^N\}_{j=1}^{n_N}$ denote a basis for V_N and set $\Phi^N = (\varphi_1^N, \varphi_2^N, \ldots, \varphi_{n_N}^N)^T \varepsilon \times_{j=1}^{n_N} V_N$.

Adopting the convention that [L] denotes the matrix representation with respect to the basis $\{\varphi_j^N\}_{j=1}^{n_N}$ for a linear operator L with domain and/or range in V_N, we find that $[F_N] = (R + [B_N]^T \Theta^N [B_N])^{-1}[B_N]^T \Theta^N[T_N]$ and $[\hat{F}_N] = [T_N] \hat{\Theta}^N[C_N]^T(\hat{R} + [C_N] \hat{\Theta}^N[C_N]^T)^{-1}$ where Θ^N and $\hat{\Theta}^N$ are the unique, symmetric, nonnegative solutions to the $n_N \times n_N$ matrix algebraic Riccati equations

$$(3.10) \qquad \Theta^N = [T_N]^T(\Theta^N - \Theta^N[B_N](R + [B_N]^T \Theta^N[B_N])^{-1}[B_N]^T \Theta^N)[T_N] + M^N[Q_N]$$

and

$$(3.11) \qquad \hat{\Theta}_N = [T_N](\hat{\Theta}^N - \hat{\Theta}^N[C_N]^T(\hat{R} + [C_N]\Theta^N[C_N]^T)^{-1}[C_N]\Theta^N)[T_N]^T + [\hat{Q}_N](M^N)^{-1} .$$

The matrix M^N is the $n_N \times n_N$ Gram matrix $<\Phi^N, (\Phi^N)^T>_V$.

If $\hat{w}^*_{N,k} = (\Phi^N)^T \hat{W}^*_{N,k}$ with $\hat{W}^*_{N,k} \in R^{n_N}$, then $\hat{u}^*_{N,k} = -[F_N]\hat{W}^*_{N,k}$, $k = 0,1,2,\ldots$ with

$$\hat{W}^*_{N,k+1} = [T_N]\hat{W}^*_{N,k} + [B_N]\hat{u}^*_{N,k} + [\hat{F}_N]\{\hat{y}^*_{N,k} - [C_N]\hat{W}^*_{N,k}\}, \ k = 0,1,2,\ldots$$

$$\hat{W}^*_{N,0} = (M^N)^{-1} < \Phi^N, \hat{w}_0>_V .$$

The approximating optimal functional feedback control gain, $f^N = (f^N_1, f^N_2, \ldots, f^N_m)^T \in \times_{j=1}^{m} V_N$ are given by $f^N = [F_N](M^N)^{-1}\Phi^N$ and the approximating optimal functional observer gain $\hat{f}^N = (\hat{f}^N_1, \hat{f}^N_2, \ldots, \hat{f}^N_p)^T \in \times_{j=1}^{p} V_N$ by $\hat{f} = [\hat{F}_N]^T\Phi^N$. If $\Pi_N P_N \to \Pi$ weakly (strongly) then $f^N_i \to f_i$, $i = 1,2,\ldots, m$ weakly (strongly) in V. If $\hat{\Pi}_N P_N \to \hat{\Pi}$ weakly (strongly) then $f^N_i \to f_i$, $i = 1,2,\ldots, p$ weakly (strongly) in V. If the injection $V \subset H$ is compact, then $f^N_i \to f_i$, $i = 1,2,\ldots, m$ and $\hat{f}^N_i \to \hat{f}_i$, $i = 1,2,\ldots,p$ strongly in H if $\Pi_N P_N$ and $\hat{\Pi}_N P_N$ converge only weakly.

4. Examples and Numerical Results

We consider the one-dimensional heat equation

$$(4.1) \qquad \frac{\partial w}{\partial t}(t,x) = a\frac{\partial^2 w}{\partial x^2}(t,x), \qquad 0 < x < 1, \qquad t > 0,$$

where $a > 0$, with the homogeneous Dirichlet boundary condition ,

$$(4.2) \qquad w(t,0) = 0, \quad t > 0,$$

and either the Neumann boundary control

$$(4.3) \qquad \frac{\partial w}{\partial x}(t,1) = v(t), \qquad t > 0,$$

or the Dirichlet boundary control

$$(4.4) \qquad w(t, 1) = v(t), \quad t > 0,$$

where $v \in L_2(0, \infty)$. For output we take a temperature measurement

$$(4.5) \qquad y(t) = w(t, \zeta), \qquad t \geq 0,$$

at some fixed point $\zeta \in (0, 1)$. Initial conditions for these systems have the form

$$(4.6) \qquad w(0, x) = w_0(x), \qquad 0 \leq x \leq 1$$

where $w_0 \in L_2(0, 1)$..

Although the two control systems above appear to be similar, they are, in fact, quite different and must be treated separately. We begin with the more straight forward of the two, Neumann boundary control. Let $H = L_2(0, 1)$, $V = H^1_L(0, 1) = \{\varphi \in H^1(0, 1) : \varphi(0) = 0\}$ and $W = H^2(0, 1) \cap H^1_L(0, 1)$. With H endowed with the usual L_2 inner product, V with the inner product $< \varphi, \psi >_V = \int_0^1 D\varphi\, D\psi$ and W with the inner product $< \varphi, \psi >_W = \sum_{j=1}^{2} \int_0^1 D^j D^j \psi$, we have the continuous and dense embeddings $W \subset V \subset H \subset V' \subset W'$. Define $\Delta \in \mathcal{L}(W, H)$, $\Gamma \in \mathcal{L}(W, R^1)$ by $\Delta\varphi = a D^2\varphi$, $\Gamma\varphi = D\varphi(1)$ and $C\varphi = \varphi(\zeta)$ respectively. With these definitions the boundary control system (4.1) - (4.3), (4.5), (4.6) has the form (2.1) - (2.4). The operator $\mathcal{A}$: Dom $(\mathcal{A}) \subset H \rightarrow H$ is given by $\mathcal{A}\varphi = a D^2\varphi$ for $\varphi \in \{\varphi \in H^2(0, 1)$: $\varphi(0) = D\varphi(1)) = 0\}$. It is densely defined, negative definite, self-adjoint and it is the infintesimal generator of a uniformly exponentially stable analytic semigroup $\{\mathcal{T}(t)$: $t \geq 0\}$ of bounded, self-adjoint linear operators on H. Also, $\{\mathcal{T}(t)$: $t \geq 0\}$ is a uniformly exponentially stable, analytic semigroup of bounded, self-adjoint operators on V with generator $\tilde{\mathcal{A}}$ given by $\tilde{\mathcal{A}}\varphi = \mathcal{A}\varphi$ for $\varphi \in \{\varphi \in H^3(0, 1)$: $\varphi(0) = D\varphi(1) = D^2\varphi(0) = 0\}$. Choosing $\Gamma^+ \in \mathcal{L}(R^1, W)$ as $(\Gamma^+u)(x) = xu$ for $x \in [0, 1]$, we have $\mathcal{R}(\Gamma^+) \subset V$, $\mathcal{R}(\Gamma^+) \subset \mathcal{N}(\Delta)$ and that conditions 1) -5) given in Section 2 are satisfied. For the optimal control and estimator problems, we take $Q = qI$, $\hat{Q} = \hat{q}\, I$, $R = r$ and $\hat{R} = \hat{r}$ where I is the identity on V, q, $\hat{q} \geq 0$ and r, $\hat{r} > 0$. The uniform exponential stability of the semigroup $\{\mathcal{T}(t)$: $t \geq 0\}$ on V implies that the algebrai Riccati equations (3.1) and (3.5) admit unique bounded, nonnegative, self-adjoint solutions Π and $\hat{\Pi}$ respectively. The optimal control (3.2) takes the form

$$(4.7) \qquad u_k^* = - \int_0^1 Df\, Dw_k^*, \qquad k = 0,1,2,...$$

where the optimal functional feedback control gain f and the optimal functional observer gain $\hat{f}$ are elements in $H_L^1(0,1)$.

We construct an approximation scheme using a linear spline based Ritz-Galerkin approach. For each N = 1,2,..., $\{\varphi_j^N\}_{j=0}^N$ denotes the usual linear spline or "hat" functions defined on the interval [0,1] with respect to the uniform mesh {0, 1/N, 2/N,...,1}. We discard the element centered at x = 0, φ_0^N, set $V_N = \text{span}\,\{\varphi_j^N\}_{j=1}^N$ and choose P_N to be the orthogonal projection of V onto V_N with respect to the V inner product. Hence V_N is an N dimensional subspace of V.

For $\varphi \in$ Dom $(\mathcal{A})$, $|\mathcal{A}\varphi|_H \geq a|\varphi|_V \geq a|\varphi|_H$ and therefore $0 \in \rho(\mathcal{A})$ and $\mathcal{A}^{-1}$: $H \rightarrow \text{Dom}(\mathcal{A})$ satisfies $|\mathcal{A}^{-1}\varphi|_V \leq a^{-1}|\varphi|_H$ for $\varphi \in H$. We define $\mathcal{A}_N : V_N \rightarrow V_N$ as the inverse of the operator $\mathcal{A}_N^{-1} = P_N\mathcal{A}^{-1}$ restricted to V_N. The operator $-\mathcal{A}_N^{-1}$ is positive definite because $< \mathcal{A}_N^{-1}\varphi_N, \varphi_N>_V = -a^{-1}|\varphi_N|_H^2$ for $\varphi_N \in V_N$, and it is self-adjoint since $< \mathcal{A}_N^{-1}\varphi_N, \psi_N>_V = <P_N\mathcal{A}^{-1}\varphi_N, \psi_N>_V = <\mathcal{A}^{-1}\varphi_N, \psi_N>_V = a^{-1} < \varphi_N, \psi_N>_H$. Hence the operator $\mathcal{A}_N$ is well defined and self-adjoint. For $\varphi_N \in V_N$ and $\psi_N = \mathcal{A}_N\varphi_N$, the estimate

$$< \mathcal{A}_N\varphi_N,\varphi_N>_V \; = < \psi_N,\mathcal{A}_N^{-1}\psi_N>_V = -a^{-1}|\psi_N|_H^2$$
$$\leq -a|\mathcal{A}^{-1}\psi_N|_V^2 \leq -a\,|P_N\mathcal{A}^{-1}\psi_N|_V^2 = -a|\,\mathcal{A}_N^{-1}\psi_N|_V^2$$
$$= -a\,|\varphi_N|_V^2$$

implies that $\mathcal{A}_N$ is the infinitesimal generator of a $\mathcal{C}_0$ semigroup $\{\mathcal{T}_N(t) : t \geq 0\}$ of bounded, self-adjoint linear operators on V_N satisfying $|\mathcal{T}_N(t)| \leq e^{-at}$, $t \geq 0$.

It can be shown that $a<\varphi\, ,\psi>_v = <(-\mathcal{A})^{1/2}\varphi, (-\mathcal{A})^{1/2}\psi>_H$. It then follows that the matrix representation for the operator $\mathcal{A}_N$ with respect to the basis $\{\varphi_j^N\}_{j=1}^{n\,N}$ is $[\mathcal{A}_N] =$ $-a<\varphi_i^N, \varphi_j^N>_H^{-1} \; <\varphi_i^N, \varphi_j^N>_v$. This agrees with the system matrix derived by a standard Ritz-Galerkin finite element approach. Note that even though $\mathcal{A}_N$ is defined to be the inverse of the operatcr $P_N(\mathcal{A})^{-1}$ restricted to the space V_N, computing its matrix representation does not require either $\mathcal{A}^{1/2}$ or $\mathcal{A}^{-1}$ explicitly. In general, the same approach can be used to obtain an operator representation for the Ritz Galerkin approximation to any self-adjoint coercive operator.

Let $\mathcal{I}_N$ denote the interpolation operator from V onto V_N defined by $(\mathcal{I}_N\,\varphi)(j/N) = \varphi(j/N)$, j = 1, 2,..., N . Then for $\varphi \in W$, elementary approximation properties of linear interpolatory spline functions (see (Schultz, 1971)) imply

$$|(P_N - I)\varphi|_V \leq |(\mathcal{I}_N - I)\varphi|_V \leq \frac{1}{N\pi}\,|D^2\varphi|_H$$

and therefore, since W is dense in V, that $P_N \to I$ strongly on V as $N \to \infty$. Also, it follows that $\mathcal{A}_N^{-1} = \; P_N\mathcal{A}^{-1} \to \; \mathcal{A}^{-1}$ strongly on V as $N \to \infty$. If we define $T_N = \mathcal{T}_N(\tau)$, then the Trotter-Kato approximation theorem yields that $T_NP_N \to T$ strongly on V as $N \to \infty$ and, since $T^* = T = \mathcal{T}(\tau)$ and $T_N^* = T_N = \mathcal{T}_N(\tau)$ that $T_N^*P_N \to T^*$ strongly on V as $N \to \infty$.

Since $\mathcal{R}(\Gamma^+) \subset V_N$ (recall that $(\Gamma^+u)(x) = ux$, $0 \leq x \leq 1$), we define the approximating input operators B_N by $B_N = (I - \mathcal{T}_N(\tau))\Gamma^+$ and set $Q_N = qI$, $\hat{Q}_N = \hat{q}\,I$ and $C_N = C$. The strong convergence of P_N to the identity and T_NP_N to T together with the finite dimensionality of the domain of B and the range of C are sufficient to conclude that $Q_NP_N \to Q$, $\hat{Q}_NP_N \to \hat{Q}$ strongly on V and that $B_N \to B$ and $C_NP_N \to C$ in norm as $N \to \infty$.

The uniform exponential stability of the semigroups $\{\mathcal{T}_N(t) : t \geq 0\}$ implies

(4.8) $\qquad |T_N^k|_V = |(T_N^*)^k|_V \le r^k, \quad k = 0,1,2,...$

with $r = e^{-a\tau} < 1$. Consequently the pairs (T_N, B_N) and (T_N, C_N^*) are uniformly exponentially stabilizable and the pairs (T_N, Q_N) and $(T_N, \hat{Q}_N)$ are detectable. It follows that there exist unique self-adjoint, nonnegative solutions Π_N and $\hat{\Pi}_N$ to the finite dimensional algebraic Riccati equations (3.7) and (3.8) respectively. The uniform exponential bound (4.8) with $r < 1$ imples that the zero control yields a uniform upper bound for Π_N and $\hat{\Pi}_N$ and therefore the uniform exponential stability of $S_N = T_N - B_N F_N$ and $\hat{S}_N = T_N - \hat{F}_N C_N$. We conclude that $\Pi_N P_N$ and $\hat{\Pi}_N P_N$ converge strongly in V to Π_N and $\hat{\Pi}_N$, respectively ,and that $F_N P_N$ and $\hat{F}_N$ converge to F and $\hat{F}$ in norm as $N \to \infty$. The approximating optimal functional feedback control and observer gains, f_N and $\hat{f}_N$, converge respectively to f and $\hat{f}$ in the H^1 norm as $N \to \infty$.

In implementing the approximation scheme just outlined above, eigenvector decomposition of the associated Hamiltonian matrix was used to solve the matrix algebraic Riccati equations (3.10) and (3.11) (see Pappas, et. al., 1980). The required matrix exponentials also were computed using eigenvalue/eigenvector decomposition. All calculations were carried out via Fortran codes on an IBM PC AT. We set $a = \sqrt{.1}$, $q = \hat{q} = r = \hat{r} = 1.0$, $\xi = \sqrt{2}/2$ and $\tau = .01$ and obtained the functional gains plotted in Figs. 4.1 and 4.2 . We plot f_N and $\hat{f}_N$ as well as Df_N and $D\hat{f}_N$ to exhibit the H^1 convergence. We note that Df (or Df_N) appears as the feedback kernel in the optimal control law (4.7).

We also simulated the operation of the closed-loop system with an approximating compensator. Using a 20 mode model for the infinite dimensional system and N = 12, we computed the closed-loop spectrum of the approximating compensator (i.e. the eigenvalues of the operator $\mathcal{A}_N$ given by (3.9) with N = 12). These eigenvalues along with the first 20 open-loop eigenvalues (i.e. the first 20 eigenvalues of the operator $T = \mathcal{T}(\tau)$) and the approximating closed-loop control and observer eigenvalues are tabulated in Table 4.1 below. Table 4.1 reveals that the last seven open-loop eigenvalues remain essentially unchanged in the closed-loop system-i.e. these modes are neither controlled nor observed by the finite dimensional compensator. Also, as one would expect, $\sigma(\mathcal{A}_N)$ consists essentially of the union of $\sigma(S_N)$, $\sigma(\hat{S}_N)$ and the eigenvalues corresponding to the uncontrolled/unobserved modes of the open-loop system.

It is worth noting that the scheme we have outlined above for the Neumann boundary control problem is the same scheme that one would ordinarily use if the problem were formulated in the space H - i.e. if the output operator C was bounded on $L_2(0,1)$ (see Gibson and Rosen, 1986). This is possible primarily because the space $V = H^1_L(0,1)$ is the natural energy space for the underlying homogeneous or open-loop system. Consequently, the inherent self-adjointness and coercivity in the problem is preserved when it is formulated in the stronger space. In the case of Dirichlet boundary control, the situation is quite different.

For the Dirichlet boundary control system (4.1), (4.2), (4.4) - (4.6), we choose the spaces H, V and W and their corresponding inner products to be the same as they were in the Neumann case. The operators $\Delta \in \mathcal{L}(W,H)$ and $C \in \mathcal{L}(V,R^1)$ also remain unchanged, however now we have $\Gamma \in \mathcal{L}(W,R^1)$ given by $\Gamma\varphi = \varphi(1)$. It then follows that the operator $\mathcal{A} : \mathrm{Dom}(\mathcal{A}) \subset H \to H$ is given by $\mathcal{A}\varphi = aD^2\varphi$ for $\varphi \in H^2(0,1) \cap H^1_0(0,1)$. It is well known that $\mathcal{A}$ is densely defined, negative definite and self-adjoint and that it is the infinitesimal generator of the uniformly exponentially stable analytic semigroup $\{\mathcal{T}(t) : t \geq 0\}$ of bounded, self-adjoint linear operators on H. However this time the operators $\mathcal{T}(t)$ for $t > 0$ are neither self-adjoint nor a semigroup on V. Indeed, since $\mathcal{R}(\mathcal{T}(t)) \subset H^1_0(0,1)$ for all $t > 0$ and $H^1_0(0,1)$ is a closed proper subspace of $H^1_L(0,1)$, $\mathcal{T}(t)$ is not strongly continuous in the V-norm at $t = 0$. (The fact that our general framework requires $\Gamma\Gamma^+ = 1$ and $\mathcal{R}(\Gamma^+) \subset V$ precludes our choosing V to be $H^1_0(0,1)$.) On the other hand, $\{\mathcal{T}(t) : t \geq 0\}$ an analytic semigroup implies (see Pazy, 1983) that there exists a constant $\mu > 0$ for which $|\mathcal{A}\mathcal{T}(t)|_H \leq \mu t^{-1}$ for $t > 0$. Consequently, if we define $T = \mathcal{T}(\tau)$, then it follows that $T \in \mathcal{L}(V)$ and moreover, that

$$|T^k\varphi|^2_V = -a^{-1}\langle \mathcal{A}\mathcal{T}(k\tau)\varphi, \mathcal{T}(k\tau)\varphi\rangle \leq a^{-1}|\mathcal{A}\mathcal{T}(k\tau)\varphi|_H |\mathcal{T}(k\tau)\varphi|_H$$

$$\leq \frac{\mu e^{-ak\tau}}{ak\tau}|\varphi|^2_H \leq \frac{\mu e^{-ak\tau}}{ak\tau}|\varphi|^2_V$$

for $k = 1,2,\ldots$ and $\varphi \in V$. We have therefore

$$(4.9) \qquad |T^k|_V = |(T^*)^k|_V \le Mr^k, \; k = 0,1,2,\ldots$$

where $M > 0$ and $r < 1$.

We again choose $\Gamma^+ \in \mathcal{L}(R^1, W)$ as $(\Gamma^+ u)(x) = xu$ for $x \in [0,1]$. Then $\mathcal{R}(\Gamma^+) \subset \mathcal{N}(\Delta)$ and we have reformulated the boundary control system (4.1), (4.2), (4.4) - (4.6) in the general form of (2.1) - (2.4) and conditions 1) - 5) are satisfied.

We formulate the optimal control problem with the performance index

$$J(u) = \sum_{k=0}^{\infty} q < w_k, w_k >_H + ru_k^2$$

where $q \ge 0$ and $r > 0$. That is, we take Q to be the bounded, self-adjoint nonnegative operator on $H_L^1(0,1)$ given by $(Q\varphi)(x) = q\int_0^x \int_y^1 \varphi(z)dzdy$ and R to be r. For the estimator problem we set $\hat{Q} = \hat{q}\, I$ and $\hat{R} = \hat{r}$ with $\hat{q} \ge 0$ and $\hat{r} > 0$.

The uniform exponential bound (4.9) implies the existence of unique, nonnegative, self-adjoint solutions Π and $\hat{\Pi}$ to the algebraic Riccati equations (3.1) and (3.5). The optimal control is again of the form (4.7) with the optimal functional gains f and $\hat{f}$ in H_L^1.

The fact that $\{\mathcal{T}(t) : t \ge 0\}$ is not a semigroup on V precludes the use of a semigroup - theoretic approach to approximation. We therefore employ modal subspaces and approximate the open-loop state transition operator T directly as a bounded linear operator on V.

For each $N = 1,2,\ldots$ let $V_N = \text{span}\,\{\varphi_j\}_{j=0}^N$ where for $x \in [0,1]$, $\varphi_0(x) = x$ and $\varphi_j(x) = \sin j\pi x$, $j = 1,2,\ldots,N$. Let p_N denote the orthogonal projection of $H = L_2(0,1)$ onto span $\{\varphi_j\}_{j=1}^N$ and let P_N denote the orthogonal projection of V onto V_N. Using the fact that $V = H_0^1(0,1) \oplus \varphi_0$, it is not difficult to see that $P_N\varphi = \varphi(1)\varphi_0 + p_N(\varphi - \varphi(1)\varphi_0)$ for $\varphi \in V$ and hence, via elementary properties of Fourier series (see Tolstov, 1962), that $|(P_N - I)\varphi|_V = |(p_N - I)(\varphi - \varphi(1)\varphi_0)|_V \to 0$ as $N \to \infty$ for each $\varphi \in V$.

We define $T_N \in \mathcal{L}(V_N)$ by $T_N = P_N T$. Then, since $\mathcal{R}(T) = \mathcal{R}(\mathcal{T}(\tau)) \subset H_0^1(0,1)$, for $\psi_N = \sum_{j=0}^{N} \psi_N^j \varphi_j \in V_N$ we have

$$T_N \psi_N = P_N T \psi_N = P_N \mathcal{T}(\tau)\psi_N = p_N \mathcal{T}(\tau)\psi_N = \mathcal{T}(\tau) p_N \psi_N = \sum_{j=1}^{N} \{ \frac{2(-1)^j}{j\pi} \psi_N^0 + \psi_N^j \} e^{-aj^2\pi^2\tau} \varphi_j .$$

It follows that $T_N^* = P_N T^*$, $|T_N^k|_V = |(T_N^*)^k|_V \leq Mr^k$, $k = 0,1,2,...$ with $M > 0$ and $r < 1$ independent of N, and that

$$\begin{aligned} |(T_N P_N - T)\varphi|_V &\leq |(P_N T P_N - P_N T)\varphi|_V + |(P_N - I)T\varphi|_V \\ &\leq Mr|(P_N - I)\varphi|_V + |(P_N - I)T\varphi|_V \to 0 \end{aligned}$$

as $N \to \infty$ for $\varphi \in V$. Similarly, $T_N^* P_N \to T^*$ strongly on V as $N \to \infty$.

The approximating input, output, and state penalization operators B_N, C_N, Q_N and $\hat{Q}_N$ take the form

$$B_N u = (I - T_N)\Gamma^+ u = \varphi_0 u + \sum_{j=1}^{N} \frac{2(-1)^j}{j\pi} e^{-aj^2\pi^2\tau} \varphi_j u ,$$

$C_N = C, Q_N = qP_N Q$ and $\hat{Q}_N = \hat{q}\, I$. Reasoning as we did in the Neumann case, the approximating algebraic Riccati equations (3.7) and (3.8) admit unique, nonnegative, self-adjoint solutions Π_N and $\hat{\Pi}_N$ respectively, $\Pi_N P_N \to \Pi$ and $\hat{\Pi}_N P_N \to \hat{\Pi}$ strongly on V and $F_N P_N \to F$ and $\hat{F}_N \to \hat{F}$ in norm as $N \to \infty$. The approximating functional feedback control and observer gains f_N and $\hat{f}_N$ converge to f and $\hat{f}$ respectively, strongly in H^1 as $N \to \infty$.

With $a = 1.0$, $q = \hat{q} = r = 1.0$, $\hat{r} = 5.0$, $\xi = \sqrt{2}/2$ and $\tau = .01$ and the scheme outlined above we obtained the approximating optimal functional feedback control and observer gains plotted in Figs. 4.3 and 4.4 below. The first 12 open-loop and the approximating closed-loop control and observer eigenvalues for N = 12 are tabulated in Table 4.2.

Table 4.2 reveals an interlacing of the closed-loop control and open-loop eigenvalues. That is, the closed-loop control eigenvalues (i.e. the elements in the spectrum of S) are alternately more and less stable than the corresponding open-loop eigenvalues. We also have observed this phenomenon in other numerical studies we are carrying out involving LQG boundary control for flexible structures. In additon, in the Dirichlet boundary control system discussed above, if Q is chosen as the identity operator on $V = H^1_L(0,1)$, virtually all of the closed-loop control eigenvalues are less stable than the corresponding open-loop eigenvalues. It is clear that this non-standard behavior results from the presence of the one dimensional subspace represented by $\mathcal{R}(\Gamma^+)$. Indeed, the behavior of the closed-loop spectrum in the case of Neumann boundary control is as would be expected. We feel that what we are seeing can most likely be explained via infinite dimensional analogs of existing results relating the asymptotic properties of the closed-loop spectrum of a linear regulator and the zeros of the corresponding open-loop transfer function (see Kwakernaak and Sivan, 1972 and Harvey and Stein, 1978). However, as of yet, we have been unable to establish this conjecture satisfactorily and we consider it to be beyond the scope of this paper, which is primarily concerned with approximation. We leave it as an interesting open question.

5. Concluding Remarks

We have developed a framework for the finite dimensional approximation of optimal discrete-time LQG compensators for distributed parameter systems with boundary input and unbounded measurement. Our theory applies to the class of boundary control problems which can be formulated in a state space in which both the discrete-time input and output operators are continuous. We have used a functional analytic treatment to develop a convergence theory and have demonstrated the feasibility of our approach via examples involving either the Neumann or Dirichlet boundary control of a one dimensional heat equation with point measurement of temperature. We have shown that while both problems outwardly appear to be quite similar, they in fact require very different approaches to approximation. Also in the Dirichlet case the observed behavior of the resulting closed-loop spectrum is, in some ways unexpected and its explanation remains open.

Finally, we have been looking at the application of our schemes to LQG problems for flexible structures with boundary inputs and unbounded measurement and systems with control and/or observations delays. We have been considering vibration suppression for cantilevered beams via shear or moment inputs at the free end and pointwise observation of strain or acceleration. These studies are currently underway with the results to be reported elsewhere.

