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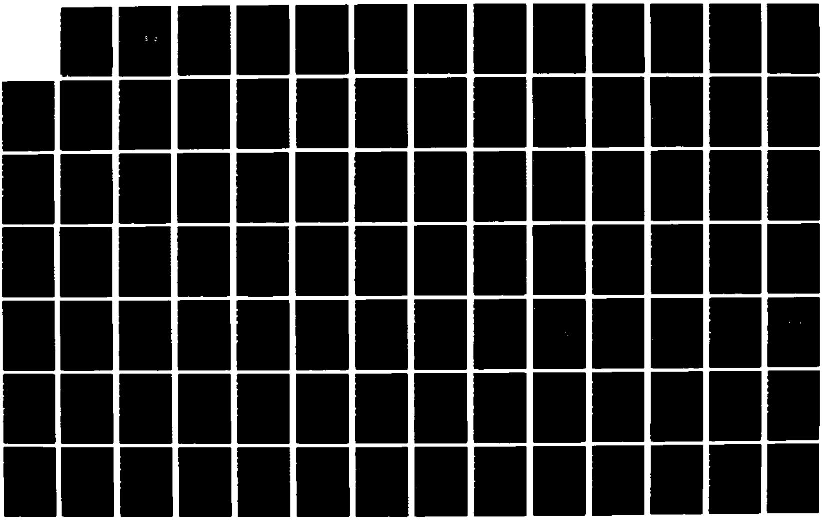

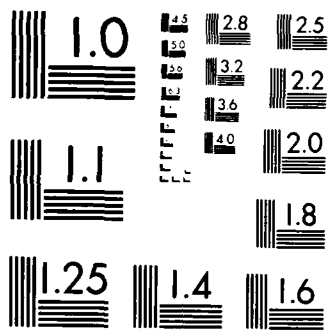

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THE TOTAL INTERVAL OF A GRAPH

Thomas Martin Kratzke

UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

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An interval representation (or simply representation) F of a graph G is a collection of finite sets $\{F(v): v \in V(G)\}$ of closed bounded intervals so that $u \leftrightarrow v$ if and only if there exist $I_u \in F(u)$, $I_v \in F(v)$ with $I_u \cap I_v \neq \emptyset$. The size of a representation is the number of intervals in the entire collection.

The total interval number of G is the size of the smallest representation of G and is denoted $I(G)$. This thesis studies $I(G)$ by proving best possible upper bounds for several classes of graphs. For some of the classes, the bounds are in terms of the number of vertices and for some of the classes, the bounds are in terms of the number of edges. The main result is that for planar graphs, $I(G) \leq 2n(G) - 3$.

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THE TOTAL INTERVAL OF A GRAPH

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THESIS

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THE TOTAL INTERVAL NUMBER OF A GRAPH

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University of Illinois at Urbana-Champaign, 1988
Douglas B. West, Advisor

An **interval representation** (or simply **representation**) R of a graph G is a collection of finite sets $\{R(v) : v \in V(G)\}$ of closed bounded intervals so that $u \leftrightarrow v$ if and only if there exist $\theta_u \in R(u)$, $\theta_v \in R(v)$ with $\theta_u \cap \theta_v \neq \emptyset$. The **size** of a representation is the number of intervals in the entire collection.

The **total interval number** of G is the size of the smallest represention of G and is denoted $I(G)$. This thesis studies $I(G)$ by proving best possible upper bounds for several classes of graphs. For some of the classes, the bounds are in terms of the number of vertices and for some of the classes, the bounds are in terms of the number of edges. The main result is that for planar graphs, $I(G) \leq 2n(G) - 3$.

To my patient loving wife, Trân

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I would also like to thank my parents. They have been unfailing in their support, dedication, and sacrifice, and I hope that a little of their spirit is evident in this work, and that it brings them some joy.

To my wife Trân, I give my gratitude as well as my devotion. She not only endured this project, but she sparked me when I was discouraged and rejoiced with me when I was successful. Her support, her friendship, and her love comprise a reward that goes beyond professional achievement.

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TABLE OF CONTENTS

I. INTRODUCTION AND PRELIMINARY RESULTS

1. Introduction

A **graph** G is a set of elements $V(G)$, called **vertices** and a multiset $E(G)$ of unordered pairs of vertices, called **edges**. If u and v are the vertices of the edge e, then this relation is denoted by $u \leftrightarrow v$, which can be read as "u is adjacent to v."

A **multiple interval representation** (or simply **representation**) R of a graph G is a collection $\{R(u) : u \in V\}$ of finite sets of closed bounded intervals of the real line so that $u \leftrightarrow v$ if and only if there exists $\theta_u \in R(u)$, $\theta_v \in R(v)$ with $\theta_u \cap \theta_v \neq \emptyset$. Unless otherwise specified, "interval" is assumed to mean "interval of the real line." We will use R to mean either the representation or the entire collection of intervals. The **size** of R is the number of intervals in the entire collection and is denoted $|R|$. The **total interval number** of G is the minimum size of a represention of G and is denoted $I(G)$. We will assume, without affecting I, that no two endpoints of intervals in any representation coincide and that no intervals corresponding to the same vertex have a non-empty intersection.

Below we give two representations of a graph. We will use the coloring of the edges in Figure I.1.1(a) and the coloring of the intervals in Figure I.1.1(b) in §I.6.

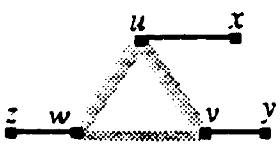


Figure I.1.1(a): G

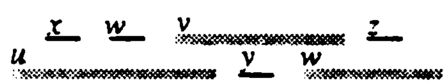


Figure I.1.1(b): R

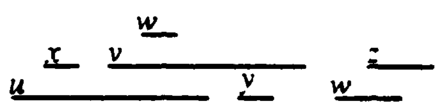


Figure I.1.1(c)

In §I.3, we will show that $I(G) \geq 7$ and so both representations are optimal. We will make further use of this example in §I.6 and §II.3.

This thesis discusses the total interval number. In particular, we find upper bounds on I for various classes of graphs. There are two types of upper bounds. The first is in terms of the number of vertices and the second is in terms of the number of edges.

There are five remaining divisions of §I. In §I.2, we introduce some standard graph theoretic terms and results that are not specifically related to the study of the total interval number. These are well known to most graph theorists and are included to make the thesis accessible to any mathematician. In §I.3, we give a brief background of the total interval number. In particular, we introduce the related parameter, interval number. In §I.4 and §I.5, we summarize the results that are proved in §II and §III and relate these results to the analogous results for the interval number. Many of the definitions that are given in §I.2 are required for §I.3, §I.4, and §I.5. In §I.6, we give additional terms and results. These are less standard and more intimately related to the study of I than the terms of §I.2. They are included in §I because they are important for almost all of §II and §III.

2. Standard Definitions

Most of these definitions are found in [6]. For a graph G, let $n(G)$ denote the number of vertices and $m(G)$ denote the number of edges of G. If $e = \{u,v\}$ is an edge, then we say that e **joins** u and v, u is **adjacent** to v, e is **incident** to both u and v, and the **endpoints** of e are u and v. If $e' = \{u,v'\}$, then we say that e and e' are **incident.** Furthermore, we will use $\{u,v\}$ and uv interchangeably to denote e. If there is no edge of the form uu and no two edges have the same pair of endpoints, then the graph is called **simple.** If we wish to emphasize that a graph is not necessarily simple, then we call it a **multigraph.** If the edge uv is repeated, then we call uv a **multiple edge.** Most of the graphs in this thesis are simple, and we consider I only for simple graphs. An **isolated** vertex is one that appears in no edge. Note that isolated vertices do not affect the total interval number. In particular, the total interval number of any graph that has no edge is zero.

The **trivial** graph is the graph that consists of one vertex and no edge [6 p. 3]. A subgraph H of G is a graph for which $V(G) \supseteq V(H)$ and $E(G) \supseteq E(H)$. A **vertex-induced** (or simply **induced**) subgraph H of G is a subgraph for which $E(H) = \{e \in E(G) : V(H) \supseteq e\}$, i.e., all edges of G that have both endpoints in $V(H)$. An **independent set** of vertices is a set of vertices whose induced subgraph has no edge.

A **walk** is a finite non-null sequence $<u_1,\ldots,u_k>$ of vertices such that for $1 \le i \le k-1$, $u_i \leftrightarrow u_{i+1}$ [6 p. 12]. The **ends** of a walk $<u_1,\ldots,u_k>$ are u_1 and u_k. A walk is **closed** if its ends are equal. A **trail** is a walk $<u_1,\ldots,u_k>$ for which $i \neq j$ implies that $\{u_i,u_{i+1}\} \neq \{u_j,u_{j+1}\}$. The **set of vertices of** a trail $<u_1,\ldots,u_k>$ is the set $\{u_i : i = 1,\ldots,k\}$. Note that these are not necessarily distinct. The **set of edges of** a trail $<u_1,\ldots,u_k>$ is the set $\{u_iu_{i+1} : i = 1,\ldots,k-1\}$ and these *are* distinct. Since we consider only simple graphs, a trail is specified equally well by its sequence of vertices or edges, so we may consider either of these or the subgraph consisting of both to be the trail. A **sub-**

trail of $<u_1,\ldots,u_k>$ is a trail $<u_l,u_{l+1},\ldots,u_{l+l'}>$ where $1 \le l$ and $l + l' \le k$. The **length** of a trail $<u_1,\ldots,u_k>$ is k - 1, the number of edges in the trail.

A **path** is a trail $<u_1,\ldots,u_k>$ for which all vertices are distinct. The **distance** in a graph G between two vertices u and v is the length of the shortest path in G between u and v. A **cycle** is a trail $<u_0,\ldots,u_{k-1}>$ for which $u_0 = u_{k-1}$ and, other than this, all vertices of the trail are distinct. To reflect the cyclic structure, we will use $(u_0u_1\ldots u_{k-1})$ for a cycle whose trail notation is $<u_1,\ldots,u_{k-1}>$. Thus, if u, v, and w are distinct vertices, then $<u,v,w,u>$ means the same as (uvw). The graph that consists entirely of a single cycle with k edges is denoted C_k. A **chord** of the cycle $(u_0u_1\ldots u_{k-1})$ is an edge $u_lu_{l'}$ where $0 \le l < l' \le k - 1$ and $|l - l'| \not\equiv 1(\text{mod } p)$.

A **Hamiltonian cycle** is a cycle that contains every vertex, and a graph is **Hamiltonian** if it contains such a cycle. A **Hamiltonian path** is a path that contains every vertex.

If $<u_1,\ldots,u_k>$ and $<u_k,u_{k+1},\ldots,u_{k+l}>$ are two trails, then the **concatenation** of these two trails is the trail $<u_1,\ldots,u_k,u_{k+1},\ldots,u_{k+l}>$ [6, p. 12]. If T_1 and T_2 are two trails, then we write T_1T_2 to denote their concatenation.

If G is a graph, V' is a subset of $V(G)$, and E' is a subset of $E(G)$, then the graph G - V' is the graph obtained by deleting the vertices of V' and any edge incident to any vertex of V', and G - E' is the graph obtained by deleting E'. If $V' = \{u\}$, then we refer to $G - \{u\}$ by $G - u$, and if $E' = \{e\}$, then we refer to to $G - \{e\}$ by $G - e$.

A graph G is **connected** if, for any $u,v \in V$, there exists a path $<u_1,\ldots,u_k>$ such that $u = u_1$ and $v = u_k$. If G is connected but $G - u$ $(G - e)$ is not connected, then u (e) is a **cut-vertex (cut-edge)**. Cut-edges of a connected graph are precisely those edges that are not in any cycle. A **block** of a graph is a maximal induced subgraph with no cut-vertex; note that it is possible that the only block of G is G itself.

If u is a cut-vertex, then let the vertex-sets of $G - u$ be $V_1,\ldots,V_p$ and define the u-**components** [6 p. 119] to be the subgraphs induced by $V_1 \cup \{u\}$, $V_2 \cup \{u\},\ldots,$ and $V_p \cup \{u\}$.

A graph whose vertices are pairwise adjacent is called a **complete graph** or a **clique** and, if it has n vertices, is denoted K_n.

We say that two sets **intersect** if their intersection is non-empty. If $\{S_i\}_{i=1}^{n}$ is a collection of sets, then the **intersection graph** that corresponds to $\{S_i\}_{i=1}^{n}$ is the graph with vertices $\{S_i\}_{i=1}^{n}$ and edges given by $v_i \leftrightarrow v_j$ if and only if $S_i \cap S_j \neq \emptyset$. If the S_i's are the vertex-sets of the blocks of the graph G, then the resulting intersection graph is called the **block graph of** G [5 p. 6] and is denoted $B(G)$. Every block of a block graph is a complete graph [5, p. 46] and from this it follows that there is a unique shortest path between any two vertices of a block graph.

A **neighbor** of a vertex u is a vertex v for which $u \leftrightarrow v$. The set of neighbors of u is denoted $N(u)$. The **valence** or **degree** of a vertex u is the number of edges that are incident to u and is denoted $d(u)$. If necessary, we will write $N_G(u)$ or $d_G(u)$ to emphasize which graph is under consideration. This notation is particularly useful when there are subgraphs that are of interest. If all of the vertices of a graph (or multigraph) have the same degree, then it is called **regular**, and if this degree is k, then it is called k-**regular**.

Suppose that G is connected and α is the number of vertices that are of odd degree. It is well known that α is even and, if $\alpha \geq 1$, then the edges can be partitioned into $\alpha/2$ trails and no fewer. [6 p. 53]. If no vertex is of odd degree, then there is a trail that contains all of the edges and starts and ends at the same vertex. In this case, G is called **Eulerian** and the trail is called an **Euler Tour** [6 p. 51].

A **leaf** is a vertex of degree one and a **leaf-edge** is an edge that is incident to a leaf. A **bivalent** vertex is a vertex of degree two. A **triangle** is a set of three vertices that are pairwise adjacent.

An **independent set** of vertices is a set of vertices that is pairwise non-adjacent. A **bipartite graph** is a graph G for which V can be partitioned into two independent sets, called its **partite sets**. A **complete bipartite graph** is a bipartite graph with partite sets V' and V'' and for which $u \in V'$ and $u' \in V''$ imply that u is adjacent to u'. A complete bipartite graph with partite sets that have sizes p and q is denoted $K_{p,q}$, and $K_{1,q}$ is called a **star** with q edges.

