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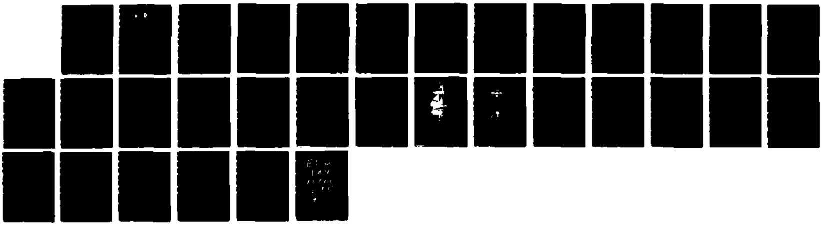

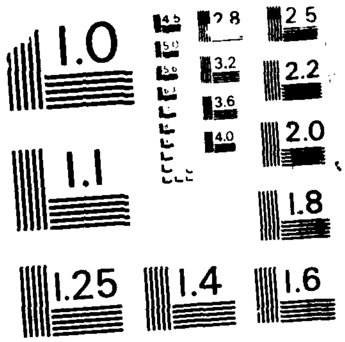

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The p-version of the finite element method is a new, important, computationally efficient, approach to finite element analysis. It is more robust than the conventional h-version and its rate of convergence, for domains with corners and for other singularity problems, is twice that of the h-version.

Hierarchic elements which implement the p-version efficiently have been formulated so as to enforce C^0 or C^1 continuity in the planar case, and so as to enforce C^0 continuity in three dimensions.

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Recent research accomplishments include:

1. Development of an algorithm that finds all roots of an analytic function in a finite domain.

2. Preprocessing procedures to restrict the search in unbounded domains which contain roots to bounded domains,

3. A reliable numerical argument principle algorithm to compute number of zeros within a closed contour.

4. Formulation of equations which determine the nature of stress singularity at a corner of a plate composed of n isotripic materials.

All of the above are used in the extraction method for p-version finite element analysis of composite materials with corners.

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1. INTRODUCTION

There are now three basic approaches to finite element analysis. In all approaches the domain Ω is divided into simple convex subdomains (usually triangles or rectangles in two dimensions, and tetrahedra or bricks in three dimensions) and over each subdomain the unknown is approximated by a (local) basis function (usually a polynomial of degree $\leq p$). Basis functions are required to meet continuously at boundaries of subdomains in the case of planar or 3 dimensional elasticity, or smoothly in the case of plate bending. The approaches are:

1. The <u>h-version</u> of the finite element method. In this approach the degree p of the approximating polynomial is kept fixed, usually at some low number such as 2 or 3. Convergence is achieved by allowing h, the maximum diameter of the convex subdomains, to go to zero. Estimates for the error in energy have long been known [1, 2, 3]. In all of these estimates p is assumed to be fixed and the error estimate is asymptotic in h, as h goes to zero.
2. The <u>p-version</u> of the finite element method. In this approach the subdivision of the domain Ω is kept fixed but p is allowed to increase until a desired accuracy is attained. The p-version is reminiscent of the Ritz method for solving partial differential equations but with a crucial distinction between the two methods. In the Ritz method a single polynomial approximation is used over the <u>entire</u> domain Ω (Ω, in general, is not convex). In the p-version of the finite element method polynomials are used as approximations over <u>convex subdomains</u>. This critical difference gives the p-version a more rapid rate of convergence than either the Ritz method or the h-version.

3. The h-p version of the finite element method. In this approach both the degree p of the approximating polynomial and the maximum diameter h of the convex subdomains are allowed to change.

The p-version of the finite element method requires families of polynomials of arbitrary degree p defined over different geometric shapes. Polynomials defined over neighboring elements join either continuously (are in C^0) for planar or three dimensional elasticity, and smoothly (are in C^1) for plate bending. In order to implement the p-version efficiently on the computer, these families should have the property that computations performed for an approximation of degree p are re-usable for computations performed for the next approximation of degree $p + 1$. We call families possessing this property <u>hierarchic</u> families of finite elements.

The h-version of the finite element method has been the subject of intensive study since the early 1950's and perhaps even earlier. Study of the p-version of the finite element method, on the other hand, began at Washington University in St. Louis in the early 1970's and led to a more recent study of the h-p version. Research in the p-version (formerly called The Constraint Method) has been supported in part of the Air Force Office of Scientific Research since 1976.

2. SUMMARY OF EARLIER RESEARCH ACCOMPLISHMENTS

2.1 <u>Rate of Convergence of the p-version</u>

Extensive computational experiments have long furnished empirical evidence that the rate of convergence of the p-version is significantly higher than that of the h-version (see for example [4, 5, 6]).

The two theorems described in detail in [5, 6] provide a mathematical explanation for the efficiency of the p-version.

Roughly speaking, these theorems state that if the domain is a bounded polygon in two dimensions, and if the criterion used to measure efficiency is the number of degrees of freedom required to achieve a given error in energy, then the rate of convergency of the p-version is twice that of the h-version.

Also, it is shown in [7, 8] if optimal mesh refinement (not necessarily quasi-uniform) is combined with optimal p-distribution then the asymptotic convergence is exponential.

Figure 1 illustrates graphically the rates of convergence of the h-, p-, and h-p-versions.

2.2 Hierarchic Families of Solid Finite Elements

In order to implement the p-version efficiently, families of finite elements are needed with the hierarchic property: computations performed for an approximation of order p should be re-usable when raising the order to $p+1$. More specifically, the stiffness matrix corresponding to the polynomial approximation of degree p should be a submatrix of the polynomial approximation of degree $p + 1$. In terms of basis functions, this implies that the basis functions for a pth order approximation should be a subset of the basis functions for a $(p + 1)$st order approximation.

Hierarchic families for triangles both in the C^0 and in the C^1 case are described in detail in [9, 10, 11, 12, 13, 14].