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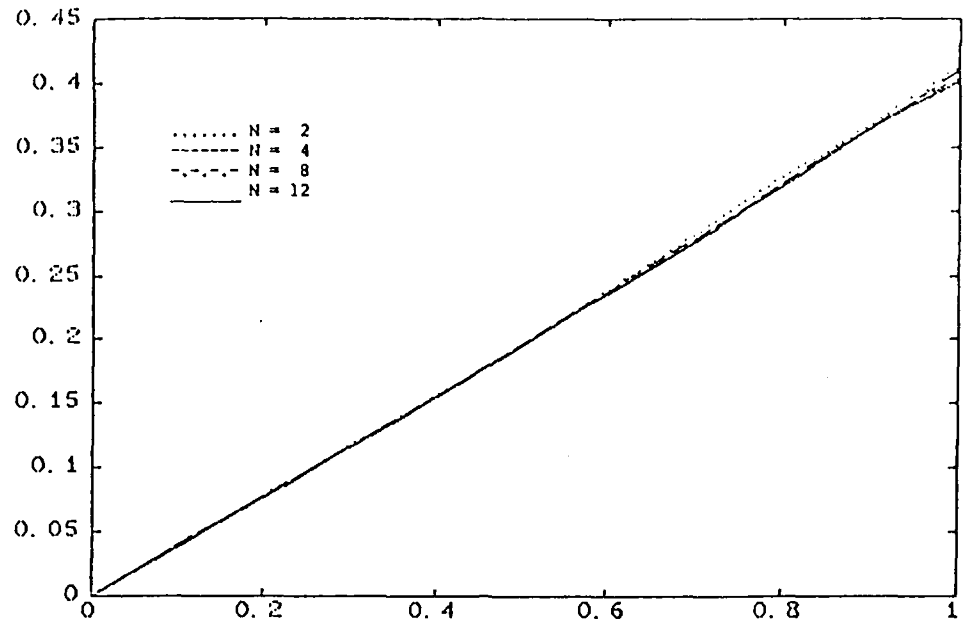
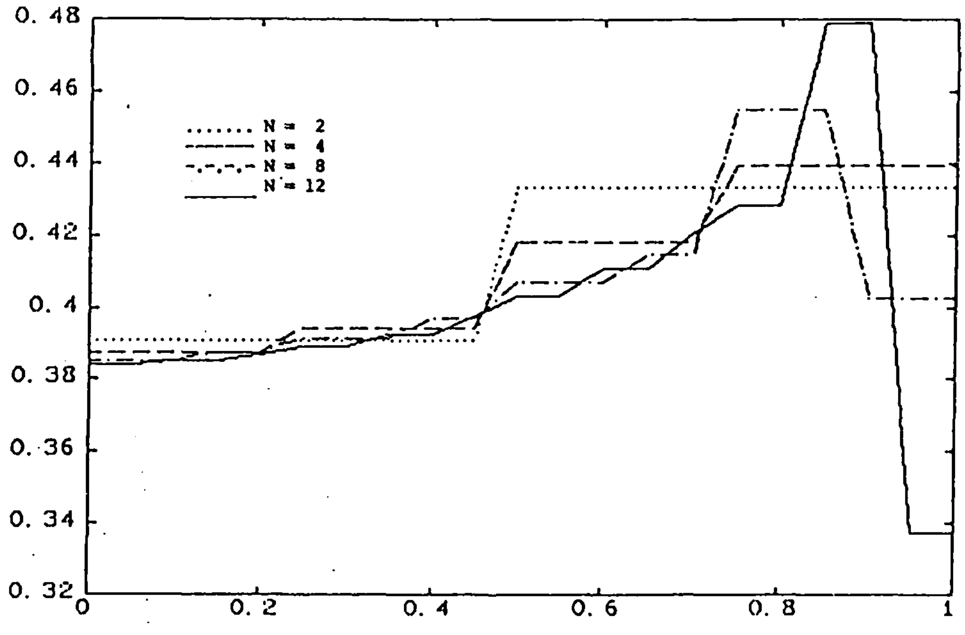
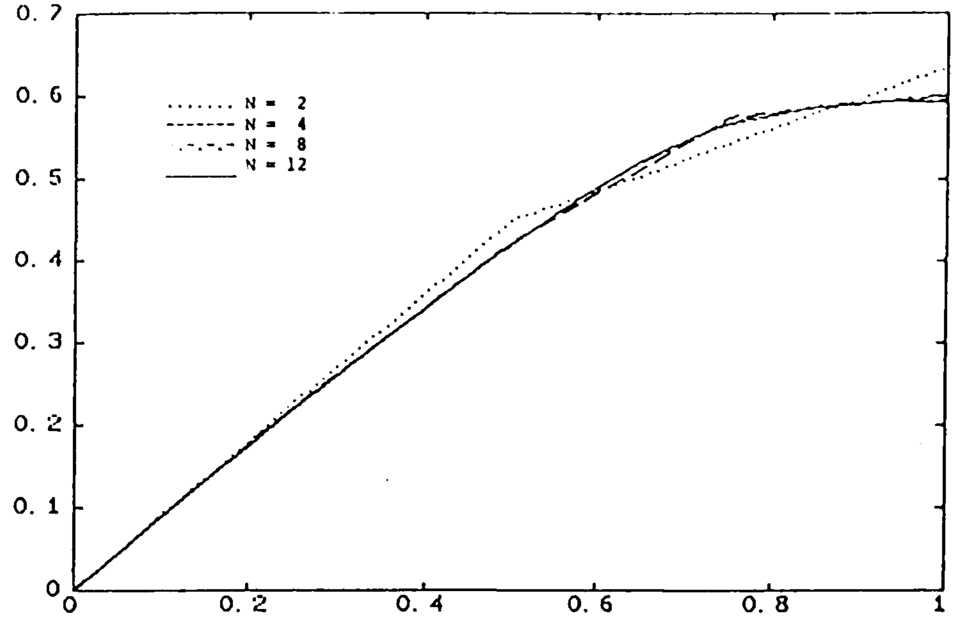


Neumann boundary control; approximating optimal functional feedback control gain, f_N.

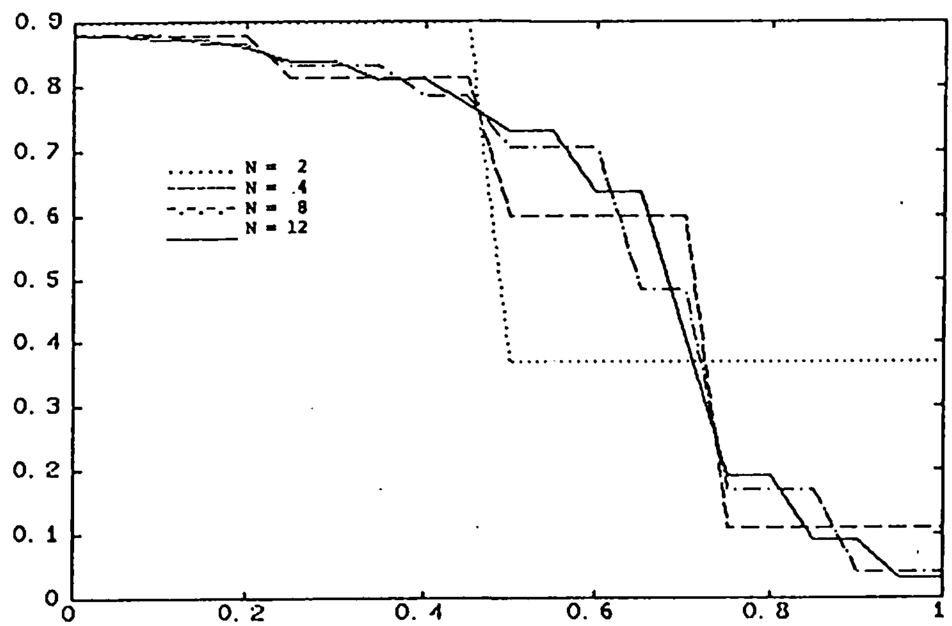


Neumann boundary control; first derivative of approximating optimal functional feedback control gain, Df_N.

Figure 4.1



Neumann boundary control; approximating optimal functional observer gain, $\hat{f}_N$.



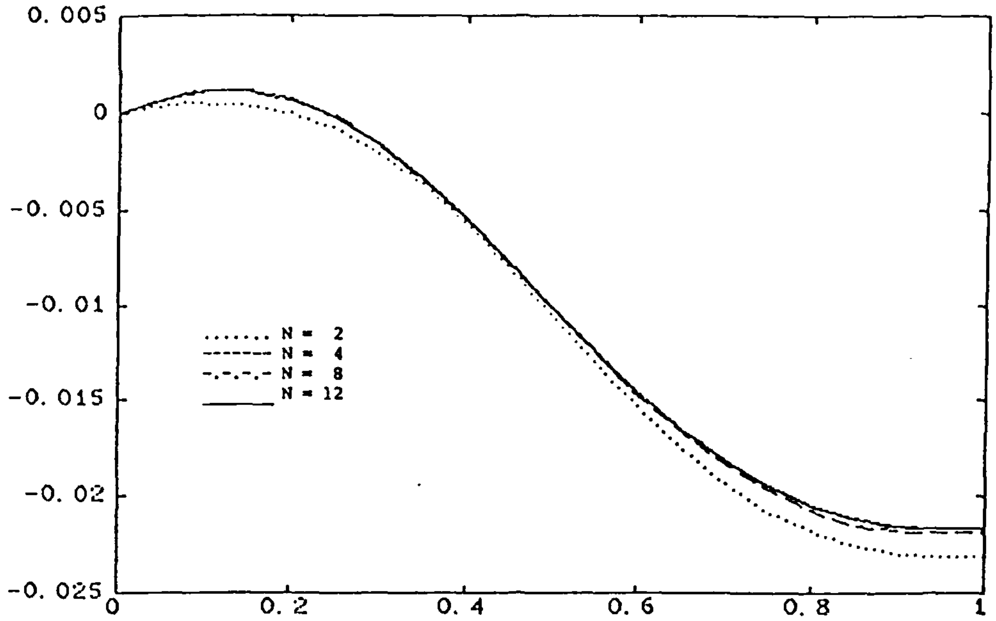
Neumann boundary control; first derivative of approximating optimal functional observer gain, $D\hat{f}_N$.

Figure 4.2

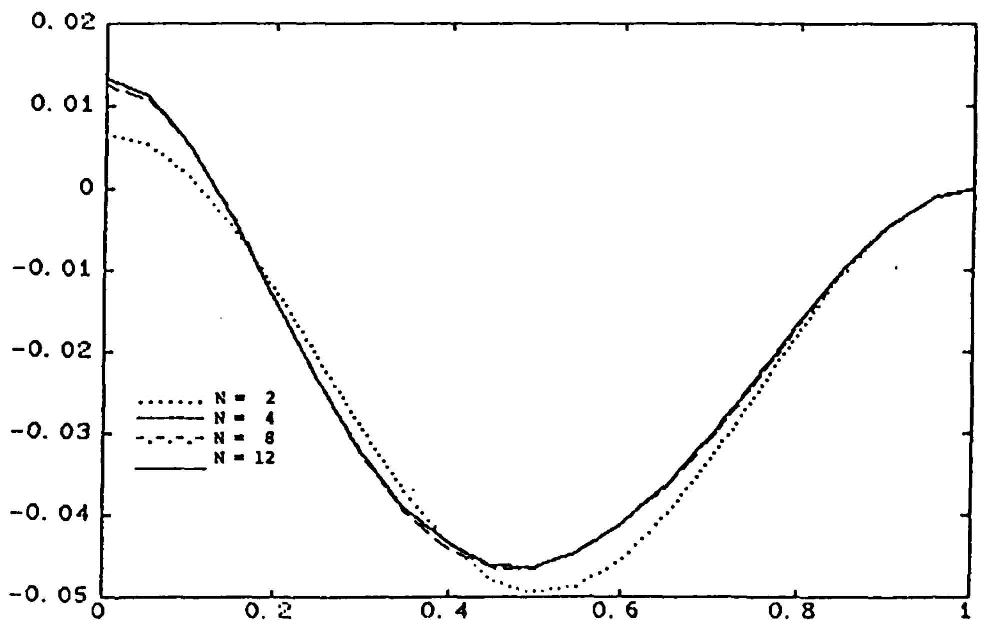
	Open-Loop	$\sigma(\mathcal{S}_{12})$	$\sigma(S_{12})$	$\sigma(\hat{S}_{12})$
1	.9975	.9968	.9968	
2	.9780	.9780		.9778
		.9769	.9768	
3	.9402	.9408		.9387
		.9371	.9371	
4	.8861	.8872		
		.8778	.8775	.8798
5	.8188	.8194		
		.7982	.7985	.7998
6	.7419	.7414		
		.7026	.7019	.7030
7	.6590	.6573		
		.5960	.5921	.5946
8	.5740	.5718		
		.4891	.4769	.4804
9	.4901	.4875		
		.4433		.4412
10	.4104	.4041		
		.3675	.3675	.3682
11	.3368	.3341		
		.2772	.2763	.2768
12	.2711	.2705		
		.2145	.2129	.2133
13	.2139	.2134		
		.1811	.1811	.1816
14	.1655	.1663		
15	.1255	.1260		
16	.0934	.0933		
17	.0681	.0677		
18	.0482	.0483		
19	.0341	.0340		
20	.0235	.0236		

Neumann boundary control; simulation results

Table 4.1

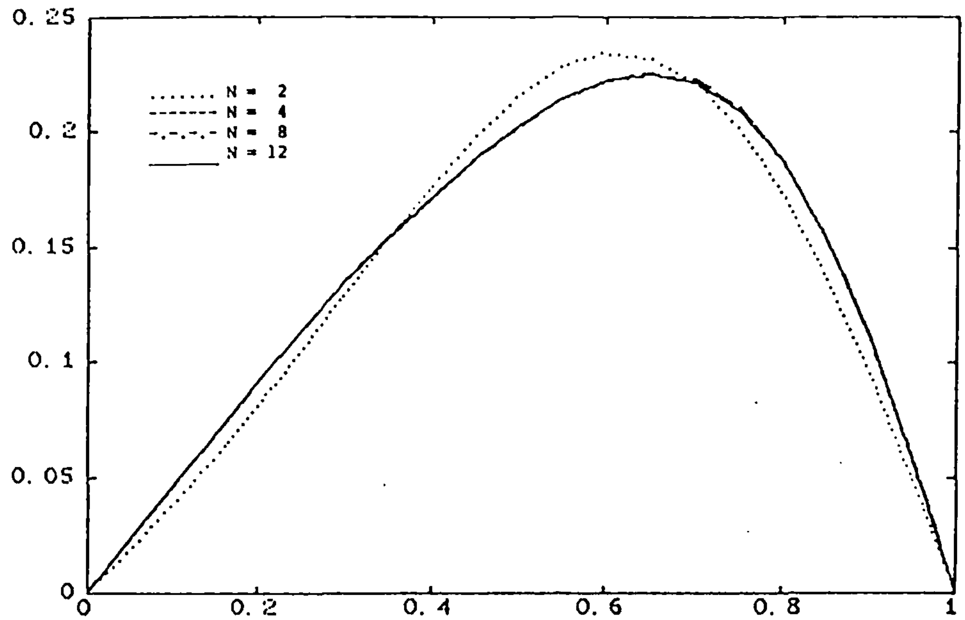


Dirichlet boundary control; approximating optimal functional feedback control gain, f_N.

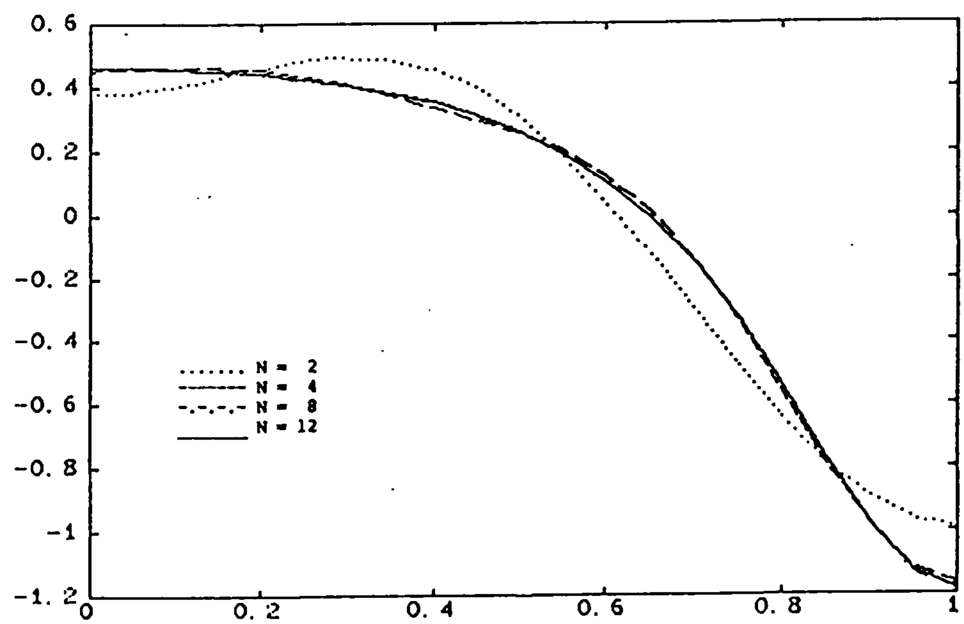


Dirichlet boundary control; first derivative of approximating optimal functional feedback control gain, Df_N.

Figure 4.3



Dirichlet boundary control; approximating optimal functional observer gain, $\hat{f}_N$.



Dirichlet boundary control; first derivative of approximating optimal functional observer gain, $D\hat{f}_N$.
Figure 4.4

	Open-Loop	$\sigma(S_{12})$	$\sigma(\hat{S}_{12})$
1	.90601806	.90569591	.78573771
2	.67382545	.68243047	.57981918
3	.41136911	.40961171	.40082268
4	.20615299	.20758391	.20936323
5	.08480497	.08447005	.08636884
6	.02863695	.02873534	.02892353
7	.00793790	.00791793	.00792193
8	.00180617	.00180978	.00178763
9	.00033753	.00033682	.00033414
10	.00005172	.00005179	.00005162
11	.00000651	.00000650	.00000654
12	.00000067	.00000067	.00000068
13	——	.00000000	.00000000

Dirichlet boundary control; open and closed-loop spectrum

Table 4.2

References:

Curtain, R. F. (1984). Finite dimensional compensators for parabolic distributed systems with unbounded control and observation. SIAM J. Control and Opt., 22, pp. 255-276.

Curtain, R. F. and D. Salamon (1986). Finite dimensional compensators for infinite dimensional systems with unbounded input operators. SIAM J. Control and Opt., 24, pp. 797-816.

Gibson, J. S. and I. G. Rosen (1985). Numerical approximation for the infinite dimensional discrete-time optimal linear quadratic regulator problem. ICSAE Report No. 86-15, Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, VA, and SIAM J. Control and Opt., to appear.

Gibson, J. S. and I. G. Rosen (1986). Computational methods for optimal linear quadratic compensators for infinite dimensional discrete-time systems. ICASE Report No. 86-72, Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, VA, and Proceedings of Conf. on Control and Identification of Distributed Parameter Systems, Vorau, Austria, July, 1986, to appear as a volume in the Springer-Verlag Lecture Notes in Control and Information Sciences.

Gibson, J.S. and I.G. Rosen (1987). Approximation in discrete-time boundary control of flexible structures. Proceedings of The 26th IEEE Conference on Decision and Control, December 9-11, 1987, Los Angeles, California.

Harvey, C.A. and G. Stein (1978). Quadratic weights for asymptotic regulator properties. IEEE Trans. Auto. Control, AC-23, pp. 378-387.

Hille, E. and R.S. Phillips (1957). Functional Analysis and Semi-Groups. American Mathematical Society, Providence, RI.

Kato, T. (1966). Perturbation Theory for Linear Operators. Springer-Verlag, New York.

Kwakernaak, H. and R. Sivan (1972). Linear Optimal Control Systems. Wiley-Interscience, New York.

Lasiecka, I. and R. Triggiani (1983a). Dirichlet boundary control problem for parabolic equations with quadratic cost: analyticity and Riccati feedback synthesis. SIAM J. Control and Opt., 21, pp. 41-67.

Lasiecka I. and R. Triggiani (1983b). An L_2 theory for the quadratic optimal cost problem of hyperbolic equations with control in the Dirichlet boundary conditions. In F. Kappel, K. Kunisch and W. Schappacher (Eds.), Control Theory for Distributed Parameter Systems and Applications, Lecture Notes in Control and Information Sciences 54, Springer-Verlag, Berlin, pp. 138-152.

Lasiecka, I. and R. Triggiani (1986). Riccati equations for hyperbolic partial differential equations with $L_2(0, T; L_2(\Gamma))$ - Dirichlet boundary terms. SIAM J. Control and Opt., 24, pp. 884-925.

Lasiecka, I. and R. Triggiani (1987a). The regulator problem for parabolic equations with Dirichlet boundary control Part I: Riccati's synthesis and regularity of optimal solution. J. Appl. Math. and Opt., to appear.

Lasiecka, I. and R. Triggiani (1987b). The regulator problem for parabolic equations with Dirichlet boundary control Part II: Galerkin approximation. J. Appl. Math. and Opt., to appear.

Pappas, T., A. J. Laub and N. R. Sandell, Jr. (1980). On the numerical solution of the discrete-time algebraic Riccati equation. IEEE Trans. Auto. Control, AC-25, pp. 631-641.

Pazy, A. (1983). Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York.

Pritchard, A. J. and D. Salamon (1987). The linear quadratic control problem for infinite dimensional systems with unbounded input and output operators. SIAM J. Control and Opt., 25, pp. 121-144.

Schultz, M. (1971). Spline Analysis. Prentice-Hall, Englewood Cliffs, NJ.

Tolstov, G. P. (1962). Fourier Series. Prentice-Hall, Englewood Cliffs, NJ

COMPUTATIONAL METHODS FOR THE IDENTIFICATION OF SPATIALLY VARYING STIFFNESS AND DAMPING IN BEAMS+

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ABSTRACT

A numerical approximation scheme for the estimation of functional parameters in Euler-Bernoulli models for the tansverse vibration of flexible beams with tip bodies is developed. The method permits the identification of spatially varying flexural stiffness and Voigt-Kelvin viscoelastic damping coefficients which appear in the hybrid system of ordinary and partial differential equations and boundary conditions describing the dynamics of such structures. An inverse problem is formulated as a least squares fit to data subject to constraints in the form of a vector system of abstract first order evolution equations. Spline-based finite element approximations are used to finite dimensionalize the problem. Theoretical convergence results are given and numerical studies carried out on both conventional (serial) and vector computers are discussed.

1. Introduction

We develop here numerical approximation methods for the estimation of functional or more precisely, spatially varying parameters that describe material properties in continuum models for elastic structures. In particular, we consider the identification of the flexural stiffness and Voigt-Kelvin viscoelastic damping coefficients in Euler-Bernoulli models for the transverse vibration of long, slender, flexible beams with tip appendages. The primary motivation for the study we report on here is the modeling and ultimately the control of the dynamics of large flexible spacecraft. The type of structures to which we are referring includes satellites with flexible appendages (solar panels and the like) antennas (reflectors as well as supporting structures) and trussed masts and platforms, both shuttle attached and free flying.

The difficulties involved in the design of efficient and practical control laws and in particular the need for extremely high fidelity models for structures of these types are well documented (see, for example, [1], [8], [21], [22]). Their high flexibility, light damping, construction with new and relatively untested composite materials (usually graphite-epoxy) and overall complexity together with their use in a fuel limited and highly variable environment all contribute to making space structure stabilization and control a formidable task. It is becoming increasingly clear that the use of continuum or distributed models with spatially and / or temporally varying functional parameters has the potential to offer several distinct and significant advantages. Included among them is the ability to, in some sense, capture the physics and inherent infinite dimensionality of the dynamics while at the same time greatly reducing the number of unknown or experimentally indeterminable material parameters which have to be identified (see [15], [18], [23], [28], [35]).

In our study we have considered exclusively Voigt-Kelvin viscoelastic damping which is based on the hypothesis that the damping moment is proportional to strain rate. There exists considerable evidence to suggest that damping mechanisms in composite materials are significantly more complex than the one described by the Voigt-Kelvin model. For example, it has been conjectured by some investigators that an appropriate model might involve hysteretic or hereditary effects. However, since there are a number of materials for which the Voigt-Kelvin assumption is appropriate and moreover, since at present many questions regarding the modeling of structural damping mechanisms remain open, we feel that the Voigt-Kelvin model leads to a reasonable class of examples and problems on which we can begin to develop, test, and evaluate identification schemes.

Our treatment here is similar in spirit to some of our earlier efforts and the work of others on inverse problems for elastic structures (see [2], [3], [4],[5], [6], [14], [17], [26], [31]). Formulating the identification problem as a least squares fit to data, the scheme we develop involves a spline based finite element approximation to the hybrid system of coupled ordinary and partial differential equations describing the dynamics of the structure together with a spline based discretization of the admissible parameter set.

Our approach here specifically differs from the one taken in [5], [6] in that the present scheme is derived from an alternative state space formulation for the underlying dynamical equations. We consider the higher order analog of the classical conservative formulation for a second order hyperbolic equation as a first order vector system in the natural states of strain u_x and velocity u_t. We have considered identification schemes based upon this formulation previously in [31]. However by replacing the semigroup theoretic convergence arguments used there with weak or variational arguments (in the spirit of those commonly found in the finite element literature) as used in [5], we are able to significantly weaken the hypotheses necessary to ensure convergence. We point out below that the weakening of these hypotheses has both theoretical and computational significance.

Along with reporting theoretical convergence results, we discuss numerical findings. Our computational results are based upon extensive numerical studies which involved a variety of examples and two machines. In addition to testing our scheme on a conventional serial computer (an IBM 3081) we vectorized our codes for the Cray 1-S and then benchmarked some of our runs in order to explore the potential of vector architectures in the context of inverse problems for systems described by distributed parameter models.

We provide a brief outline and summary of the remainder of the paper. In Section 2 we specify the ordinary and partial differential equations which govern the underlying dynamics of the structure and precisely formulate the identification problem. We reformulate the initial-boundary value problem as an abstract second order evolution equation and then as a first order vector system. Existence, uniqueness and regularity results for solutions are summarized. Section 3 contains the abstract approximation theory and convergence results. A spline-based scheme is discussed in detail in Section 4 and our numerical findings are reported and summarized in Section 5.

We use standard notation throughout. For X and Y Banach spaces, the Banach space of continuous linear transformations from X into Y is denoted by $\mathcal{L}(X,Y)$. When X = Y we use the shortha. .notation $\mathcal{L}(X)$. The spaces of (equivalence classes of) functions f from an interval $\mathcal{I}$ into X which satisfy

$$\int_{\mathcal{I}} |f(\theta)|_X^2 \, d\theta < \infty \quad \text{or} \quad \operatorname{ess\,sup}_{\mathcal{I}} |f(\theta)|_X < \infty$$

are denoted respectively by $L_2(\mathcal{I}\,;X)$ and $L_\infty(\mathcal{I}\,;X)$. For k = 0,1,2... the space of X-valued functions with k continuous derivatives on $\mathcal{I}$ are denoted by $C^k(\mathcal{I}\,;X)$. When k = 0 we use $C(\mathcal{I}\,;X)$. The completion of the space $C^k(\mathcal{I}\,;X)$ with respect to the norm

$$|f|_k = \left(\sum_{j=0}^{k} \int_{\mathcal{I}} |f^{(j)}(\theta)|_X^2 \, d\theta \right)^{\frac{1}{2}}$$

is denoted by $H^k(\mathcal{I};X)$. When X = R we use simply $L_2(\mathcal{I})$, $L_\infty(\mathcal{I})$, $C^k(\mathcal{I})$ and $H^k(\mathcal{I})$.

2. The Identification Problem

We consider the identification, or estimation, of the mass and/or material properties of a long, slender, flexible, viscoelastic beam of length ℓ and spatially varying mass density ρ which is clamped at one end and free at the other with a body rigidly attached at the free end (see Figure 2.1 below).