A **forest** is a cycle free graph and a **tree** is a connected forest. It is easy to see by induction that if G is a tree with n vertices, then it has n - 1 edges. We emphasize that the definitions of leaf and leaf-edge apply to *any* graph and not just to trees. However, it is true that any tree with at least two vertices has at least two leaves, and that the only trees that have *exactly* two leaves are paths. A **peripheral** vertex of a tree G is a leaf that is part of a maximum length path. A **branchpoint** is a vertex with degree at least three.

If, for any subset V' of V, where $|V'| < k$, $G - V'$ is connected, then we say that G is **k-connected.** If, for any subset E' of E, where $|E'| < k$, $G - E'$ is connected, then we say that G is **k-edge-connected.** The **connectivity** of G, denoted $\kappa(G)$, is defined to be the maximum k such that G is k-connected if $G \neq K_2$, and 1 if $G = K_2$. The **edge-connectivity** of G, denoted $\kappa'(G)$, is the maximum k such that G is k-edge-connected. The minimum of the vertex degrees is denoted $\delta(G)$. It is easy to show that $\kappa \leq \kappa' \leq \delta$ [6 p. 43]. If $\delta(G) \geq 2$, then we say that G is **leafless.** We also define $\Delta(G)$ to be the degree of a vertex with maximum degree.

A graph is often drawn by associating a point in R^2 with each vertex and, for each edge uv, drawing a continuous curve between u and v. A **planar graph** is a graph that can be drawn in R^2 in such a way that no pair of edges intersects except at a vertex. Such a drawing is called an **embedding.** A **plane** graph is a planar graph, together with a fixed embedding; we identify the points and curves of the embedding with vertices and edges of the graph. A **face** of a plane graph is a maximal connected region of the plane that does not intersect any edge or vertex. The boundary of any face consists of edges and vertices and we say that these are **incident** to the face. We will identify a face F with the graph induced by the edges that are incident to F. For a plane graph G, let $\phi(G)$ be the number of faces. It is easy to use induction on the number of edges of G to prove **Euler's formula,** which states that if G is connected, then $n(G) - m(G) + \phi(G) = 2$. The **degree** of a face is the number of edges that are incident to the face, where cut-edges are counted twice. The following examples illustrate this idea.

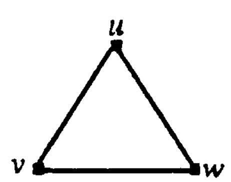


Figure I.2.1(a): G

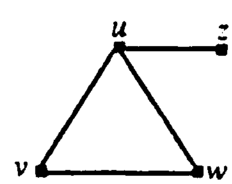


Figure I.2.1(b): H

Both faces of G are of degree three. The unbounded face of H is of degree five and the bounded face is of degree three.

For any plane graph, a cut-edge belongs to only one face and will be counted twice when computing the degree of that face. Other edges belong to two faces and will be counted once each when computing the degrees of these faces. By adding up the degrees of the faces, each edge is counted twice and so $2m = \sum_{i=2}^{n} if_i$, where f_i is the number of faces of degree i.

The **dual** of a plane graph G is the graph G^* defined as follows [6 p. 140]. Corresponding to each face F of G is a vertex F^* of G^*. If $e \in E(G)$ is incident to two faces F_1 and F_2, then there is a corresponding edge between F_1^* and F_2^*. If e is a cut-edge, then it is only incident to one face F and there is a corresponding edge $\{F^*,F^*\} \in E(G^*)$. Note that the dual might not be simple.

A planar graph having an embedding in which all of the vertices lie on a single face is called **outerplanar**. An outerplanar graph with no cut-vertex is either K_2 or a cycle with some non-crossing chords.

We **subdivide** an edge uv if we replace it by a new vertex w and new edges uw and wv. We **contract** an edge uv by replacing u, v, and their incident edges by a new vertex w with $N(w) = N_G(u) \cup N_G(v) - \{u,v\}$. If G is a graph with an edge e, then we use $G \bullet e$ to denote the graph G with the edge e contracted.

3. History

A graph G is an **interval graph** if it is the intersection graph of $\{S_i\}_{i=1}^{n}$, where each S_i is an interval. In other words, an interval graph is simply a graph whose total interval number equals its number of non-isolated vertices. The total interval number is one of several parameters used to generalize the concept of an interval graph.

Interval graphs and multiple interval representations arise in many "natural" contexts. Writing about genetics, Benzer [3] discussed systems of intersecting intervals in a genetics context but did not discuss the resulting graph. Other applications for interval representations include scheduling and avoiding interference in a cellular phone system. Hajos [14] wrote about them in a mathematical context. A thorough treatment of multiple interval representations, including applications, is given by Roberts [21].

There are several well-known characterizations of interval graphs, one of which we will find useful. An **asteroidal triple** is a set of three vertices x, y, and z with the following property: For any two vertices of $\{x,y,z\}$, there exists a path having no vertex adjacent to the remaining vertex of $\{x,y,z\}$. A **triangulated graph** is one for which any cycle with at least four edges has a chord.

Theorem I.3.1 [4]. A graph is an interval graph if and only if it is triangulated and has no asteroidal triple.

This theorem often allows us to deduce that some graphs are not interval graphs. For example, the graph of Figure I.1.1 has an asteroidal triple $\{x,y,z\}$ and so it is not an interval graph. Since the graph has six vertices, its total interval number must therefore be at least seven and the representations of Figures I.1.1 show that it is at most seven.

It is not surprising that most graphs are not interval graphs. Given a graph G, we would like to measure how close G is to being an interval graph. Differing criteria of "close" lead to different graph parameters. For example, the **boxicity** is defined as the minimum k such that G is the intersection graph of k-dimensional real intervals. More closely related to the topic of this thesis is the **interval number**, $i(G)$ of a graph G, which is defined as the smallest number of real intervals that must be assigned to some vertex in order to obtain a multiple interval representation of G. This is expressed below.

$$i(G) = min\{max\{|R(v)|: v \in V(G)\} : R \text{ is a represention of } G\} \tag{I.3.1}$$

Note the fundamental relationship $I \leq ni$, to which we shall return shortly.

The history of the interval number is much longer than that of the total interval number. The first results on i appeared in 1979, when Trotter and Harary [15] computed i for trees and complete bipartite graphs. Although the notion of total interval number is suggested in several papers (e.g. [13]) on interval number, the first paper dealing with total interval number, by Andreae and Aigner [2], will not appear until 1988. In the past decade, many results on i that are analogous to the results of this thesis on I have been established. In §I.4 and §I.5, we shall state what our results are and compare them with these results for i.

4. The Total Interval Number and the Number of Vertices

In §II, we will compare I to n for various classes of graphs. For each class, we will find an upper bound on I in terms of n and give examples to show that the bound is best possible. We have not yet given definitions for all of these clases. A **Husimi tree** is a graph for which every block is a clique. A **cactus** is a graph for which every edge is in at most one cycle and it is **dense** if every edge is in exactly one cycle.

The classes and bounds are given in Table I.4.1. For comparison, we include the analogous results for *ni* in the same table.

Class	$\max\{I(G) : n(G) = n\}$	Reference	$\max\{n(G)i(G) : n(G) = n\}$	Reference
Trees	$\frac{5n - 3}{4}$	[2], §II.1	$2n$	[15]
Dense Cacti	$\frac{11n - 4}{8}$	§II.2	$2n$	[23]
Cacti	$\frac{18n - 12}{13}$	§II.2†	$2n$	[23]
Husimi trees	$\frac{3n - 4}{2}$	§II.3	$2n$	§I.4
Outerplanar Graphs	$\frac{3n - 2}{2}$	§II.4#	$2n$	[23]
Planar Graphs	$2n - 3$	§II.5#	$3n$	[23]
Simple Graphs	$\frac{n^2 + 4}{4}$	§II.6#	$n\lceil\frac{n+1}{4}\rceil$	[10], [1]

†$\frac{11n - 4}{8}$ is conjectured in [2].

#Triangle-free case proved, general case conjectured, and extremal examples given in [2].

Table I.4.1

Our results for trees are more extensive than for other classes where we just obtain the upper bound. We obtain an algorithm for constructing an optimal representation and we also characterize those trees for which the contraction of any edge reduces the total interval number by two.

It is of interest to note that the only classes of graphs for which no extremal graph is triangle-free are dense cacti and Husimi trees; for the other classes, it is sufficient to consider triangle-free graphs to show that the results are best possible.

The sequence of classes in Table I.4.1 is, except for Husimi trees, an ascending sequence of classes. For any n, there are additional graphs to consider at each step of the sequence and so the bounds grow with the classes. Husimi trees is included as another way of generalizing trees.

Griggs [10] proved the bound on i for simple graphs and used $K_{\lfloor n/2\rfloor\lceil n/2\rceil}$ to show that it is best possible. Andreae [1] showed uniqueness for this extremal graph when the number of vertices is divisible by four, and later showed that this graph is also the unique triangle-free extremal graph when considering I [2]. He also obtained bounds for i in terms of the maximum size ω of a clique in triangulated graphs. Comparing I with ω has not yet been tried and it may be a rich topic. The reason for this is that ω limits the number of intervals that can overlap at a certain point and so it limits the number of edges that a fixed number of intervals can represent. Using this idea, it is not hard to derive the bound $I \geq \frac{m}{\omega - 1} + \omega/2$ and this would be a good place to start an investigation.

The cacti and outerplanar results for *ni* are corollaries of the proof for planar graphs. We complete the comments on Table I.4.1 by showing that if G is a Husimi tree, then $i(G) \leq 2$.

Theorem I.4.1. If G is a Husimi tree, then there is a representation R that assigns at most two intervals to each cut-vertex and exactly one interval to each other vertex.

Proof. Let γ be the number of cut-vertices; we use induction on γ. If $\gamma = 0$, then G is a clique and we can assign [0,1] to each vertex.

If $\gamma \geq 1$, then pick a cut-vertex u such that, except for at most one u-component of G, all u-components of G are cliques. If all u-components are cliques, then choose one of the u-components and call it G'. Otherwise, let G' be the u-component that is not a clique. Note that u is not a cut-vertex of G'. Let G'' be the union of rest of the u-components An example is given below.

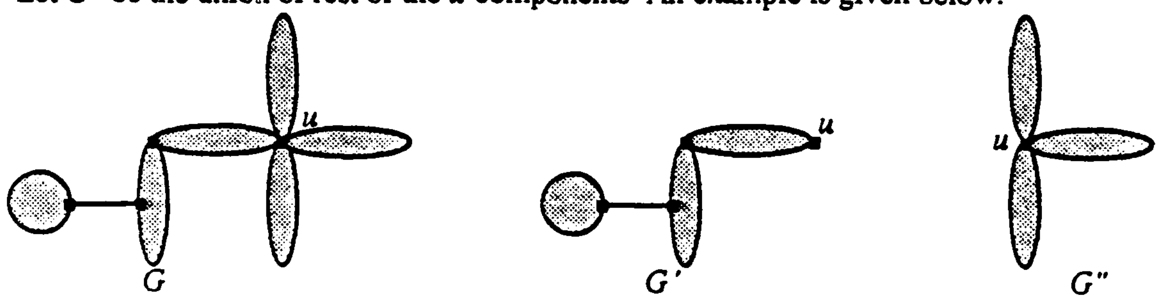


Figure I.4.2: Gray areas represeent cliques.

By induction, G' has a representation R' that assigns only one vertex to u. We can then represent the edges that are not in G' by using one interval for for each vertex in G''. Using the above example to demonstrate this, the representation for G'' would be as below.

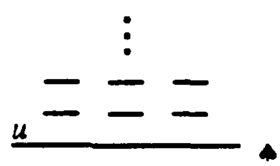


Figure I.4.3

We close §I.4 with a discussion of I in terms of n for random graphs, which were introduced by Erdös and Rényi [8] and are treated in the book by Palmer [21]. We define a probability model for each n by considering all $2^{\binom{n}{2}}$ labelled graphs on n vertices to be equally likely. If the probability in this model that a graph has property P approaches one as n approaches infinity, then we say that **almost all graphs have property** P. Let $\lg(n)$ denote $\log_2(n)$.

We know that almost all graphs satisfy $\frac{n^2}{4\lg(n)} \leq ni \leq \frac{n^2}{2\lg(n)}$. The lower bound was obtained by Erdös and West [9], and Scheinerman [25] recently proved the upper bound. Both bounds hold for I as well. The upper bound holds because $I \leq ni$. We follow the method of the Erdös-West proof on i to establish the lower bound.

Suppose that the vertex-set is $\{u_i : i = 1,\ldots,n\}$ and that we have a representation with p intervals. Let $\xi_1,\ldots,\xi_{2p}$ be the ascending sequence of endpoints of the intervals. For $i = 1,\ldots,2p$, let v_i be the vertex corresponding to the interval with an endpoint at ξ_i. Call $\langle v_1,\ldots v_{2p}\rangle$ the **derived** sequence of vertices. Because no vertex has two intervals that intersect each other, the odd occurrences of any vertex in the derived sequence of vertices correspond to left endpoints and the even occurrences correspond to right endpoints of intervals. Because the sizes of the intersections of intervals is immaterial, two representations with the same derived sequence of vertices represent the same graph. Since there are at most n^{2p} different derived sequences of vertices, we can represent only n^{2p} graphs.

Hence, to represent all graphs with at least n vertices, we need to choose p such that $n^{2p} \geq 2^{\binom{n}{2}}$. Taking the logarithm base 2 of both sides, we get $p \geq \frac{\binom{n}{2}}{2\lg(n)}$ and we denote the right side as $h(n)$. If $q(n) \leq h(n) - \varepsilon$ for some fixed small ε, then the proportion of graphs with n vertices that we can represent with $q(n)$ intervals is at most $n^{-2\varepsilon}$ and this approaches zero as n gets large. Hence the probability of a graph having total interval number of $q(n)$ or less goes to zero and we now have shown the first inequality.

Although $I \leq ni$, and Table I.4.1 suggests that, for sparse graphs, I and ni are far apart, we believe that the difference is small for most graphs. In particular, we believe that, for any $\varepsilon > 0$, $P(|I - ni| > n\varepsilon) \to 0$ as $n \to \infty$. This means that, for almost all graphs, there exists a minimum representation that assigns about i intervals to each vertex.

5. The Total Interval Number and the Number of Edges

In §II, we will compare I with m for various classes of graphs. For some of these classes, we will give upper bounds on I in terms of m and give examples to show that the bound is best possible. For other classes, we can only give examples to show how big I can be for fixed m. The classes and bounds are given in Table I.5.1.