2.3 Quantities of Special Interest - Extraction Techniques

Often the main purpose of a finite element analysis is to obtain values of a few important quantities with a high accuracy. In structural mechanics, for example, the values of displacements or stresses in a small number of designated areas, or the stress intensity factor at a small number of points is of critical importance for design. Both the h- and p-versions of the finite element method give approximations for these values. However, it is much more efficient to use a post-processing technique which uses weighted averages of values taken directly from the finite element approximation. The post-processor determines these quantities of special interest much more accurately. In particular, when stresses are computed pointwise as derivatives of displacements in the p-version, they may exhibit (some times severe) oscillatory behavior. In the case of the centrally cracked panel shown in Figure 2(h), for example the normal stresses σ_y along the x-axis computed for polynomial orders ranging from 1 to 7 are shown. The oscillatory behavior near the crack tip singularity is evident. Two techniques are being developed for the post-processing of quantities of special interest. Both techniques are very accurate, yielding approximations that are of the same order of accuracy as the strain energy. (This is the square of the error in energy norm).

2.3.1 Use of Functional Forms

This approach is based on the idea that the functional forms of the quantities of interest are generally known. In the case of a centrally cracked panel, for example, the displacements u(z), v(z), and the stresses $\sigma_x(z)$, $\tau_{xy}(z)$ and $\sigma_y(z)$ $(z = x + iy)$ are given in terms of two functions.

$$\Phi(z) = \Phi_0(z) + \frac{\Phi_1(z)}{\sqrt{z}} = \sum_{j=0}^{\infty} b_j z^j + \sum_{j=0}^{\infty} a_j z^{j - 1/2}$$

$$\Omega(z) = -\Phi_0(z) + \frac{\Phi_1(z)}{\sqrt{z}} = -\sum b_j z^j + \sum_{j=0}^{\infty} a_j z^{j - 1/2}$$

where

$$\Phi_0(z) = \sum_{j=0}^{\infty} b_j z^j \qquad \Phi_1(z) = \sum_{j=0}^{\infty} a_j z^j$$

are holomorphic functions. Approximate values for the coefficients a_j, b_j are determined using the displacements $u_p(z)$, $v_p(z)$ computed by the p-version of the finite element method.

This technique was used in [15] to obtain improved values of σ_y along the x-axis in the case of a centrally cracked panel. Sample results are shown in Figure 3 (R in Figure 3 is the radius of the circle surrounding the crack tip within which values of $u_p(z)$ and $v_p(z)$ were taken). Figure 4 shows the convergence of this post processing technique for the stress intensity factor K^* as a function of 1/NDF.

2.3.2 Extraction Techniques

A general form for post processing calculations is given in [16, 17, 18]. Green's theorem and generalized influence functions are used together with smooth cut-off functions and blending functions in order to calculate higher derivatives of the unknown function and also stress intensity factors.

These techniques have been applied to post processing of the u_p and v_p displacement fields obtained from the p-version. See, for example, [19, 20].

3. SUMMARY OF RECENT RESEARCH ACCOMPLISHMENTS

3.1 Extraction Techniques for Composite Materials-Formulation of Equations

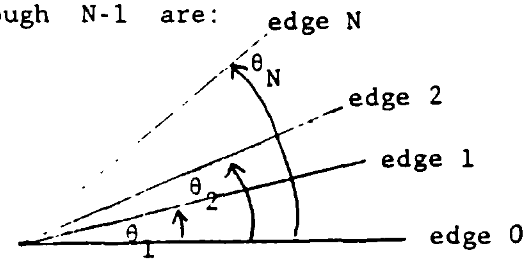
In the techniques discussed in 2.3.1 and 2.3.2 for the post-process calculation of stresses and stress intensity factors it is essential that the behavior of the solution at the singular point be known in advance. It is well known [21] that the behavior of the displacements of a thin plate in a state of plain stress or plain strain at a vertex of a sector is of the form

$$\sum_k a_k r^{\lambda k} \log^{q_k} \psi_k(\theta)$$

where (r,θ) are polar coordinates centered at the vertex and $\psi_k(\theta)$ are smooth functions (usually sines or cosines). The (complex) numbers λ_k, and their multiplicities are determined by enforcing boundary conditions at both sides meeting at the vertex. The values of λ_k with smallest real part have been determined in a well known paper by Williams [22] for different sets of boundary conditions in the case that the sector is composed of a single material.

In many modern applications the sector is composed of N materials each with its own Poisson's ratio ν_i and Young's modulus E_i. In this case the determination of the eigenvalues λ_k and corresponding eigenfunctions $r^{\lambda k} \log^{q_k} \psi_k(\theta)$ is considerably more difficult. However, it is important to determine the eigenvalues λ_k and corresponding eigenfunctions in order to use the post processing capabilities of the p-version.

Consider N materials meeting at a point as shown in the adjoining figure. The interface conditions at edges 1 through N-1 are:



$$\tau_{r\theta,i}(\theta_i) = \tau_{r\theta,\ i+1}(\theta_i)$$

$$\sigma_{\theta,i}(\theta_i) = \sigma_{\theta,\ i+1}(\theta_i)$$

$$U_{\theta,i}(\theta_i) = U_{\theta,\ i+1}(\theta_i)$$

$$U_{r,i}(\theta_i) = U_{r,\ i+1}(\theta_i)$$

Let $\mu_i = \dfrac{E_i}{2(1+\nu_i)}$ = the shear modulus in material i

$$\sigma_i = \frac{\nu_i}{1+\nu_i}$$

and let λ be the unknown eigenvalue.

Define the 2×2 matrices for i=1, ..., N-1

$$H_i = \begin{bmatrix} \cos(\lambda+1)\theta_i & -\sin(\lambda+1)\theta_i \\ \sin(\lambda+1)\theta_i & \cos(\lambda+1)\theta_i \end{bmatrix}$$

$$G_i = \begin{bmatrix} \cos(\lambda-1)\theta_i & -\sin(\lambda-1)\theta_i \\ \sin(\lambda-1)\theta_i & \cos(\lambda-1)\theta_i \end{bmatrix}$$

$$T_i = \begin{bmatrix} \lambda+1 & 0 \\ 0 & 1 \end{bmatrix} \qquad T_2 = \begin{bmatrix} \lambda-1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$M_i = \begin{bmatrix} \dfrac{\lambda - 1}{\mu_i} + \dfrac{8}{E_i} & 0 \\ 0 & \dfrac{\lambda + 1}{\mu_i} - \dfrac{8}{E_i} \end{bmatrix}$$

and define the boundary matrices B_j for $j = 0, N$ by

$$B_0 = T_2 \quad \text{(free edge)} \qquad\qquad B_N = T_2 \quad \text{(free edge)}$$

$$B_0 = T_2 + \frac{\mu_1}{E_1} T_3 \quad \text{(clamped edge)} \qquad\qquad B_N = T_2 + \frac{\mu_N}{E_N} T_3 \quad \text{(clamped edge)}$$