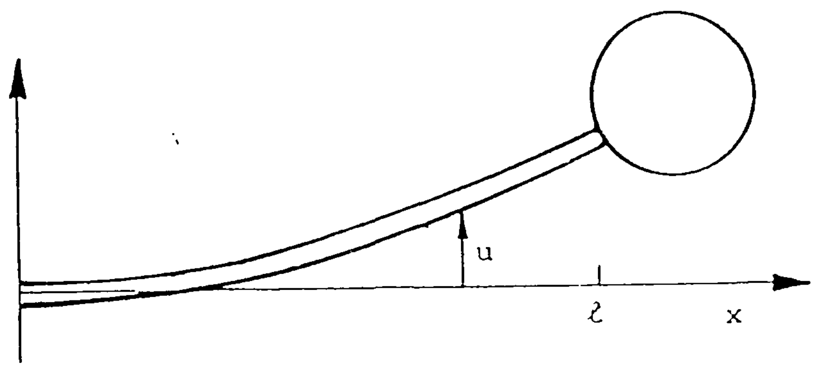

Figure 2.1

We assume that the material behavior of the beam is that of an idealized Voigt-Kelvin solid with modulus of elasticity E and coefficient of viscosity C_D (see [30]). We assume further that E, C_D and the cross sectional moment of inertia I of the beam are in general spatially varying. We take the mass properties of the tip body to be mass m and moment of inertia J about the center of mass O which is assumed to be located at a distance c from the tip of the beam directed along the beam's tip tangent (see Figure 2.2 below).

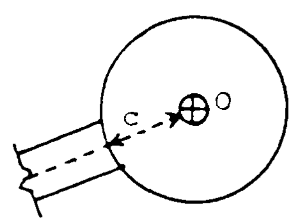

Figure 2.2

We note that there is no essential loss of generality in assuming that the mass center of the tip body is not offset from the tip tangent of the beam. We refer the interested reader to [31] where the more general situation is treated. Also, the problem with non-zero mass center offset can be transformed into a problem of the general form of the one which will be considered here. See [32] for details.

Letting $u = u(t,x)$ denote the transverse displacement of the beam at position x at time t and assuming only small deformations ($|u(t,x)| << \ell$, $|\frac{\partial u}{\partial x}(t,x)| << 1$), the Euler-Bernoulli theory and elementary Newtonian mechanics yield the hybrid system of ordinary and partial differential equations (see [19], [34])

$$(2.1) \qquad \rho\frac{\partial^2 u}{\partial t^2}(t,x) + \frac{\partial^2}{\partial x^2}\{EI\frac{\partial^2 u}{\partial x^2}(t,x) + C_D I\frac{\partial^3 u}{\partial x^2 \partial t}(t,x)\} =$$

$$\frac{\partial}{\partial x}\,\sigma\frac{\partial u}{\partial x}(t,x) + f(t,x), \qquad x \,\varepsilon\, (0,\ell),\ t > 0$$

$$(2.2) \qquad m\frac{\partial^2 u}{\partial t^2}(t,\ell) + mc\,\frac{\partial^3 u}{\partial t^2 \partial x}(t,\ell) - \frac{\partial}{\partial x}(EI\frac{\partial^2 u}{\partial x^2} +$$

$$C_D I\frac{\partial^3 u}{\partial x^2 \partial t})(t,\ell) = \sigma\frac{\partial u}{\partial x}(t,\ell) + g(t), \qquad t > 0$$

$$(2.3) \qquad mc\frac{\partial^2 u}{\partial t^2}(t,\ell) + (J + mc^2)\,\frac{\partial^3 u}{\partial t^2 \partial x}(t,\ell) + EI\frac{\partial^2 u}{\partial x^2}(t,\ell) +$$

$$C_D I\frac{\partial^3 u}{\partial x^2 \partial t}(t,\ell) = -c\sigma\frac{\partial u}{\partial x}(t,\ell) + h(t), \qquad t > 0$$

$$(2.4) \qquad u(t,0) = 0, \qquad \frac{\partial u}{\partial x}(t,0) = 0, \qquad t > 0$$

$$(2.5)\quad u(0,x) = \phi(x), \qquad \frac{\partial u}{\partial t}(0,x) = \psi(x), \qquad x \in [0,\ell].$$

Equation (2.2) and (2.3) are derived from the usual transverse and rotational equilibrium considerations at the free end. The geometric boundary conditions (2.4) are the zero displacement and zero slope constraints at the clamped end. The functions $f = f(t,x)$, $g = g(t)$, $h = h(t)$ and $\sigma = \sigma(t,x)$ denote externally applied loads in the form of moments (h) and transversally (f and g) or axially (σ) directed forces exerted on the beam or tip body. (In fact, $h(t) = \tilde{h}(t) + cg(t)$ where $\tilde{h}$ is an externally applied torque on the tip body). The temporal boundary conditions (2.5) reflect the initial displacement and velocity distributions which are assumed to be given by the functions ϕ and ψ respectively.

We treat the initial-boundary value problem (2.1) - (2.5) in the form of an abstract second order evolution equation which we then rewrite as an equivalent first order vector system. The particular state space formulation we choose forms the basis for the finite dimensional approximation schemes we develop in the next section. It also allows us to easily establish existence, uniqueness and regularity results for solutions to (2.1) - (2.5) using the theory of abstract parabolic systems.

Let H denote the Hilbert space $R^2 \times L_2(0,\ell)$ with inner product

$$<(\eta_1,\xi_1,\theta_1),(\eta_2,\xi_2,\theta_2)>_H = \eta_1\eta_2 + \xi_1\xi_2 + <\theta_1,\theta_2>_0$$

and let V denote the Hilbert space

$$V = \{(\eta,\xi,\theta) \in H : \theta \in H^2(0,\ell),\ \theta(0) = D\theta(0) = 0,\ \eta = \theta(\ell),\ \xi = D\theta(\ell)\}$$

with inner product

$$<\hat{\theta}_1, \hat{\theta}_2>_V = <EI(D^2\theta_1), D^2\theta_2>_0$$

for $\hat{\theta}_i = (\theta_i(\ell), D\theta_i(\ell), \theta_i) \in V$, $i = 1,2$. In the above definitions the inner product $<\cdot,\cdot>_0$ is the standard one on $L_2(0,\ell)$ and D denotes the spatial differentiation operators $\frac{d}{dx}$ or $\frac{\partial}{\partial x}$. With H as the pivot space, we obtain the usual dense embeddings $V \subset H = H' \subset V'$.

We consider the system (2.1) - (2.5) in the form of the abstract second order initial value problem

(2.6) $\mathcal{M}_0 \hat{u}_{tt}(t) + \mathcal{C}_0\hat{u}_t(t) + \mathcal{K}_0\hat{u}(t) = \mathcal{B}_0(t)\hat{u}(t) + \mathcal{F}_0(t), \quad t > 0$

(2.7) $\hat{u}(0) = \hat{\phi}, \qquad \hat{u}_t(0) = \hat{\psi}$

in the state $\hat{u}(t) = (u(t,\ell),Du(t,\ell),u(t,\cdot)) \in H$. The abstract mass, damping and stiffness operators $\mathcal{M}_0$, $\mathcal{C}_0$ and $\mathcal{K}_0$ are given formally by

$$\mathcal{M}_0(\eta,\xi,\theta) = (m\eta + mc\xi,\ mc\eta + (J + mc^2)\xi,\ \rho\theta)$$

$$\mathcal{C}_0(\eta,\xi,\theta) = (-D(C_DI(D^2\theta))(\ell),\ C_DI(D^2\theta)(\ell), D^2(C_DI(D^2\theta)))$$

and

$$\mathcal{K}_0(\eta,\xi,\theta) = (-D(EI(D^2\theta))(\ell),\ EI(D^2\theta)(\ell),\ D^2(EI(D^2\theta)))$$

respectively. For each $t > 0$, the operator valued function $\mathcal{B}_0$ and input or forcing function $\mathcal{F}_0$ take on the values

$$\mathcal{B}_0(t)(\eta,\xi,\theta) = (-\sigma(t,\ell)(D\theta(\ell)),\ -c\sigma(t,\ell)(D\theta(\ell)),\ D(\sigma(t,\cdot)(D\theta)))$$

and

$$\mathcal{F}_0(t) = (g(t),\ h(t),\ f(t,\cdot)).$$

The initial conditions $\hat{\phi}$ and $\hat{\psi}$ are given by

$$\hat{\phi} = (\phi(\ell),\ D\phi(\ell),\ \phi)$$

and

$$\hat{\psi} = (\psi(\ell),\ D\psi(\ell), \psi).$$

The formal definitions given above can be made precise and the existence and uniqueness of solutions to the initial value problem (2.6), (2.7) can be established if we make the following assumptions.

A_1 The functions ρ, EI and C_DI are elements in $C[0,\ell]$ and there exists a positive constant α for which $\rho(x) \geq \alpha$, $EI(x) \geq \alpha$, $C_DI(x) \geq \alpha$, $x \in [0, \ell]$.

A_2 The mapping $t \to \sigma(t,\cdot)$ is an element in $L_\infty((0,T); H^1(0,\ell))$ for some $T > 0$.

A_3 The mapping $t \to f(t,\cdot)$ is an element in $L_2((0,T); L_2(0,\ell))$ and $g,h \in L_2(0,T)$.

A_4 The function ϕ is an element in $H^2(0,\ell)$ with $\phi(0) = D\phi(0) = 0$ and $\psi \in L_2(0,\ell)$ with $\psi(\ell)$ and $D\psi(\ell)$ defined.

Under the hypotheses A_1 - A_4 above, the operator $\mathcal{M}_0$ is a bounded linear operator from H onto H and $\mathcal{C}_0$: Dom $(\mathcal{C}_0) \subset H \to H$ and $\mathcal{K}_0$: Dom $(\mathcal{K}_0) \subset H \to H$ are densely defined, nonnegative, self -adjoint operators defined on $\mathrm{Dom}(\mathcal{C}_0) = \{\hat{\theta} \in V : C_D I(D^2\theta) \in H^2(0,\ell)\}$ and $\mathrm{Dom}\,(\mathcal{K}_0) = \{\hat{\theta} \in V: EI(D^2\theta) \in H^2(0,\ell)\}$ respectively (see [32]). For each $t \in (0,T)$, $\mathcal{B}_0(t) \in \mathcal{L}(V,H)$ and $\mathcal{F}_0(t) \in H$ while $\hat{\phi} \in V$ and $\hat{\psi} \in H$. It also follows that $\mathcal{B}_0 \in L_\infty((0,T); \mathcal{L}(V,H))$ and $\mathcal{F}_0 \in L_2((0,T); H)$.

We shall call a mapping $t \to \hat{u}(t)$ from $[0,T]$ into H a strong solution to (2.6), (2.7) if

$$\hat{u} \in C([0,T]; V) \cap C^1((0,T]; V) \cap C^1([0,T]; H) \cap C^2((0,T]; H),$$

$\hat{u}(t) \in \mathrm{Dom}\,(\mathcal{K}_0)$, $\hat{u}_t(t) \in \mathrm{Dom}\,(\mathcal{C}_0)$, $t \in (0,T]$, and $\hat{u}$ satisfies (2.6) and (2.7) where the time derivatives are interpreted in a strong (norm) sense in H. We shall call a mapping $t \to \hat{u}(t)$ from $[0,T]$ into H a weak solution to (2.6), (2.7) if

$$\hat{u} \in C([0,T]; V) \cap H^1((0,T); V) \cap C^1([0,T]; H) \cap H^2((0,T); V')$$

and it satisfies the initial value problem (2.6), (2.7) with the operators $\mathcal{C}_0$ and $\mathcal{K}_0$ replaced by their natural extensions to operators in $\mathcal{L}(V,V')$ and the time derivatives are interpreted in a weak or distributional sense (see [20], [27]). A function $u = u(t,x)$ will be called a strong (weak) solution to the initial-boundary value problem (2.1) - (2.5) if the mapping $t \to \hat{u}(t)$ given by $\hat{u}(t) = (u(t,\ell), Du(t,\ell), u(t,\cdot))$ is a strong (weak) solution to (2.6), (2.7).

Our approximation theory for the estimation problem to be developed below is based upon the reformulation of the initial value problem (2.6), (2.7) as a first order vector system. This reformulation is formally equivalent to rewriting the initial-boundary value problem (2.1) - (2.5) as a first order system in the states D^2u (strain) and u_t (velocity) (see [3], [31]). We note that since the stiffness operator $\mathcal{K}_0$ is nonnegative and self-adjoint it has a unique nonnegative, selfadjoint square root $\mathcal{K}_0^{1/2}: V \subset H \to H$. It can be written in factored form as

$$\mathcal{K}_0 = L_{EI}^* L$$

where $L : V \subset H \to L_2(0,\ell)$ is given by

$$L\hat{\theta} = D^2\theta,$$

for $\hat{\theta} = (\theta(\ell), D\theta(\ell), \theta) \varepsilon V$, and $L_{EI}^* : \mathrm{Dom}\,(L_{EI}^*) \subset L_2(0,\ell) \to H$ by

$$\mathrm{Dom}\,(L_{EI}^*) = \{\theta \,\varepsilon\, L_2(0,\ell) \;:\; EI\theta \,\varepsilon\, H^2(0,\ell)\}$$

(2.8) $$L_{EI}^*\theta = (-D(EI\theta)(\ell), EI\theta(\ell), D^2(EI\theta)).$$

If, for $\tau \,\varepsilon\, C[0,\ell]$ with $\tau(x) \geq \alpha > 0$, $x \,\varepsilon\, [0,\ell]$, we let $L_{2,\tau}$ denote the Hilbert space $L_2(0,\ell)$ endowed with the inner product

$$<\theta_1,\theta_2>_{0,\tau} = <\tau\theta_1,\theta_2>_0$$

then L_τ^* given by (2.8) with EI replaced by τ is the Hilbert space adjoint of L as a mapping from $V \subset H$ into $L_{2,\tau}$.

We note that $L \,\varepsilon\, \mathcal{L}(V,L_{2,EI})$ is a Hilbert space isomorphism with

$$<\hat{\theta}_1, \hat{\theta}_2>_V = <\mathcal{K}_0^{1/2}\hat{\theta}_1, \mathcal{K}_0^{1/2}\hat{\theta}_2>_H = <L\hat{\theta}_1, L\hat{\theta}_2>_{0,EI}$$

and $L^{-1} : L_2(0,\ell) \to V$ given by

$$L^{-1}\theta = (\int_0^\ell \int_0^x \theta(y)dydx, \int_0^\ell \theta(x)dx, \int_0^{\cdot} \int_0^x \theta(y)dydx).$$

We also have

$$\mathcal{C}_0 = L_{C_DI}^* L\,.$$

Letting $\mathcal{H} = L_2(0,\ell) \times H$ with inner product

(2.9) $\quad <(\theta_1,(\eta_1,\xi_1,\chi_1)),(\theta_2,(\eta_2,\xi_2,\chi_2))>_{\mathcal{H}} = <\theta_1,\theta_2>_{0,EI} + <\mathcal{M}_0(\eta_1,\xi_1,\chi_1),(\eta_2,\xi_2,\chi_2)>_H$

and $\mathcal{V} = L_2(0,\ell) \times V$ with inner product

$$<(\theta_1,\hat{\chi}_1),(\theta_2,\hat{\chi}_2)>_{\mathcal{V}} = <\theta_1,\theta_2>_{0,EI} + <\hat{\chi}_1,\hat{\chi}_2>_V$$

we have the dense imbeddings $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$. We consider the initial value problem for $z(t) = (w(t), \hat{v}(t))\, \varepsilon \mathcal{H}$ given by

(2.10) $\quad w_t(t) = L\hat{v}(t)$

(2.11) $\quad \mathcal{M}_0\, \hat{v}_t(t) = -L^{*}_{EI}\, w(t) - L^{*}_{C_DI}L\hat{v}(t) + \mathcal{B}_0(t)L^{-1}w(t) + \mathcal{F}_0(t) \qquad 0 < t \le T$

(2.12) $\quad w(0) = L\hat{\phi}, \qquad \hat{v}(0) = \hat{\psi}$

which we rewrite as

(2.13) $\quad z_t(t) = \mathcal{A}(t)z(t) + \mathcal{F}(t), \quad 0 < t \le T,$

(2.14) $\quad z(0) = z_0$

where

(2.15) $\quad \mathcal{A}(t) = \tilde{\mathcal{A}} + \mathcal{B}(t)$

with $\tilde{\mathcal{A}} : \mathcal{D} \subset \mathcal{H} \to \mathcal{H}$, $\mathcal{B}\, \varepsilon\, L_\infty((0,T); \mathcal{L}(H))$, $\mathcal{F}\, \varepsilon\, L_2((0,T); \mathcal{H})$ and $z_0\, \varepsilon\, \mathcal{H}$ given by

$$\tilde{\mathcal{A}}(\theta,\hat{\chi}) = (L\hat{\chi}, -\mathcal{M}_0^{-1}L^{*}_{EI}\theta - \mathcal{M}_0^{-1}L^{*}_{C_DI}L\hat{\chi})$$

for $(\theta,\hat{\chi})\, \varepsilon\, \mathcal{D} = \mathrm{Dom}(L^{*}_{EI}) \times \mathrm{Dom}(\mathcal{C}_0)$,

$$\mathcal{B}(t)(\theta,(\eta,\xi,\chi)) = (0, \mathcal{M}_0^{-1}\mathcal{B}_0(t)L^{-1}\theta),$$

for $(\theta,(\eta,\xi,\chi))\, \varepsilon\, \mathcal{H}$,

$$\mathcal{F}(t) = (0, \mathcal{M}_0^{-1}\, \mathcal{F}_0(t))$$

and

$$z_0 = (L\hat{\phi},\hat{\psi})\,.$$

In formulating the inverse problem, we keep technical details to a minimum by considering only the estimation of the beam's spatially varying flexural stiffness EI and viscous damping coefficient C_DI. Extending the finite dimensional approximation methods and corresponding convergence theory which are developed below so as to be applicable to the identification of other structural or input parameters, for example mass properties (of the beam and/or tip body), initial conditions or loading, is, at least in principle, routine (see [7] [12] [14] [16] [31]).

Let $\mathcal{Q} = C[0,\ell] \times C[0,\ell]$ with norm

$$|q|_{\mathcal{Q}} = |(q_1,q_2)|_{\mathcal{Q}} = |q_1|_\infty + |q_2|_\infty = \sup_{x\varepsilon[0,\ell]} |q_1(x)| + \sup_{x\varepsilon[0,\ell]} |q_2(x)|. \tag{2.16}$$

We take the admissible parameter space Q to be a compact subset of $\mathcal{Q}$ (compact with respect to the metric topology induced by the norm (2.16)). Recalling assumption (i) we assume further that the set Q has the property that all $q = (q_1,q_2) \varepsilon Q$ satisfy $q_1(x) \geq \alpha$ and $q_2(x) \geq \alpha$, $x \varepsilon [0,\ell]$. We formulate the identification problem as a least-squares fit-to-data over the admissible parameter space Q. We assume that the structure has undergone a time varying elastic deformation in response to the initial conditions described by ϕ and ψ and the input loads represented by f,g,h and σ. Denoting the observation space by $\mathcal{Z}$, we assume that at times t_i, $i = 1,2,...,\nu$ measurements $\zeta(t_i) \varepsilon \mathcal{Z}$ (e.g. displacement, velocity, slope, strain, etc.) were taken from the structure.

We require that $\mathcal{Z}$ be a linear space endowed with a norm $|\cdot|_{\mathcal{Z}}$ and let Γ denote an appropriately defined continuous mapping from $\mathcal{H}$ into $\mathcal{Z}$. For example, suppose that displacement measurements have been taken at the points x_j, $j = 1,2,...,\mu$ along the span of the beam. We choose $\mathcal{Z}$ as Euclidean μ-space, R^μ, and take Γ to be

$$\Gamma(z) = (\theta(x_1), \theta(x_2),...,\theta(x_\mu))^T$$

where $z = (w,\hat{v}) \varepsilon \mathcal{H}$ and

$$\hat{\theta} = (\theta(\ell), D\theta(\ell),\theta) = L^{-1}w \varepsilon V.$$

With distributed strain or velocity observations, we would take $\Gamma(z) = w$ or $\Gamma(z) = \hat{v}$ respectively. We formulate the identification problem as follows

(ID) Given $\zeta(t_i) \varepsilon \mathcal{Z}$, $i = 1,2,\ldots,\nu$, find $q^* \varepsilon Q$ which minimizes

$$\mathcal{J}(q) = \sum_{i=1}^{\nu} |\Gamma(z(t_i\,;q)) - \zeta(t_i)|^2_{\mathcal{Z}}$$

where for each $q = (q_1,q_2) \varepsilon Q$, $z(\cdot\,;q) = (w(\cdot\,;q), \hat{v}(\cdot\,;q))$ is the solution to the initial value problem (2.13), (2.14) or (2.10) - (2.12) with EI set equal to q_1, and $C_D I$ set equal to q_2.

It is immediately clear that the optimization problem given above is inherently infinite dimensional. The admissible parameter set Q is a subset of a function space and the evaluation (and therefore minimization) of the least-squares performance index $\mathcal{J}$ requires the solution of an infinite dimensional evolution equation. The introduction of finite dimensional approximations is essential to the development of practical computational methods. Fundamental to the approach we take here is a weak, distributional, or variational formulation of the initial value problem (2.13), (2.14). We derive the weak form and briefly outline existence, uniqueness and regularity results for solutions.

In the usual manner, we extend the operator $\mathcal{A}(t)$ given by (2.15) to an operator in $\mathcal{L}(\mathcal{V},\mathcal{V}')$ via

$$(\mathcal{A}(t)(v))(\tilde{v}) = a(t)(v,\tilde{v}), \qquad v,\tilde{v} \varepsilon \mathcal{V}$$

where the bilinear form $a(t)(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to R$ is given by

$$a(t)((\theta_1,\hat{\chi}_1),(\theta_2,\hat{\chi}_2)) = \langle EI\, L\, \hat{\chi}_1,\theta_2\rangle_0 - \langle EI\, \theta_1, L\hat{\chi}_2\rangle_0 - \langle C_D I\, L\hat{\chi}_1, L\hat{\chi}_2\rangle_0 -$$

(2.17)

$$c\sigma(t,\ell)\int_0^{\ell} \theta_1(x)dx(D\chi_2(\ell)) - \langle\sigma(t,\cdot)\int_0^{\cdot} \theta_1(x)dx,\ \chi_2\rangle_0 .$$

Standard estimates can be used to demonstrate the existence of positive constants k, λ and β for which

$$|a(t)(v_1,v_2)| \le k\, |v_1|_{\mathcal{V}}\, |v_2|_{\mathcal{V}}, \qquad v_i \varepsilon \mathcal{V},\ i = 1,2,$$

and

$$a(t)(v,v) + \lambda |v|^2_{\mathcal{H}} \ge \beta |v|^2_{\mathcal{V}}, \qquad v \varepsilon \mathcal{V},\ t \varepsilon [0,T] .$$

Consequently (see [27]) the system (2.13), (2.14) interpreted as an initial value problem in $\mathcal{V}'$ or equivalently, written in weak form as

(2.18) $\quad <z_t(t),v>_{\mathcal{H}} = a(t)(z(t),v) + <\mathcal{F}(t),v>_{\mathcal{H}} \qquad v \in \mathcal{V},\ t \in (0,T]$

(2.19) $\quad z(0) = z_0$

admits a unique solution z with $z(t) \in \mathcal{V}$, $t \in (0,T]$ and

$$z \in L_2((0,T);\mathcal{V}) \cap C([0,T];\ \mathcal{H}) \cap H^1((0,T);\ \mathcal{V}').$$

If $z = (w,\hat{v})$ is the unique solution to (2.18), (2.19) then

$$\hat{u}(t) = L^{-1}w(t), \quad t \in [0,T]$$

is a weak solution to (2.6), (2.7) and it is unique.

Under somewhat stronger hypotheses than those given in A_2 and A_3 above, the existence of strong solutions can be established. Indeed, if in addition to A_1 and A_4, we assume

A_2' The mapping $t \to \sigma(t,\cdot)$ is an element in $C^1([0,T];\ H^1(0,\ell))$ for some $T > 0$

A_3' The mapping $t \to f(t,\cdot)$ is an element in $C^1([0,T];\ L_2(0,\ell))$ and $g,h \in C^1[0,T]$ (in fact, Hölder continuity will suffice, see [29], [37])

then the family of operators $\{\mathcal{A}(t)\}_{t\in[0,T]}$ given by (2.15) generates a unique evolution system $\{U(t,s) : 0 \le s \le t \le T\}$ on $\mathcal{H}$ and z given by

(2.20) $$z(t) = U(t,0)z_0 + \int_0^t U(t,s)\mathcal{F}(s)ds, \qquad 0 \le t \le T,$$

is the unique solution to the inital value problem (2.13), (2.14) and satisfies $z(t) \in \mathcal{D}$, $t \in (0,T]$ with $z \in C([0,T];\ \mathcal{H}) \cap C^1((0,T];\ \mathcal{H})$. Once again, with $z = (w,\hat{v})$ now given by (2.20), $\hat{u}(t) = L^{-1}w(t)$, $t \in [0,T]$, is a strong solution to (2.6), (2.7) and it is unique.

3. An Abstract Approximation Framework

We turn next to a discussion of a general approximation framework and convergence theory for the identification problem (ID) formulated above. In the following section we formulate a specific spline-based scheme to which the general theory developed here applies.

Fundamental to our approach is the construction of a sequence of finite dimensional (with regard to both the state dynamics and the admissible parameter set) approximating identification problems each of which, under appropriate hypotheses, can be shown to have a solution that in some sense (specifically, subsequential convergence) approximates a solution to the original infinite dimensional estimation problem (ID).

In the discussion to follow, we exhibit the explicit dependence on $q = (q_1, q_2) \in \mathcal{Q}$ of the $\mathcal{H}$ inner product $\langle \cdot,\cdot \rangle_{\mathcal{H}}$ and the bilinear form $a(t)(\cdot,\cdot)$ given in (2.9) and (2.17) respectively by using the notation $\langle \cdot,\cdot \rangle_q$ and $a(t;q)(\cdot,\cdot)$. For each $N_1 = 1,2,\ldots$ and each $N_2 = 1,2,\ldots$ let W^{N_1} and V^{N_2} be finite dimensional subspaces of $L_2(0,\ell)$ and V respectively. If, for $N = (N_1,N_2)$ we define $\mathcal{V}^N = W^{N_1} \times V^{N_2}$, then $\mathcal{V}^N$ is a finite dimensional subspace of both $\mathcal{H}$ and $\mathcal{V}$. Let $\mathcal{P}^N : \mathcal{H} \to \mathcal{V}^N$ denote the projection map of $\mathcal{H}$ onto $\mathcal{V}^N$ given by

$$\mathcal{P}^N(w,\hat{v}) = (P_1^N w, P_2^N \hat{v}) \tag{3.1}$$

where P_1^N is the orthogonal projection of $L_2(0,\ell)$ onto W^{N_1} and P_2^N is the orthogonal projection of H onto V^{N_2}, both computed with respect to the standard (unweighted) inner products on the respective spaces $L_2(0,\ell)$ and H.