Class	$\max\{I(G) : m(G) = m\}$	Reference
Arbitrary	$2m$	§II.1
Connected	$\frac{5m+2}{4}$	§II.1
$\delta \geq 2$, connected	$\frac{9m+1}{8}$	§II.2
$\delta \geq k$, for $k \geq 3$, connected	†	§II.2
2-connected	$\frac{10m}{9}$	§II.3
3-connected	†	§II.3
4-connected	$m+1$	§II.3
2-edge-connected	$\frac{10m}{9}$	§II.3
3-edge-connected	†	§II.3
4-edge-connected	$m+1$	[2] §I.6

†Constructions show only that I can be greater than $(1+\varepsilon)m$ for some $\varepsilon > 0$.

Table I.5.1

The nature of this collection of classes is quite different from the collection of Table I.4.1. For fixed n, the larger classes of Table I.4.1 allow more edges. When we fix the number of *edges*, we again want a chain of classes to study, but it is not clear what classes yield interesting bounds. The impetus for such a study came from considering a question raised by Andreae and Aigner [2]; what is the maximum total interval number for a connected graph with m edges? We will obtain a solution that, in terms of Table I.4.1, seems quite surprising: If $m \equiv 2 \pmod 4$, then the *unique* connected graph with a maximum total interval number is a tree.

For a tree G, $\delta(G) = \kappa(G) = \kappa'(G) = 1$. Therefore we studied the effects of the parameters δ, κ, and κ'. By raising these parameters, we forbid the extremal example and thus obtain other interesting classes to study. Raising these parameters also decreases the classes and so it is not surprising that the upper bound for I/m decreases. Our results are not quite as complete for these classes as they are for the classes of Table I.4.1. For some values of these parameters, we simply use examples to obtain a lower bound on the upper bound and we do not claim that these examples are extremal.

The bound $I \leq m-1$ for 4-edge-connected graphs was noted by Andreae and Aigner [2], by citing a result of Jaeger [17]. In §I.6, we will show that, for triangle-free graphs, $I(G) \geq m(G) - 1$

and, for any complete bipartite graph G, $I(G) = m(G) + 1$. Thus, $\kappa'(G) \geq 3$ is the last interesting question for connectivity classes. The examples referred to in Table I.5.1 show that there is no such trivial bound for other connectivity classes.

It is more difficult to compare bounds on I with bounds on i when the bounds are in terms of m than when they are in terms of n. Spinrad, Vijayan, and West [9] proved that $i(G) \leq 1 + \sqrt{m(G)/2}$, with equality for triangle-free regular graphs. This result is hard to relate to I. However comparison is easier when we focus on a local vertex property. Griggs and West [15] proved that $i(G) \leq \lceil \frac{\Delta(G) + 1}{2} \rceil$ and we can compare ni to I here. Suppose that k is odd and that G is k-regular so that $n = 2m/k$). The above bound then gives us $ni \leq nk/2 + n/2 = m + n/2$. We will soon see that $I \leq m + n/2$ and so these bounds agree for this class of graphs.

6. Fundamental Ideas for the Study of the Total Interval Number

An edge uv is **twice subdivided** if it is subdivided and then one of the new edges is subdivided. A **penultimate** vertex of a tree is one for which all but at most one of its neighbors is a leaf (if all of its neighbors are leaves, then it is the central vertex of a star).

If F is a face and no vertex incident to F is a cut-vertex, then the edges that are incident to F are the edges of a cycle. If this is the case, then we will refer to the face by citing the cycle. If we have such a face $F = (u_0,\ldots,u_{k-1})$, then we say that it is a k-**gon** and that u_i and u_j are l **steps apart** on F where l is the distance on F between u_i and u_j. Note that a plane graph with no cut-vertex is one that is either a single edge or one for which every face is a cycle. Note that the vertices of any 3-gon form a triangle but that some triangles are not the vertex-sets of any 3-gon.

If R is a set of intervals, α is the leftmost endpoint of any interval in R, and β is the rightmost endpoint of any interval in R, then let $F(R) = [\alpha,\beta]$. We say that R is **contiguous** if $F(R) = \cup \{\theta : \theta \in R\}$. A maximal contiguous subset of $R(V)$ is called a **component** of R. If $\theta \in R(u)$, then we say that θ is a u-**interval**. If θ is a u-interval and some subinterval θ' of θ intersects no other member of $R(V)$, then both θ and u are called **displayed** and θ' is called a **displayed part** of θ. Note that we can place a small interval within θ' without changing the fact that θ is displayed.

We order the intervals in $R(V)$ by left endpoints. If the left endpoint of θ_1 is to the left of θ_2, then we say that θ_1 **is earlier than** θ_2. An interval that overlaps k - 1 earlier intervals is called a

depth-k interval. The earliest interval is necessarily a depth-1 interval. Note that since the intervals of representations have distinct endpoints, there is a gap of positive length between the left endpoint of a depth-1 interval and all earlier intervals. We say that a depth-k interval **introduces** the k - 1 edges that are accounted for by its intersecting earlier intervals. For example, if a u-interval intersects an earlier v-interval and an earlier w-interval, then we say that the u-interval introduces both uv and uw. If no edge is introduced more than once, then the representation is **irredundant**. A **component** of a representation R is a maximal real interval, every point of which is in some member of R.

Let $r_k(R)$ be the number of depth-k intervals in R. Note that R has exactly r_1 components. In an irredundant represention, each depth-k interval corresponds to k - 1 edges and so $m = \sum_{k=2}^{n-1}(k-1)r_k$. Therefore small irredundant representations will have relatively few intervals of small depth and relatively many intervals of large depth.

It is nevertheless often useful to restrict ourselves to represention with no intervals of large depth. A **depth-k representation** is a representation with no interval of depth more than k. For a graph G, let $I_k(G)$ be the minimum size of a depth-k represention of G.

We are most often interested in depth-2 representations and the corresponding parameter I_2. Note that, for triangle-free graphs, all representations are depth-2 and therefore $I_2 = I$.

Depth-2 representations are intimately connected with trail covers. A trail **visits (ends at)** a vertex u if u is in (an end of) the trail. A trail **covers** an edge if it visits one of the ends. A **trail cover** of G is a collection of edge-disjoint trails in such a way that every edge of G is covered by at least one trail. The minimum number of trails in a trail cover of G is called the **trail cover number** $t(G)$. A trail cover with a minimum number of trails is called **optimal**. If there exists a trail T such that $\{T\}$ is a trail cover, then we call T a **covering trail**. A trail cover $\mathcal{T}$ **visits (ends at)** u if some member of $\mathcal{T}$ visits (ends at) u.

Suppose that $T = <u_1,\ldots,u_k>$ is a trail. We define the **canonical representation** of T to be the depth-2 represention that has k intervals, each of which is displayed, with left endpoints in the order $u_1,\ldots,u_k$. We illustrate this in Figure I.6.1. Note that the representation is contiguous.

If R_1 and R_2 are representations for the graphs G and H, then the representation $R_1 \cup R_2$ of $G \cup H$ is defined by shifting the intervals of R_2 so that no member of R_1 intersects any member of R_2, and then taking the union of R_1 and R_2.

Figure I.6.1

Since the theme of the thesis is to minimize the number of intervals, the operation of making one interval from two is quite desirable. If R is a representation, $\theta_1, \theta_2 \in R(u)$, $\theta_1 = [\alpha,\beta]$, $\theta_2 = [\gamma,\delta]$, $\alpha < \beta < \gamma < \delta$, and $[\beta,\gamma]$ intersects no member of $R(V)$, then we can replace θ_1 and θ_2 by $[\alpha,\delta]$ to obtain a smaller representation. We call this operation a **splice**.

Theorem I.6.1. For any graph G, $I_2(G) = m(G) + t(G)$. Furthermore, any minimum depth-2 represention has exactly t components.

Proof. We first show that $I_2(G) \leq m(G) + t(G)$. Suppose that $\mathcal{T}$ is an optimal trail covering so that $|\mathcal{T}| = t$. Let R' be the union of the canonical representations of the trails in $\mathcal{T}$. Then for each edge uv that is not in any $T \in \mathcal{T}$, at least one of its vertices, say u, is in some $T \in \mathcal{T}$ and there is a displayed u-interval θ in R'. Place a small v-interval inside a displayed part of θ. Having done this for all such edges, call the resulting representation R.

Every interval in R introduces exactly one edge except for the intervals corresponding to the first vertices of each trail. Hence $I_2(G) \leq |R| = t(G) + m(G)$.

The reverse inequality is essentially proved by reversing the above construction. Suppose that R is an optimal depth-2 represention. Because R is depth-2, any non-empty intersection of intervals can be eliminated by shortening or removing intervals without affecting other intersections. Hence we may assume that R is irredundant.

Remove the intervals that are not displayed and consider the set R' of remaining intervals. Because R is irredundant, R' is a union of canonical representations for a set $\mathcal{T}$ of edge-disjoint trails with each depth-1 interval corresponding to the first vertex of a member of $\mathcal{T}$. Hence $|\mathcal{T}| = r_1(R') = r_1(R)$.

Each edge in G is either an edge of some member of $\mathcal{T}$ or it is introduced by a non-displayed interval placed within the displayed part of an interval corresponding to some vertex of a member of $\mathcal{T}$. Hence $\mathcal{T}$ is a trail cover.

We still must show that $|\mathcal{T}| = I_2(G) - m(G)$. Since R is optimal, this is the same as showing that $|\mathcal{T}| = |R| - m(G)$. Since R is irredundant, there is a one-to-one correspondence between depth-2 intervals and edges. Hence $m(G) = r_2(R) = |R| - r_1(R) = |R| - |T|$.♠

Consider the graph G and representation R of Figure I.1.1. The optimal trail cover is marked on G with a thick gray line ($t = 1$) and the corresponding R' of the above proof is marked with thick gray intervals. As a more abstract example, we can partition the edges into $n/2$ trails (fewer unless every vertex is of odd degree), and so $I \leq m + n/2$; this result was of interest in §I.5.

The previous lemma shows that $m + t$ is an upper bound on I since $m + t = I_2 \geq I$. Since the parameter t is conceptually easy to deal with, finding bounds on t is very important when studying I. We now present a few fundamental tools for doing this.

The **edge-set (vertex-set) of a trail cover** is defined to be the union of the edge-sets (vertex-sets) of the constituent trails. An edge e (vertex u) is **vital** if, for every trail cover T with edge-set S_1 (vertex-set T_1), there is a set S_2 of edges (T_2 of vertices) that contains e (contains u) such that there exists a trail cover T' with edge-set $S_1 \cup S_2$ (vertex-set $T_1 \cup T_2$) and $|T'| = |T|$.

When trying to find an optimal trail cover, one can start by assuming that all vital edges are in the edge-set, and then augment or merge trails.

Lemma I.6.2.

(i) The neighbor of a leaf is vital.

(ii) If $u \leftrightarrow v$ and $d(u) = d(v) = 2$, then uv is vital.

(iii) If $u \leftrightarrow v$, $w \leftrightarrow v$, and $u \neq w$, then we may subdivide vw and delete uv without decreasing the trail cover number.

(iv) If $N(v) = \{u,w\}$, $u \neq w$, $x \leftrightarrow u$, $y \leftrightarrow v$, and $v \notin \{x,y\}$, then we may subdivide ux and wy and remove $\{uv,vw\}$ without decreasing the trail cover number.

Proof. The first two assertions are trivial.

For (iii), let G' be the graph that results from applying (iii) to G and let the new edges be ux and xw. Note that, for any trail cover of G', there is a corresponding trail cover of G that is obtained by contracting the edge ux and replacing any subtrail $<u,x,w>$ by $<u,w>$.

To cover the edge ux in G', some trail T must visit either u or x. If T visits u, then we can use the corresponding trail cover in G. If T visits x, then the trail contains either the edge ux or xw. In either case, we may assume that T continues to w or x, and so we may assume that both u and w are visited. Therefore the corresponding trail cover in G contains u and so e is covered.

For (iv), we simply repeat the argument of (iii) one more time. ♠

We call the operation of Lemma I.6.2(iii) a **snip** and the operation of Lemma I.6.2(iv) a **double snip**.

We now use the characterization of I_2 that is given in Theorem I.6.1 to show that computing I is NP-complete. It is well-known [11] that the problem of determining if there exists a Hamiltonian path for triangle-free 3-regular planar graphs is NP-complete. Given such a graph G, replace each vertex as shown in Figure I.6.2, and call the resulting graph $K(G)$.

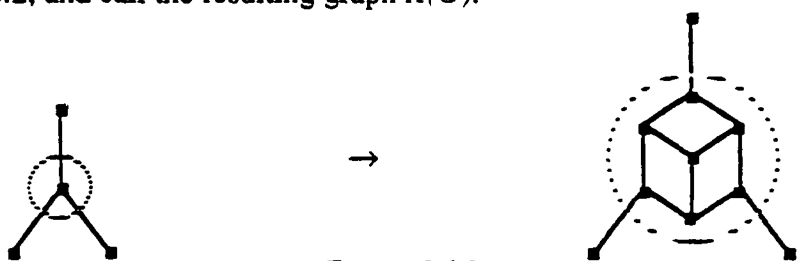


Figure I.6.2

Lemma I.6.3. A 3-regular planar graph G has a Hamiltonian path if and only if $t(K(G)) = 1$.

Proof. Let H be the seven-vertex replacement of each vertex in G. If $t(K(G)) = 1$, then let T be a covering trail. Because of the edges within each copy of H, T must enter each copy of H, and because there are only three edges from a copy of H to the rest of the graph, T can pass through H at most once. During this visit of H, it is possible for T to visit every vertex of H and so we may assume that T visits each copy of H exactly once. Contracting the edges of T that are within each copy of H gives a path through G that visits each vertex exactly once and we have a Hamiltonian path of G.

Conversely, if G has a Hamiltonian path, then, instead of passing through a vertex u of G, we can have the path touch each vertex of the copy of H that replaced u and get a covering path for $K(G)$. ♠

Theorem I.6.4. The decision problem $I \leq m + 1$ is NP-complete.

Proof. We restrict our class of graphs to the class that arises from replacing every vertex of a 3-regular planar graph with H as in Lemma I.6.3. Such graphs are triangle-free and therefore, for this class, $I \leq m - 1$ if and only if there is a covering trail. By Lemma I.6.3, a fast algorithm for testing coverability by one trail would yield a fast algorithm for deciding whether a 3-regular planar graph has a Hamiltonian path. Hence we have reduced the problem of deciding whether a 3-regular planar graph has a Hamiltonian path to deciding whether $I \leq m - 1$.