It was shown [23, 24] that each eigenvalue λ satisfies $\det S = 0$ where

$$S = \left[\begin{array}{cccccccccc}
T_1H_0 & B_0G_0 & & & & & & & & \\
T_1H_1 & T_2G_1 & -T_1H_1 & -T_2G_1 & & & & & & \\
\frac{\lambda+1}{\mu_1}H_1 & M_1G_1 & -\frac{\lambda+1}{\mu_2}H_1 & -M_2G_1 & & & & & & \\
 & & T_1H_2 & T_2G_2 & -T_1H_2 & -T_2G_2 & & & & \\
 & & \frac{\lambda+1}{\mu_2} & M_2G_2 & -\frac{\lambda+1}{\mu_3}H_2 & -M_3G_2 & & & & \\
 & & & & & & & & & \\
 & & & & & & T_1H_{N-1} & T_2G_{N-1} & -T_1H_{N-1} & -T_2G_{N-1} \\
 & & & & & & \frac{\lambda+1}{\mu_{N-1}}H_{N-1} & M_{N-1}G_{N-1} & -\frac{\lambda+1}{\mu_N}H_{N-1} & -M_NG_{N-1} \\
 & & & & & & & & T_1H_N & B_NG_N
\end{array}\right]$$

Furthermore a lengthy computation shows that

$$\det S = \det S'' \, (\lambda+1)^N (-64)^{N-1} \prod_{t=2}^{N} E_t^{-2}$$

Where $S'' =$

$$\begin{bmatrix} T_1 H_N, & B_N G_N \end{bmatrix} \begin{bmatrix} T_1 H_{N-1} & T_2 G_{N-1} \\ \frac{\lambda+1}{\mu_N} H_{N-1} & M_N G_{N-1} \end{bmatrix}^{-1} \begin{bmatrix} T_1 H_{N-1}, & T_2 G_{N-1} \\ \frac{\lambda+1}{\mu_{N-1}} H_{N-1} & M_{N-1} G_{N-1} \end{bmatrix} \cdots \begin{bmatrix} T_1 H_K, & T_2 G_K \\ \frac{\lambda+1}{\mu_{K+1}} H_K & M_{K+1} G_K \end{bmatrix}^{-1} \begin{bmatrix} T_1 H_K & T_2 G_K \\ \frac{\lambda+1}{\mu_K} H_K & M_K G_K \end{bmatrix}$$

$$\cdots \begin{bmatrix} T_1 H_1 & T_2 G_1 \\ \frac{\lambda+1}{\mu_2} H_1 & M_2 G_1 \end{bmatrix}^{-1} \begin{bmatrix} T_1 H_1 & T_2 G_1 \\ \frac{\lambda+1}{\mu_1} H_1 & M_1 G_1 \end{bmatrix} \begin{bmatrix} -T_1^{-1} & B_0 \\ I_2 & \end{bmatrix}$$

Note that this last expression gives S'' as product of a 2×4 matrix, followed by (4×4) matrices, followed by a (4×2) matrix. Therefore S'' is a (2×2) matrix.

It is of course sufficient to solve $\det S'' = 0$. The expression for S'' even in the simplified form given above is vary lengthy. The symbolic manipulator code REDUCE can be used to obtain the expression for S''. A closed from would be much more efficient for numerical computations and is currently being sought.

It is remarkable that if the shear modulus $\mu_i = \dfrac{E_i}{2(1+\nu_i)} = \mu$ is constant for all materials then a relatively simple closed form nonlinear equation can be obtained for each eigenvalue λ:

$$D(\lambda) = \frac{1}{2} \sum_{j=0}^{N} \sum_{i=0}^{N} \left[\lambda^2 (\nu_{i+1} - \nu_i)(\nu_{j+1} - \nu_j) \cos 2(\theta_j - \theta_i) - (\nu'_{i+1} - \nu'_i)(\nu'_{j+1} - \nu'_j) \cos 2\lambda(\theta_j - \theta_i) \right] = 0$$

where $\nu_0 = \nu_{N+1} = 0$, $\nu'_k = \nu_k \qquad k = 1, \ldots, N$

$$\nu'_0 = \begin{cases} -1 & \text{for edge } -0 \text{ free} \\ 3 & \text{for edge } -0 \text{ clamped} \end{cases} \qquad \nu'_{N+1} = \begin{cases} -1 & \text{for edge } N \text{ free} \\ 3 & \text{for edge } N \text{ clamped} \end{cases}$$

Once the nonlinear equations for the eigenvalues λ are formulated, they are solved for eigenvalues with smallest real parts. These eigenvalues, together with associated eigenfunctions are used in the extraction techniques discussed earlier in order to determine the constants a_k. The displacement and stress fields in the neighborhood of a vertex formed by composite materials are then computable through the post processor of the p-version. This provides a powerful new procedure for the possible prediction of crack initiation based on factors (the a_k) which are analogous to stress intensity factors (used for the prediction of crack propagation).

3.2 Globally Convergent Algorithms for Finding all the Eigenvalues λ

The determination of all of the eigenvalues λ_k themselves and their corresponding eigenfunctions is a challenging problem. We have developed algorithms with the following properties:

1. All eigenvalues in a given infinite strip $-x_0 \le \mathrm{Re}\lambda \le x_0$ are computed by the algorithms to a prespecified accuracy.
2. All of the corresponding eigenfunctions are computed.

It is important that no eigenvalues (with small real parts) be omitted in the solution of the eigenequation because this will lead to incorrect characterization of the singularity at the corner. The extraction technique described earlier will then fail.

For this purpose we have developed globally convergent algorithms. Our approach is based on an exclusion method. In this approach, a simple criterion is formulated to determine whether a root does not lie in a given region. If so, the region is excluded from further computation. This approach assures that all roots of a nonlinear function in a given starting region will be determined. Then, those with smallest real part, can be selected.

In [25], a reliable solver for all roots of a nonlinear equation is described in detail. The idea of using global information (in the form of bounds on derivatives) dynamically is introduced. Also the concepts of dominating function and dominated functions are formulated to reduce the search domain from an infinite strip to a bounded domain.