The Galerkin equations in $\mathcal{V}^N$ corresponding to the system (2.18), (2.19) and $q \in \mathcal{Q}$ are

$$\langle z_t^N(t), v^N \rangle_q = a(t\,;q)(z^N(t),v^N) + \langle \mathcal{F}(t), v^N \rangle_q, \qquad v^N \in \mathcal{V}^N,\ 0 \le t \le T \tag{3.2}$$

$$z^N(0) = \mathcal{P}^N z_0 . \tag{3.3}$$

For each $M_i \varepsilon Z_+ = \{1,2,\ldots\}$, $i = 1,2$, let $S^1_{M_1}$ and $S^2_{M_2}$ be finite dimensional subspaces of $C[0,\ell]$ and for $M = (M_1, M_2)$ define $\mathcal{Q}_M \subset \mathcal{Q}$ by $\mathcal{Q}_M = S^1_{M_1} \times S^2_{M_2}$. Let $\mathcal{J}^1_M$ and $\mathcal{J}^2_M$ denote mappings from $C[0,\ell]$ onto $S^1_{M_1}$ and $S^2_{M_2}$ respectively and define $\mathcal{J}_M$, a mapping from $\mathcal{Q}$ onto $\mathcal{Q}_M$, by

$$\mathcal{J}_M(q) = \mathcal{J}_M((q_1,q_2)) = (\mathcal{J}^1_M(q_1), \mathcal{J}^2_M(q_2)), \qquad q = (q_1, q_2) \ \varepsilon \ \mathcal{Q} .$$

We define a sequence of approximating admissible parameter spaces $\{Q_M\}$, $M \varepsilon \ Z_+ \times Z_+$ by

$$Q_M = \mathcal{J}_M(Q) \tag{3.4}$$

and formulate the sequence of approximating identification problems as follows:

(ID^N_M) Given $\zeta(t_i) \varepsilon \mathcal{Z}, i = 1,2,\ldots,\nu$, find $(q^N_M)^* \varepsilon Q_M$ which minimizes

$$\mathcal{J}^N(q) = \sum_{i=1}^{\nu} |\Gamma(z^N(t_i\,;q)) - \zeta(t_i)|^2_{\mathcal{Z}}$$

over Q_M, where $z^N(\cdot;q)$ is the solution to the initial value problem (3.2), (3.3) in $\mathcal{V}^N$.

We choose bases $\{\theta^N_i\}_{i=1}^{K_1^N}$, $\{\hat{\chi}^N_i\}_{i=1}^{K_2^N}$, $\{\phi^i_M\}_{i=1}^{L^1_M}$ and $\{\psi^i_M\}_{i=1}^{L^2_M}$ for the finite dimensional spaces W^{N_1}, V^{N_2}, $S^1_{M_1}$ and $S^2_{M_2}$ respectively. Then $q^1_M \varepsilon S^1_{M_1}$, $q^2_M \varepsilon S^2_{M_2}$ and the solution $z^N(\cdot\,;q_M)$ to the initial value problem (3.2), (3.3) with $q = q_M = (q^1_M, q^2_M)$ can be written as

$$q^1_M = \sum_{i=1}^{L^1_M} \alpha^i_M \phi^i_M ,$$

$$q^2_M = \sum_{i=1}^{L^2_M} \alpha_M^{i+L^1_M} \psi^i_M$$

and

$$z^N(t\,;q_M) = \sum_{i=1}^{K_1^N} Z_i^N(t\,;\alpha_M)\theta_i^N + \sum_{i=1}^{K_2^N} Z^N_{i+K_1^N}(t\,;\alpha_M)\hat{\chi}_i^N, \qquad t\ \varepsilon\ [0,T],$$

respectively. Moreover, $Z^N(\cdot\,;\alpha_M)$ is the solution to the initial value problem in $R^{K_1^N + K_2^N}$ given by

(3.5) $\mathcal{M}^N(\alpha_M)\dot{Z}^N(t) = A^N(t\,;\alpha_M)\, Z^N(t) + F^N(t), \qquad t\ \varepsilon\ (0,T]$

(3.6) $Z^N(0) = Z_0^N$.

Here the positive definite matrix $\mathcal{M}^N(\alpha_M)$ is of the form

$$\mathcal{M}^N(\alpha_M) = \begin{bmatrix} \mathcal{M}_1^N(\alpha_M) & O \\ & \\ O & \mathcal{M}_2^N \end{bmatrix}$$

where $\mathcal{M}_1^N(\alpha_M)$ is a K_1^N-square matrix with components

$$[\mathcal{M}_1^N(\alpha_M)]_{ij} = \sum_{k=1}^{L_M^1} \alpha_M^k <\phi_M^k\theta_i^N, \theta_j^N>_0 ,$$

and $\mathcal{M}_2^N$ is K_2^N-square matrix with entries

$$[\mathcal{M}_2^N]_{ij} = <\mathcal{M}_0\hat{\chi}_i^N, \hat{\chi}_j^N>_H .$$

For each $t \geq 0$ the matrix $A^N(t\,;\alpha_M)$ is given by $A^N(t\,;\alpha_M) = \mathcal{A}^N(\alpha_M) + B^N(t)$ with

$$\mathcal{A}^N(\alpha_M) = \begin{bmatrix} O & E^N(\alpha_M) \\ & \\ -E^N(\alpha_M)^T & -C^N(\alpha_M) \end{bmatrix}$$

and

$$B^N(t) = \begin{bmatrix} O & O \\ & \\ D^N(t) & O \end{bmatrix}$$

where $E^N(\alpha_M)$ is a $K_1^N \times K_2^N$ matrix with components

$$[E^N(\alpha_M)]_{ij} = \sum_{k=1}^{L_M^1} \alpha_M^k <\phi_M^k \theta_i^N, D^2\chi_j^N>_0 ,$$

$C^N(\alpha_M)$ is a K_2^N -square matrix with components

$$[C^N(\alpha_M)]_{ij} = \sum_{k=1}^{L_M^2} \alpha_M^{k+L_M^1} <\psi_M^k D^2\chi_i^N, D^2\chi_j^N>_0$$

and $D^N(t)$ is a $K_2^N \times K_1^N$ matrix with components

$$[D^N(t)]_{ij} = c\sigma(t,\ell)\int_0^{\ell} \theta_j^N(x)dxD\chi_i^N(\ell) - <\sigma(t\,,\cdot)\int_0^{\cdot} \theta_j^N(x)dx, D\chi_i^N>_0 .$$

The nonhomogeneous term $F^N(t)$ is given by $F^N(t) = (0, F_2^N(t))$ where $F_2^N(t)$ is a K_2^N vector with entries

$$[F_2^N(t)]_i = <\mathcal{F}_0(t), \hat{\chi}_i^N>_H .$$

The initial data is of the form

$$Z_0^N = (G^N)^{-1}(Z_{01}^N , Z_{02}^N)^T$$

where the K_1^N vector Z_{01}^N and the K_2^N vector Z_{02}^N are given component-wise by

$$[Z_{01}^N]_i = <D^2\phi, \theta_i^N>_0$$

and

$$[Z_{02}^N]_j = <\hat{\psi}, \hat{\chi}_j^N>_H$$

respectively, and

$$G^N = \begin{bmatrix} G_1^N & O \\ & \\ O & G_2^N \end{bmatrix}$$

with G_1^N a K_1^N-square matrix defined by

$$[G_1^N]_{ij} = <\theta_i^N, \theta_j^N>_0 ,$$

and G_2^N a K_2^N-square matrix with components

$$[G_2^N]_{ij} = <\hat{\chi}_i^N , \hat{\chi}_j^N>_H .$$

It is now easily seen that the finite dimensional identification problem (ID_M^N) in fact involves simply the minimization of a least-squares performance index over a subset of $R^{L_M^1 + L_M^2}$. Furthermore, the evaluation of the functional $\mathcal{J}^N$ requires only the solution of the $K_1^N + K_2^N$ dimensional, linear, non-autonomous ordinary differential equation (3.5) with initial conditions specified in (3.6). If the existence of solutions to the finite dimensional optimization problems can be established, it is immediately clear that they can, in principle, be computed using standard techniques. Conditions which guarantee the existence of solutions to problem (ID_M^N) and the fact that they in some sense approximate solutions to the original infinite dimensional estimation problem (ID) are given in the following theorem.

Theorem 3.1 Suppose

H_1 the mappings $\mathcal{I}_M$ are continuous from Q into $\mathcal{Q}$,

H_2 for each $q \in Q$, $\mathcal{I}_M(q) \to q$ as $|M| \to \infty$ with the convergence being uniform in q for $q \in Q$,

H_3 the spaces $\mathcal{V}^N$ and projections $\mathcal{P}^N$ are such that if $\{q^N\}$ is a sequence in Q with $q^N \to \bar{q} = (\bar{q}_1, \bar{q}_2) \in Q$ as $|N| \to \infty$ then $z^N(t\,;q^N) \to z(t\,;\bar{q})$ in $L_2(0,\ell) \times H$ for each $t \in [0,T]$ as $|N| \to \infty$ where $z^N(\cdot\,;q^N)$ is the solution to the initial value problem (3.2), (3.3) with $q = q^N$ and $z(\cdot\,;\bar{q})$ is the solution to the initial value problem (2.18), (2.19) corresponding to $EI = \bar{q}_1$ and $C_DI = \bar{q}_2$.

Then, each of the problems (ID_M^N) has a solution $(q_M^N)^*$. Furthermore, the sequence $\{(q_M^N)^*\}$ admits a $\mathcal{Q}$-convergent subsequence whose limit q^* is a point in Q and is a solution to problem (ID).

In the statement of the theorem, for an element $K = (K_1,K_2) \in Z_+ \times Z_+$ we have adopted the notation $|K| \to \infty$ to denote $K_1,K_2 \to \infty$. We remark that it is also true that the limit point of any $\mathcal{Q}$-convergent subsequence $\{(q_{M^j}^{N^k})^*\}$ of $\{(q_M^N)^*\}$ with $|M|, |N| \to \infty$ as $j,k \to \infty$ is a solution to problem (ID) as well. Moreover, if problem (ID) has a unique solution, q^*, then the sequence $\{(q_M^N)^*\}$ itself converges to q^*. It is also important to note that the hypotheses of the theorem do not require that $Q_M \subset Q$.

We have established results analogous to those given in Theorem 3.1 for inverse problems involving parabolic and hyperbolic systems (see, for example [12], [13], [16]) as well as for related methods for higher order equations for elastic structures (see [4], [5], [6]). For the flexible structure problems treated here, the essential features of the argument remain, for the most part, unchanged. We therefore only briefly sketch them below.

Standard continuous dependence results for linear ordinary differential systems, the continuity assumptions on $\mathcal{J}_M$ and Γ (and therefore on $\mathcal{J}^N$ as well) and the fact that Q is a compact subset of $\mathcal{Q}$ are sufficent to conclude that there exists a solution $(q_M^N)^* \varepsilon Q_M$ to problem (ID_M^N).

The definition of the space Q_M (see (3.4)) implies the existence of a $\bar{q}_M^N \varepsilon Q$ for which $(q_M^N)^* = \mathcal{J}_M(\bar{q}_M^N)$. Since Q is compact, there exists a subsequence $\{\bar{q}_{M^j}^{N^k}\}$ of $\{\bar{q}_M^N\}$ with $\bar{q}_{M^j}^{N^k} \rightarrow q^* \varepsilon Q$ as $j,k \rightarrow \infty$. The subsequence $\{\bar{q}_{M^j}^{N^k}\}$ can always be chosen with $|M^j|, |N^k| \rightarrow \infty$ as $j,k \rightarrow \infty$. It follows that

$$\mathcal{J}^{N^k}((q_{M^j}^{N^k})^*) \leq \mathcal{J}^{N^k}(q), \qquad q \varepsilon Q_{M^j}$$

and consequently that

$$(3.7) \qquad \mathcal{J}^{N^k}((q_{M^j}^{N^k})^*) \leq \mathcal{J}^{N^k}(\mathcal{J}_{M^j}(q)), \qquad q \varepsilon Q.$$

Assumption H_2 above and

$$|(q_{M^j}^{N^k})^* - q^*|_{\mathcal{Q}} \leq |\mathcal{J}_{M^j}(\bar{q}_{M^j}^{N^k}) - \bar{q}_{M^j}^{N^k}|_{\mathcal{Q}} + |\bar{q}_{M^j}^{N^k} - q^*|_{\mathcal{Q}}$$

imply $(q_{M^j}^{N^k})^* \rightarrow q^*$ as $j,k \rightarrow \infty$. Taking the limit as $j,k \rightarrow \infty$ in (3.7) above with an application of assumption H_3, we find $\mathcal{J}(q^*) \leq \mathcal{J}(q)$, $q \varepsilon Q$, and hence that q^* is a solution to problem (ID).

4. A Scheme Using Polynomial Splines

In this section we outline a scheme which uses piecewise polynomial spline functions and show that it satisfies the conditions and hypotheses of Theorem 3.1. We first treat the discretization of the admissible parameter set Q.

For each $M = (M_1, M_2) \in Z_+ \times Z_+$ let Δ_M^1 and Δ_M^2 denote the uniform partitions of the interval $[0, \ell]$ determined by the meshes $\{0, \ell/M_1, 2\ell/M_1, \ldots, \ell\}$ and $\{0, \ell/M_2, 2\ell/M_2, \ldots, \ell\}$ respectively. For $m = 1,2,\ldots$ and Δ a partition of $[0, \ell]$ let $Sp(m,\Delta)$ denote the usual spline space of functions in $C^{2m-2}[0,\ell]$ which are polynomials of degree $2m-1$ on each subinterval of Δ (see [36]). We then define $S_{M_i}^i = Sp(1, \Delta)$, $i = 1,2$. In this case we have $\dim S_{M_i}^i = L_M^i = M_i + 1$, $i = 1,2$, with the usual "hat" functions forming a cardinal basis for each of the spaces $S_{M_i}^i$, $i = 1,2$. For $i = 1,2$, let $\mathcal{I}_M^i : C[0,\ell] \to S_{M_i}^i$ be the interpolation operator defined by

$$\left(\mathcal{I}_M^i \gamma\right)\left(\frac{j\ell}{M_i}\right) = \gamma\left(\frac{j\ell}{M_i}\right), \quad j = 0,1,2,\ldots,M_i$$

for $\gamma \in C[0,\ell]$. The theory of interpolatory splines (see [33]) yields the continuous dependence result

$$|\mathcal{I}_M^i\gamma_1 - \mathcal{I}_M^i\gamma_2|_\infty \le |\gamma_1 - \gamma_2|_\infty, \; i = 1,2$$

where $\gamma_1, \gamma_2 \in C[0,\ell]$ and consequently that hypothesis H_1 of Theorem 3.1 is satisfied. Also, the approximation result (see [36])

$$|\mathcal{I}_M^i\gamma - \gamma|_\infty \le \omega(\gamma, 1/M_i)$$

where $\omega(\gamma,\delta)$ is the usual modulus of continuity of $\gamma \in C[0,\ell]$ with respect to δ, together with the assumption that Q is a compact subset of $\mathcal{Q} = C[0,\ell] \times C[0,\ell]$ and the Arzela-Ascoli theorem yield that hypothesis H_2 is satisfied as well.

Next we define a state approximation and verify that hypothesis H_3 holds. As above, given $N = (N_1, N_2) \in Z_+ \times Z_+$, we define the uniform partitions Δ_i^N of the interval $[0,\ell]$ determined by the meshes $\{0, \ell/N_i, 2\ell/N_i, \ldots, \ell\}$, $i = 1,2$. We may then choose either

$$W^{N_1} = Sp(1, \Delta_1^N)$$

or

$$W^{N_1} = Sp(2,\Delta_1^N).$$

In the first case, once again the "hat" functions may be chosen as a basis with $\dim W^{N_1} = K_1^N = N_1 + 1$. In the second, the standard cubic B-splines (see [33]), $\{B_j^{N_1}\}_{j=-1}^{N_1+1}$, corresponding to the partition Δ_1^N form an appropriate finite element basis with

$\dim W^{N_1} = N_1 + 3$. In either case, approximation results for interpolatory splines can be used to obtain

$$|P_1^N\theta - \theta|_0 \to 0 \text{ as } N_1 \to \infty \tag{4.1}$$

for $\theta \in L_2(0,\ell)$.

We set

$$V^{N_2} = \{(\chi(\ell), D\chi(\ell), \chi) \in H : \chi \in Sp(2,\Delta_2^N), \chi(0) = D\chi(0) = 0\}.$$

Then $V^{N_2} \subset V$ and defining

$$\beta_i^{N_2} = B_0^{N_2} - 2B_1^{N_2} - 2B_{-1}^{N_2},$$

$$\beta_i^{N_2} = B_i^{N_2}, \; i = 2,3,\ldots,N_2+1$$

and

$$\hat{\beta}_i^{N_2} = (\beta_i^{N_2}(\ell), D\beta_i^{N_2}(\ell), \beta_i^{N_2}), \quad i = 1,2,\ldots,N_2+1,$$

the collection $\{\hat{\beta}_i^{N_2}\}_{i=1}^{N_2+1}$ forms a basis for V^{N_2} with $\dim V^{N_2} = K_2^N = N_2 + 1$. With V^{N_2} as defined above, it is not difficult to show (using arguments similar to those in [31])

$$|P_2^N(\eta,\xi,\chi) - (\eta,\xi,\chi)|_H \to 0 \quad \text{as } N_2 \to \infty \tag{4.2}$$

for $(\eta,\xi,\chi) \, \varepsilon \, H$ and

$$(4.3) \qquad |LP_2^N\hat{\chi} - L\hat{\chi}|_H \to 0 \qquad \text{as } N_2 \to \infty$$

for $\hat{\chi} \, \varepsilon \, V$.

So as to avoid obscuring the essential features of our argument with technical details, we verify hypothesis H_3 for the spline-based scheme described above in the case $\sigma \equiv 0$. The term which results from axial loading is a bounded perturbation and does not involve the unknown parameters. Showing that the desired convergence continues to hold in the presence of a non-zero axially directed acceleration requires only a routine extension of the proof which we give below (see [14]).

Suppose that $\{q^N\}$ is a sequence in Q with $q^N \to \bar{q} \, \varepsilon \, Q$ as $|N| \to \infty$. Let $z^N = (w^N, \hat{v}^N)$ denote the solution to (3.2), (3.3) with $q = q^N$ and let $z = (w,\hat{v})$ denote the solution to (2.18), (2.19) corresponding to $\bar{q}$. We shall require the assumption that z is a strong solution.

In the estimates which follow, we simplify our notation by referring to the inner products (norms) $<\cdot,\cdot>_{q^N}$ $(|\cdot|_{q^N})$ and $<\cdot,\cdot>_{\bar{q}}$ $(|\cdot|_{\bar{q}})$ on $\mathcal{H}$ by $<\cdot,\cdot>_N$ $(|\cdot|_N)$ and $<\cdot,\cdot>$ $(|\cdot|)$ respectively. Also note that with $\sigma = 0$, we have $a(t;q)(\cdot,\cdot) = a(q)(\cdot,\cdot)$.

Since

$$\|z^N - z\| \leq \|z^N - \mathcal{P}^N z\| + \|(\mathcal{P}^N - I)z\|$$

where $\|\cdot\|$ denotes the usual (unweighted) product norm on $\mathcal{H} = L_2(0,\ell) \times H$, (3.1), (4.1) and (4.2) imply that we need only to consider the term $\|z^N - \mathcal{P}^N z\|$. Letting $y^N(t) = z^N(t) - \mathcal{P}^N z(t)$, using (2.18), (2.19), (3.2), (3.3) and the fact that $V^N \subset V$ we find

$$\begin{aligned} <y_t^N, v^N>_N &= <(z - \mathcal{P}^N z)_t, v^N>_N + <z_t, v^N> - <z_t, v^N>_N \\ &+ a(q^N)(y^N,v^N) - a(q^N)(z - \mathcal{P}^N z, v^N) + a(q^N)(z,v^N) - a(\bar{q})(z,v^N) \\ &+ <\mathcal{F},v^N>_N - <\mathcal{F},v^N> \qquad v^N \, \varepsilon \, \mathcal{V}^N, \; 0 < t \leq T \end{aligned}$$

$$(4.4) \qquad y^N(0) = 0.$$

Choosing $v^N = y^N \varepsilon \mathcal{V}^N$, we obtain

$$\frac{1}{2}\frac{d}{dt}\,|y^N|_N^2 + \left|\sqrt{q_2^N}\, L(\hat{v}^N - P_2^N \hat{v})\right|_0^2$$

$$= \langle (z - \mathcal{P}^N z)_t\,,\, y^N\rangle_N + \langle (\bar{q}_1 - q_1^N) w_t\,,\, w^N - P_1^N w\rangle_0$$

$$- \langle q_1^N L(\hat{v} - P_2^N \hat{v}),\, w^N - P_1^N w\rangle_0 + \langle q_1^N (w - P_1^N w),\, L(\hat{v}^N - P_2^N \hat{v})\rangle_0 -$$

$$\langle q_2^N L(\hat{v} - P_2^N \hat{v}),\, L(\hat{v}^N - P_2^N \hat{v})\rangle_0 + \langle (q_1^N - \bar{q}_1) L\hat{v},\, w^N - P_1^N w\rangle_0 -$$

$$\langle (q_1^N - \bar{q}_1) w,\, L(\hat{v}^N - P_2^N \hat{v})\rangle_0 - \langle (q_2^N - \bar{q}_2) L\hat{v}, L(\hat{v}^N - P_2^N \hat{v})\rangle_0 .$$

Recalling that Q is a compact subset of $\mathcal{Q}$ and that for $q = (q_1, q_2) \varepsilon Q$, $EI = q_1$ and $C_D I = q_2$ are assumed to satisfy assumption A_1 of Section 2, we find

$$\frac{d}{dt}\|y^N\|^2 + |L(\hat{v}^N - P_2^N \hat{v})|_0^2 \le \kappa_0 \{\|(I - \mathcal{P}^N) z_t\|^2$$

$$+ \|y^N\|^2 + |\bar{q}_1 - q_1^N|_\infty^2\, |w_t|_0^2 + |w^N - P_1^N w|_0^2 +$$

$$|L(I - P_2^N)\hat{v}|_0^2 + |w^N - P_1^N w|_0^2 + \frac{1}{4\varepsilon}|(I - P_1^N) w|_0^2$$

$$+ \varepsilon\, |L(\hat{v}^N - P_2^N \hat{v})|_0^2 + \frac{1}{4\varepsilon}\, |L(I - P_2^N)\hat{v}|_0^2 +$$

$$\varepsilon |L(\hat{v}^N - P_2^N \hat{v})|_0^2 + |q_1^N - \bar{q}_1|_\infty^2 |L\hat{v}|_0^2 +$$

$$|w^N - P_1^N w|_0^2 + \frac{1}{4\varepsilon}\, |q_1^N - \bar{q}_1|_\infty^2\, |w|_0^2 +$$

$$\varepsilon |L(\hat{v}^N - P_2^N \hat{v})|_0^2 + \frac{1}{4\varepsilon}|q_2^N - \bar{q}_2|_\infty^2 |L\hat{v}|_0^2$$

$$+ \varepsilon |L(\hat{v}^N - P_2^N \hat{v})|_0^2\}$$

where κ_0 is a positive constant and ε is an arbitrary positive constant. Gathering up like terms and choosing $\varepsilon < \frac{1}{4}$ we obtain

$$\frac{d}{dt}\|y^N\|^2 \le \kappa_1\{\|(I - \mathcal{P}^N)z_t\|^2 + |q_1^N - \bar{q}_1|_\infty^2\, |w_t|_0^2 + |L(I - P_2^N)\hat{v}|_0^2 +$$

$$|(I - P_1^N)w|_0^2 + |q_1^N - \bar{q}_1|_\infty^2\, |L\hat{v}|_0^2 + |q_1^N - \bar{q}_1|_\infty^2 |w|_0^2 + |q_2^N - \bar{q}_2|_\infty^2 |L\hat{v}|_0^2\} + \kappa_2\|y^N\|^2$$

where κ_1 and κ_2 are positive constants. Integrating both sides of the above inequality from 0 to t and recalling (4.4) we obtain

$$\text{(4.5)} \qquad \|y^N(t)\|^2 \le \delta + \kappa_2 \int_0^t \|y^N(s)\|^2 ds$$

where

$$\delta = \kappa_1 \int_0^T \{\|(I - \mathcal{P}^N)z_s(s)\|^2 + |q_1^N - \bar{q}_1|_\infty^2\, |w_s(s)|_0^2 + |L(I - P_2^N)\hat{v}(s)|_0^2 + |(I - P_1^N)w(s)|_0^2 +$$

$$|q_1^N - \bar{q}_1|_\infty^2 |L\hat{v}(s)|_0^2 + |q_1^N - \bar{q}_1|_\infty^2 |w(s)|_0^2 + |q_2^N - \bar{q}_2|_\infty^2 |L\hat{v}(s)|_0^2\} ds\ .$$

Since $q^N \to \bar{q}$ as $|N| \to \infty$ and $z = (w,\hat{v})^T$ was assumed to be a strong solution, (4.1), (4.2), (4.3) and (4.5) together with an application of the Gronwall inequality yield the desired result, $\|y^N(t)\| \to 0$.

A close inspection of the estimates above reveals that they depend, to a large extent, on the presence of the viscous damping term $\langle C_D I\, L\hat{\chi}_1, L\hat{\chi}_2\rangle_0$ in the bilinear form $a(t)(\cdot,\cdot)$ given in (2.17). That is, we require that $q_2 \ge \alpha > 0$ for some $\alpha > 0$ for all $q = (q_1, q_2) \in Q$. In the absence of the Voigt-Kelvin damping we can still argue the convergence of z^N to z; however, we must assume that Q is H^2-compact. If one is to enforce the compactness constraint on Q when solving the finite dimensional optimization problems (a desirable implementation feature in many cases - see [10],[11]), this stronger assumption becomes especially unappealing. On the other hand, by employing a somewhat different (but closely related) factorization of the stiffness operator $\mathcal{K}_0$ than the one which was used here (one which is formally equivalent to rewriting the initial-boundary value problem (2.1) - (2.5) as a first order system in the states EI D^2u and u_t as opposed to D^2u and u_t) hypothesis H_3 of Theorem 3.1 can be verified for the resulting spline-based scheme under the present assumptions on Q. Unfortunately this scheme is also difficult to implement and from a numerical standpoint, has not performed as satisfactorily as the one based on the formulation given in this paper. The present scheme performed well whether or not damping was present in the equation and hence the assumption that $C_D I \geq \alpha > 0$ may be an artifact of our proof of convergence (see Example 5.3 below).

5. Numerical Findings

We present and discuss some of the results which we obtained from our numerical studies of the scheme that was described in Section 4. All codes were written in FORTRAN, and tested and run on the IBM 3081 at either Brown University or the University of Southern California. The same codes were, with only minor modification, run on the Cray 1-S at Boeing Computer Services in Seattle with support made available to us through the National Science Foundation's Super Computer Initiative program. Examples were benchmarked so that the potential benefits of vectorization to our research program could be accurately and effectively assessed. Our findings are described below. This information will become especially important to us when we begin to consider the extension of our general approach to inverse problems involving the vibration of two dimensional structures, such as flexible plates or platforms, or vibrations of structures in which nonlinearities play a significant role. The finite dimensional optimization problems (ID_M^N) were solved using the IMSL routine ZXSSQ, an implementation of the iterative Levenberg-Marquardt quasi-Newton algorithm. The finite dimensional initial value problems (3.5), (3.6) were solved in each iteration of the minimization procedure (for the evaluation of the least-squares performance index $\mathcal{J}$ and its gradient with respect to the parameters) using Gear's method for stiff systems (IMSL routine DGEAR).

Our codes were written to take full advantage of the banded structure of the generalized mass, stiffness and damping matrices afforded by the use of polynomial B-spline elements. All necessary inner products were computed using a two point composite Gauss-Legendre quadrature scheme.

All of the examples presented here involve fits based upon displacement measurements obtained through simulation. "True" values (which, in the examples below will be denoted with an asterisk, for example EI^*, C_DI^*, etc.) for the unknown parameters were chosen. The resulting initial boundary value problem (2.1) - (2.5) was then solved using an independent integration scheme. (We used a seven element, quintic spline based Galerkin method applied directly to the second order system (2.6), (2.7)). This procedure produced sufficient noise in the data so that the use of a random noise generator was not required.