The decision problem $I \leq m + 1$ is in NP since, given a graph G, a non-deterministic algorithm can guess a set of $m(G) + 1$ intervals and verify in polynomial time that this is an interval representation of G. ♠

II. THE TOTAL INTERVAL NUMBER AND THE NUMBER OF VERTICES

1. Trees

The first main result of §II.1 is a proof of the correctness of an algorithm for finding a minimum trail cover. Since trees are triangle-free, $I = I_2 = m + t$ and so this algorithm yields an algorithm for computing the total interval number of trees. The second main result of §II.1 is a characterization of trees for which the contraction of any edge decreases the trail cover number. Note that the contraction of any edge of any tree does not increase the trail cover number. We will use this characterization to give a new proof of the upper bound on the total interval number of trees and to show that if $n \equiv 3 (mod\ 4)$, then there is a unique extremal graph.

We now start on the first main result. Suppose that G is a tree and that $u \in V(G)$. A vertex u is **partially useful (useful)** if some optimal trail cover ends at u (contains $\langle u \rangle$). Note that a useful vertex is a partially useful vertex. Relative to a vertex u, we need to define two kinds of optimal trail covers. The definitions of these kinds depend on whether u is useful, partially useful but not useful, or not partially useful. A trail cover is **partially u-optimal** or **u-optimal** as described in the following table:

If u is:	then a trail cover $\mathcal{T}$ is a partially u-optimal trail cover if $\mathcal{T}$ is:	then a trail cover $\mathcal{T}$ is a u-optimal trail cover if $\mathcal{T}$ is:
useful	optimal and ends at u.	optimal and contains $\langle u \rangle$.
partially useful	optimal and ends at u.	optimal and ends at u.
not partially useful	optimal.	optimal.

Table II.1.1

Note that a u-optimal trail cover is a partially u-optimal trail cover.

Fix G and some $u \in V(G)$. The algorithm that we describe will find a u-optimal trail cover of G. Let $N(u) = \{u_1, \ldots, u_d\}$. For each i, let G_i be the component of $G - u$ that contains u_i. Let:

$$\mathcal{C} = \{(\mathcal{T}_1, \ldots, \mathcal{T}_d) : \mathcal{T}_i \text{ is a partially } u_i\text{-optimal trail cover of } G_i\} \qquad \text{(II.1.1)}$$

If $\mathcal{T}$ is a trail cover of G, then deleting u and all edges incident to u from each $T \in \mathcal{T}$ that contains u results in a set $\mathcal{T}'$ of trails for which each trail is within one G_i. Let $\mathcal{T}_i = \{T : V(G_i) \supseteq V(T)\}$. It is easy to see that $\mathcal{T}_i$ is a trail cover of G_i. Let $A(\mathcal{T}) = (\mathcal{T}_1, \ldots, \mathcal{T}_d)$.

Theorem II.1.1. If $\mathcal{T}$ is u-optimal, then $A(\mathcal{T}) \in \mathcal{C}$.

Proof. Let $(\mathcal{T}_1^*,\ldots,\mathcal{T}_d^*) = A(\mathcal{T})$. If the theorem is false, then we may assume that $\mathcal{T}_1^*$ is not a partially u_1-optimal trail cover of G_1. We now construct a trail cover $\mathcal{U}$ that will demonstrate that $\mathcal{T}$ is not u-optimal. Let $\mathcal{T}_1 = \{T \in \mathcal{T} : V(G_1) \supseteq V(T)\}$ and let $\mathcal{U}_1^*$ be a partially u_1-optimal trail cover of G_1.

Case i. No member of $\mathcal{T}$ contains uu_1 and hence $\mathcal{T}_1^* = \mathcal{T}_1$. Suppose that $|\mathcal{U}_1^*| < |\mathcal{T}_1|$. Let $\mathcal{U}' = \mathcal{T} - \mathcal{T}_1 + \mathcal{U}_1^*$. Then $|\mathcal{U}'| < |\mathcal{T}|$ and, since $\mathcal{T}$ is optimal, $\mathcal{U}'$ is not a trail cover of G. But the only possible edge that is not covered by $\mathcal{U}'$ is uu_1. This implies that $\mathcal{T}$ does not visit u. But then $\mathcal{U} = \mathcal{U}' \cup \{<u>\}$ is a trail cover of G, $|\mathcal{U}| \leq |\mathcal{T}|$, and $\mathcal{U}$ ends at u, so $\mathcal{T}$ cannot be u-optimal.

Hence we may assume that $|\mathcal{U}_1^*| = |\mathcal{T}_1|$. Since $\mathcal{T}_1$ is not partially u_1-optimal, $\mathcal{T}_1$ does not end at u_1 and $\mathcal{U}_1^*$ does end at u_1. Let U' be the member of $\mathcal{U}_1^*$ that ends at u_1. Extend U' to u and call the resulting trail U. Let $\mathcal{U}_1 = \mathcal{U}_1^* - U' + U$ and let $\mathcal{U}' = \mathcal{T} - \mathcal{T}_1 + \mathcal{U}_1$. If $\mathcal{T}$ ends at u, then the corresponding trail can be concatenated with U to obtain a smaller trail cover $\mathcal{U}$. If $\mathcal{T}$ does not end at u, then let $\mathcal{U} = \mathcal{U}'$. In each case $\mathcal{U}$ demonstrates that $\mathcal{T}$ is not u-optimal. This completes Case i.

Case ii. There exists $T \in \mathcal{T}$ such that $E(T) \supseteq \{uu_1\}$. Let T' be the trail induced by the vertices of T that are not in $V(G_1)$. Note that T' ends at u. Since $\mathcal{T}_1^*$ ends at u_1 and is not partially u_1-optimal, $|\mathcal{U}_1^*| < |\mathcal{T}_1^*|$. Let $\mathcal{U}' = \mathcal{T} - \mathcal{T}_1 - T + T' + \mathcal{U}_1^*$. Note that $|\mathcal{U}'| = |\mathcal{T}|$ and $\mathcal{U}'$ is a trail cover of G.

If T ends at u, then $T' = <u>$, $\mathcal{T}$ does not contain $<u>$, and $\mathcal{U}'$ *does* contain $<u>$. Hence $\mathcal{T}$ is not u-optimal.

If T does not end at u, then there are two cases. If $\mathcal{T}$ ends at u, then, in $\mathcal{U}'$, concatenate the member other than T' of $\mathcal{T}$ that ends at u with T' and call the resulting trail cover $\mathcal{U}$. Because of the concatenation, $|\mathcal{U}| < |\mathcal{T}|$. If $\mathcal{T}$ does not end at u, then let $\mathcal{U} = \mathcal{U}'$. Since T' ends at u, $\mathcal{U}$ ends at u.

In either case, the existence of $\mathcal{U}$ proves that $\mathcal{T}$ is not partially u-optimal and hence is not u-optimal. ♠

We now demonstrate that both types of optimality are necessary by giving two examples. In Figure II.1.2(a), we have a graph with a *partially* u-optimal trail cover (indicated with thick gray lines). If we remove u from the trail cover, then we are left with a non-optimal trail cover for G_1. In Figure

II.1.2(b), we have a u-optimal trail cover, but if we remove u, then the resulting trail cover of G_2 is only *partially* u_2-optimal and not u_2-optimal.

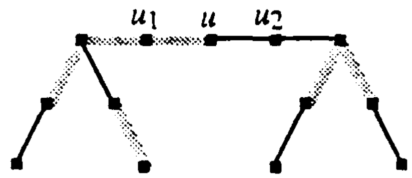


Figure II.1.2(a)

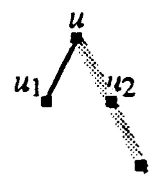


Figure II.1.2(b)

If we adopt the convention that the trail $\langle u \rangle$ has two ends at u, then the optimality, u-optimality, and partial u-optimality of a trail cover are all determined by the number of and placement of the trail cover's ends. For example, if two trails intersect at some vertex u, then which continuations from u are assigned to which trails is irrelevant for our purposes. Henceforth, we will ignore these distinctions and simply consider them to be the same trail cover.

The algorithm for finding a u-optimal trail cover is essentially the reverse of the removal of u from a u-optimal trail cover. It takes as input a member of $\mathcal{C}$ and gives as output a trail cover of G.

Algorithm II.1.2. Suppose that $(\mathcal{T}_1,\ldots,\mathcal{T}_d) \in \mathcal{C}$. If $d = 0$ (i.e. $G = K_1$), then $\mathcal{T} = \emptyset$. Otherwise, use the following algorithm.

a. Let $\mathcal{U}_1 = \mathcal{T}_1 \cup \ldots \cup \mathcal{T}_d$.
b. Extend any member of $\mathcal{U}_1$ that ends at any u_i to u. Call the resulting set of trails $\mathcal{U}_2$.
c. If $\mathcal{U}_2$ does not vist u (i.e., there were no extensions in step b.) and some $\mathcal{T}_i$ does not visit u_i, then let $\mathcal{U}_3 = \mathcal{U}_3 \cup \langle u \rangle$. Otherwise, let $\mathcal{U}_3 = \mathcal{U}_2$.
d. If two members of $\mathcal{U}_3$ end at u, then concatenate them to form one trail. Repeat this until at most one trail ends at u. Call the resulting set of trails $\mathcal{T}$.

It is clear that if $\mathcal{T}$ is u-optimal, then applying Algorithm II.1.2 to $A(\mathcal{T})$ produces $\mathcal{T}$, a u-optimal trail cover of G. Our next goal is to show that we can start with *any* member of $\mathcal{C}$ and apply Algorithm II.1.2 to obtain a u-optimal trail cover. The following lemma is phrased in terms of u and G but its first application is for each u_i, G_i combination.

Lemma II.1.3. Suppose that some partially u-optimal trail cover $\mathcal{T}$ of G visits but does not end at u. Then every partially u-optimal trail cover of G visits but does not end at u.

Proof. Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are partially u-optimal trail covers of G, with $\mathcal{T}$ visiting but not ending at u, and $\mathcal{T}'$ not visiting u. Since $\mathcal{T}$ does not end at u, no optimal trail cover ends at u and the set of optimal trail covers is the same as the set of partially u-optimal trail covers which is the same a the set of u-optimal trail covers. Hence $\mathcal{T}$ and $\mathcal{T}'$ are u-optimal.

We apply Theorem II.1.1 to each of $\mathcal{T}$ and $\mathcal{T}'$. The number of trails produced by removing u from $\mathcal{T}'$ is $|\mathcal{T}'|$ since $\mathcal{T}'$ does not visit u. But the number of trails produced by removing u from $\mathcal{T}$ is at least $|\mathcal{T}| + 1$ since $\mathcal{T}$ visits but does not end at u. This is impossible since Theorem II.1.1 shows that these numbers must be the same.♠

Theorem II.1.4. If $(\mathcal{T}_1,\ldots,\mathcal{T}_d) \in \mathcal{C}$, then applying Algorithm II.1.2 to $(\mathcal{T}_1,\ldots,\mathcal{T}_d)$ produces a u-optimal trail cover of G.

Proof. Let $\mathcal{T}'$ be a u-optimal trail cover and $(\mathcal{T}_1',\ldots,\mathcal{T}_d') = A(\mathcal{T}')$. By Theorem II.1.1, each $\mathcal{T}_i'$ is a partially u_i-optimal trail cover of G_i and so $\mathcal{T}_i'$ ends at u_i if and only if $\mathcal{T}_i$ does. If, for at least one i, $\mathcal{T}_i$ ends at u_i, then the result is straightforward.

If no $\mathcal{T}_i$ ends at u_i, then the only way that the theorem can fail is if, for some i, $\mathcal{T}_i'$ visits u_i and $\mathcal{T}_i$ does not. (If this happens, then Algorithm II.1.2 applied to $(\mathcal{T}_1,\ldots,\mathcal{T}_d)$ will produce the trail $<u>$ in step c). But this is impossible by Lemma II.1.3.♠

It is clear that $t(K_1) = 0$ and so the only partially u-optimal trail cover of K_1 is the empty set of trails. Furthermore, Algorithm II.1.2 produces a u-optimal trail cover of G and therefore a *partially* u-optimal trail cover. We now have the following recursive algorithm for finding a u-optimal trail cover $\mathcal{T}$ of G.

Algorithm II.1.5.

a. If $G = K_1$, then $\mathcal{T} = \emptyset$. Otherwise, go to step b.
b. a) For each i, apply this algorithm to G_i to obtain a u_i-optimal trail cover $\mathcal{T}_i$ of G_i.
   b) Apply Algorithm II.1.2 to $(\mathcal{T}_1,\ldots,\mathcal{T}_d)$ to get $\mathcal{T}$.

By modifying Algorithm II.1.5 slightly, we obtain an algorithm for computing the trail cover number without storing the trails. For a tree G and a vertex u, let $\mathcal{T}$ be a u-optimal trail cover of G. Define $\upsilon(u,G)$ as follows:

$\upsilon(u,G) = 0$ if $\mathcal{T}$ does not visit u.
$\upsilon(u,G) = 1$ if $\mathcal{T}$ visits u but does not end at u.
$\upsilon(u,G) = 2$ if $\mathcal{T}$ ends at u but does not contain $<u>$.
$\upsilon(u,G) = 3$ if $\mathcal{T}$ contains $<u>$.

By Lemma II.1.3 and the definitions of u-optimal, the value of $\upsilon(u,G)$ is independent of the choice of the u-optimal trail cover $\mathcal{T}$. Define d, $\{u_1,\ldots,u_d\}$, and $\{G_1,\ldots,G_d\}$ as before. For $j = 0, 1, 2$, and 3, let $\alpha_j(u,G) = |\{i : \upsilon(u_i,G_i) = j\}|$ and $\beta(G) = \alpha_3(G) + \alpha_4(G)$. Algorithm II.1.6 computes $t(G)$, $\upsilon(u,G)$, $\alpha_0(u,G),\ldots, \alpha_3(u,G)$, and $\beta(u,G)$.