3.3 A Numerical Argument Principle

In order to assure that all roots have indeed been obtained, we have studied the use of the principle of the argument for functions of a complex variable, to determine the number of zeros of the eigenequations in a given (bounded) domain. It is crucial that the steps taken in numerically integrating along a contour be small enough to assure that there are no spurious jumps in the change of the argument which may be caused by inaccuracies. A criterion has been developed to compute the size of these steps. Details are given in [26].

These algorithms may have wider application to the problem of finding all zeros of any analytic functions in a given domain.

4. COMPUTER IMPLEMENTATIONS AND TECHNOLOGY TRANSFER

Our earlier work on research and development of the p-version have resulted in several computer implementations. As our research progresses, new procedures and ideas are continually incorporated into these codes making them more efficient. Two of the codes (FIESTA/3D and PROBE) are in commercial use. A detailed description of the efficient technology transfer was reported in SIAM NEWS Vol. 19, No. 6, (November 1986) and Vol. 20, No. 2 (March 1987).

4.1 COMET-X

COMET-X is an experimental computer code which implements the p-version of the finite element method by using the hierarchic families which have been constructed. COMET-X is maintained by the Center for Computational Mechanics at Washington University. COMET-X can be used as a code to implement the h-version as well simply by fixing the polynomial order p and refining the mesh.

COMET-X currently has the following capabilities:

A. Element types: Stiffeners, triangular elements, triangular elements with one side curved, rectangular elements, solid elements of the shapes described earlier.

B. Types of Analysis: Laplace and Poisson equations, plane elasticity, temperature distribution in 3-dimensions.

C. Special Capabilities: non-uniform p-distribution, elastic fracture mechanics computations in two dimensions, nearly incompressible solids, linear boundary layer problems.

D. Pre- and Post processing capabilities including graphics and visual displays. The capabilities and usage of COMET-X are described in detail in [27].

4.2 FIESTA/3D

FIESTA/3D is a software system for static analysis of solid structures based on the p-version of the finite element method. It is marketed by

McDonnell-Douglas Automation Company. Some of the advanced features available on FIESTA/3D are:

- controllability of the quality of the solution
- efficient modeling
- treatment of stress singularities
- a method for surface identification.

4.3 PROBE

PROBE is an advanced computer implementation of the p-version for 2-dimensional analysis currently under development at NOETIC Technologies in St. Louis, Missouri. Some of PROBE's unique features are:

- automatic error estimators. These estimators provide initial feedback on solution quality by computing element-by-element and edge-by-edge equilibrium checks.
- interactive error estimators. These estimates offer immediate feedback on solution quality by computing action/reaction checks overall equilibrium checks and a convergence trajectory of stresses and strains at user-selected points.
- load tracking, for extracting free-body diagrams
- pinpoint solutions which give specific results anywhere in the model
- Mode I (Symmetric) and Mode II (antisymmetric) Stress Intensity Factors for Fracture Mechanics
- Precise curve definitions which eliminate mesh refinement near cutouts and provide more precise solutions in critical regions.

5. Figures

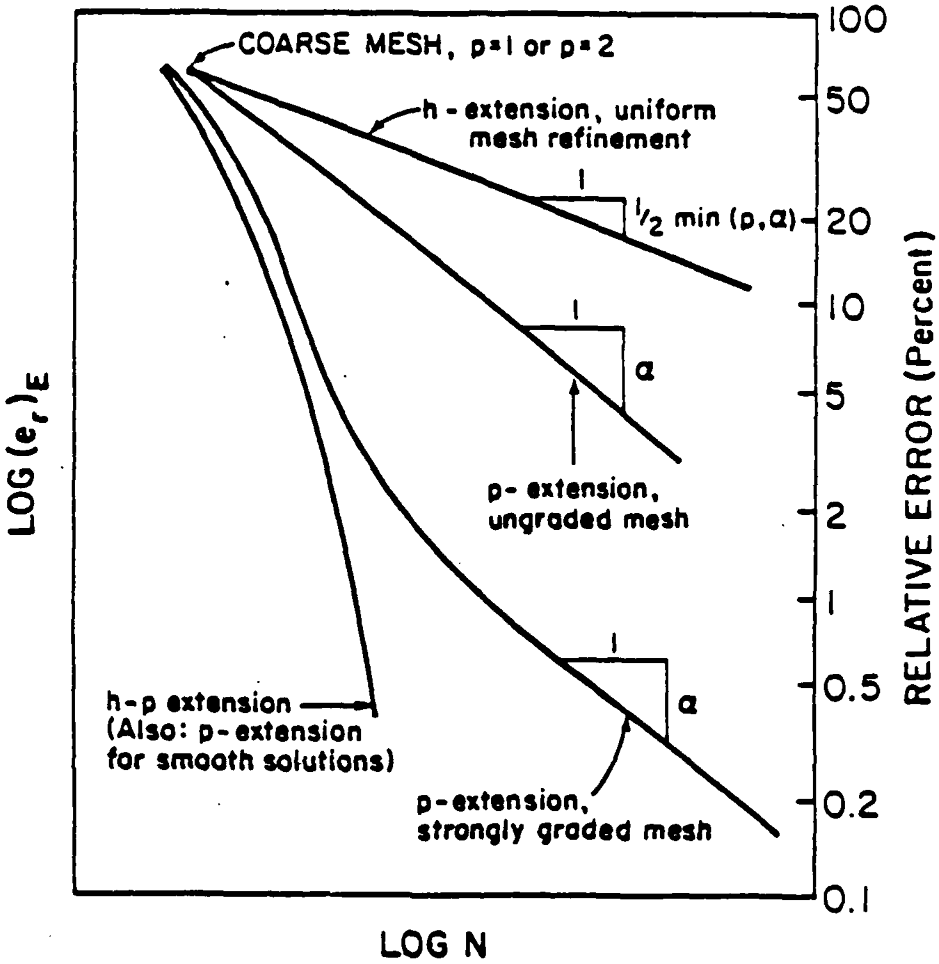


Fig. 1

Performance of the h-, p- and h-p extension processes.
(The relative error values shown are typical for certain engineering problems).