In addition to the test example numerical studies we report on here we have successfully used methods similar to those developed above with experimental data. These results are presented in detail in [9].

In the examples which follow we took the axial loading to be induced by an acceleration of the base or root of the structure in the positive x-direction. In this case we have (see [34])

$$\sigma(t,x) = -a_0(t)\,\{m + \int_0^{\ell} \rho(y)dy\}$$

where m is the mass of the tip body, ρ is the linear mass density of the beam and a_0 is the time dependent base acceleration.

In Examples 5.1 thru 5.4 below we took $\ell = 1$, $\rho(x) = 3 - x$ for $0 \le x \le 1$, $f(t,x) = e^x \sin 2\pi t$, $g(t) = 2e^{-t}$, $h(t) = e^{-2t}$, $a_0(t) = 1$ for $0 \le t \le 1.5$, $a_0(t) = 0$ for $t > 1.5$, $m = 1.5$, $c = .1$ and $J = .52$ and considered the estimation of the flexural stiffness coefficient EI and/or the viscoelastic damping coefficient C_DI only. In Example 5.1, 5.2 and 5.4, the fits we describe are based upon observations at times $t_i = .2i$, $i = 1,2,\ldots,5$ at locations $x_j = .5, .75$ and 1. In Example 5.3 observations at times $t_i = .5i$, $i = 1,2,\ldots,10$ at locations $x_j = .75$ and 1 were used. In all of the examples we discuss here the space W^{N_1} was generated by cubic splines (i.e. as $Sp\,(2,\Delta_1^N)$) with $N_1 = N_2 = N$. This corresponds to the approximation of the first and second components of z with respectively $N + 3$ and $N + 1$ piecewise cubic C^2 elements.

The compactness constraints on the spaces Q_M were not enforced when the finite dimensional optimization problems (ID_M^N) were solved. When M_1 and M_2 became large, the inherent ill-posedness of the inverse problem became apparent as the performance of our schemes deteriorated. There is evidence strongly suggesting that this situation can be remedied either by imposing the compactness constraints on the admissible parameter space and then solving the minimization problem using a constrained optimization procedure (see [10], [11]) or by regularizing the least squares performance index (see [24], [25]). We intend to direct our attention to these ideas in the near future.

Example 5.1

In this example we consider the simultaneous estimation of a constant flexural stiffness coefficient, $EI^* = .15$, and a damping coefficient given by $C_DI^*(x) = \gamma(1.5 - \tanh(3x - 1.5))$, $x \varepsilon [0,1]$, with $\gamma = .01$. With $N = 4$, $M_1 = 1$ and $M_2 = 3$ and taking start up values (for the least squares minimization algorithm) $EI^0 = .1$ and $C_DI^0(x) = .015$, $x \varepsilon [0,1]$ we obtained the results shown in Figure 5.1 below. This particular run required approximately 30 seconds of CPU time on the IBM 3081

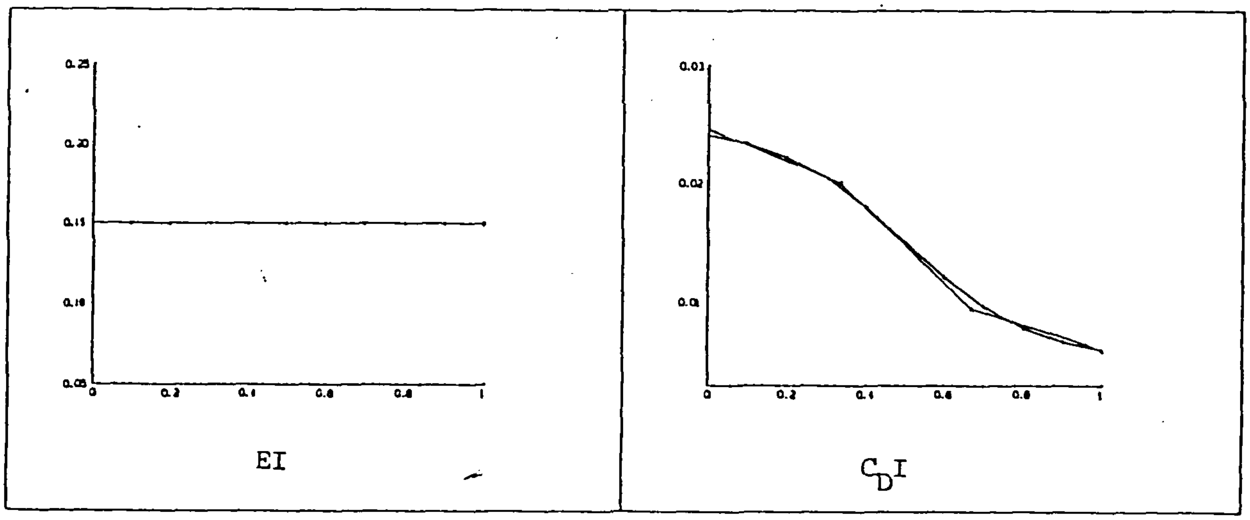

Figure 5.1

We observed that how well the scheme performed depended upon the magnitude of the scaling factor γ. As γ was decreased, so too did the "sensitivity" of the least squres performance index to the damping coefficient. Results similar to those shown in the figures above were obtained with $\gamma = .005$. With $\gamma = .001$, on the other hand, we were unable to simultaneously identify both of the unknown parameters. However, again with $\gamma = .001$, but this time fixing EI at the true value, we were able to identify $C_D I$ alone.

When we replaced the constant EI^* with the linear function $EI^*(x) = 1 - \frac{1}{2}x$ and took $\gamma = 1$, the performance of the scheme, from a qualitative point of view, remained unchanged.

Example 5.2

We again consider the simultaneous estimation of the stiffness and damping coefficients. We again set $EI^* = .15$ but this time choose $C_D I^*(x) = .01\ (1.5 - \tanh\ (20x - 10))$, $x \in [0,1]$. The identification of this steeper hyperbolic tangent function has, in past test examples, proven to be a somewhat stiffer challange for our methods (see [5],[6]). With $N = 4$, $M_1 = 1$, $M_2 = 3$, $EI^0 = .1$ and $C_D I^0(x) = .015$ for $0 \le x \le 1$, we obtained the estimates which are plotted along with the true parameters in Figure 5.2.

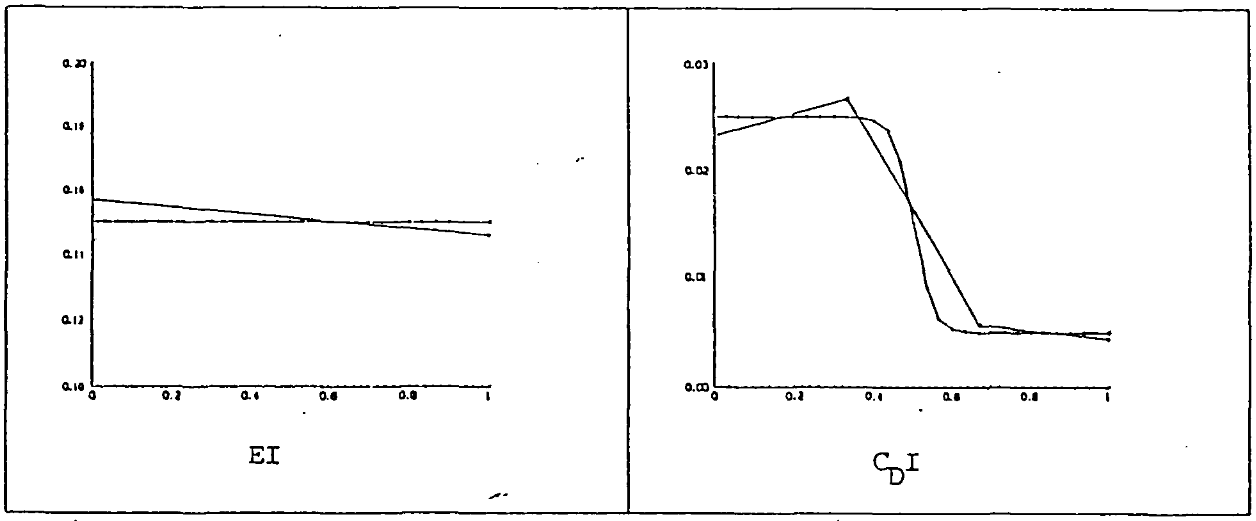

Figure 5.2

Also, although the theory was not explicitly treated here, we note that elements other than linear splines can be used to discretize the admissible parameter space. Our investigations have included numerical studies with 0-order splines (i.e. piecewise constant functions) and cubic spline elements. Using two linear elements to approximate EI (i.e. $M_1 = 1$) and nine cubic elements to discretize $C_D I$ we obtained the estimates,shown in Figure 5.3. We have obtained an acceptable estimate for $C_D I$ with as few as six cubic elements.

In the tests reported on for the present example, residuals were typically in the range 10^{-6} to 10^{-8} with CPU times from 25 to 40 seconds.

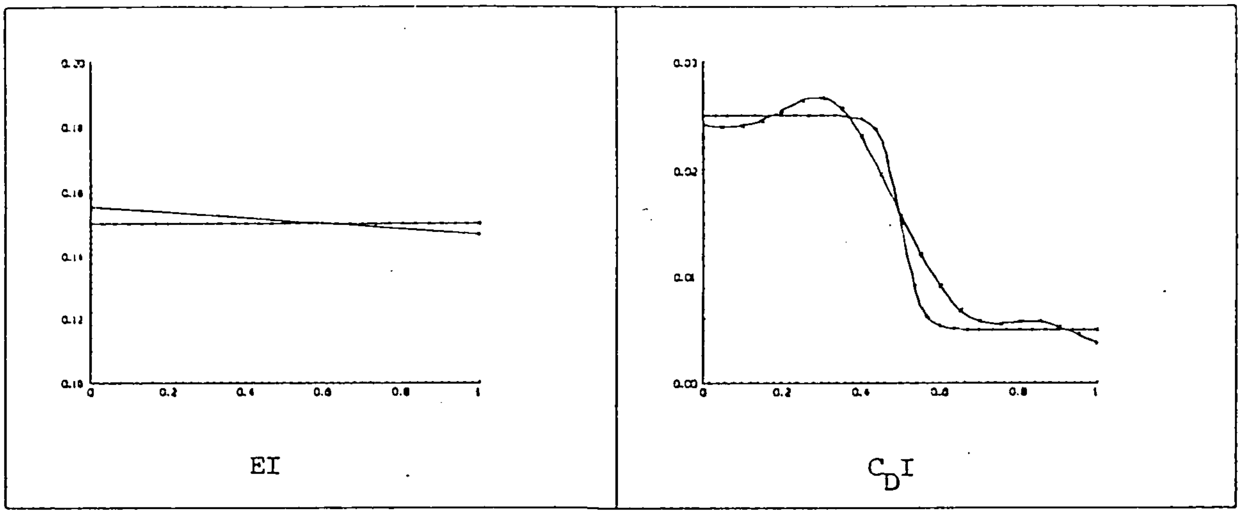

Figure 5.3

<u>Example 5.3</u>

In this example we identify only the spatially varying flexural stiffness coefficient $EI^*(x) = 1.5 - \tanh(3x - 1.5)$, $x \in [0,1]$, in a model with no viscoelastic damping ($C_D I = 0$). In Section 4 we remarked that our convergence arguments required either the presence of viscoelastic damping in the model or that the admissible parameter set Q be compact in the stronger H^2 topology. The results shown in the figure below suggest that this is only an artifact of our proof and not a fundamental requirement for the convergence of our approximation (i.e. the absence of damping does not appear to effect the overall performance of our scheme).

Taking $N = 4$ and M_1 equal to 1 thru 8 we produced the results shown in the series of graphs in Figure 5.4. The initial estimate or start-up value for EI^* was taken to be the constant function $EI^0(x) = 1$ for $0 \leq x \leq 1$.

Recalling our earlier remarks, the oscillations which appear in the graphs corresponding to

M_1 = 6, 7 and 8 due to the inherent ill posedness of the estimation problem are not unexpected. In fact, as M_1 or $M_2 \to \infty$, the appearance of the undesirable oscillations in our final estimates occurred in virtually every test we ran. As we have noted earlier however, preliminary findings in related studies [10] and [11] regarding the enforcing of the compactness constraints and the subsequent use of constrained optimization techniques to solve the approximating finite dimensional identification problems suggest that this difficulty can be overcome. Our investigations in these directions are continuing.

In addition, the series of tests corresponding to the graphs in Figure 5.4 were benchmarked on the IBM 3081 and the Cray 1-S. The same estimates were obtained on both machines. However, we were able to achieve a speed-up factor *f*, of 7 - 10 on the vector machine. The CPU times are reported in Table 5.1. In comparing the CPU times on the 3081 for this example with the times reported for the previous examples it is important to note that the results here were based upon observations taken over the longer time interval, [0,5], versus the interval [0,1] for examples 5.1 and 5.2.

Example 5.4

In Figure 5.5 below we plot the final estimates obtained when we attempted to use our scheme to simultaneously identify the spatially varying flexural stiffness coefficient $EI^*(x) = .5 + 4x(1 - x)$, $x \in [0,1]$, and viscoelestic damping coefficient, $C_DI^*(x) = .1\ (1.5 - \tanh(3x - 1.5))$, $x \in [0,1]$. The start-up values for the iterative least squares minimization routine were taken to be the constant functions $EI^0(x) = 1$ and $C_DI^0(x) = .15$ for $0 \leq x \leq 1$. The graphs in the figure were obtained with N = 4 and a linear spline discretization of the admissible parameter set Q with $M_1 = M_2 = 3$. In all of our tests with this example the minimum sum of the squares of the residuals was in the range 10^{-7} - 10^{-8} with the optimization typically requiring 50 - 70 seconds of CPU time.

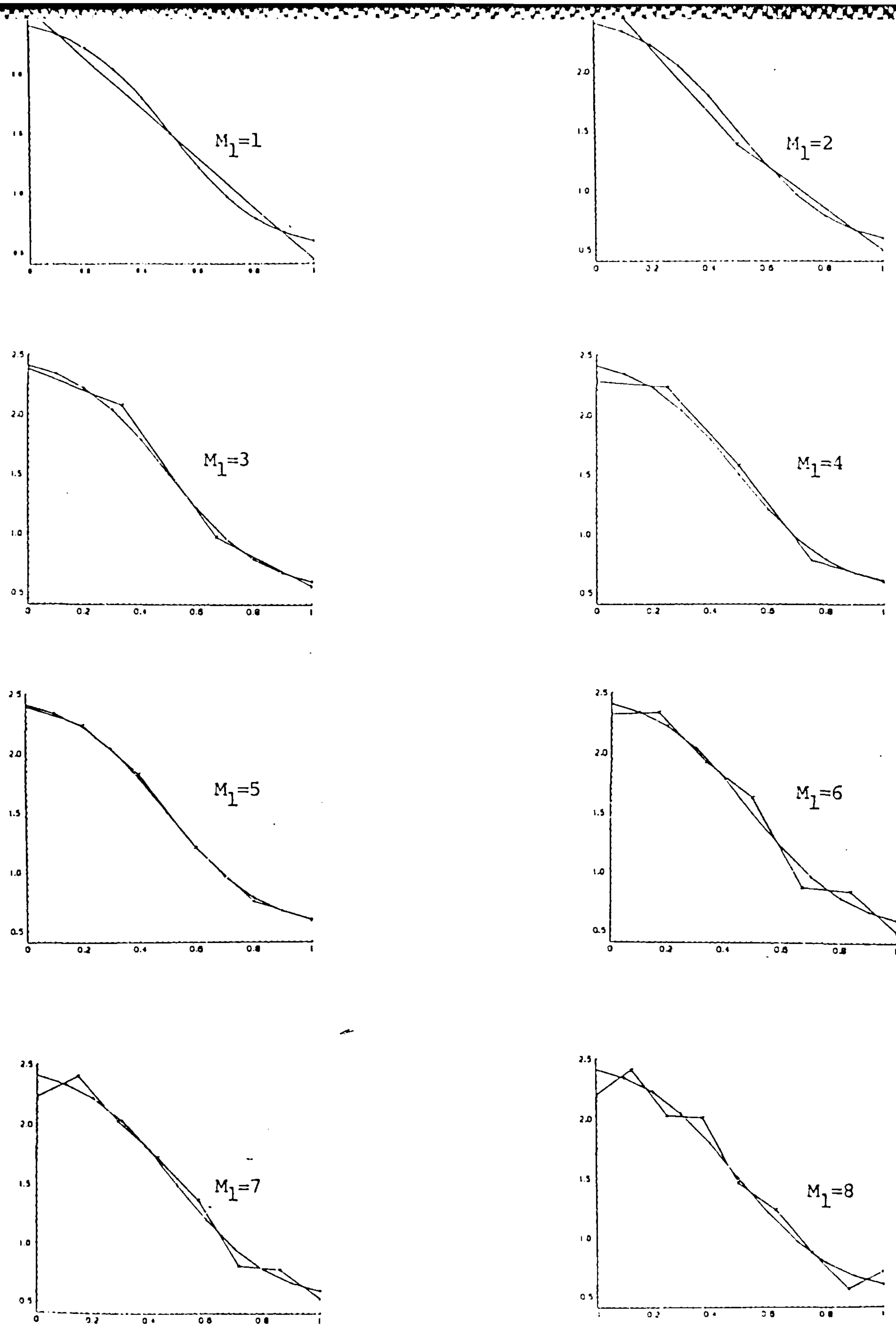

Figure 5.4

M_1	IBM 3081 (CPU sec.)	CRAY 1-S (CPU sec.)	f
1	110	12.5	8.8
2	164.1	20.8	7.9
3	207	23	9
4	249	32	7.8
5	245.5	36	6.8
6	404.4	41.6	9.7
7	346.5	45.6	7.6
8	437.9	49.1	8.9

Table 5.1

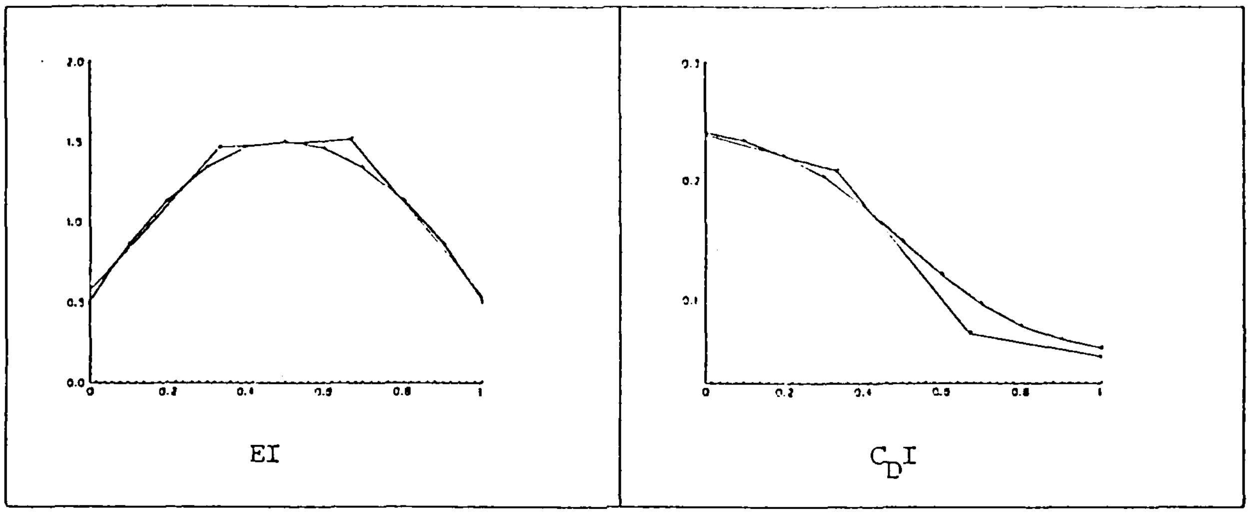

Figure 5.5

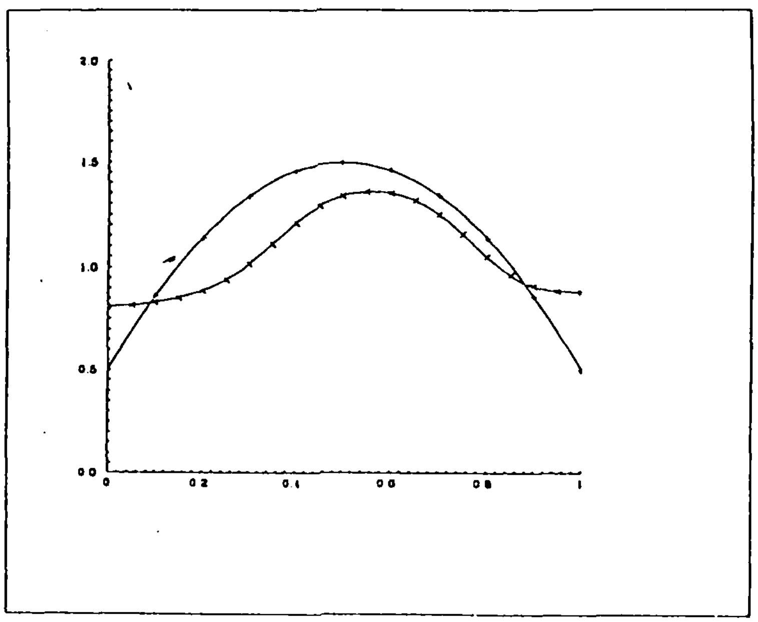

Figure 5.6

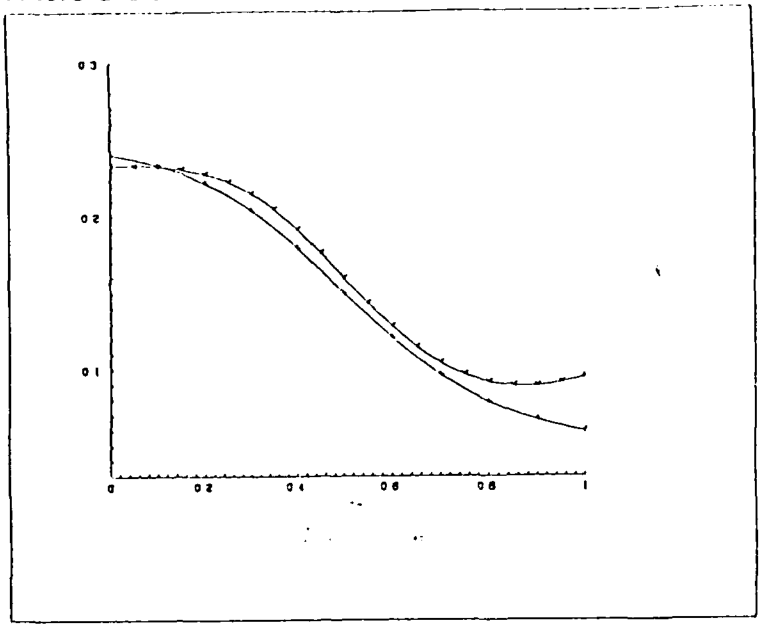

Figure 5.7

For this example we also tried a cubic-spline based discretization for Q. We considered all possible combinations, linear splines for EI^* together with cubic splines for C_DI^*, cubic splines for EI^* together with linear splines for C_DI^*, etc. Although small values for the sum of the squares of the residuals were obtained in each instance, our by far best approximation to the true parameters is the one shown in Figure 5.5 which corresponds to a linear spline based discretization for both components of the admissible parameter set.

Holding C_DI fixed at the true value and using cubic splines to identify EI and then holding EI fixed at the true value and using cubic splines to identify C_DI^* we were able to obtain the estimates plotted in Figures 5.6 and 5.7 respectively. The estimate for EI^* graphed in Figure 5.6 was obtained with 10 cubic elements while the estimate for C_DI^* in Figure 5.7 is a linear combination of 6 cubic elements. An inspection of these figures reveals that while the approximations obtained are at least marginally acceptable, it is also not surprising that our scheme had some difficulty when we attemped to identify both parameters simultaneously with a cubic spline-based discretization for either one or both components of Q.

For this example we also looked at the robustness of our iterative scheme with respect to the initial values chosen (i.e., EI^0 and C_DI^0). In Figure 5.8 we plot those points in the C_DI^0 - EI^0

plane which correspond to the start up values we tried. The point marked with " * " corresponds to the startup values which produced the approximations shown in Figure 5.5. The points marked with " × " correspond to start-up values which led to essentially the same extimates as those shown in the figures. The points marked with an " ⊕ " correspond to start-up values for which the scheme did not converge. The region whose boundary is denoted with dashed lines corresponds to a "convergence envelope" for the vector valued function ($C_D I^*$, EI^*). An analogous study was carried out for Example 5.2, for which similar robustness results were obtained.

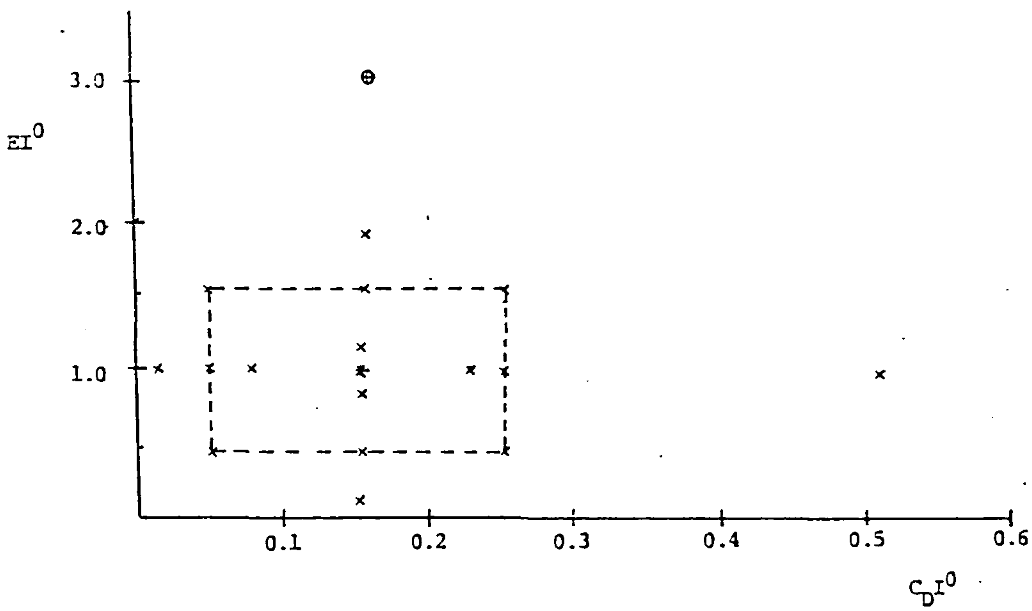

Figure 5.8

Finally we offer several summary comments on some of our other numerical findings. In virtually all examples we tried, we found that the estimates yielded by the scheme which we develop here based on state space coordinates (D^2u, u_t) and the ones yielded by the scheme based on a state space formulation in coordinates (u, u_t) described in [5] and [6] were comparable. Although in any given example one scheme or the other may produce a somewhat better approximation to the true parameters, we found it impossible to designate or identify a clear favorite among the two methods.