Algorithm II.1.6.

a. If $G = K_1$, then $\upsilon(u,G) = t(G) = 0$. Otherwise, go to step b.

b. a) For $i = 1,\ldots,d$, apply this algorithm to (u_i,G_i) to obtain $t(G_i)$ and $\upsilon(u_i,G_i)$. Use these to compute $\alpha_0(u,G),\ldots, \alpha_3(u,G)$, and $\beta(u,G)$.

   b) Let $t' = \sum_{i=1}^{d} t(G_i) - \lfloor \beta(u,G)/2 \rfloor$.

   c) If $\beta(u,G) = 0$ and $\alpha_2(u,G) < d$, then $t(G) = 1 + t'$ and $\upsilon(u,G) = 4$.

   d) If $\beta(u,G) = 0$ and $\alpha_2(u,G) = d$, then $t(G) = t'$ and $\upsilon(u,G) = 1$.

   e) If $\beta(u,G) > 0$ and β is even, then $t(G) = t'$ and $\upsilon(u,G) = 2$.

   f) If $\beta(u,G)$ is odd, then $t(G) = t'$ and $\upsilon(u,G) = 3$.

We now concentrate on the second main result of §II.1. Recall that the result of contracting $e \in E(G)$ is denoted $G \bullet e$. An edge e is **contractible** if $t(G \bullet e) = t(G)$ and a tree is **critical** if it has no contractible edge. A critical tree G is k-**critical** if $t(G) = k$. Let $\mathcal{G}_k$ be the set of k-critical trees.

The next lemma is a collection of simple observations.

Lemma II.1.7.

(i) If e is incident to a penultimate vertex and e is not a leaf-edge, then e is vital.

(ii) If an optimal trail cover of a tree G contains intersecting trails $\langle u_1,\ldots,u_p,u,\ldots\rangle$ and $\langle\ldots,u,w_1,\ldots,w_q\rangle$, then some optimal trail cover contains the trail $\langle u_1,\ldots,u_p,u,w_1,\ldots,w_q\rangle$. ♠

Note that Lemma II.1.7(ii) is not necessarily true of graphs with cycles.

Lemma II.1.8. Each non-leaf neighbor of a penultimate vertex is not useful.

Proof. Let u be a penultimate vertex in a tree and let v be its non-leaf neighbor. Suppose that $\mathcal{T}$ is a trail cover that contains $\langle v\rangle$. The leaf edges that are incident to u must be covered and this requires another trail T. Replacing T and $\langle v\rangle$ by $\langle u,v\rangle$ produces a trail cover that is smaller than $\mathcal{T}$ and so $\mathcal{T}$ is not optimal. ♠

Lemma II.1.9. In a critical tree, every penultimate vertex u is bivalent.

Proof. Let G be a tree containing a penultimate vertex u, and let G' be the tree obtained from G

by contracting all but one leaf-edge incident to u. Note that u is still a penultimate vertex in G' and so there is an optimal trail cover T of G' that ends at u. Then T is also a trail cover of G. Hence G is not critical. ♠

We now consider 2-colored graphs, i.e., graphs in which each vertex is assigned black or white. If G is 2-colored. then we define an **augmentation** of G at u to be a larger 2-colored graph obtained as follows. Add a path $<w_1,\ldots,w_5>$ on five new vertices, making w_3 white and w_1, w_2, w_4, and w_5 black. Then if u is black, add an edge uw_3 and if u is white, identify u with w_3. These operations are illustrated in Figures II.1.3(a) and Figure II.1.3(b).

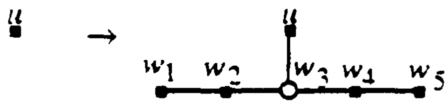


Figure II.1.3(a)

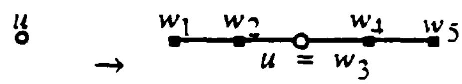


Figure II.1.3(b)

We say that H is **an augmentation** of G if, for some $u \in V(G)$, H is the augmentation of G at u. Let $\mathcal{H}_1 = \{K_2\}$, where both vertices are black. Let $\mathcal{H}_k$ be the augmentations of the members of $\mathcal{H}_{k-1}$. We list $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$ below.

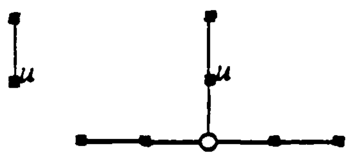


Figure II.1.4

Let $\mathcal{H}$ be the union of the $\mathcal{H}_i$'s.

Theorem II.1.10. If $G \in \mathcal{H}_k$, then $t(G) = k$, each white vertex of G is partially useful but not useful, and each black vertex of G is useful.

Proof. We use induction on k. For $k = 1$, the claim is clear. Suppose that $k > 1$ and that G is an augmentation of $G' \in \mathcal{H}_{k-1}$ at u. We will apply Algorithm II.1.6 to G at w_3.

Suppose that u is black. The w_3 has three neighbors and the subtrees of algorithm II.1.6 are G' rooted at u and two copies of K_2. By the induction hypothesis, u is useful in G', and so $\upsilon(u,G') = 3$. Since $\upsilon(w_4,w_4w_5) = \upsilon(w_2,w_2w_1) = 3$, we obtain $\upsilon(w_3,G) = 2$ (i.e., w_3 is partially useful) and $t(G) = 1 + t(G')$ from Algorithm II.1.6.

To show that any black vertex v of G' is useful in G, note that, by induction, v is useful in G' and so there exists a trail cover T' of G' that contains $<v>$. Let $T = T' \cup \{<w_2,w_3,w_4>\}$. Since

$t(G) = t(G') + 1 = |\mathcal{T}|$, $\mathcal{T}$ is optimal.

We can use a similar argument to show that the white vertices of G' are partially useful in G and no edge of G' is contractible. We now show that each white vertex v of G' is not useful. Suppose that $\mathcal{T}$ is a trail cover that contains $<v>$. Let $\mathcal{T}'$ be the subset of trails of $\mathcal{T}$ that intersect G'; $|\mathcal{T}'| \geq t(G') + 1$ since $<v>$ is not useful in G'. Moreover, since there is only one edge from G' to the rest of the graph, at most one member of $\mathcal{T}'$ can visit $\{w_1, w_2, w_3, w_4, w_5\}$ and it must visit u. Such a trail cannot cover both w_1w_2 and w_4w_5 and hence $\mathcal{T}$ must contain a trail that does not intersect G'. Hence $|\mathcal{T}| \geq t(G') + 2$ and is therefore not optimal.

We must show that no new edge is contractible and that each new black vertex is useful. To show that w_1 or w_2 is useful, or that w_1w_2 or w_2w_3 is not contractible, extend the w_4-optimal trail cover of w_4w_5 and the u-optimal trail cover of G' in Algorithm II.1.6. A similar argument works to show that w_4 and w_5 are useful, and that w_1w_2 and w_2w_3 are not contractible. To show that uw_3 is not contractible, let $\mathcal{T}$ be an optimal trail cover of G' that contains $<u>$. Now contract uw_3 and replace $<u>$ by $<w_2,u,w_4>$; we now have a trail cover of G that has $t(G')$ trails and so uw_3 is not contractible.

Now suppose that u is white in G'. From Algorithm II.1.6, we obtain $\beta(u,G) = \beta(u,G') + 2$ and so $t(G) = t(G') + 1$ and $\upsilon(u,G) = 2$. Moreover, since u is white in G', $\upsilon(u,G') = 2$ and there is an optimal trail cover $\mathcal{T}$ of G' that contains a trail T that ends at u. Extending T to w_2 or w_1 shows that w_3 and w_4 are useful and that uw_4 and w_4w_5 are not contractible. A similar argument shows that w_1 and w_2 are useful, and that uw_2 and w_1w_2 are not contractible. By starting with the trail $<w_2,u,w_4>$, we see that every vertex in G' is partially useful, black vertices of G' are useful, and edges of G' are not contractible. We must still show that no white vertex of G' is useful.

We already know that u is not useful. Let v be a white vertex of G' that is not equal to u. Let $\mathcal{T}$ be a trail cover of G' that contains $<v>$. Let $\mathcal{T}'$ be the set of trails of $\mathcal{T}$ that intersect G'; $|\mathcal{T}'| \geq t(G') + 1$ since $<v>$ is not useful in G'. If there are two members of $\mathcal{T}'$ that visit $\{w_1, w_2, w_4, w_5\}$, then one must visit w_2 and one must visit w_4. We can redistribute the edges of the two trails so that one of them is $<w_2,u,w_4>$ and the other does not visit $\{w_1, w_2, w_4, w_5\}$; call this trail cover $\mathcal{U}$. Note that $\mathcal{U} - \{<w_2,u,w_4>\}$ contains $<v>$ and visits u. Hence it is a trail cover of G' and, since v is not useful in G', $|\mathcal{U} - \{<w_2,u,w_4>\}| \geq t(G') + 1$ and $|\mathcal{T}| \geq t(G') + 2$ and is therefore not optimal. ▲

Theorem II.1.11. For any $k \geq 1$, $\mathcal{G}_k = \mathcal{H}_k$.

Proof. From the previous theorem, we only need to show that $\mathcal{H}_k \supseteq \mathcal{G}_k$. We use induction on k. It is easy to verify the theorem for $k = 1$. Assume that the theorem holds for $1, \ldots, k - 1$.

Suppose that $G \in \mathcal{G}_k$, u_1 and u are the leaves on a path of maximum length, and that $u_1 \leftrightarrow v_1$. Since v_1 is penultimate, it is bivalent and we can define w by $N(v_1) = \{u_1, w\}$. Let w' be the neighbor of w besides v_1 that is on the path between u_1 and u. If $d(w) = 2$, then the edge u_1v_1 is contractible. If w is adjacent to any leaf, then that leaf edge is contractible. Hence $d(w) \geq 3$ and, except for w', every neighbor of w is penultimate (or else u would have a vertex at greater distance that u_1). Since all penultimate vertices in any critical tree are bivalent, we have the situation illustrated below.

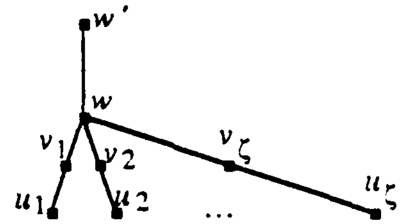


Figure II.1.5

Let $\zeta = d(w) - 1$; the vertex w has the ζ penultimate and bivalent neighbors $v_1, \ldots, v_\zeta$.

If $\zeta = 2$, then, by two applications of Lemma II.1.7(i) and one application of Lemma II.1.7(ii), we have one trail T that covers the edges that are incident to w, v_1, and v_2. Let G' be the graph obtained by deleting these edges from G. Since G is k-critical, G' must be $(k - 1)$-critical and, by induction, $G' \in \mathcal{H}_{k-1}$.

We now show that w' is black. Suppose that w' is white. Contract the edge ww' in G. The trail T now contains the vertex w'. By Theorem II.1.10, w' is not useful in G' and therefore the fact that T visits w' does not affect the number of trails that are still required to cover $E(G')$. Hence $t(G \bullet ww') = 1 + t(G') = k$, contradicting the criticality of G. Hence w' is black in G' and G is an augmentation of G' at w'.

If ζ is odd, then υ applications of Lemma II.1.7(i), followed by $\frac{\zeta - 1}{2}$ applications of Lemma II.1.7(ii) results in $\frac{\zeta - 1}{2}$ trails and an additional trail T that we may assume is $v_\zeta w$. The construction of Lemma II.1.7 does not forbid extending T to w'. Therefore, the edge ww' is contractible, contradicting the criticality of G. Hence ζ cannot be odd.

If ζ is even and at least four, then let $G' = G - \{v_1, v_2\}$. By two applications of Lemma II.1.7(i) and one application of Lemma II.1.7(ii), we obtain one trail $\langle v_1, w, v_2 \rangle$ that covers $E(G) - E(G')$. Repeating this procedure, we may assume that there is another trail $\langle v_3, w, v_4 \rangle$, making the fact that w has already been visited meaningless. Hence we can safely ignore the fact that the first trail visits w and so $G' \in \mathcal{G}_{k-1}$ and, by induction, $G' \in \mathcal{H}_{k-1}$. By Lemma II.1.8, w is not useful in G'. By induction, w must be white and so G is an augmentation of G' at w.♠

Corollary II.1.12. For any tree G with at least three vertices, $I(G) \leq \frac{5n(G) - 3}{4}$. Furthermore, for any n, there exists a tree G_n such that $I(G_n) = \lfloor \frac{5n - 3}{4} \rfloor$ and so the result is best possible.

Proof. Let $f(G) = n(G) - 4t(G) + 1$. It suffices to show that $f \geq 0$ for all trees. Suppose that G is such that f is minimized. If the contraction of an edge does not decrease the trail cover number, then it decreases f and so we may assume that no edge is contractible. Hence $G \in \mathcal{G}_k = \mathcal{H}_k$ for some k. But then G is created by a sequence of augmentations from K_2. Each augmentation from a black vertex increases t by one, n by five, and hence f by one. Each augmentation from a white vertex increases t by one, n by four, and hence does not affect f. To minimize f, we therefore use as few augmentations from black vertices as possible.

The first augmentation must come from a black vertex because initially, there are no white vertices. However the remaining augmentations can come from the one white vertex. This construction yields graphs with minimum values of f and this value is zero. This establishes the lower bound on f and the sequence of graphs constructed this way shows that it is best possible.♠

Let $\mathcal{F}$ be the set of graphs indicated in Figure II.1.6. The above proof shows that members of $\mathcal{F}$ are the *only* graphs for which $f = 0$ and therefore, the only graphs for which $I = \frac{5n - 3}{4}$ (note that there are no "floor" marks in this statement).

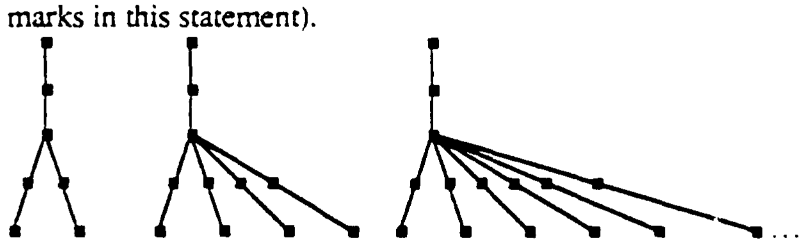


Figure II.1.6

2. Cacti and Dense Cacti

We will define a finite set Ξ of **exceptional** cacti. The goal of §II.2 is to prove Theorem II.2.1.

Theorem II.2.1. If G is a cactus that is not in Ξ, then $l(G) \leq \frac{18n(G) - 12}{13}$. Furthermore, for any n, there exists a cactus G_n with n vertices for which $l(G_n) = \lfloor\frac{18n - 12}{13}\rfloor$, and so the result is best possible.