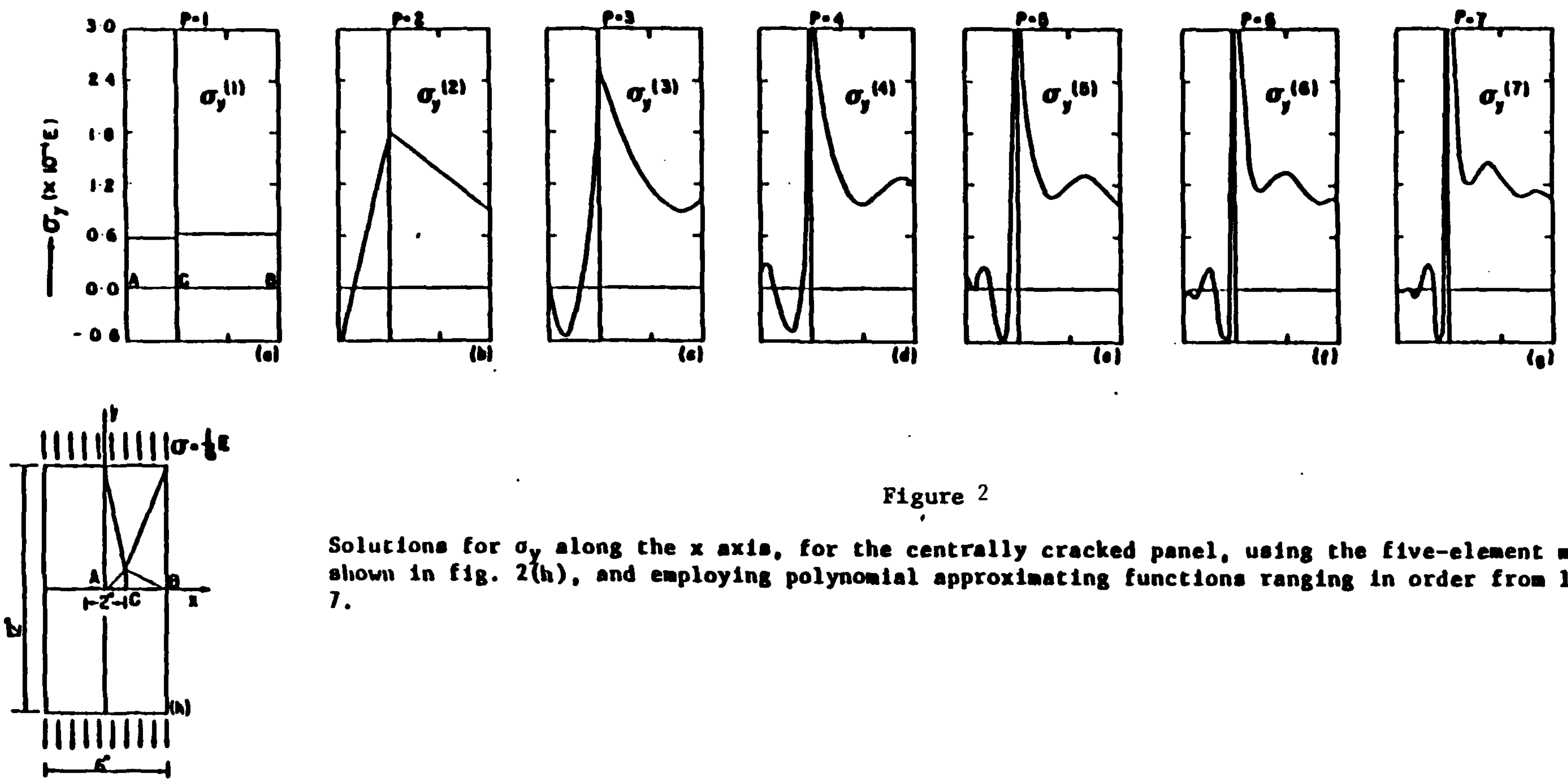


Figure 2

Solutions for σ_y along the x axis, for the centrally cracked panel, using the five-element mesh shown in fig. 2(h), and employing polynomial approximating functions ranging in order from 1 to 7.