We also ran a series of tests in which we varied the boundary conditions at the free end of the beam. That is, in addition to the tip body end condition we considered a beam which is clamped at one end and free at the other with either a point mass ($c = J = 0$) or no mass ($m = c = J = 0$) rigidly attached at the tip. We also studied the effect that the presence or absence of external forces and/or moments at the tip of the beam (i.e. g and h) has on the performance of our scheme. Based upon these tests, we found it difficult to make definitive statements regarding "best" experimental procedures for identification of structural parameters with our schemes. However, we are able to offer several observations. For example, with a point mass at the tip, the schemes performance was enhanced when an external moment was applied at the tip (i.e. $h \neq 0$). On the other hand, the presence of an externally applied force in the transverse direction (i.e. $g \neq 0$) did not appear to have any effect at all. Also, with no mass at the tip, the scheme was most effective when $g = h = 0$. In general we found the scheme to be most dependable with tip body end conditions.

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References

[1] M.J. Balas, Trends in large space structure control theory: fondest hopes, wildest dreams, IEEE Trans. Auto. Control., Vol. AC-27, (June, 1982), pp. 522-535.

[2] H.T. Banks and J.M. Crowley, Parameter estimation for distributed systems arising in elasticity, Proc. Symposium on Engineering Sciences and Mechanics, (National Cheng Kung University, December 28-31, 1981) pp. 158-177; LCDS Report No. 81-24, Brown University, November 1981.

[3] H.T. Banks and J.M. Crowley, Parameter esitmation in Timoshenko beam models, LCDS Report No. 82-14, Brown University, June 1982; J. Astronaut. Sci., 31, (1983), pp. 381-397.

[4] H.T. Banks and J.M. Crowley, Parameter identification in continuum models, LCDS Report No. 83-1, Brown University, March 1983; J. Astronautical Science, 33, (1985), pp. 85-94.

[5] H.T. Banks and J.M. Crowley, Estimation of material parameters in elastic systems, LCDS Report No. 84-20, Brown University, June 1984.

[6] H.T. Banks, J.M. Crowley and I.G. Rosen, Methods for the identification of material parameters in distributed models for flexible structures, ICASE Report No. 84-66, Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, VA, and Mat. Applicada e Computacional, 5 (1986).

[7] H.T. Banks and P.L. Daniel (Lamm), Estimation of variable coefficients in parabolic distributed systems, LCDS Report No. 82-22, Brown University, September 1982; IEEE Trans. Auto. Control AC-30, (1985), pp. 386-398.

[8] H.T. Banks, P.L. Daniel and E.S. Armstrong, A spline-based parameter and state estimation technique for static models of elastic surfaces, ICASE Report No. 83-25, NASA Langley Research Center, Hampton, VA, June 1983.

[9] H.T. Banks, S.S. Gates, I.G. Rosen and Y. Wang, The identification of a distributed parameter model for a flexible structure, ICASE Report No. 86- , Institute for Computer Applications in Science and Engineering, NASA Langley Reseach Center, Hampton, VA, August, 1986; SIAM J. Control and Opt., Submitted.

[10] H.T. Banks and D.W. Iles, A Comparison of stability and convergence properties of techniques for inverse problems, LCDS Report No. 86-3. Brown University, January, 1986.

[11] H.T. Banks and D.W. Iles, On compactness of admissible parameter sets: convergence and stability in inverse problems for distributed parameter systems, ICASE Report No. 86-38, Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, VA, June, 1986.

[12] H.T. Banks, P. Kareiva and P.K. Lamm, Estimation techniques for transport equations, in Mathematics in Biology and Medicine (Proceedings, Bari 1983), V. Capasso, et. al., eds., LN in Biomath Vol. 57, Springer, New York, 1985, pp. 428-438.

[13] H.T. Banks and K.A. Murphy, Estimation of coefficients and boundary parameters in hyperbolic systems, LCDS Report No. 84-5, Brown University, February, 1984; SIAM J. on Control and Optimization, 24 (1986), pp. 926-950.

14] H.T. Banks and I.G. Rosen, A Galerkin method for the estimation of parameters in hybrid systems governing the vibration of flexible beams with tip bodies, CSDL-R-1724, Charles Stark Draper Laboratory, Cambridge, MA, June 1984.

[15] H.T. Banks and I.G. Rosen, Approximation techniques for estimation and feedback control for distributed models of large flexible structures, Proc. NASA/ACC Workshop on Identification and Control of Flexible Space Structures (June 4-6, 1984, San Diego); ICASE Report No. 84-26, NASA Langley Research Center, Hampton, VA., June, 1984.

[16] H.T. Banks and I.G. Rosen, Numerical schemes for the estimation of functional parameters in distributed models for mixing mechanisms in lake and sea sediment cores, LCDS Report No. 85-27, Brown University , October, 1985; Inverse Problems, to appear.

[17] Burns, J.A. and E.M. Cliff, An approximation technique for the control and identification of hybrid systems, Proceedings of the Third VPI&SU/AIAA Symposium, on the Dynamics and Control of Large Flexible Spacecraft, Blacksburg, VA, June 15-17, 1981.

[18] C.C. Chen and C.T. Sun, Transient analysis of large frame structures by simple models, Proc. Symposium on Engineering Science and Mechanics, (National Cheng Kung Univ., Dec. 28-31, 1981), pp. 753-775.

[19] R.W. Clough and J. Penzien, Dynamics of Structures, McGraw-Hill, New York, 1975.

[20] A. Friedman, Partial Differential Equations of Parabolic Type, Prentice-Hall, Englewood Cliffs, 1964.

[21] R. Gran and M. Rossi, A survey of the large structures control problem, Presented at IEEE Decision and Control Conference, Ft. Lauderdale, FL, 1979.

[22] P.C. Hughes, Passive dissipation of engergy of large space structures, J. Guidance Contr., Vol. 3, (1980), pp. 380-382.

[23] J.N. Juang and C.T. Sun, System identification of large flexible structures by using simple continuum models, J. Astronautical Sciences, 31, (1983), pp. 77-98.

[24] C. Kravaris and J.H. Seinfeld, Identification of parameters in distributed parameter systems by regularization, SIAM J. Control and Optimization, 23 (1985), pp. 217-241.

[25] Kravaris, C. and Seinfeld, J.H. "Identification of spatially-varying parameters in distributed parameter systems by discrete regularization," J. Mathematical Analysis and Appl, to appear.

[26] K. Kunisch and E. Graif, Parameter estimation for the Euler-Bernoulli beam, Inst. fur Math. Bericht 83-26, Techn. Universitat Graz, December 1983; Mat. Aplicada e Computacional 4 (1985), pp. 95-124.

[27] J.L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, Springer-Verlag, New York, 1971.

[28] A.K. Noor and C.M. Anderson, Analysis of beam-like lattices, Comp. Math. Appl. Mech. Engrg., 20 (1979), pp. 53-70.

[29] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.

[30] E.P. Popov, Introduction to Mechanics of Solids, Prentice-Hall, Englewood Cliffs, 1968.

[31] I.G. Rosen, A numerical scheme for the identification of hybrid systems describing the vibration of flexible beams with tip bodies, CSDL-P-1893, Charles Stark Draper Laboratory, 1984; J. Math. Anal. Appl., 116 (1986), pp. 262-288.

[32] I.G. Rosen, "Spline-based Rayleigh-Ritz methods for the approximation of the natural modes of vibration for flexible beams with tip bodies," Quarterly of Applied Mathematics, Vol. XLIV, No. 1, (1986), pp. 169 - 185.

[33] M.H. Schultz, Spline Analysis, Prentice-Hall, Englewood Cliffs, 1973.

[34] J. Storch and S. Gates, Transverse vibration and buckling of a cantilevered beam with tip body under axial acceleration, J. Sound and Vibration, 99 (1985), pp. 43-52.

[35] C.T. Sun, B.J. Kim, and J.L. Bagdanoff, On the derivation of equivalent simple models for beam and plate-like structures in dynamic analysis, Proceedings AIAA Specialists Conference, Atlanta, Georgia, April 6-8, 1981, pp. 523-532.

[36] B.K. Swartz and R.S. Varga, Error bounds for spline and L-spline interpolation, J. Approx. Theory, 6 (1972), pp 6-49.

[37] H. Tanabe, Equations of Evolution, Pitman, London, 1979.

THE IDENTIFICATION OF A DISTRIBUTED PARAMETER MODEL FOR A FLEXIBLE STRUCTURE

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ABSTRACT

We develop a computational method for the estimation of parameters in a distributed model for a flexible structure. The structure we consider (part of the "RPL experiment") consists of a cantilevered beam with a thruster and linear accelerometer at the free end. The thruster is fed by a pressurized hose whose horizontal motion effects the transverse vibration of the beam. We use the Euler-Bernoulli theory to model the vibration of the beam and treat the hose-thruster assembly as a lumped or point mass-dashpot-spring system at the tip. Using measurements of linear acceleration at the tip, we estimate the hose parameters (mass, stiffness, damping) and a Voigt-Kelvin viscoelastic structural damping parameter for the beam using a least squares fit to the data.

We consider spline based approximations to the hybrid (coupled ordinary and partial differential equations) system; theoretical convergence results and numerical studies with both simulation and actual experimental data obtained from the structure are presented and discussed.

1. Introduction

The difficulties involved in the design of practical and efficient control laws for large flexible spacecraft (e.g. the inherent infinite dimensionality of the system, a large number of closely spaced modal frequencies, high flexibility, light damping, a fuel-limited, hostile, highly variable environment, etc.) have stimulated research into the development of system identification and parameter estimation procedures which will yield high fidelity models. A particular area of interest involves schemes for the estimation of material parameters describing, for example, mass, inertia, stiffness or damping properties in distributed models for the vibration of viscoelastic systems-specifically, mechanical beams, plates and the like. In addition, since the resulting inverse problems are often infinite dimensional, substantial attention has been focused on approximation; see, for example, [1], [2], [3], [4], [8] and [12]. In these treatments, the parameter estimation problem is formulated as a least squares fit to measurements of either displacement or velocity. Although significant gains have been made in the development of instrumentation to measure displacement and velocity (e.g. laser technology, etc.), one of the least expensive, most reliable and most commonly used sensors is the linear accelerometer. While in principle it is possible to integrate acceleration measurements once or twice to obtain respectively velocity or displacement data, in practice this task can pose significant challenges. For example, integration of the signal could result in the amplification of low frequency measurement noise or dynamic effects which have not been included in the underlying model. In light of this, we have undertaken to show here, both theoretically and computationally, that a scheme in the spirit of those developed in the previously cited references can also be effectively used with acceleration measurements. In particular we note, this involves the nontrivial extension of the familiar variational arguments which are used to demonstrate the convergence of the finite element state approximations upon which the identification schemes are based. Indeed, it must be shown that in addition to the convergence of the displacement and velocity, the convergence of acceleration can be obtained as well.

The other primary motivation for the present effort is that while these methods have been extensively tested and evaluated with simulation data, they have never been tried with actual experimental data. We have tested our scheme with data obtained from an experimental structure which was designed and constructed at the Charles Stark Draper Laboratory in Cambridge, Massachusetts with funding provided by the United States Air Force Rocket Propulsion Laboratory (RPL). The RPL structure (as it will henceforth be referred to as) was designed to serve as a test bed for the implementation and evaluation of control algorithms for large angle slewing of spacecraft with flexible appendages. The structure was specifically designed to exhibit structural modes and damping characteristics representative of realistic large flexible space structures.

In Section 2 we describe the RPL structure (its geometry, instrumentation, etc.) and formulate an inverse problem involving a distributed system. In Section 3, we use the resulting infinite dimensional estimation problem to motivate the development of a finite dimensional, finite element based approximation scheme. We also discuss our theoretical convergence results. In Section 4 we present numerical findings.

We use standard notation throughout. For X a normed linear space, $L(X)$ denotes the space of bounded linear operators from X into X. For Ω an interval and $k = 0,1,2,\cdots$, $C^k(\Omega;X)$ denotes the space of functions from Ω into X which are k times continuously strongly differentiable on Ω. When $k = 0$ we shall simply write $C(\Omega;X)$. A function f from Ω into X will be said to belong to $L_2(\Omega;X)$ if $\int_\Omega |f(t)|_X^2\, dt < \infty$. For $k = 0,1,2,\cdots$, $H^k(\Omega;X)$ denotes the completion of $C^k(\Omega;X)$ with respect to the norm

$$|f|_k = \Big(\sum_{j=0}^{k} \int_\Omega |f^{(j)}(t)|_X^2\, dt\Big)^{1/2}.$$

If, in addition, X is a Hilbert space with inner product $\langle\cdot,\cdot\rangle_X$ then $H^k(\Omega;X)$is a Hilbert space with inner product

$$\langle f,g\rangle_k = \sum_{j=0}^{k} \int_\Omega \langle f^{(j)}(t),\ g^{(j)}(t)\rangle_X\, dt.$$

When X = R, we use the abbreviated notations $C^k(\Omega)$, $L_2(\Omega)$ and $H^k(\Omega)$. Note that $H^0(\Omega) = L_2(\Omega)$ and $\langle\cdot,\cdot\rangle_0$ is the standard inner product on $L_2(\Omega)$.

2. The Identification Problem

The RPL structure (see Figure 2.1 below) consists of four flexible appendages which are cantilevered at right angles to one another from a rigid central hub. The hub is mounted on an air bearing table thus permitting the near frictionless rotation of the structure about the vertical axis.

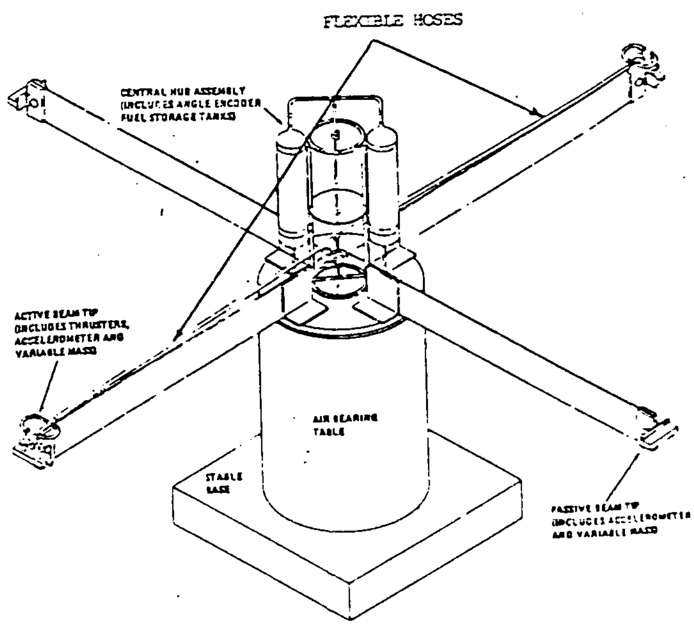

Figure 2.1

Two of the appendages (which are mounted to the hub 180° apart) are "active"; each has two nitrogen cold gas thrusters mounted in opposing directions at its tip. The remaining two appendages are "passive" with only counter-balancing masses affixed to their free ends. The presence of the tip masses on the passive arms serves to preserve the

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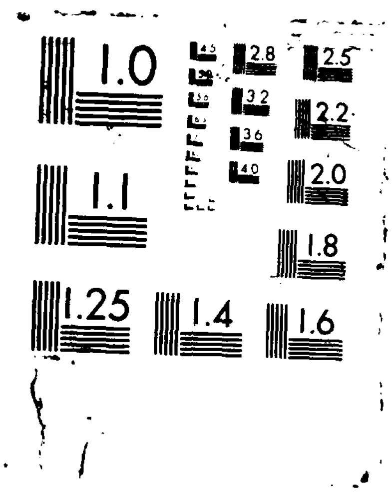
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overall symmetry of the structure. Nitrogen gas from tanks mounted on the central hub is supplied to the thrusters via two stainless steel mesh-wrapped high pressure hoses. The expulsion of propellant from the thruster nozzles is controlled by electro-mechanical or solenoidal valves. Each of the four appendages is equipped with a sensor in the form of a linear accelerometer attached at its tip. Data from the accelerometers is processed and recorded and control input signals to the thrusters are generated by a MINC 11/23 microcomputer. A detailed description of the structure's design specifications can be found in [6] and [15].

The problem which is of primary concern to us here involves the modeling of the effects of the nitrogen supply hoses on the transverse vibration of the active members. We consider therefore, the structure with the central hub immobilized and look only at the vibration of one of the active appendages and view it as a simple cantilevered beam (see Figure 2.2).

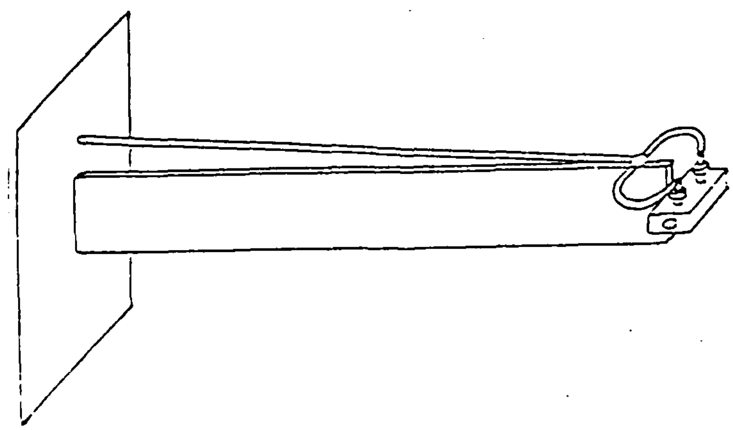

Figure 2.2

We treat the thruster assembly as a point mass that is rigidly attached to the beam at the tip and propose a model for the hose effects in the form of a proof mass which reacts against the tip mass. In effect, we consider the idealized, simplified structure depicted in Figure 2.3 below involving a single, cantilevered, flexible, uniform beam with a two-mass-dashpot-spring system affixed to its free end.

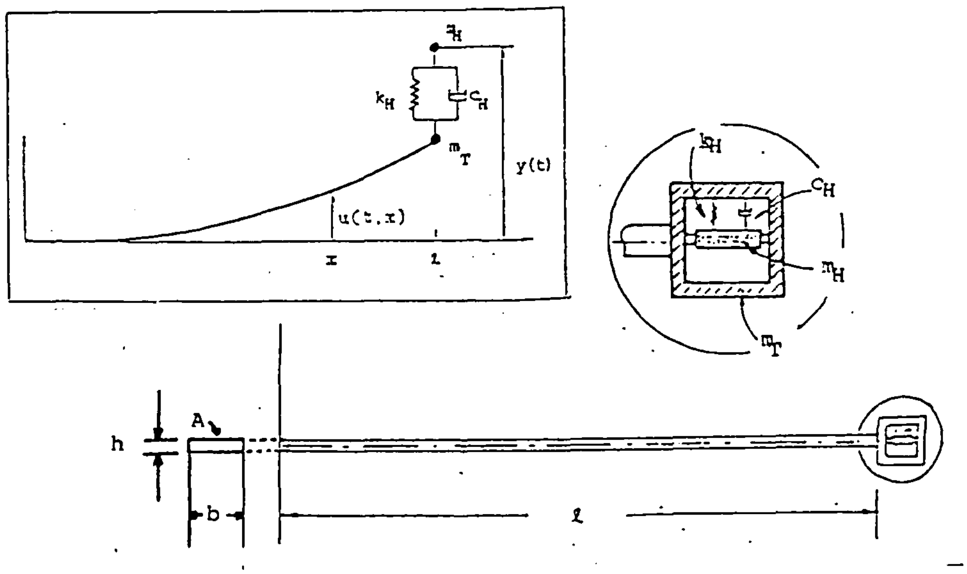

Figure 2.3

In formulating a mathematical model for the structure shown in Figure 2.3 above, we assume that the beam is of length ℓ with uniform rectangular cross section of height h and width b. We let u(t,x) and y(t) denote respectively the transverse displacement of the beam at position x along its span and the displacement of the proof or hose mass, each at time t. Both are measured relative to the x-axis in the coordinate frame determined by the longitudinal axis of the beam in its undeformed state with origin located at the beam's root or fixed end. Assuming the beam undergoes only small deformations (i.e. $\left|u(t,x)\right| \ll \ell$ and $\left|\frac{\partial u}{\partial x}(t,x)\right| \ll 1$) and has a small height to span length ratio, the Euler-Bernoulli theory (see [5]) including Voigt-Kelvin viscoelastic structural damping (see [10]) yields the partial differential equation

$$(2.1) \qquad \rho \frac{\partial^2 u}{\partial t^2}(t,x) + c_D I \frac{\partial^4}{\partial x^4} \frac{\partial u}{\partial t}(t,x) + EI \frac{\partial^4 u}{\partial x^4}(t,x) = 0, \qquad 0 < x < \ell, \quad t > 0$$

where ρ is the linear mass density of the beam, E is the modulus of elasticity, c_D is the coefficient of viscosity and I is the second moment or moment of inertia of the cross sectional area A about the neutral axis. For the beam we consider here with constant rectangular cross section, $I = bh^3/12$. Since the beam is assumed to be uniform, the parameters ρ, E and c_D are taken to be constant in time and space.

Balancing forces at the free end, elementary Newtonian mechanics yields the equations of motion

$$(2.2) \qquad m_T \frac{\partial^2 u}{\partial t^2}(t,\ell) - c_D I \frac{\partial^3}{\partial x^3} \frac{\partial u}{\partial t}(t,\ell) - EI \frac{\partial^3 u}{\partial x^3}(t,\ell)$$
$$= c_H \left(\frac{dy}{dt}(t) - \frac{\partial u}{\partial t}(t,\ell)\right) + k_H \left(y(t) - u(t,\ell)\right) + f(t), \qquad t > 0$$

and

$$(2.3) \qquad m_H \frac{d^2 y}{dt^2}(t) + c_H \left(\frac{dy}{dt}(t) - \frac{\partial u}{\partial t}(t,\ell)\right) + k_H \left(y(t) - u(t,\ell)\right) = 0, \qquad t > 0$$

for the tip and hose masses m_T and m_H respectively. Here k_H is the hose stiffness, c_H is the hose damping coefficient and $f(t)$ is the externally applied force at time t due to the firing of the thrusters mounted at the tip.

Making the assumption that the rotatory inertia of the proof mass system is negligible, rotational equilibrium at the tip can be expressed as

$$(2.4) \qquad c_D I \frac{\partial^2}{\partial x^2}\frac{\partial u}{\partial t}(t,\ell) + EI\frac{\partial^2 u}{\partial x^2}(t,\ell) = 0, \qquad t > 0.$$

The zero displacement and zero slope constraints at the fixed end are given by

$$(2.5) \qquad u(t,0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(t,0) = 0, \qquad t > 0$$

respectively. Taking the structure to be initially at rest we have the initial conditions

$$(2.6) \qquad u(0,x) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(0,x) = 0, \qquad 0 \le x \le \ell$$

and

$$(2.7) \qquad y(0) = 0 \quad \text{and} \quad \frac{dy}{dt}(0) = 0.$$

In the mathematical model given by (2.1) - (2.7) above the parameters ρ, m_T and I can be measured or computed directly. The modulus of elasticity E is typically determined in the laboratory. For the most commonly used materials (including aluminum which is the material from which the structure of interest to us here is made) its value can be readily looked up in standard engineering tables. The parameters c_D, m_H, c_H and k_H on the other hand, must be determined experimentally; that is, they will have to be identified based upon the observed response of the structure to a given input disturbance. This is one class of inverse problems which we formulate and consider below. In the system of equations (2.1) - (2.7) we explicitly modeled (albeit, in a rather simple fashion) the dynamical effects of the hose. The unknown hose parameters are then determined as the solution to an inverse problem.

An alternative approach to obtaining a model which exhibits a reasonable degree of fidelity involves a technique which is sometimes referred to as model adjustment. Starting with a simple model, the parameters are then "adjusted" so as to compensate for unmodeled dynamics. The choice of parameters to be adjusted and the resulting variations may or may not be motivated by physical considerations.

In our problem for example, we might consider a simple cantilevered beam with tip mass (i.e. $m_H = c_H = k_H = 0$) and then adjust the theoretical or measured values of E and m_T to compensate for the dynamical effects which result from the hose mass and motion. A value for the parameter c_D could also be identified if damping effects are considered significant. Model adjustment was used in [6] to obtain a model for the RPL structure upon which control design could be based.

We define an inverse problem which encompasses both of the general approaches which have been outlined above. We assume that an input disturbance described by the function f(t), $t \in [0,T]$ is applied to the structure via the tip thrusters and that the linear acceleration at the free end of the beam, $z(t)$, is measured and recorded for each $t \in [t_0, t_1]$ where $0 \le t_0 \le t_1 \le T$. (Of course, in actual practice, z could in fact only be sampled discretely). Let R_+ denote the positive real numbers and let Q be a closed and bounded subset of R_+^6. We seek a $\bar{q} \in Q$ which minimizes

$$J(q) = \int_{t_0}^{t_1} \left|\frac{\partial^2 u}{\partial t^2}(t,\ell;q) - z(t)\right|^2 dt$$

where $u(\cdot,\cdot;q)$ denotes the solution to the initial-boundary value problem (2.1) - (2.7) corresponding to $q = (m_T, E, c_D, m_H, c_H, k_H) \in Q$.

Our primary concerns in the next section will include well-posedness of the system (2.1) - (2.7), existence of a minimizer for J, and development of approximation techniques to find this minimizer.

3. Approximation Theory

A computational method for the solution of the estimation problem posed above will invariably involve finite dimensional approximation of the initial-boundary value problem (2.1) - (2.7). We have been successful in solving inverse problems for distributed parameter models for flexible structures (see, for example, [1], [2], [3], [4], [12]) using spline-based Ritz-Galerkin techniques. We apply those ideas here and derive finite element approximations based upon an abstract Hilbert space formulation of the hybrid system of ordinary and partial differential equations and boundary conditions given in (2.1) - (2.7). This abstract formulation is also useful in establishing existence, uniqueness and necessary regularity results for solutions. We briefly outline the essential features of our general approach (including theoretical convergence results) in the context of the particular problem of interest to us here.

Let $H = R^2 \times L_2(0,\ell)$ be endowed with the usual product space inner product

$$\langle(\zeta,\eta,\phi),(\lambda,\mu,\psi)\rangle_H = \zeta\lambda + \eta\mu + \langle\phi,\psi\rangle_0$$

and let

$$V = \{(\zeta,\eta,\phi) \in H\colon \phi \in H^2(0,\ell),\ \phi(0) = D\phi(0) = 0,\ \eta = \phi(\ell)\}$$

be endowed with the inner product

$$\langle(\zeta,\phi(\ell),\phi),(\lambda,\psi(\ell),\psi)\rangle_V = (\zeta-\phi(\ell))(\lambda-\psi(\ell)) + \langle D^2\phi,D^2\psi\rangle_0$$

where the symbol D is used here and below to denote the spatial differentiation operator $\frac{d}{dx}$. The space V together with the inner product $\langle\cdot,\cdot\rangle_V$ form a Hilbert space which is densely and compactly embedded in H.

We rewrite the system (2.1) - (2.7) as the abstract second order initial value problem in H

$$(3.1) \qquad M\hat{u}_{tt}(t) + C\hat{u}_t(t) + K\hat{u}(t) = F(t), \qquad t > 0$$

$$(3.2) \qquad \delta(\hat{u}(t) + \epsilon\hat{u}_t(t)) = 0, \qquad t > 0$$

$$(3.3) \qquad \hat{u}(0) = 0 \qquad u_t(0) = 0$$

in the states $\hat{u}(t) = (y(t), u(t,\ell), u(t,\cdot))$. The operators $M \in L(H)$, $C: \mathcal{D} \subset H \to H$ and $K: \mathcal{D} \subset H \to H$ are given by

$$(3.4) \qquad M(\zeta,\eta,\phi) = (m_H\zeta, m_T\eta, \rho\phi),$$

$$C(\zeta,\eta,\phi) = (c_H(\zeta-\eta), c_H(\eta-\zeta) - c_D I\, D^3\phi(\ell), c_D I\, D^4\phi),$$

and

$$K(\zeta,\eta,\phi) = (k_H(\zeta-\eta), k_H(\eta-\zeta) - EI\, D^3\phi(\ell), EI\, D^4\phi)$$

where $\mathcal{D} = \{(\zeta,\eta,\phi) \in V: \phi \in H^4(0,\ell)\}$. For each $t > 0$, $F(t) = (0, f(t), 0) \in H$, $\delta: \mathcal{D} \subset H \to R$ is given by $\delta((\zeta,\eta,\phi)) = D^2\phi(\ell)$ and $\epsilon = c_D/E$.