Let $f(G) = 18n(G) - 13l(G) - 12$. Theorem II.2.1 can be restated in the following more convenient form.

Theorem II.2.1'. If G is a cactus that is not in Ξ, then $f(G) \geq 0$. Furthermore, for any n, there exists a cactus G_n with n vertices for which $0 \leq f(G_n) < 13$, and so the result is best possible.

After defining Ξ, we will construct an infinite sequence of cacti that shows that Theorem II.2.1' is best possible. We will then prove Theorem II.2.1' for triangle-free cacti and then use this result to prove Theorem II.2.1' for arbitrary cacti.

We first define an infinite set Ξ' of graphs, a finite subset of which will be the exceptional subset Ξ. The set Ξ' is built by starting with the two small graphs K_2 and C_4; $f(K_2) = -2$ and $f(C_4) = -5$. We then define two enlargement procedures. Each procedure will increase f but by such a small amount that we can apply the procedures a few times and still have a negative value of f.

The enlargement procedures depend on the graphs Γ_1 and Γ_2 shown below.

Figure II.2.1 Γ_1 Γ_2

For $i = 1,2$, an i-**operation** on a graph G is the identification of $u \in V(\Gamma_i)$ with some $v \in V(G)$. Examples of these operations appear below.

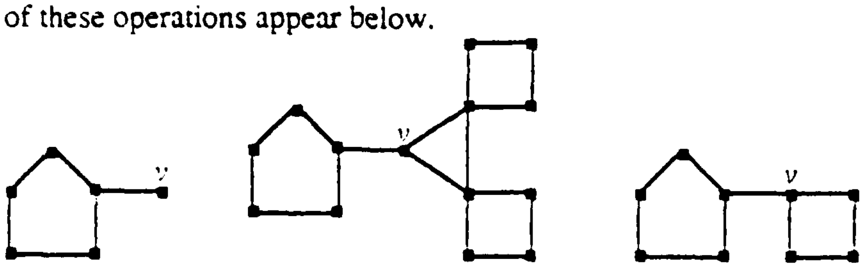


Figure II.2.2 G $\Gamma_1(G)$ $\Gamma_2(G)$

The vertices and edges added to the graph during the operations are called **new** and the rest of the vertices and edges are called **old**.

Lemma II.2.2. If H is any graph obtained by a 1-operation from a graph G, then $f(H) = f(G) + 1$.

Proof. Since $n(H) = n(G) + 8$, it is easy to verify that the lemma is true if and only if $I(H) = I(G) + 11$. The representation below shows that $I(\Gamma_1) \leq 11$ and so, by representing G and Γ_1 separately, we see that $I(H) \leq I(G) + 11$.

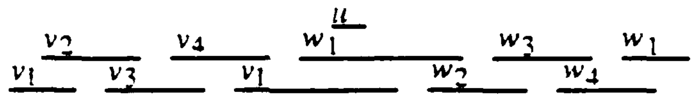


Figure II.2.3

We must show that $I(H) \geq I(G) + 11$. Suppose that R is a represention of H. Partition the intervals of R into $R_1 \cup R_2$ where $R_1 = \cup\{R(u) : u \text{ is old}\}$ and $R_2 = \cup\{R(u) : u \text{ is new}\}$. Since the graph induced by the new vertices is triangle-free, we can use Lemma I.6.1 to show that $|R_2| \geq 10$ and that, if $|R_2| = 10$, then R_2 is contiguous. Since R_1 is a represention of G, $|R_1| \geq I(G)$. Therefore, $|R| = |R_1| + |R_2| \geq I(G) + 10$ *and equality exists only if* $|R_1| = I(G)$ and R_2 is contiguous.

Hence we may assume that R_2 is contiguous and that it corresponds to a single covering trail of the graph that is induced by the new vertices. Any such trail must contain the edge v_1w_1 and this edge can be neither the first nor the last edge of the trail. Therefore, any interval of R_1 that intersects both a v_1-interval and a w_1-interval must either intersect no other interval or at least one other member of R_2.

In particular, if some u-interval θ intersects both a v_1-interval and a w_1-interval then, since u is not adjacent to any other new vertex, θ must not intersect any other interval. But then the removal of θ leaves a represention of G. Hence $|R_1| \geq I(G) + 1$ and $|R| \geq I(G) + 11$.

If two different u-intervals θ_1 and θ_2 intersect members of R_2, then, since R_2 is contiguous, removing R_2 and splicing θ_1 to θ_2 results in a representation of G with $|R_1|$-1 intervals. Hence $|R_1| \geq I(G) + 1$ and $|R| \geq I(G) + 11$.♠

Lemma II.2.3. If H is any graph obtained by a 2-operation from a graph G, then $f(H) \geq f(G) + 2$.

Proof. Since $n(H) = n(G) + 3$, it is easy to verify that the lemma is true if and only if $I(H) \geq I(G) + 4$. Suppose that R is a represention of H. We partition R as in Lemma II.2.2 and use the same arguments to show that $I(H) \leq I(G) + 3$ implies that $|R_1| = I(G)$ and $|R_2| = 3$.

If it is possible to remove the intervals of R_2 and rearrange the components of R_1 so that there are two u-intervals θ_1 and θ_2, with no other interval between them, then θ_1 and θ_2 can be spliced together to obtain a represention of G that has fewer intervals than R_1. But then $|R_1| > I(G)$ and $|R| \geq I(G) + 4$. Since u is the *only* old vertex with an interval that intersects *any* member of R_2, only one u-interval intersects any member of R_2. But there must be at least *five* intervals involved in the representation of the new edges. Since there is only one u interval involved, we must have $|R_2| \geq 4$ and $|R| \geq I(G) + 4$ ♠

Let Ξ' be the set of graphs that can be built from $\{K_2, C_4\}$ by a sequence of 1-operations and 2-operations.

Lemma II.2.4. For any $G \in \Xi'$, $t(G) = 1$.

Proof. Neither operation changes the number of vertices that are of odd degree. Since K_2 has only two vertices of odd degree and C_4 has none, each member of Ξ' has at most two vertices of odd degree and all of the edges can be *traversed* with a single trail.♠

For $G \in \Xi'$ and $i \in \{1,2\}$, let $k_i(G)$ be the number of i-operations applied to a member of $\{K_2, C_4\}$ to obtain G. We say that G has **base** K_2 (C_4) if G is the result of 1-operations and 2-operations applied to K_2 (C_4).

Corollary II.2.5. Suppose that $G \in \Xi'$. If G has base B, then $f(G) = f(B) + k_1(G) + 2k_2(G)$.

Proof. Lemmas II.2.2 and II.2.3 show that 1-operations and 2-operations increase the total interval number by at least eleven and four. Lemma II.2.4 shows that, when restricted to members of Ξ', they increase it by at *most* eleven and four. The rest is simple arithmetic.♠

We define Ξ to be those members of Ξ' that are obtained by one of the following.

i. Start with K_2 and apply up to one 1-operation.
ii. Start with C_4 and apply one of the following.
   a. Two 2-operations
   b. One 2-operation and up to two 1-operations
   b. Zero 2-operations and up to four 1-operations

The order of the operations in ii.a., ii.b., and ii.c. is irrelevant. By Corollary II.2.5, the set Ξ is precisely those members of Ξ' for which $f < 0$. After establishing the best possible assertion of Theorem II.2.1', we will show that the members of Ξ are the *only* cases for which $f < 0$.

To prove that Theorem II.2.1' is best possible, we need to construct a sequence of cacti for which we can find a suitably large *lower* bound. This bound will be established by actually computing I for the members of the sequence.

There are two types of subgraphs that are particularly useful when computing the total interval number of triangle-free cacti. A **cluster** of a non-Eulerian subgraph is a maximal induced subgraph C that satisfies the following:

i. $n(C) \geq 2$
ii. C is Eulerian.
iii. Exactly one component of $G - E(C)$ is non-trivial.

If C is a cluster, then let $D(C)$ be the non-trivial component of $G - E(C)$. There is exactly one vertex of $V(C)$ that is also in $D(C)$ and this vertex is called the **base** of the cluster. If C is a cluster with base u and $N_{D(C)}(u) = \{v\}$, then C is called an **appendage** and v is called the **neighbor** of C. Examples appear in Figures II.2.4(a) and II.2.4(b).

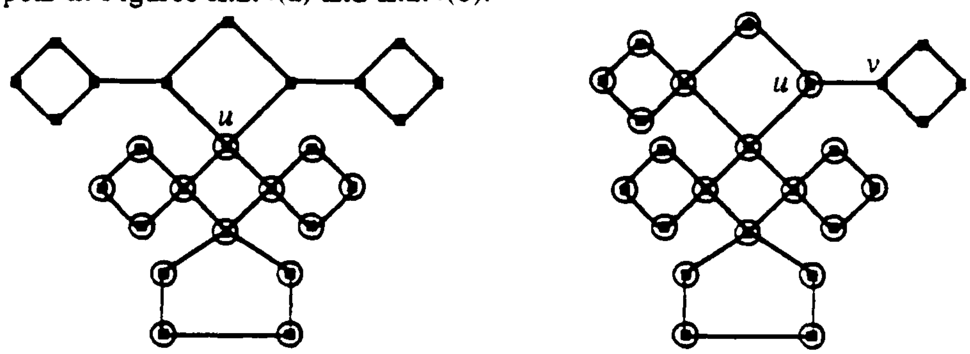


Figure II.2.4(a) Figure II.2.4(b)

For each graph, the circled vertices induce clusters with base u. In Figure II.2.4(b), this cluster is an appendage with neighbor v. Note that v is also the base of a (different) cluster and its neighbor is u.

The next lemma and its corollary are the keys to computing t for cacti.

Lemma II.2.6. There exists an optimal trail cover $\mathcal{T}$ such that, for each cluster C, there exists $T_C \in \mathcal{T}$ such that some subtrail of T_C is an Euler tour of C.

Proof. Let $\mathcal{T}$ be a trail cover. Fix some cluster C and let u be the base of C. Because some edge of C is not incident to u, some trail must contain a vertex of C other than u. If such a trail contains u, then it is an **entering** trail and if it does not contain u, then it is an **interior** trail.

If there is no entering trail, then remove all interior trails and add a trail that is an Euler tour. Now suppose that there is at least one entering trail. We first remove all interior trails. Each entering trail T contains at least one subtrail that starts and ends at u and whose edge-set is a subset of $E(C)$. We remove the edges of these subtrails from the edge-set of T, leaving us with a set of edges of a trail T' that contains u and covers all of the edges outside of C that T does Repeat this for every entering trail. Now select some trail U that contains u. If $U = \langle u \rangle$, then replace U with an Euler tour of C. Otherwise, there is a subtrail $\langle u,v \rangle$ of U. Replace this by an Euler tour of C concatenated with $\langle u,v \rangle$.

For each case, the number of trails does not increase. Repeat this procedure for each cluster.♠

Corollary II.2.7. Let G be a cactus. Then there is an optimal trail cover $\mathcal{T}$ such that, for each appendage C with base u and neighbor v, some $T_C \in \mathcal{T}$ has an end at u. In particular, a lower bound on t is half of the number of appendages.

Proof. Consider the optimal trail cover of Lemma II.2.6. Because there is only one edge incident to u and not in $\mathcal{T}(C)$, both ends of the trail cannot be outside of $V(C)$.♠

In Figure II.2.5, there are thirteen graphs $\{G_i : i = 1,\ldots,13\}$; one for each congruence class modulo thirteen.

Let G' be the graph in Figure II.2.6.

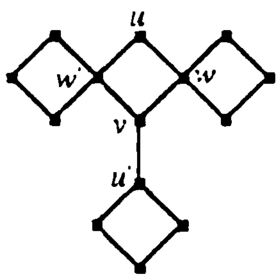
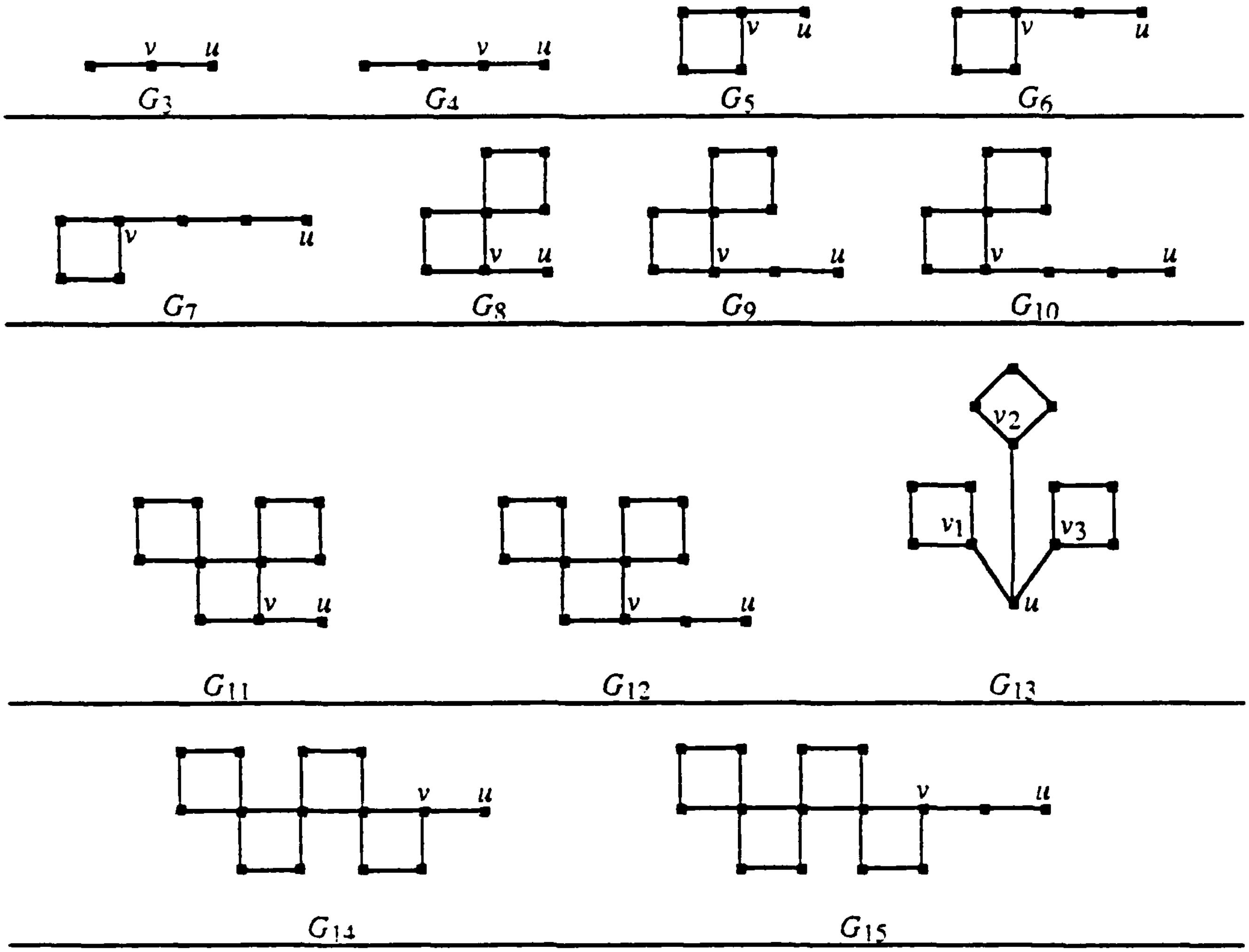


Figure II.2.6: G'

We will construct graphs by identifying a vertex of G_i with copies of G'. The next lemma shows that each copy of G' increases the trail cover number by one.