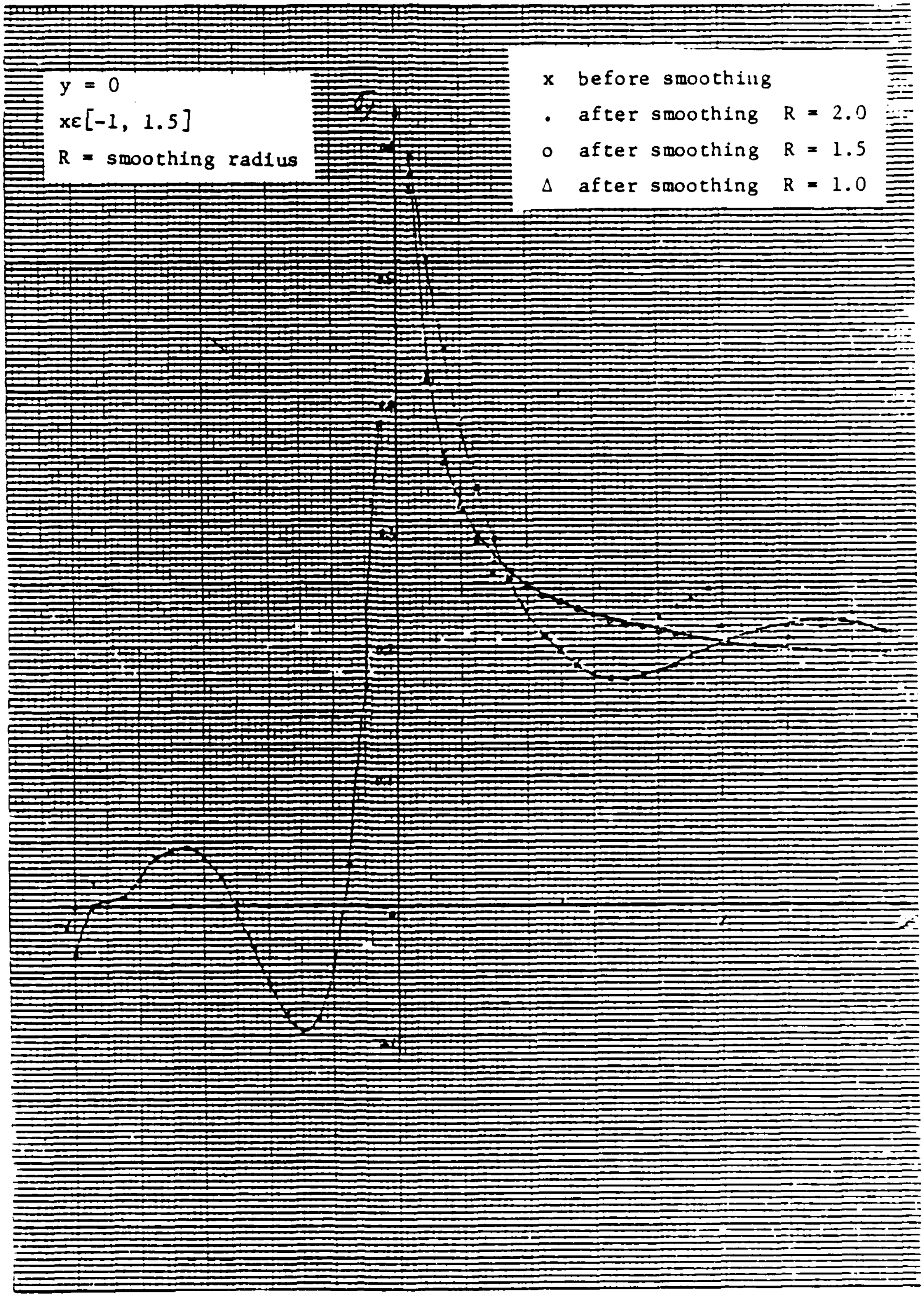
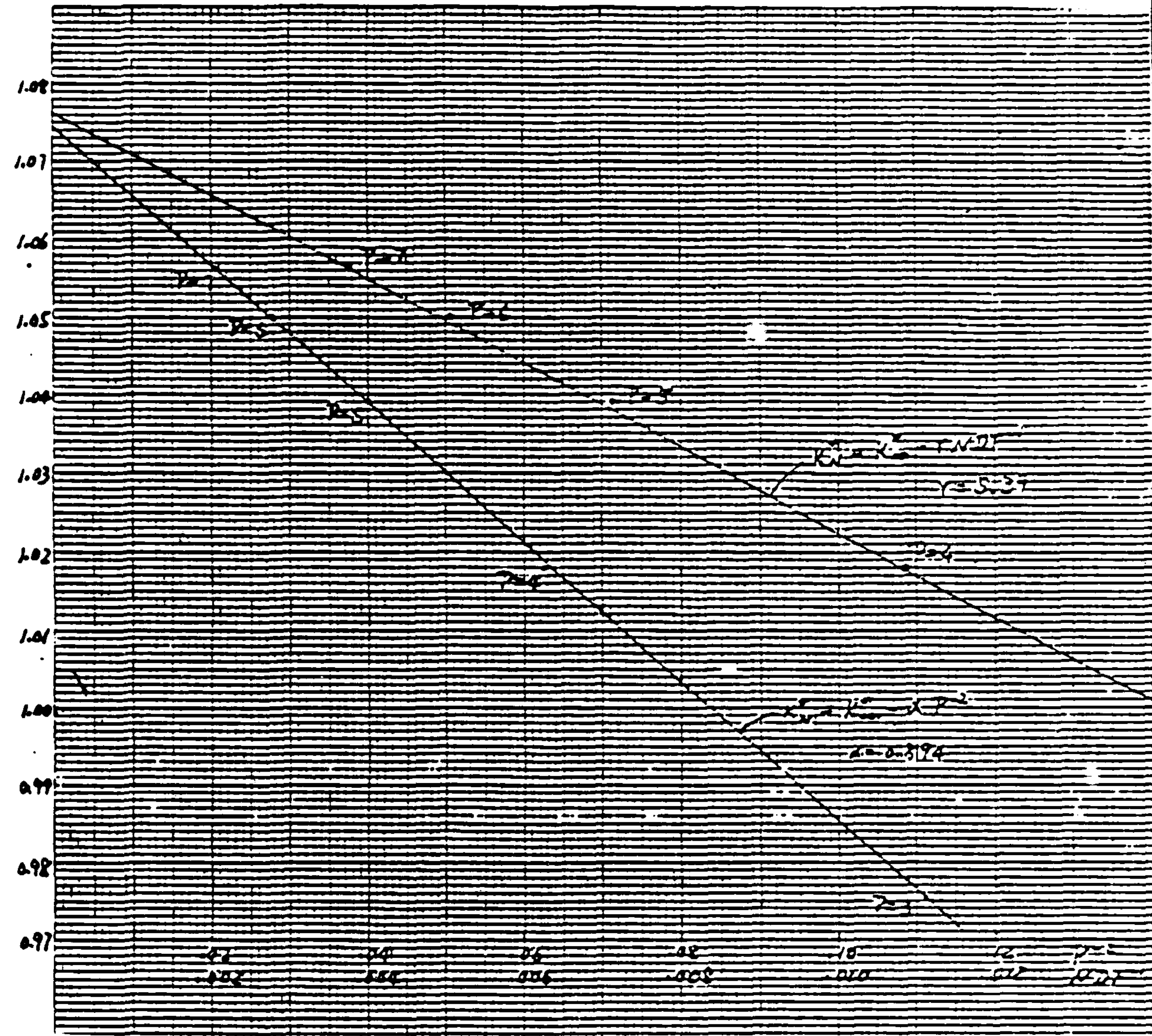


Figure 3 Smoothed and Unsmoothed Stresses Near the Crack Tip of a Centrally Cracked Plate



$$K^x = \frac{K_I}{\sigma\sqrt{\pi a}}$$ analytical value of K^* is 1.075

Figure 4 Computation of the Stress Intensity Factor in a Centrally Cracked Plate

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7. PRINCIPAL PERSONNEL

Dr. I.N. Katz is Professor of Applied Mathematics and Systems Science and a member of the Center for Computational Mechanics at Washington University. He served as principal investigator.

Dr. B.A. Szabo is a A.P. Greensfelder Professor of Mechanical Engineering and Director of the Center for Computational Mechanics at Washington University. He served as co-principal investigator.

Dr. X-R. Ying was supported as a graduate research assistant. He was awarded his D.Sc. degree in December, 1986.

8. PAPERS PUBLISHED AND PRESENTED SINCE THE START OF THE PROJECT (1977)

8.1 Published Papers:

1. "Hierarchical Finite Elements and Precomputed Arrays", by M.P. Rossow and I.N. Katz, Int. J. for Num. Method in Engr., Vol. 13, No. 6 (1978) pp. 977-999.