The restrictions $\tilde{C}$ and $\tilde{K}$ of the operators C and K that appear in equation (3.1) above to $\mathcal{N}(\delta)$, the null space of the operator δ, have natural extensions to bounded operators from V (which is the V-closure of $N(\delta)$) into V′, the dual of V. The extensions are defined in terms of the bilinear forms $c(\cdot,\cdot): V \times V \to R$ and $k(\cdot,\cdot): V \times V \to R$ given by

$$(3.5) \qquad (\tilde{C}\hat{\phi})(\hat{\psi}) = c(\hat{\phi},\hat{\psi}) = c_H(\zeta-\phi(\ell))(\lambda-\psi(\ell)) + c_D I\langle D^2\phi, D^2\psi\rangle_0$$

and

$$(3.6) \qquad (\tilde{K}\hat{\phi})(\hat{\psi}) = k(\hat{\phi},\hat{\psi}) = k_H(\zeta-\phi(\ell))(\lambda-\psi(\ell)) + EI\langle D^2\phi, D^2\psi\rangle_0$$

for $\hat{\phi} = (\zeta,\phi(\ell),\phi) \in V$ and $\hat{\psi} = (\lambda,\psi(\ell),\psi) \in V$.

The finite element method we develop below could be derived from standard energy considerations. While this is not the approach we take, it is worth noting that the usual energy expressions can be given in terms of the forms, operators and inner products defined above. The kinetic energy is given by

$$T_0 = \frac{1}{2}\langle M\hat{u}_t(t), u_t(t)\rangle_H,$$

the potential or strain energy by

$$U_0 = \frac{1}{2} k(\hat{u}(t),\hat{u}(t))$$

and the Rayleigh dissipation function by

$$F_0 = \frac{1}{2} c(\hat{u}_t(t),\hat{u}_t(t)).$$

Written in its weak, variational or distributional form

$$(3.7) \qquad \langle M\hat{u}_{tt}(t),\hat{\phi}\rangle_H + c(\hat{u}_t(t),\hat{\phi}) + k(\hat{u}(t),\hat{\phi}) = \langle F(t),\hat{\phi}\rangle_H, \qquad t > 0, \quad \hat{\phi} \in V$$

$$(3.8) \qquad \hat{u}(0) = 0 \qquad\qquad \hat{u}_t(0) = 0$$

the initial value problem (3.1) - (3.2) in H becomes an initial value problem in V'. If we assume that $f \in L_2(0,T)$ and rewrite (3.7), (3.8) as an equivalent first order vector system, the theory of abstract parabolic systems (see [9], [14]) yields the existence of a unique mapping

$$\hat{u} \in C([0,T];V) \cap H^1((0,T);V) \cap C^1([0,T];H) \cap H^2((0,T);V')$$

which satisfies (3.7), (3.8). If we are willing to assume further that f is Hölder continuous then there exists a

$$(3.9) \qquad \hat{u} \in C([0,T];V) \cap C^1((0,T];V) \cap C^1([0,T];H) \cap C^2((0,T];H)$$

with $\hat{u}(t) + \varepsilon\hat{u}_t(t) \in D$, $t > 0$ which uniquely satisfies the initial value problem (3.1) - (3.3).

In order to demonstrate the convergence of the approximation schemes we develop below, we shall require a somewhat more regular solution to the initial value problem (3.7), (3.8) than either of the conditions on f stated above can guarantee. In addition to (3.9), we shall require that $\hat{u} \in H^2((0,T);V)$. This can be guaranteed (see [7]) if we assume that $f \in H^1(-\tau,T)$ for some $\tau > 0$ with $f(t) = 0$, $t < 0$ and we modify our original mathematical model so that

$$(3.10) \qquad F(t) = f(t)\hat{\theta}, \qquad\qquad t \in [-\tau,T]$$

for some $\hat{\theta} = (0,\theta(\ell),\theta)$, a fixed element in V. We note that with $\hat{\theta}$

chosen appropriately in V, F given by (3.10) may in fact represent an improved model of reality when compared with our present choice of F where $\hat{\theta} = (0,1,0) \in H$.

Central to our approach is a cubic spline based Galerkin approximation to the initial value problem (3.7), (3.8). For each $N = 1,2,\cdots$ let Δ^N denote the uniform mesh $\{0, \frac{\ell}{N}, \frac{2\ell}{N}, \cdots, \ell\}$ on $[0,\ell]$ and let $\{B_j^N\}_{j=-1}^{N+1}$ denote the usual cubic B-splines defined with respect to the nodal set Δ^N (see [1_], [13]). Briefly, each B_j^N is a C^2 function on $[0,\ell]$ which is a cubic polynomial on each subinterval $[(k-1)\frac{\ell}{N}, k\frac{\ell}{N}]$, $k = 1,2,\cdots,N$. The support of B_j^N is $[(j-2)\frac{\ell}{N}, (j+2)\frac{\ell}{N}] \cap [0,\ell]$ with $B_j^N(j\frac{\ell}{N}) = 4$, $DB_j^N(j\frac{\ell}{N}) = 0$, $B_j^N((j\pm1)\frac{\ell}{N}) = 1$, and $DB_j^N((j\pm1)\frac{\ell}{N}) = \mp \frac{N}{\ell}$. Defining $\{\beta_j^N\}_{j=1}^{N+1}$ by $\beta_1^N = B_0^N - 2B_1^N - 2B_{-1}^N$ and $\beta_j^N = B_j^N$, $j=2,3,\cdots,N+1$, we have $\beta_j^N(0) = D\beta_j^N(0) = 0$, $j = 1,2,\cdots,N+1$. With $\hat{\beta}_0 = (1,0,0)$ and $\hat{\beta}_j^N = (0, \beta_j^N(\ell), \beta_j^N)$, $j = 1,2,\cdots,N+1$, $V^N = \text{span}\,\{\hat{\beta}_j^N\}_{j=0}^{N+1}$ is an N+2 dimensional subspace of V.

The Galerkin equations in V^N corresponding to (3.7), (3.8) for $\hat{u}^N(t) \in V^N$ are given by

$$\text{(3.11)} \quad \langle M\hat{u}_{tt}^N(t), \hat{\beta}_j^N\rangle_H + c(\hat{u}_t^N(t), \hat{\beta}_j^N) + k(\hat{u}^N(t), \hat{\beta}_j^N) = \langle F(t), \hat{\beta}_j^N\rangle_H,$$

$$t > 0, \quad j = 0,1,2,\cdots,N+1$$

$$\text{(3.12)} \quad \hat{u}^N(0) = 0 \qquad \hat{u}_t^N(0) = 0.$$

Setting

$$\hat{u}^N(t) = \sum_{j=0}^{N+1} w_j^N(t)\hat{\beta}_j^N, \qquad t \geq 0,$$

the initial value problem (3.11), (3.12) in V^N is equivalent to the linear, nonhomogeneous, second order N+2 - vector system

$$(3.13) \qquad M^N \frac{d^2w^N}{dt^2}(t) + C^N \frac{dw^N}{dt}(t) + K^N w^N(t) = F^N(t), \qquad t > 0$$

$$(3.14) \qquad w^N(0) = 0 \qquad\qquad \frac{dw^N}{dt}(0) = 0$$

where $w^N(t) = (w_0^N(t), w_1^N(t), \cdots, w_{N+1}^N(t))^T$. The entries in the (N+2) $\times$ (N+2) matrices M^N, C^N and K^N are given by

$$M_{i,j}^N = \langle M\hat{\beta}_i^N, \hat{\beta}_j^N\rangle_H,$$

$$C_{i,j}^N = c(\hat{\beta}_i^N, \hat{\beta}_j^N),$$

and

$$K_{i,j}^N = k(\hat{\beta}_i^N, \hat{\beta}_j^N),$$

$i,j = 0,1,2,\cdots,N+1$ respectively. For each $t > 0$ the components in the N+2 - vector $F^N(t)$ are given by $F_i^N(t) = \langle F(t), \hat{\beta}_i^N\rangle_H = f(t)\beta_i^N(\ell)$ or, recalling (3.10), by

$$F_i^N(t) = f(t)\langle\hat{\theta},\hat{\beta}_i^N\rangle_H = f(t)(\theta(\ell)\beta_i^N(\ell) + \langle\theta,\beta_i^N\rangle_0),$$

$i = 0,1,2,\cdots,N+1.$

We consider the sequence of approximating finite dimensional identification problems which consist of finding $\bar{q}^N \in Q$ which minimizes

$$(3.15)\qquad J^N(q) = \int_{t_0}^{t_1}\left|\frac{\partial^2 u^N}{\partial t^2}(t,\ell;q) - z(t)\right|^2 dt$$

where for each $q \in Q$, $\hat{u}^N(t;q) = (y^N(t;q),u^N(t,\ell;q),u^N(t,\cdot;q))$ is the unique solution to the initial value problem (3.11), (3.12) in V^N corresponding to $q = (m_T,E,c_D,m_H,c_H,k_H) \in Q$. In actual practice, for a given $q \in Q$, $J^N(q)$ is computed as

$$J^N(q) = \int_{t_0}^{t_1}\left|w_{N-1}^N(t;q) + 4w_N^N(t;q) + w_{N+1}^N(t;q) - z(t)\right|^2 dt$$

where $w^N(\cdot;q) = (w_0^N(\cdot;q),\cdots,w_{N+1}^N(\cdot;q))^T$ is the unique solution to the $N+2$ - vector system (3.13), (3.14) corresponding to $q \in Q$.

With finite dimensional state constraints, the solution of the N^{th} estimation problem above is, <u>at least in principle</u>, routine. For inverse problems which are closely related to the one we treat here, our earlier numerical studies have shown that satisfactory results can be obtained using any one of a number of standard computational techniques for least squares minimization (for example, Newton's method, conjugate gradient, steepest descent, Levenberg-Marquardt, etc., see [2]).

Our fundamental theoretical result is that each of the approximating identification problems and the original problem have solutions. Moreover, we show that the solutions to the approximating problems, in some sense, approximate solutions to the original problem. We require the following lemma.

<u>Lemma 3.1</u> Suppose $\{q^N\} \subset Q$ with $q^N \to q^0$ as $N \to \infty$. Let $\hat{u}^N(\cdot;q^N)$ denote the unique solution to the initial value problem (3.11), (3.12) corresponding to q^N and let $\hat{u}(\cdot;q^0)$ denote the unique solution to the initial value problem (3.7), (3.8) corresponding to q^0. If $u(\cdot;q^0) \in H^2((0,T);V)$ then

$$(3.16) \qquad \int_0^T \left|\hat{u}^N_{tt}(t;q^N) - \hat{u}_{tt}(t;q^0)\right|^2_H dt \to 0$$

as $N \to \infty$.

<u>Proof</u>

For each $N = 1,2,\cdots$ let P^N denote the orthogonal projection of H onto V^N defined with respect to the standard inner product on H, $\langle\cdot,\cdot\rangle_H$. Using the approximation theoretic properties of interpolatory splines, it is not difficult to show that (see [3])

$$(3.17) \qquad \lim_{N\to\infty} \left|(P^N - I)(\zeta,\eta,\phi)\right|_H = 0$$

for each $(\zeta,\eta,\phi) \in H$ and that

$$(3.18) \qquad \lim_{N\to\infty} \left|(P^N - I)\hat{\phi}\right|_V = 0$$

for each $\hat{\phi} \in V$.

For $q = (m_T, E, c_D, m_H, c_H, k_H) \in Q$ it is immediately clear that M, $c(\cdot,\cdot)$ and $k(\cdot,\cdot)$, the operator and forms defined in (3.4), (3.5) and (3.6) respectively depend upon q. For $q^0 = (m_T^0, E^0, c_D^0, m_H^0, c_H^0, k_H^0) \in Q$ and $q^N = (m_T^N, E^N, c_D^N, m_H^N, c_H^N, k_H^N) \in Q$ we adopt the shorthand notation $M^0 = M(q^0)$, $c^0(\cdot,\cdot) = c(q^0)(\cdot,\cdot)$, $k^0(\cdot,\cdot) = k(q^0)(\cdot,\cdot)$, $M^N = M(q^N)$, $c^N(\cdot,\cdot) = c(q^N)(\cdot,\cdot)$ and $k^N(\cdot,\cdot) = k(q^N)(\cdot,\cdot)$. Similarly, we denote $\hat{u}(\cdot;q^0)$ and $\hat{u}^N(\cdot;q^N)$ by $\hat{u}^0$ and $\hat{u}^N$ respectively.

From (3.17), the assumption that $\hat{u}^0 \in H^2((0,T);V)$ and the inequality

$$\int_0^T \left|\hat{u}_{tt}^N(t) - \hat{u}_{tt}^0(t)\right|_H^2 dt$$
$$\leq 2\int_0^T \left|\hat{u}_{tt}^N(t) - P^N\hat{u}_{tt}^0(t)\right|_H^2 dt + 2\int_0^T \left|(I-P^N)\hat{u}_{tt}^0(t)\right|_H^2 dt$$

it is clear that we need only to consider the first term on the right hand side of the above estimate.

Letting $\hat{v}^N(t) = \hat{u}^N(t) - P^N\hat{u}^0(t)$ for $t \geq 0$, (3.7), (3.8), (3.11), (3.12) and $V^N \subset V$ imply

$$(3.19)\quad \langle M^N\hat{v}_{tt}^N, \hat{\phi}^N\rangle_H + c^N(\hat{v}_t^N, \hat{\phi}^N) + k^N(\hat{v}^N, \hat{\phi}^N)$$
$$= \langle M^N(I-P^N)\hat{u}_{tt}^0, \hat{\phi}^N\rangle_H + \langle (M^0 - M^N)\hat{u}_{tt}^0, \hat{\phi}^N\rangle_H$$
$$+ c^N((I-P^N)\hat{u}_t^0, \hat{\phi}^N) + c^0(\hat{u}_t^0, \hat{\phi}^N) - c^N(\hat{u}_t^0, \hat{\phi}^N)$$
$$+ k^N((I-P^N)\hat{u}^0, \hat{\phi}^N) + k^0(\hat{u}^0, \hat{\phi}^N) - k^N(\hat{u}^0, \hat{\phi}^N), \quad t > 0,\ \hat{\phi}^N \in V^N$$

$$(3.20)\qquad \hat{v}^N(0) = 0 \qquad\qquad \hat{v}^N_t(0) = 0.$$

Choosing $\hat{\phi}^N = \hat{v}^N_{tt}(t) \in V^N$, from (3.19) we obtain

$$\begin{aligned}
&\langle M^N\hat{v}^N_{tt}, \hat{v}^N_{tt}\rangle_H + c^N(\hat{v}^N_t, \hat{v}^N_{tt}) \\
&\quad = \langle M^N(I-P^N)\hat{u}^0_{tt}, \hat{v}^N_{tt}\rangle_H + \langle (M^0-M^N)\hat{u}^0_{tt}, \hat{v}^N_{tt}\rangle_H \\
&\quad + \frac{d}{dt}\, c^N((I-P^N)\hat{u}^0_t, \hat{v}^N_t) - c^N((I-P^N)\hat{u}^0_{tt}, \hat{v}^N_t) \\
&\quad + \frac{d}{dt}\, \{c^0(\hat{u}^0_t, \hat{v}^N_t) - c^N(\hat{u}^0_t, \hat{v}^N_t)\} - \{c^0(\hat{u}^0_{tt}, \hat{v}^N_t) - c^N(\hat{u}^0_{tt}, \hat{v}^N_t)\} \\
&\quad + \frac{d}{dt}\, k^N((I-P^N)\hat{u}^0, \hat{v}^N_t) - k^N((I-P^N)\hat{u}^0_t, \hat{v}^N_t) \\
&\quad + \frac{d}{dt}\, \{k^0(\hat{u}^0, \hat{v}^N_t) - k^N(\hat{u}^0, \hat{v}^N_t)\} - \{k^0(\hat{u}^0_t, \hat{v}^N_t) - k^N(\hat{u}^0_t, \hat{v}^N_t)\} \\
&\quad - \frac{d}{dt}\, k^N(\hat{v}^N, \hat{v}^N_t) + k^N(\hat{v}^N_t, \hat{v}^N_t), \qquad\qquad t > 0.
\end{aligned}$$

Integrating the above expression from 0 to t and recalling (3.20), we find

$$\begin{aligned}
(3.21)\qquad &\int_0^t \langle M^N\hat{v}^N_{ss}, \hat{v}^N_{ss}\rangle_H\, ds + \frac{1}{2}\, c^N(\hat{v}^N_t, \hat{v}^N_t) \\
&\quad = \int_0^t \{\langle M^N(I-P^N)\hat{u}^0_{ss}, \hat{v}^N_{ss}\rangle_H - \langle (M^0-M^N)\hat{u}^0_{ss}, \hat{v}^N_{ss}\rangle_H \\
&\qquad - c^N((I-P^N)\hat{u}^0_{ss}, \hat{v}^N_s) - (c^0(\hat{u}^0_{ss}, \hat{v}^N_s) - c^N(\hat{u}^0_{ss}, \hat{v}^N_s)) \\
&\qquad - k^N((I-P^N)\hat{u}^0_s, \hat{v}^N_s) - (k^0(\hat{u}^0_s, \hat{v}^N_s) - k^N(\hat{u}^0_s, \hat{v}^N_s)) \\
&\qquad + k^N(\hat{v}^N_s, \hat{v}^N_s)\}ds \\
&\qquad + c^N((I-P^N)\hat{u}^0_t, \hat{v}^N_t) + (c^0(\hat{u}^0_t, \hat{v}^N_t) - c^N(\hat{u}^0_t, \hat{v}^N_t))
\end{aligned}$$

$$+ k^N((I-P^N)\hat{u}^0, \hat{v}_t^N) + (k^0(\hat{u}^0, \hat{v}_t^N) - k^N(\hat{u}^0, \hat{v}_t^N)) - k^N(\hat{v}^N, \hat{v}_t^N).$$

We recall that Q has been assumed to be a closed and bounded subset of R_+^6 and observe therefore that the forms $c^0(\cdot,\cdot)$, $c^N(\cdot,\cdot)$, $k^0(\cdot,\cdot)$ and $k^N(\cdot,\cdot)$ are uniformly bounded. These two facts together with the repeated application of the inequality

$$\langle a,b\rangle \le |a||b| \le \alpha|a|^2 + \frac{1}{4\alpha}|b|^2, \qquad \alpha > 0$$

in (3.21) yield the estimate

$$\int_0^t |\hat{v}_{ss}^N|_H^2 ds + |\hat{v}_t^N|_V^2$$

$$\le \gamma_0\{ \int_0^t (\frac{1}{4\alpha} |(I-P^N)\hat{u}_{ss}^0|_H^2 + \alpha|\hat{v}_{ss}^N|_H^2$$

$$+ \frac{1}{4\alpha}(|m_H^N - m_H^0|^2 + |m_T^N - m_T^0|^2)|\hat{u}_{ss}^0|_H^2 + \alpha|\hat{v}_{ss}^N|_H^2 + |(I-P^N)\hat{u}_{ss}^0|_V^2$$

$$+ |\hat{v}_s^N|_V^2 + (|c_H^N - c_H^0|^2 + |c_D^N - c_D^0|^2)|\hat{u}_{ss}^0|_V^2 + |\hat{v}_s^N|_V^2$$

$$+ |(I-P^N)\hat{u}_s^0|_V^2 + |\hat{v}_s^N|_V^2 + (|k_H^N - k_H^0|^2 + |E^N - E^0|^2)|\hat{u}_s^0|_V^2 + |\hat{v}_s^N|_V^2$$

$$+ |\hat{v}_s^N|_V^2)ds + \frac{1}{4\alpha} |(I-P^N)\hat{u}_t^0|_V^2 + \alpha|\hat{v}_t^N|_V^2$$

$$+ \frac{1}{4\alpha}(|c_H^N - c_H^0|^2 + |c_D^N - c_D^0|^2)|\hat{u}_t^0|_V^2 + \alpha|\hat{v}_t^N|_V^2 + \frac{1}{4\alpha}|(I-P^N)\hat{u}^0|_V^2$$

$$+ \alpha|\hat{v}_t^N|_V^2 + \frac{1}{4\alpha}(|k_H^N - k_H^0|^2 + |E^N - E^0|^2)|\hat{u}^0|_V^2 + \alpha|\hat{v}_t^N|_V^2$$

$$+ \frac{1}{4\alpha}\left|\hat{v}^N\right|_V^2 + \alpha\left|\hat{v}_t^N\right|_V^2 \}$$

where γ_0 is a positive constant. Choosing $\alpha > 0$ sufficiently small, we find

$$(3.22) \qquad \int_0^t \left|\hat{v}_{ss}^N(s)\right|_H^2 ds + \left|\hat{v}_t^N(t)\right|_V^2 \le \sigma_0(t) + \int_0^t \sigma_1(s)ds + \gamma_1\int_0^t \left|\hat{v}_s^N(s)\right|_V^2 ds$$

where

$$\sigma_0(t) = \gamma_2\{\left|(I-P^N)\hat{u}^0(t)\right|_V^2 + \left|(I-P^N)\hat{u}_t^0(t)\right|_V^2$$

$$+ \left|q^N - q^0\right|^2(\left|\hat{u}^0(t)\right|_V^2 + \left|\hat{u}_t^0(t)\right|_V^2) + \left|\hat{v}^N(t)\right|_V^2\}$$

$$\sigma_1(t) = \gamma_3\{\left|(I-P^N)\hat{u}_t^0(t)\right|_V^2 + \left|(I-P^N)\hat{u}_{tt}^0(t)\right|_V^2$$

$$+ \left|q^N - q^0\right|^2(\left|\hat{u}_t^0(t)\right|_V^2 + \left|\hat{u}_{tt}^0(t)\right|_V^2) \}$$

and γ_i, $i = 1,2,3$ are positive constants which do not depend on N.

Choosing $\hat{\phi}^N = \hat{v}_t^N(t) \in V^N$ in (3.19), arguments similar to those used above (see [2], [3]) yield

$$(3.23) \qquad \lim_{N\to\infty} \left|\hat{v}^N(t)\right|_V^2 = 0$$

for each $t \in [0,T]$. Using $\hat{u}^0 \in H^2((0,T);V)$, (3.18) and an application of the Gronwall inequality to (3.22) we obtain the desired result.

We note that we also obtain

$$\lim_{N\to\infty} \left|\hat{v}_t^N(t)\right|_V^2 = 0 \tag{3.24}$$

for each $t \in [0,T]$. From (3.23) and (3.24) we find $\left|\hat{u}^N(t;q^N)-\hat{u}(t;q^0)\right|_V \to 0$ and $\left|\hat{u}_t^N(t;q^N)-\hat{u}_t(t;q^0)\right| \to 0$ as $N \to \infty$ for each $t \in [0,T]$.

We remark that it is the L_2 convergence (more precisely, H convergence) in (3.16) which necessitates, at least in theory, that we be provided with distributed time observations (i.e. observations which are continuous in time). It is clear from (3.23) and (3.24) that for fits based upon displacement, velocity or slope, time-sampled measurements are sufficient. Of course when the approximating optimization problems are solved, the integral least squares performance indices (3.15) are discretized. Consequently, in practice, only discrete measurements of linear acceleration at the tip are required.

<u>Theorem 3.1</u> Each of the approximating identification problems has a solution $\bar{q}^N$. The sequence $\{\bar{q}^N\} \subset Q$ admits a convergent subsequence $\{\bar{q}^{N_j}\}$ with $\bar{q}^{N_j} \to \bar{q} \in Q$ as $j \to \infty$. If for each $q \in Q$, $\hat{u}(\cdot;q)$, the unique solution to the initial value problem (3.7), (3.8) corresponding to q, is an element in $H^1((0,T);V)$ then $\bar{q}$ is a solution to the original identification problem. In addition, the limit point of any convergent subsequence of $\{\bar{q}^N\}$ is a solution to the original identification problem as well.

<u>Proof</u> Standard continuous dependence results for linear ordinary differential equations, the fact that Q has been assumed to be a

closed and bounded subset of R^6 and the form of J^N are sufficient to conclude that a solution $\bar{q}^N \in Q$ to the N^{th} approximating identification problem exists. Once again since Q is a closed and bounded (and therefore compact) subset of R^6, the sequence $\{\bar{q}^N\} \subset Q$ admits a convergent subsequence. If $\{\bar{q}^{N_j}\} \subset \{\bar{q}^N\}$ with $\bar{q}^{N_j} \to \bar{q} \in Q$ as $j \to \infty$ and q is any point in Q, then two applications of Lemma 3.1 (the second one with the constant sequence {q}) yield

$$J(\bar{q}) = \lim_{j\to\infty} J^{N_j}(\bar{q}^{N_j}) \le \lim_{j\to\infty} J^{N_j}(q) = J(q)$$

and the theorem is proved.

Although Theorem 3.1 above guarantees only subsequential convergence, in all test and simulation examples we have considered, we in fact observe the convergence of the sequence $\{\bar{q}^N\}$ itself to the optimal parameters $\bar{q}$. Also, it is not difficult to verify that with only minor modification (see [2]) the approximation scheme reported on here (together with the convergence theory outlined in the lemma and theorem above) could be applied to inverse problems involving the estmation of spatially varying parameters (such as linear mass density ρ, flexural stiffness EI, or damping coefficient $c_D I$) which appear in the equations (2.1) - (2.4). We note of course that when either EI or $c_D I$ are spatially varying, the Euler-Bernoulli equation and corresponding boundary conditions are of a slightly different form than those given in (2.1) - (2.4) (see [3]).

4. Numerical Results

We used our scheme to attempt to solve the inverse problem which was posed above with data obtained from an experiment on the RPL structure. We report on our findings and observations here.

All computer codes were written in Fortran and run on the IBM 3081 at the University of Southern California. The approximating finite dimensional least-squares minimization problems were solved using the IMSL implementation of the Levenberg-Marquardt algorithm (routine ZXSSQ), an iterative Newton's method-steepest descent hybrid (see[2]). The second order N+2 - vector systems (3.13), (3.14) were solved (integrated) in each iteration (for the evaluation of J^N and its gradient) using Gear's method for stiff systems (IMSL routine DGEAR). The integral least squares performance index was approximated by a discrete sum over a uniform mesh on $[t_0, t_1]$. The integral inner products in the definitions of the matrices M^N, C^N and K^N were computed using a composite two point Gauss-Legendre quadrature rule.

The second time derivative of w^N, or generalized acceleration, $\frac{d^2 w^N}{dt^2}$, was computed using a second order centered difference on the generalized displacement,

$$\frac{d^2 w^N}{dt^2}(t) \approx \frac{w^N(t+\Delta) - 2w^N(t) + w^N(t-\Delta)}{\Delta^2}. \tag{4.1}$$

We found this to be a somewhat more stable method for computing acceleration (an unbounded measurement) than was a first order centered difference on the generalized velocity.

$$(4.2)\qquad \frac{d^2w^N}{dt^2}(t) = \frac{\frac{dw^N}{dt}(t + \frac{\Delta}{2}) - \frac{dw^N}{dt}(t - \frac{\Delta}{2})}{\Delta}.$$

Either of the time differencing formulas (4.1) or (4.2) proved to be significantly more stable than using the differential equation (3.13) directly to compute $\frac{d^2w^N}{dt^2}(t)$ via an inversion of M^N. As to why this was so, we can only offer the conjecture that the time differencing provided, at least to a certain extent, some filtration of the signal.