Lemma II.2.8. Let $G_{i,r}$ be the graph obtained by identifying the vertex u of r copies of G' with the vertex u of G_i. Let $t_{i,r} = r + 1$ if $i \neq 13$ and $r + 2$ if $i = 13$. Then $t(G_{i,r}) = t_{i,r}$.

Proof. To establish $t(G_{i,r}) \leq t_{i,r}$, note that the number of vertices that are of odd degree shows that we can *partition* (and not merely cover) the edges into $t_{i,r}$ trails.



Fgure II.2.5

Let $G = G_{i,r}$. We will show that $t(G) \geq t_{i,r}$ We will do this by finding a subset U of V for which $|U| \geq 2t_{i,r}$-1 and there exists some optimal trail cover T that ends at each member of U. Note that each vertex labelled v or v_j in Figure II.2.5 is the base of an appendage. Therefore, by Corollary II.2.6, we may put each of these vertices into U. We will be done if we can find $2r$ more vertices to include in U and this will be accomplished if we can find two vertices from each copy of G' in G.

Fix some copy of G' and assume that it is labelled as in Figure II.2.6. Consider the appendage with base u'. By Corollary II.2.6, we have an end at the base of this appendage. If this copy of G' does not have another end, then this trail must continue out of the appendage to v and, by symmetry, we may assume that it continues to w.

After removing the edges traversed so far, v is a leaf of the remaining graph. If some trail contains v, then it has an end at v, providing the second end within G'. If not, then the edge $w'v$ is not in any trail and can be removed when searching for an optimal trail cover. The left 4-cycle is then an appendage and we may assume that some trail has an end at w' since no trail can traverse the edge uw' twice. Hence each copy of G' contains two ends of trails. ♠

The reader may wonder why attaching copies of G' to the triangle-free members of Ξ does not increase the trail cover number. The reasoning of the above argument breaks down since each triangle-free member of Ξ has *no* appendage and hence the *first* copy of G' does not increase the trail cover number. Subsequent copies *do* increase t by one.

Corollary II.2.9. Theorem II.2.1' is best possible.

Proof. By using the fact that $t(G_i) = 1$ if $i \neq 13$ and 2 if $i = 13$, we have $0 \leq f(G_i) < 13$. From the previous theorem, we have $t_{i,r+1} = t_{i,r} + 1$. It is clear that $m(G_{i,r+1}) = m(G_{i,r}) + 17$ and, since $G_{i,r}$ is triangle-free, that $l(G_{i,r+1}) = 18 + l(G_{i,r})$. Moreover it is immediate that $n(G_{i,r+1}) = n(G_{i,r}) + 13$. From these facts and the definition of f, we have $f(G_{i,r+1}) = f(G_{i,r})$. ♠

We now present the proof of the upper bound for triangle-free graphs. We seek to find triangle-free cacti with minimum values of f and show that, for cacti not in Ξ, this minimum is zero.

If G is a cactus and some operation on G results in a cactus H, then we define $\Delta n = n(G) - n(H)$, $\Delta m = m(G) - m(H)$, $\Delta t = t(G) - t(H)$, $\Delta l = l(G) - l(H)$, and $\Delta f = f(G) - f(H)$. Note that $\Delta f = 18\Delta n - 13\Delta l$ and $\Delta l = \Delta m + \Delta t$. These **difference operators** allow us to focus on hypothetical graphs with negative values of f. It may seem more natural to define the difference operaters to be the negative of what they are; the choice of sign is made to make the most critical part of the proof more natural.

A triangle-free cactus G is **abnormal** if $G \notin \Xi$ and $f(G) < 0$. Our goal is to prove that there are no abnormal cacti. The following lemma follows immediately from the definitions and simple arithmetic.

Lemma II.2.10 Suppose that G is an abnormal graph and H is the result of some operation on G. If H is a non-exceptional triangle-free cactus and $\Delta f \geq 0$, then H is abnormal. ♠

By Lemma II.2.10, it is sufficient to establish the following three properties for an operation in order to prove that it preserves abnormality. We continue to use the convention that G is the graph being operated on and H is the result of the operation.

If G is abnormal, then H is a triangle-free cactus. (II.2.1)

If $H \in \Xi$, then G is normal. (II.2.2)

$\Delta f \geq 0$ (II.2.3)

For the operations that we will discuss, (II.2.1) will always be easy to establish because the operation will be defined to ensure it. A tedious but simple inspection will be necessary to establish (II.2.2). For this, it is convenient to have the triangle-free members of Ξ listed explicitly. There are only six and they are shown below.

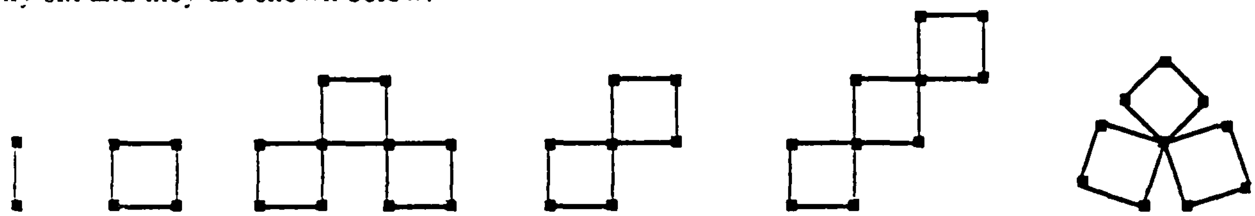


Figure II.2.7

We will work out the details for establishing (II.2.2) only for the first operation.

The key step in establishing (II.2.3) is to bound Δt by some appropriate constant. We do this by transforming an optimal trail cover for H into a trail cover for G. It is because of *this* transformation that we define the difference operators as we do.

There are two abnormality preserving operations A and A' (defined in Lemmas II.2.11 and II.2.12) that increase the number of vertices. We will assume that these operations are performed until they cannot be performed any more. It is easy to see that this process terminates in a finite number of steps.

There are then ten operations $\{A_i' : i = 1,\ldots,10\}$, each of which preserves abnormality and decreases the number of vertices. But each of these operations could result in a graph on which A or A' could operate. Therefore we define ten "companion" operations $\{A_i : i = 1,\ldots,10\}$, each of which preserves abnormality, decreases the number of vertices, *and* leaves a graph on which neither A and A' can operate.

Lemma II.2.11. If $u \in V$ and $N(u) = \{v\}$, then let $A(G)$ be the graph obtained from G by adding two new vertices u_1 and u_2 and the edges $\{uu_1, u_1u_2, u_2v\}$. Then A preserves abnormality.

Proof. It is clear that A satisfies (II.2.1).

For (II.2.3), let $H = A(G)$. Let α be the minimum size of trail covers of $G - u$ that visit v. It is easy to see that $t(G) = t(H) = \alpha$. Hence $\Delta t = 0$ and $\Delta l = \Delta m$. Since $\Delta m = -3$ and $\Delta n = -2$, we have $18\Delta n - 13\Delta l = 3 > 0$ and (II.2.3) is established.

For (II.2.2), note that if H is exceptional, then, by inspection, we have the candidates for the pairs (G,H) that are listed below.

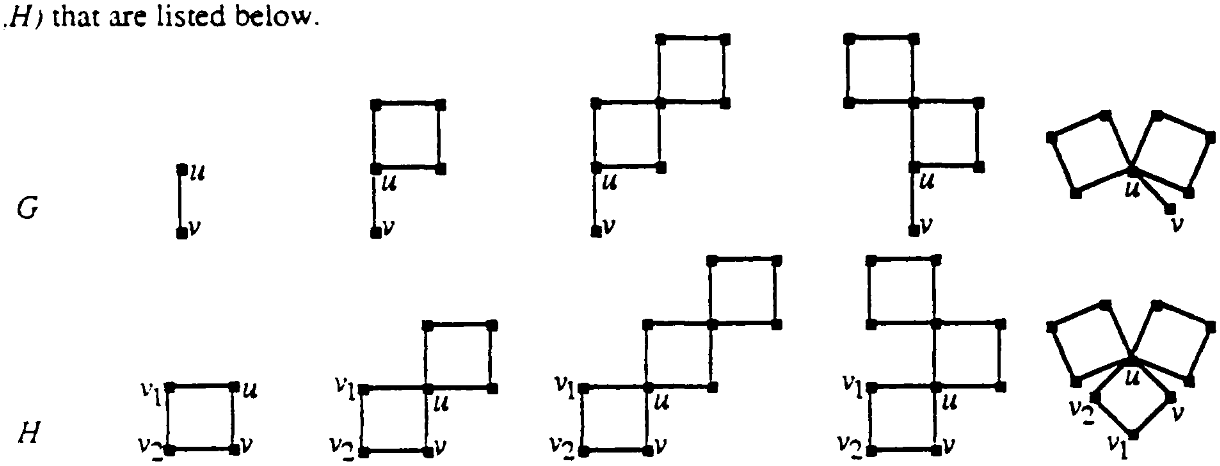


Figure II.2.8

For each candidate for G that is not exceptional, it is easy to verify that $t(G) = 1$ and $f(G) \geq 0$. ♠

Lemmas II.2.12 through II.2.20 similar to Lemma II.2.11. For each operation, we must verify properties (II.2.1) and (II.2.2). These are always straightforward so the proofs will concentrate on the verification of (II.2.3).

Lemma II.2.12. If u and v are bivalent and adjacent to each other, and no 4-cycle contains uv, then let $H = A'(G)$ be the graph obtained from G by contracting uv to form the new vertex u', and adding vertices $\{v', v_1, v_2\}$ and edges $\{u'v', v'v_1, v_1v_2, v_2u'\}$. Then A' preserves abnormality.

Proof. The restriction concerning a 4-cycle ensures that H is triangle-free. It is easy to verify (II.2.1) and (II.2.2) (the same candidates for H exist as for Lemma II.2.11). For (II.2.3), note that Δt is again zero since uv is vital by Lemma I.6.2(ii). Then the same calculations used in Lemma II.2.11 apply. ♠

Henceforth, we restrict ourselves to triangle-free cacti for which neither A nor A' applies. Let C be the abnormal triangle-free cacti for which neither A nor A' applies.

In the proofs of the following lemmas, we will assume that H is the result of the operation being discussed.

Lemma II.2.13. If C is a cluster with base u, and C is not a 4-cycle, then let $A_1^*(G)$ be the graph obtained from G by removing $V(C) - u$ and making u the base of a new cluster that *is* a 4-cycle. Then A_1^* preserves abnormality.

Proof. Properties (II.2.1) and (II.2.2) are easy to verify.

For (II.2.3), Lemma II.2.2 guarantees that $\Delta t = 0$ and therefore that $\Delta l = \Delta m$. Now let α_k be the number of k-cycles in C. By hypothesis, $(\alpha_4 - 1) + \sum_{k=5}^{n} \alpha_k \geq 1$.

Since each k-cycle in C increases the number of vertices by $k - 1$ and the number of edges by k, we have the following identities.

$$n(C) = \sum_{k=4}^{n}(k-1)\alpha_k + 1 \text{ and } m(C) = \sum_{k=4}^{n} k\alpha_k$$
$$\Delta n = \sum_{k=4}^{n}(k-1)\alpha_k - 3 \text{ and } \Delta m = \sum_{k=4}^{n} k\alpha_k - 4$$
$$\Delta f = 18\Delta n - 13\Delta l = 2\alpha_4 - 2 + \sum_{k=5}^{n} 5k\alpha_k - 18\alpha_k$$

Note that each summand in the final formula of Δf is nonnegative. If $\alpha_4 \leq 1$, then $\sum_{k=5}^{n}\alpha_k \geq 1$, $\sum_{k=5}^{n} 5k\alpha_k - 18\alpha_k \geq 7$, and $\Delta f \geq 5$. If $\alpha_4 \geq 2$, then $2\alpha_4 - 2 \geq 2$ and $\Delta f \geq 2$. Either way, (II.2.3) is established.♠

Lemma II.2.14. If C_1 is an appendage with neighbor u and C_2 is a cluster with base u, then let $A_2^*(G)$ be the graph obtained from G by removing $V(C_2) - \{u\}$. Then A_2^* preserves abnormality.

Proof. Properties (II.2.1) and (II.2.2) are easy to verify.

From Lemma II.2.13, both C_1 and C_2 are 4-cycles. By Lemma II.2.2 and corollary II.2.3, $t(H) = t(G)$. Hence $\Delta t = 0$, $\Delta m = 4$, $\Delta l = 4$, $\Delta n = 3$, and $18\Delta n - 13\Delta l = 2 > 0$.♠

Lemma II.2.15. If C_1 and C_2 are appendages that have a common neighbor u, then let $A_3^*(G)$ be the graph obtained from G by removing $V(C_1) \cup V(C_2)$ Then A_3^* preserves abnormality.

Proof. Properties (II.2.1) and (II.2.2) are easy to verify.

Suppose that we have an optimal trail cover for H. By using one additional trail to cover the edges of the appendages and the edges from u to the appendages, we see that $\Delta t \leq 1$ (Note that if $H \in \Xi$, then $\Delta t = 0$). Since $\Delta m = 10$, we have $\Delta l \leq 11$. Since $\Delta n = 8$, we have $18\Delta n - 13\Delta l > (18)(8) - (13)(11) = 1 > 0$.♠

For the next two lemmas, properties (II.2.1) and (II.2.2) are easy to verify and property (II.2.3) follows directly from calculations and the easily verified fact that $\Delta t = 0$.