2. "Nodal Variables for Complete Conforming Finite Elements of Arbitrary Polynomial Order", by I.N. Katz, A.G. Peano, and M.P. Rossow, Computers and Mathematics with Applications, Vol. 4, No. 2, (1978), pp. 85-112.

3. "Hierarchic Solid Elements for the p-version of the Finite Element Method", by I.N. Katz, B.A. Szabo and A.G. Peano (in preparation).

4. "P-convergence Finite Element Approximations in Linear Elastic Fracture Mechanics", by A.K. Mehta (doctoral dissertation), Department of Civil Engineering, Washington University (1978).

5. "An Improved p-version Finite Element Algorithm and a Convergence Result for the p-version", by A.G. Kassos, Jr. (doctoral dissertation) Department of Systems Science and Mathematics, Washington University, (August, 1979).

6. "Hierarchic Families for the p-version of the Finite Element Method", I. Babuska, I.N. Katz and B.A. Szabo, invited paper presented at the Third IMACS International Symposium on Computer Methods for Partial Differential Equations, published in <u>Advances in Computer Methods for Partial Differential Equations - III</u> (1979) pp. 278-286.

7. "The p-version of the Finite Element Method", I. Babuska, B.A. Szabo and I.N. Katz, SIAM J. of Numerical Analysis, Vol. 18, No. 3, June 1981, pp. 515-545.

8. "Hierarchic Triangular Elements with one Curved Side for the p-version of the Finite Element Method", by I.N. Katz (in preparation).

9. "The p-version of the Finite Element Method for Problems Requiring C^1-Continuity", by D.W. Wang (doctoral dissertation), Department of Systems Science and Mathematics, Washington University, August 1982.

10. "Implementation of a C^1 Triangular Element based on the p-version of the Finite Element Method", by I.N. Katz, D.W. Wang and B. Szabo, Proceedings of the Symposium in Advances and Trends in Structural and Solid Mechanics, October 4-7, 1982, Washington, D.C.

11. "Implementation of a C^1-triangular Element based on the p-version of the Finite Element Method", by Douglas W. Wang, I.N. Katz and B.A. Szabo, Computers and Structures, Vol. 19, No. 3, pp. 381-392, (1984).

12. "h- and p-version Analyses of a Rhombic Plate", by Douglas W. Wang, I.N. Katz and B.A. Szabo, International Journal for Numerical Methods in Engineering, Vol. 20, pp. 1399-1405, (1984).

13. "The p-version of the Finite Element Method for Problems Requiring C^1-Continuity", by I.N. Katz and D.W. Wang, SIAM Journal on Numerical Analysis, Vol. 22, No. 6, December 1985, pp. 1082-1106.

14. "A Reliable Root Solver for Automatic Computation with Application to Stress Analysis of a Composite Plane Wedge," by X-R. Ying (doctoral dissertation), Department of Systems Science and Mathematics, Washington University, December, 1986.

15. "Computation of Stress Field Parameters in Areas of Steep Stress Gradients", by B.A. Szabo, Communications in Applied Numerical Methods, Vol. 2, pp. 133-137, (1986).

16. "Implementation of a Finite Element Software System with h- and p-extension Capabilities", by B.A. Szabo, Finite Elements in Analysis and Design, Vol. 2, pp. 177-194, (1986).

17. "Computation of the Amplitude of Stress Singular Terms for Cracks and Re-entrant Corners", by B.A. Szabo and I. Babuska (to appear in American Society for Testing Materials, Special Technical Publication).

18. "A Uniform Formulation for the Calculation of Stress Singularities in the Plane Elasticity of a Wedge Composed of Composite Material", by X-R. Ying and I.N. Katz, International Journal for Computers and Mathematics with Applications, Vol. 14, No. 6, pp. 437-458, (1987).

19. "A Reliable Argument Principle Algorithm to find the Number of Roots of an Analytic Function in a Bounded Domain", (to appear in Numerische Mathematik).

8.2 Presented Papers:

1. "Hierarchical Approximation in Finite Element Analysis", by I.N. Katz, International Symposium on Innovative Numerical Analysis in Applied Engineering Science, Versailles, France, May 23-27, 1977.

2. "Efficient Generation of Hierarchical Finite Elements Through the Use of Precomputed Arrays", by M.P. Rossow and I.N. Katz, Second Annual ASCE Engineering Mechanics Division Speciality Conference, North Carolina State University, Raleigh, NC, May 23-25, 1977.

3. "C^1 Triangular Elements of Arbitrary Polynomial Order Containing Corrective Rational Functions", by I.N. Katz, SIAM 1977 National Meeting, Philadelphia, PA, June 13-15, 1977.

4. "Hierarchical Complete Conforming Tetrahedral Elements of Arbitrary Polynomial Order", by I.N. Katz, presented at SIAM 1977 Fall Meeting, Albuquerque, NM, October 31-November 2, 1977.

5. "A Hierarchical Family of Complete Conforming Prismatic Finite Elements of Arbitrary Polynomial Order", by I.N. Katz, presented at SIAM 1978 National Meeting, Madison, WI, May 24-26, 1978.

6. "Comparative Rates of h- and p-Convergence in the Finite Element Analysis of a Model Bar Problem", by I.N. Katz, presented at the SIAM 1978 Fall Meeting, Knoxville, Tennessee, October 20-November 1, 1978.

7. "Smooth Approximation to a Function in $H_0^2(D)$ by Modified Bernstein Polynomials over Triangles", by A.G. Kassos, Jr. and I.N. Katz, presented at the SIAM 1979 Fall Meeting, Denver, Colorado, November 12-14, 1979.

8. "Triangles with one Curved Side for the p-version of the Finite Element Method", by I.N. Katz, presented at the SIAM 1980 Spring Meeting, Alexandra, VA, June 5-7, 1980.

9. "Hierarchic Square Pyramidal Elements for the p-version of the Finite Element Method", by I.N. Katz, presented at the SIAM 1980 Fall Meeting, Houston, TX, November 6-8, 1980.

10. "The Rate of Convergence of the p-version of the Finite Element Method for Plate Bending Problems", by Douglas W. Wang and I.N. Katz, presented at SIAM 1981 Fall Meeting, October 6-8, 1981, Cincinatti, Ohio.

11. "The p-version of the Finite Element Method", by I.N. Katz, 1982 Meeting of the Illinois Section of the Mathematical Association of America, Southern Illinois University at Edwardsville, April 30-May 1, 1982.