Before turning our attention to the experimental data, we tested our scheme with simulated data. "True" values for the unknown parameters c_D (actually c_DI), m_H, c_H and k_H were chosen and a quintic spline-based semi-discrete Galerkin scheme applied to the initial value problem (3.7), (3.8) was used to generate data.

Setting ρ .03, m_T = .15, EI = 80.0, ℓ = 4.0 and

$$f(t) = \begin{cases} 1.0 & 0 \le t \le 0.05 \\ 0.0 & 0.05 < t \le 5.0, \end{cases}$$

the fit was carried out based upon observations of linear acceleration at the tip at times $t_i = .1i$, $i = 2,3,\cdots,50$. We note that this is equivalent to taking $t_0 = .1$, $t_1 = 5.0$ and using a standard rectangle rule with uniform mesh spacing .1 to discretize the integral appearing in the definition of the least squares performance index J^N. The initial estimates $c_DI = .0035$, $m_H = .035$, and $k_H = .4$ were used to start the iterative optimization procedure. In (4.1), Δ was taken to be .1. Our results are summarized in Table 4.1 below.

N	$\bar{c}_D^N I$	$\bar{m}_H^N$	$\bar{c}_H^N$	$\bar{k}_H^N$	$J^N(\bar{q}^N)$
2	.037537	.039471	.003428	.298626	2.57×10^{-1}
3	.066997	.039485	.003907	.298875	4.37×10^{-2}
4	.005063	.039777	.003997	.299455	5.06×10^{-3}
5	.005667	.039899	.003971	.299787	7.66×10^{-4}
6	.005049	.040035	.004006	.300087	4.63×10^{-5}
True value	.005000	.040000	.004000	.300000	
Initial Estimate	.003500	.035000	.003500	.400000	

Table 4.1

The experiment which we describe below was carried out for us on the RPL structure by Dr. Michel A. Floyd, formerly of the Control and Flight Dynamics Division of the Charles Stark Draper Laboratory and the Department of Aeronautics and Astronautics, MIT.

The air bearing table was clamped so that the central hub could not rotate. The thruster lines for one of the active appendages was set to 300 psi and the thruster was fired for .05 seconds (50 milliseconds). With the appendage initially at rest, the firing of the thruster was assumed to have begun at time $t = 0$. Linear acceleration at the tip was observed over the time interval 0 to 5 seconds. With a sampling period of .005 seconds (5 milliseconds) a total of 1000 measurements were recorded. The data is plotted in Figure 4.1 below. The scale factor for the accelerometer is 5 volts/g ($g = 32$ ft/sec^2).

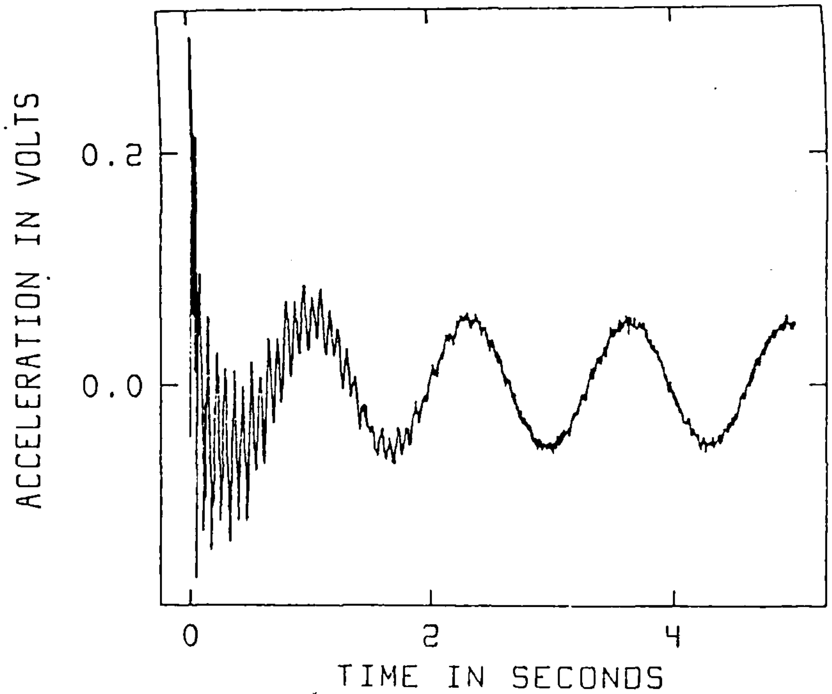

Figure 4.1

The noticeably higher frequency (≈ 14 Hz) component of the data is a torsional mode of the arm excited by the motion of the thruster valve mechanisms and inertial and elastic forces applied to the tip of the arm by the nitrogen supply hose. The opening or closing of the solenoidal valve in the thruster generates an inertial force which acts as a torque on the tip of the arm. Consequently, torsional modes are excited. Also, in addition to modifying transverse bending characteristics, since the hose is attached to the top of the arm, its horizontal motion will tend to generate torques which have a "twisting" effect. Although the accelerometer is mounted at the center of the arm (and therefore on a nodal line of the longitudinal torsional modes, if we assume vertical symmetry), as the arm twists, the accelerometer picks up a component of the earth's gravitational force. Since the first torsional mode has a much higher frequency than either of the first two flexible modes (.75 Hz and 7.5 Hz, as identified from an FFT of the data) and since it is rapidly damped, we neglected its contribution to the accelerometer signal, treating it as white noise, and left it unmodeled. A detailed discussion of the causes of the excitation of the torsional modes and its effect on the transverse bending characteristics of the active appendages can be found in [6].

The physical characteristics of the structure are as follows. The arm is made of aluminum and is 4 feet in length, 6 inches in width and .125 inches in height. From this we obtain $\ell = 4.0$ ft, $\rho = .027$ slug/ft and $I = 4.71 \times 10^{-8}\ (\text{ft})^4$. The theoretically predicted value for E is $15.84 \times 10^{8}\ \text{lb}/(\text{ft})^2$. The mass of the thruster assembly was determined to be $m_T = .149$ slug. From the calibration table in [6], we find that a hose pressure of 300 psi is equivalent to a force of .297 lb. We set therefore

$$f(t) = \begin{cases} 0.297 \text{ lb} & 0 \le t \le 0.05 \\ 0.0 & 0.05 < t \le 5.0 \end{cases}$$

To serve as a basis for comparison, we neglected the hose effects and structural damping (i.e. we chose $c_D = m_H = c_H = k_H = 0$) and used the standard Euler-Bernoulli model with the parameters ρ, E, I and m_T and input f as specified above to generate the plot of linear acceleration at the tip given in Figure 4.2.

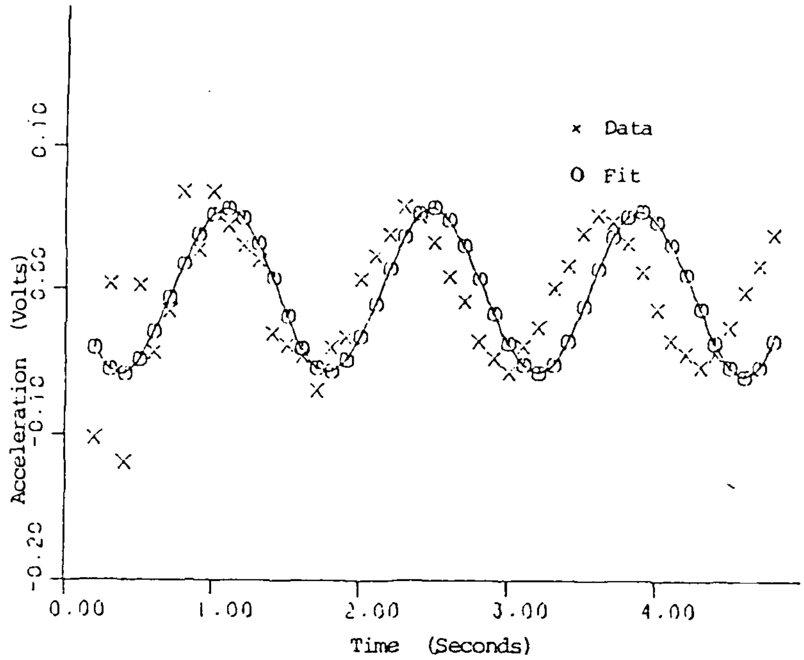

Figure 4.2

The plot was obtained by integrating the initial value problem (3.13), (3.14) with N = 4 and then using (4.1) to compute the acceleration at the free end. The residuals $\left(\frac{\partial^2 u}{\partial t^2}(t,\ell) - \frac{\partial^2 u^N}{\partial t^2}(t,\ell)\right)$ over the time interval [0,5] are plotted in Figure 4.3. The sum of the squares of the residuals (at intervals of .1 seconds) was found to be 3.03.

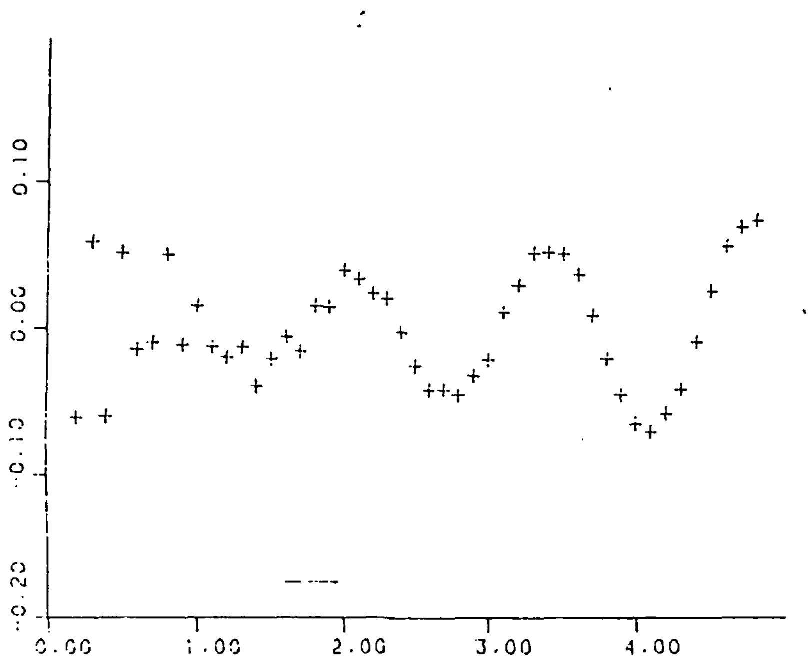

Figure 4.3

Using the data on the interval 3.0 to 5.0 (where the contribution from the torsional modes has been significantly damped) with a sampling period of .1 seconds we used our scheme with N = 4 to obtain optimal estimates for the coefficient of viscosity c_D and the hose parameters m_H, c_H and k_H. In the set of runs we are about to describe the values of E and m_T were held fixed at their theoretically predicted values. A rough calculation based upon "matching" the first two observed natural frequencies of the data with the first two modal frequencies of the model was used to obtain a crude initial estimate for the ratio k_H/m_H. Then, using our scheme to minimize over the parameters m_H and k_H only, we obtained the optimal values shown in Table 4.2 below. Integrating the system (3.13), (3.14) over the time interval [0,5] with m_H and k_H set to the values in the table and c_D = c_H = 0 the sum of the squares of the residuals (at intervals of .1 seconds) was found to be .73.

m_H(slug)	k_H(lb/ft)
.039269	.339935

Table 4.2

Next, holding m_H and k_H fixed at the values shown in Table 4.2, a search on c_H was carried out (the initial estimate for c_H was taken to be zero and c_D was held fixed at zero). Then using the resulting values of m_H, c_H and k_H as initial estimates, a fit over all three parameters was performed. The result is shown in Table 4.3. The sum of the squares of the residuals was found to be .728.

m_H(slug)	c_H(lb·sec/ft)	k_H(lb/ft)
.043431	.004056	.351385

Table 4.3

Continuing to use the same procedure to generate "start up" values, we eventually used our scheme to search over all four parameters c_D, m_H, c_H and k_H simultaneously obtaining the values given in Table 4.4 and the fit plotted in Figure 4.4. The residuals are plotted in Figure 4.5. The sum of their squares was computed to be .70.

c_D(lb·sec/(ft)^2)	m_H(slug)	c_H(lb·sec/ft)	k_H(lb/ft)
127.40	.0801	.007804	.412977

Table 4.4

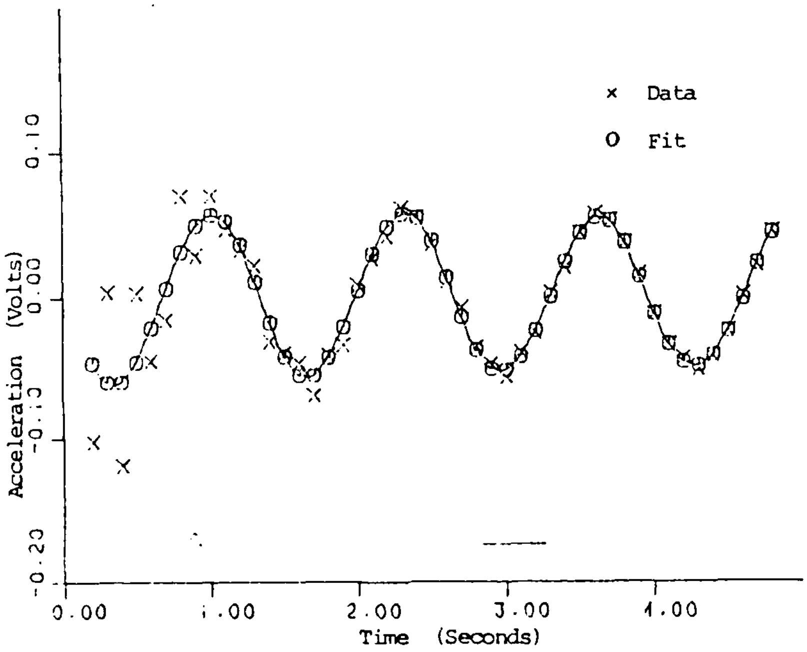

Figure 4.4

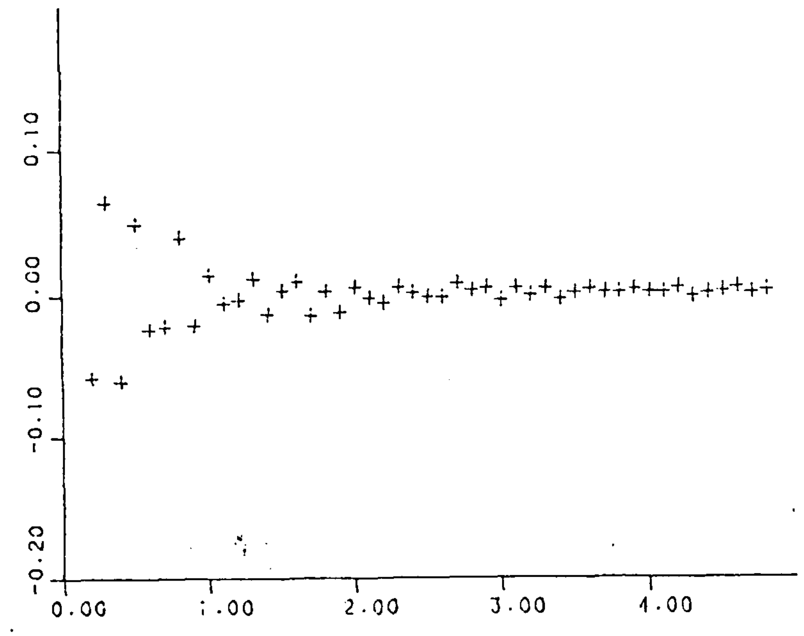

Figure 4.5

In designing a controller for the RPL experiment, Floyd in [6] used model adjustment to tune a simple, undamped, cantilevered beam with tip mass model for the active arms (i.e. the arms with the hoses) of the structure. He used the following procedure. The air bearing table was locked in a stationary position. With the hose depressurized, an impulsive force was applied to the beam and linear acceleration at the tip was measured and recorded. Based upon the physical assumption that with the hose depressurized, the presence of the hose serves only to add mass to the tip of the arm, the parameter m_T was adjusted so that the first mode or frequency of the model agreed with the first observed cantilever mode (obtained via an FFT) of the data. Then, with the hose pressurized, the same experimental procedure was carried out. This time however, the modulus of elasticity E of the beam was adjusted to compensate for the variation in stiffness which results from the presence of the hose. The adjusted values of the tip mass, $\bar{m}_T$, and modulus of elasticity, $\bar{E}$, obtained by Floyd are given in Table 4.5 below.

$\bar{m}_T$ (slug)	$\bar{E}$ (lb/(ft)^2)
.254	17.31×10^8

Table 4.5

We integrated the system (3.13), (3.14) using the adjusted values of m_T and E given in the table (and $c_D = m_H = c_H = k_H = 0$) and obtained the plot shown in Figure 4.6. The corresponding residuals are plotted in Figure 4.7. The sum of the squares of the residuals was computed to be 5.1.

Starting with the same basic model, we used our scheme to determine the values of m_T and E which minimize the sum of the squares of the residuals over the time interval [3.0, 5.0] with a sampling period of .1 seconds. Taking the theoretically predicted values of m_T and E (m_T = .149 slug, E = 15.84×10^8 lb/(ft)^2) as start up values for the optimization routine yielded the results given in Table 4.6.

The corresponding fit and residuals are plotted in Figures 4.8 and 4.9 respectively below. The sum of the squares of the residuals (over the interval [0,5]) was computed to be .73.

m_T (slug)	E $(lb/(ft)^2)$
.185	21.95×10^8

Table 4.6

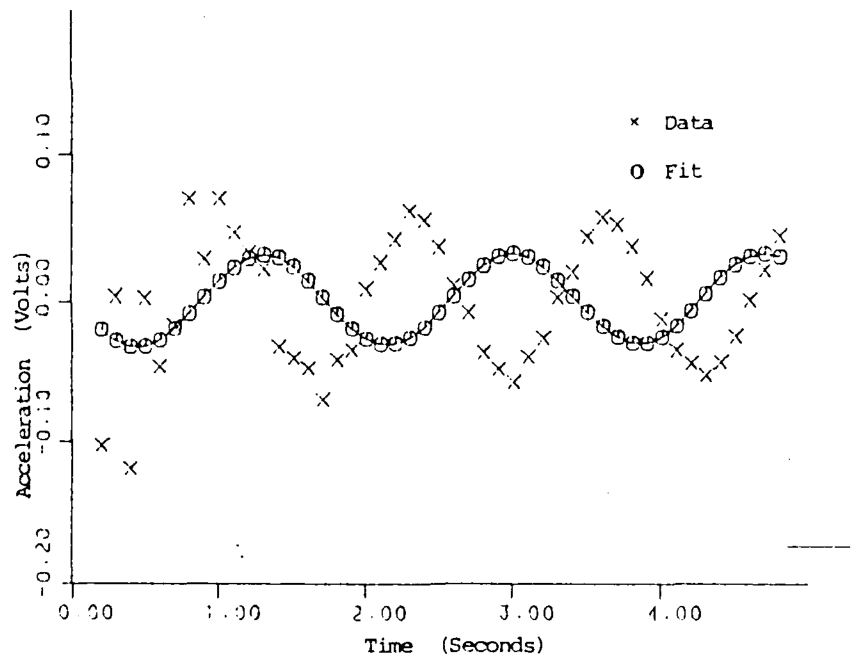

Figure 4.6

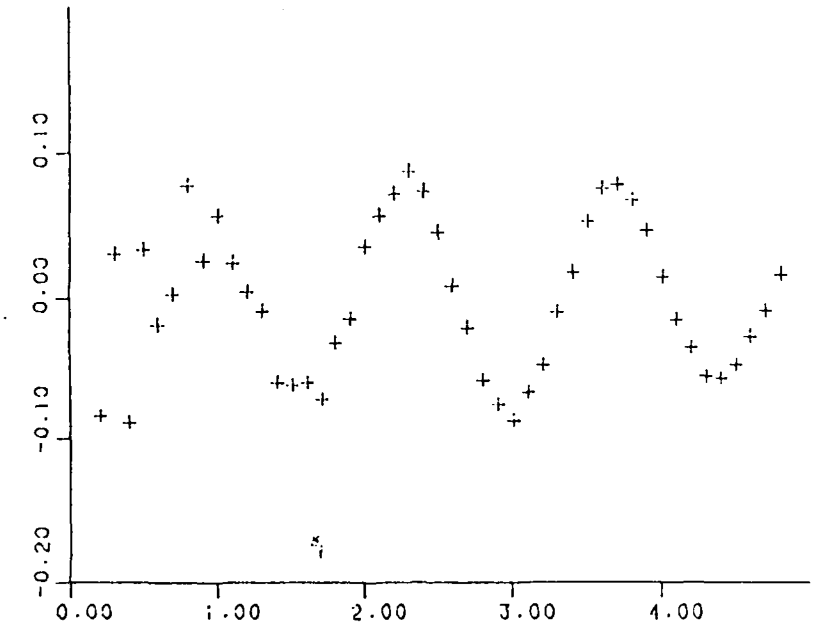

Figure 4.7

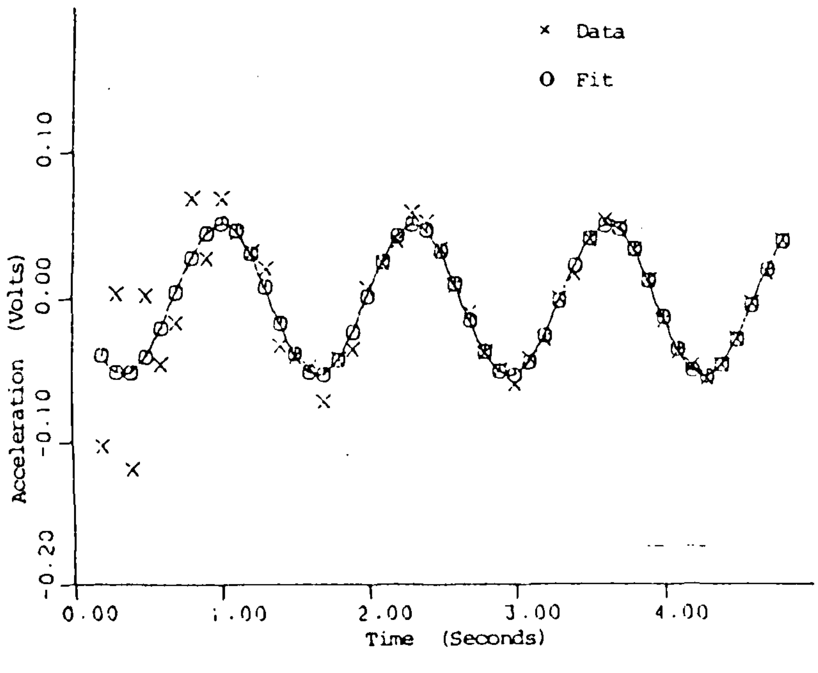

Figure 4.8

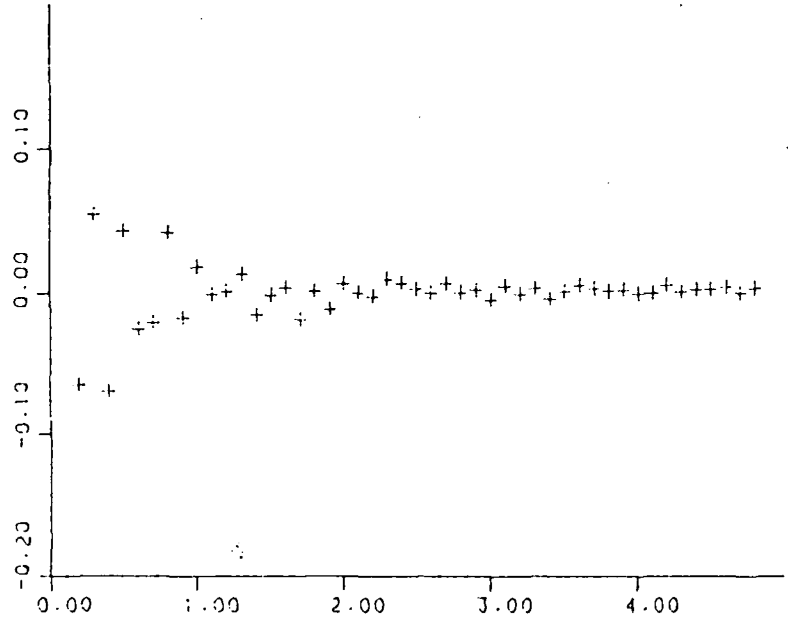

Figure 4.9

In summary, we have seen that analysis of the RPL experimental data can be carried out in several ways with a number of different models. Our techniques can be used to provide reasonable fits of the data to models with or without hose and/or beam damping. Even if one attempts to leave the physics of the hose - beam dynamic interaction unmodeled and perform "model adjustment" (by adjusting the values of the tip mass m_T and beam modulus of elasticity E), our estimation techniques provide a much better fit than that obtained using "modal matching" methods common in engineering practice.

One of the primary objectives of our effort here was to demonstrate the efficacy of our scheme and in particular, to assess its effectiveness when provided with actual experimental data. While we are pleased with the results obtained for the RPL data, we are careful to point out that to provide a fair and complete evaluation of the usefulness of our models for the RPL experimental structure, a more complete and in-depth study involving extensive experimental work and statistical analysis would necessarily be required.

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References

[1] H. T. Banks and J. M. Crowley, Parameter identification in continuum models, J. Astronautical Sciences, 33 (1985), pp.85-94.

[2] H. T. Banks, J.M. Crowley amd I. G. Rosen, Methods for the identification of material parameters in distributed models for flexible structures, ICASE Report No. 84-66, Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, VA, 1984, Mat. Aplicada e Computacional, to appear.

[3] H. T. Banks and I. G. Rosen, A Galerkin method for the estimation of parameters in hybrid systems governing the vibration of flexible beams with tip bodies, ICASE Report No. 85-7, Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, VA, 1985.

[4] H. T. Banks and I. G. Rosen, Computational methods for the identification of spatially varying stiffness and damping in beams, ICASE Report No. 86- , Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, VA, 1986.

[5] R. W. Clough and J. Penzien, Dynamics of Structures, McGraw-Hill, New York, 1975.

[6] M. A. Floyd, Single-step optimal control of Large Space Structures, Ph.D. Thesis, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, MA, 1984 and Report CSDL-T-840,The Charles Stark Draper Laboratory, Cambridge, MA, 1984.

[7] A. Friedman, Partial Differential Equations of Parabolic Type, Prentice Hall, Englewood Cliffs, New Jersey, 1964.

[8] K. Kunisch and E. Graif, Parameter estimation for the Euler-Bernoulli beam, Mat. Aplicada e Computacional, 4, (1985), pp.95-124.

[9] J. L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, Springer-Verlag, New York, 1971.

[10] P. P. Popov, Introduction to Mechanics of Solids, Prentice-Hall, Englewood Cliffs, New Jersey, 1968.

[11] P. M. Prenter, Splines and Variational Methods, Wiley-Interscience, New York, 1975.

[12] I. G. Rosen, A numerical scheme for the identification of hybrid systems describing the vibration of flexible beams with tip bodies, J. Math. Anal. Appl., 116, (1986), pp.262-288.

[13] M. H. Schultz, Spline Analysis, Prentice Hall, Englewood Cliffs, New Jersey, 1973.

[14] R. E. Showalter, Hilbert Space Methods for Partial Differential Equations, Pitman, London, 1977.

[15] R. Strunce, et. al., Verification of RCS Controller Methods for Flexible Spacecraft (RPL-EXP), Report CSDL-P-1653, The Charles Stark Draper Laboratory, Cambridge, MA, 1982.

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