12. "Computer Implementation of a C^1 Triangular Element based on the p-version of the Finite Element Method," by D.W. Wang and I.N. Katz, SIAM 30th Anniversary Meeting, July 19-23, 1982, Stanford, California.

13. "Implementation of a C^1 Triangular Element Based on the p-version of the Finite Element Method," Symposium on Advanced and Trends in Structural and Solid Mechanics, October 4-7, 1982, Washington, D.C.

14. "P-Convergent Polynomial Approximations in $H_0^2(\Omega)$," by D.W. Wang and I.N. Katz, Fourth Texas Symposium on Approximation Theory, Department of Mathematics, Texas A&M University, College Station Texas 77843, January 17-21, 1983.

15. "Design Aspects of Adaptive Finite Element Codes," by D.W. Wang, I.N. Katz and M.Z. Qian, ASCE-EMD (American Society of Civil Engineers-Engineering Mechanics Division) Speciality Conference, Purdue University, May 25-28, 1983.

16. "Smoothing Stresses Computed Pointwise by the p-version of the Finite Element Method," by I.N. Katz and X-R. Ying, SIAM 1983 National Meeting, Denver, Colorado, June 6-8, 1983.

17. "The Use of High Order Polynomials in the Numerical Solution of Partial Differential Equations," a Mimi Symposium, I.N. Katz, Organizer and Chairman; "The h-p version of the Finite Element Method," I. Babuska, B. Szabo, K. Izadpanah, W. Gui, and B. Guo; "A Pseudospectral Legendre Method for Hyperbolic Equations," D. Gottlieb and H. Tal-Ezer; "The Approximation Theory for the p-version of the Finite Element Method," Milo Dorr; "On the Robustness of Higher Order Elements," M. Vogelius; SIAM Summer Meeting, University of Washington, Seattle, Washington, July 16-20, 1984.

18. "Implementation of a Finite Element Software System with h- and p-extension Capabilities," by B.A. Szabo, 8th Invitational Symposium on the Unification of Finite Element-Finite Differences and the Calculus of Variations, University of Connecticut, Storrs, Connecticut, May 3, 1985.

19. "Stress Singularities at Angular Corners of Composite Plates," X-R. Ying and I.N. Katz, SIAM Spring Meeting, Pittsburgh, PA, June 1985.

20. "Computation of the Amplitude of Stress Singular Terms for Cracks and Reentrant Corners," B.A. Szabo and I. Babuska presented at the 19th National Symposium on Fracture Mechanics, San Antonio, Texas, June 30-July 2, 1986.

21. "A Global Algorithm for Finding all Roots of an Analytic Function in a Finite Domain," X-R. Ying and I.N. Katz, presented at the SIAM 1986 National Meeting, Boston, Massachusetts, July 21-25, 1986.

22. "A Reliable Argument Principle Algorithm to Find the Number of Zeros of an Analytic Function in a Bounded Domain," X-R. Ying and I.N. Katz, presented at the Symposium on the Impact of Mathematical Analysis on the Solution of Engineering Problems, University of Maryland, September 17-19, 1986.

23. "On Stress Analysis with Large Length Rations," by B.A. Szabo, presented at First World Congress on Computational Mechanics, Austin, Texas, September 24, 1986.

24. "A Reliable Argument Principle Algorithm to Find the Number of Zeros of an Analytic Function in a Bounded Domain," X-R. Ying and I.N. Katz, presented at the First International Conference on Industrial and Applied Mathematics, June 29-July 3, 1987, Paris.

8.3 Visit and Seminars at Government Laboratories

1. "Advanced Stress Analysis Technology", by B.A. Szabo and I.N. Katz, presented on September 8, 1977 at the Air Force Flight Dynamic Laboratory, Wright-Patterson Air Force Base.

abstract

With one exception, all finite element software systems have element libraries in which the approximation properties of elements are frozen. The user controls only the number and distribution of finite elements. The exception is an experimental software system, developed at Washington University. This system, called COMET-X, employs conforming elements based on complete polynomials of arbitrary order. The elements are hierarchic, i.e. the stiffness matrix of each element is embedded in the stiffness matrices of all higher order elements of the same kind. The user controls not only the number and distribution of finite elements but their approximation properties as well. Thus convergence can be achieved on fixed mesh. This provides for very efficient and highly accurate approximation and a new method for computing stress intensity factors in linear elastic fracture mechanics. The theoretical developments are outlined, numerical examples are given and the concept of an advanced self-adaptive finite element software system is presented.

2. "The Constraint Method for Finite Element Stress Analysis," by I.N. Katz, presented at the National Bureau of Standards, Applied Mathematics Division on October 19, 1977.

abstract

In conventional approaches to finite element stress analysis accuracy is obtained by fixing the degree p of the approximating polynomial and by allowing the maximum diameter h of elements in the triangulation to approach zero. An alternate approach is to fix the triangulation and to increase the degrees of approximating polynomials in those elements where more accuracy is required. In order to implement the second approach efficiently it is necessary to have a family of finite elements of arbitrary polynomial degree p with the property that as much information as possible can be retained from the pth degree approximation when computing the $(p+1)$st degree approximation. Such a HIERARCHIC family has been formulated with $p \geq 2$ for problems in plane stress analysis and with $p \geq 5$ for problems in plate bending. The family is described and numerical examples are presented which illustrate the efficiency of the new method.

3. "The p-version of the Finite Element Method," by I.N. Katz and B.A. Szabo, to be presented at Air-Force Flight Dynamics Laboratory Wright-Patterson Air Force Base on April 23, 1981 (tentative date).

abstract

The theoretical basis of the p-version of the finite element method has been established only quite recently. Nevertheless, the p-version is already seen to be the most promising approach for implementing adaptivity in

practical computations. The main theorems establishing asymptotic rates of convergence for the p-version, some aspects of the algorithmic structure of p-version computer codes, numerical experience and a posteriori error estimation will be discussed from the mathematical and engineering points of view.

4. Attended the meeting sponsored by AFOSR on "The Impact of Large Scale Computing on Air Force Research and Development," at Kirtland Air Force Base, Albuquerque, New Mexico, April 4-6, 1984.

5. Attended the meeting sponsored by AFOSR on "The Impact of Supercomputing on Air Force Research and Development II," at Eglin Air Force Base, Florida, November 3-5, 1986.

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