Lemma II.2.16. Suppose that C is a cluster with base u_1, and some $u_2 \in N(u_1) - V(C)$ is bivalent. Let $A_4'(G) = G - (V(C) - u_1)$. Then A_4' preserves abnormality.♠

Lemma II.2.17. Suppose that C_1 and C_2 are clusters with bases u_1 and u_2, $u_1 \leftrightarrow u_2$, and each u_i has exactly two neighbors that are not in its cluster. Let $A_5'(G) = G - (V(C_2) - \{u_2\})$. Then A_5' preserves abnormality.♠

The next three lemmas deal with appendages. The operations and proofs are almost identical to each other. Therefore we summarize the operations in Figure II.2.9. Lemmas II.2.18, II.2.19, and II.2.20 deal with the configurations shown in Figures II.2.9(a), II.2.9(b), and II.2.9(c). For all three configurations, the corresponding operation results in Figure II.2.9(d).

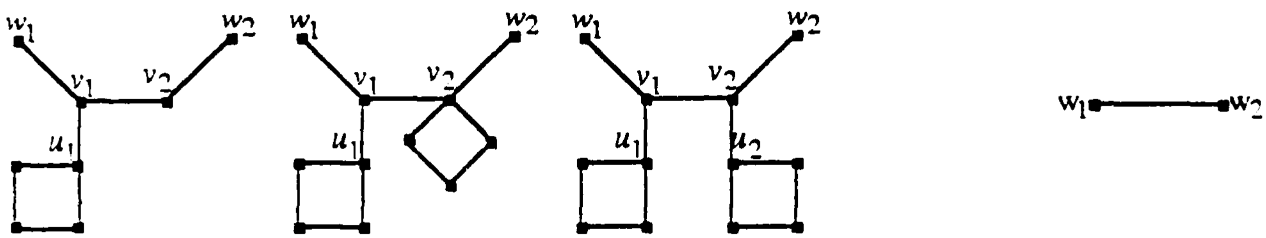


Figure II.2.9(a) Figure II.2.9(b) Figure II.2.9(c) A_6', A_7', or A_8' → Figure II.2.9(d)

We write out the proof only for Lemma II.2.18. The other proofs are similar.

Lemma II.2.18. Suppose that C is an appendage with base u_1 and neighbor v_1, $N(v_1) = \{u_1, w_1, v_2\}$, $N(v_2) = \{v_1, w_2\}$ and v_1v_2 is not part of a 4-cycle or a 5-cycle. Let $A_6'(G) = G - (V(C) \cup \{v_1, v_2\}) \cup \{w_1w_2\}$. Then A_6' preserves abnormality.

Proof. Properties (II.2.1) and (II.2.2) are easy to verify. Note that the restriction concerning 4-cycles and 5-cycles included ensure that $A_6(G)$ contains no triangles. For (II.2.3), arithmetic shows that it is sufficient to establish that $\Delta t \leq 1$.

Let $e = w_1w_2$ and let $\mathcal{T}$ be an optimal trail cover of H. If $e \in E(G)$ or if no trail in $\mathcal{T}$ contains e, then use $\mathcal{T}$ and add a trail that covers $E(C)$ and the path $\langle w_1, v_1, v_2, w_2 \rangle$ thus showing that $\Delta t \leq 1$. If some $T \in \mathcal{T}$ contains e, then adjust $\mathcal{T}$ by the following:

i. Remove e, resulting in two trails T_1 and T_2, where T_1 ends at w_1 and T_2 ends at w_2.
ii. Extend T_1 to v_1, then u_1, and then an Euler tour of C.
iii. Extend T_2 to v_2.

We have replaced T by $\{T_1,T_2\}$, again establishing that $\Delta t \leq 1$.♠

For Lemmas II.2.19 and II.2.20, it is sufficient to show that $\Delta t \leq 1$ in order to establish (II.2.3). The adjustment of T in the last part of the proof can easily be modified to do this. We state these lemmas without additional proof.

Lemma II.2.19. Suppose that C is an appendage with base u_1 and neighbor v_1, C' is a cluster with base v_2, $v_1 \leftrightarrow v_2$, $N(v_1) = \{u_1,w_1,v_2\}$, $N(v_2) - V(C') = \{v_1,w_2\}$, and v_1v_2 is not part of a 4-cycle or a 5-cycle. Let $A_7'(G) = G - (V(C) \cup V(C')) \cup \{w_1w_2\}$. Then A_7' preserves abnormality.♠

Lemma II.2.20. Suppose C is a an appendage with base u_1 and neighbor v_1, C' is an appendage with base u_2 and neighbor v_2, $v_1 \leftrightarrow v_2$, $N(v_1) = \{u_1,w_1,v_2\}$, $N(v_2) - V(C') = \{v_1,w_2\}$, and v_1v_2 is not part of a 4-cycle or a 5-cycle. Let $A_8'(G) = G - (V(C) \cup V(C') \cup \{v_1,v_2\}) \cup \{w_1w_2\}$. Then A_8' preserves abnormality.♠

For each of $i = 1,\ldots,8$, let A_i be defined as A_i' followed by applications of A or A' until neither A nor A' applies. We state the following lemma without proof. At this point, the only assertion that needs verification is that the number of vertices decreases. For each of the eight cases, this is easy to verify.

Lemma II.2.21. For each $i = 1,\ldots,8$, if $G \in \mathcal{C}$ and G is abnormal, then $A_i(G) \in \mathcal{C}$, $A_i(G)$ is abnormal, and $n(A_i(G) < n(G)$.♠

Now assume that G is such that none of $\{A_i : i = 1,\ldots,8\}$ applies. We will define two more operations A_9' and A_{10}' that also preserve abnormality and decrease the number of vertices, and then modify them as we modified $A_1,\ldots,A_8$ to form A_9 and A_{10}.

Recall that every block is either a single edge or a cycle. We call these types of blocks **edge-blocks** and **cycle-blocks** respectively. If there is at most one edge-block, then we can traverse the entire graph with just one trail and, by the argument of Lemma II.2.13, G is normal. Hence we may assume that there are at least two edge-blocks.

Lemma II.2.22. There exists a cycle Θ in G and a vertex w in Θ such that Θ has the following property:

With the exception of w, every vertex u of Θ satisfies exactly one of the following conditions.
(i) $d(u) = 2$
(ii) $d(u) = 3$ and u is the neighbor of an appendage. (II.2.4)
(iii) $d(u) = 4$ and u is the base of a cluster that is not Θ.

Furthermore, some vertex of Θ other than w is the neighbor of an appendage.

Proof. Let $B(G)$ be the block graph of G. Let e and e' be maximally distant edge-blocks. Since e is not part of any cycle, it is a cut-edge. Let X and Y be the two components of $G - e$, where $e' \in E(Y)$. Define u and v by $e = uv$, $u \in X$, and $v \in Y$. Let X' be the block that is on the minimum path in $B(G)$ between e and e' and is adjacent to e.

If X has an edge-block e'', then e' and e'' are a greater distance from each other in $B(G)$ than e and e' are, a contradiction. Hence, X is Eulerian. Since G has no leaves, $n(X) \geq 2$ and it follows that X is a cluster and therefore a 4-cycle.

If there is an edge-block $e'' = u'v$, where $u' \neq u$, then e'' is also a maximum distance from e' and so the same argument shows that u' is the base of a 4-cycle cluster. But then we have a configuration on which A_3 could act. Therefore the only blocks that contain v are e, X', and perhaps some other cycle-blocks. Let H be the graph induced by the union of the vertices of these other cycle-blocks.

If H contains a vertex x of some edge-block g, and $x \neq v$, then g is farther from e' than e, a contradiction. Therefore H is a cluster and we have a configuration on which A_2 can act, a contradiction. Therefore, the only blocks that contain v are e and X'.

If X' is an edge-block, then we have a configuration on which A_6 can act. Therefore, X' is a cycle-block. Let Θ be the cycle corresponding to the block X'. Let X'' be the block that is not e, is on the minimum path between e and e', and shares a vertex w with X' (See Figure II.2.10).

Now suppose that $y \in V(\Theta)$ and $y \neq w$. Let Z be the union of the y-components that do not contain $E(\Theta)$. If Z contains exactly one cut-edge g', then, either y satisfies (ii) or else g' is farther from e' than e is, a contradiction. If Z contains more than one cut-edge, then either we have a configuration on which A_2 can act or we again have an edge-block that is farther from e' than e is. Hence Z is Eulerian and is therefore a cluster, and (iii) holds for y. ♠

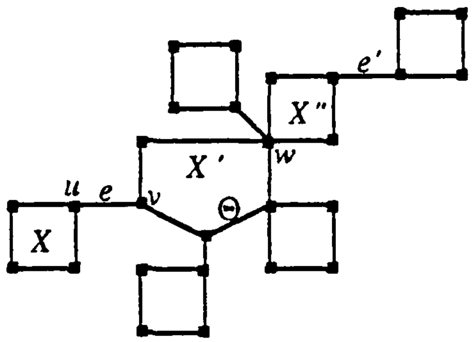


Figure II.2.10

We continue with the assumption that none of A, A', $A_1, \ldots A_8$ applies. We now define the two operations A_9' and A_{10}', one of which *will* apply.

Define Θ and w as in Lemma II.2.22. Let u_1 and u_2 be members of Θ that are not equal to w. We summarize the restrictions on the pair (u_1,u_2) from Lemmas II.2.11 to II.2.20 in the following table.

u_1	u_2	Operation that inhibits this configuration	Resulting possible lengths for Θ
bivalent	bivalent	A'	4
bivalent	base of cluster	A_4	None
bivalent	neighbor of appendage	A_6	4,5
base of cluster	base of cluster	A_5	None
base of cluster	neighbor of appendage	A_7	4,5
neighbor of appendage	neighbor of appendage	A_8	4,5

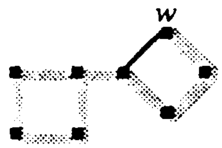
Table II.2.11

Let $A_9'(G) = G - (V(G'')-\{w\})$. Define $A_{10}'(G)$ by the following. Start by applying $A_9'(G)$. Then add a 4-cycle that consists of new vertices and edges and a new edge between w and one of the vertices of the 4-cycle. Examples of these two operations are given below.

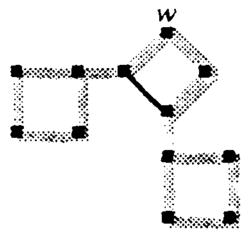
Figure II.2.12 G → $A_9'(G)$ G → $A_{10}'(G)$

Let Ξ be the union of the vertex-sets of the clusters that have bases or neighbors in $V(\Theta) - \{w\}$ and let G' be the graph that is induced by $\Xi \cup V(\Theta)$. We see from Table II.2.11 that we are restricted to our candidates for G'. A partial list of the candidates for G' is listed in Figures II.2.13. Each candidate that is not listed is a trivial modification of one that appears: e.g., one can place a cluster such that its base is any vertex that is already on a trail and simply increase Δf. Note also that there must be at least one appendage.

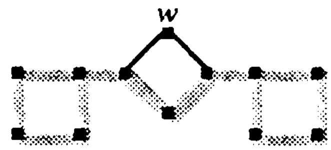
The caption of each candidate gives the operation that is to be applied, as well as bounds on the values of the difference operators. In the diagram of the candidate, we show a partial trail cover that verifies the claim for Δt. Note that, for graphs on which we wish to apply A'_{10}, we get a "free" trail as long as it starts or ends at w.



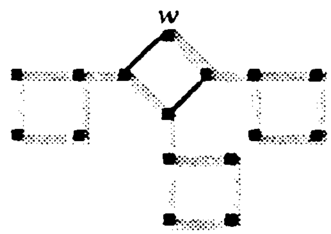
A'_{10}
$\Delta n = 3$ $\Delta m = 4$
$\Delta t = 0$ $\Delta f = 2$
Figure II.2.13(a)



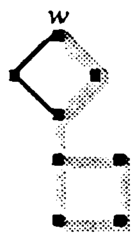
A'_{9}
$\Delta n = 11$ $\Delta m = 14$
$\Delta t \leq 1$ $\Delta f \geq 3$
Figure II.2.13(b)



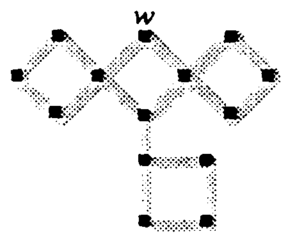
A'_{9}
$\Delta n = 11$ $\Delta m = 14$
$\Delta t \leq 1$ $\Delta f \geq 3$
Figure II.2.13(c)



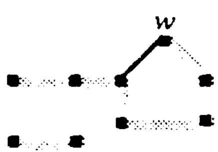
A'_{10}
$\Delta n = 11$ $\Delta m = 14$
$\Delta t \leq 1$ $\Delta f \geq 3$
Figure II.2.13(d)



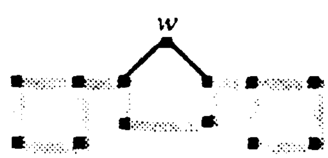
A'_{10}
$\Delta n = 3$ $\Delta m = 4$
$\Delta t = 0$ $\Delta f = 2$
Figure II.2.13(e)



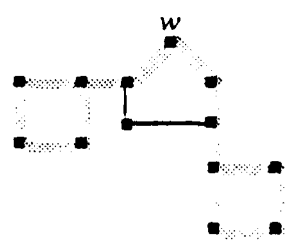
A'_{9}
$\Delta n = 13$ $\Delta m = 17$
$\Delta t \leq 1$ $\Delta f \geq 0$
Figure II.2.13(f)



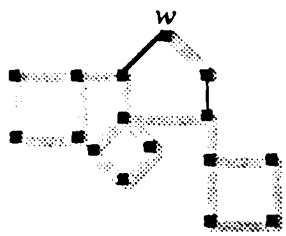
A'_{9}
$\Delta n = 8$ $\Delta m = 10$
$\Delta t \leq 1$ $\Delta f \geq 1$
Figure II.2.13(g)



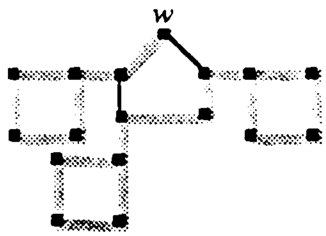
A'_{9}
$\Delta n = 12$ $\Delta m = 15$
$\Delta t \leq 1$ $\Delta f \geq 8$
Figure II.2.13(h)



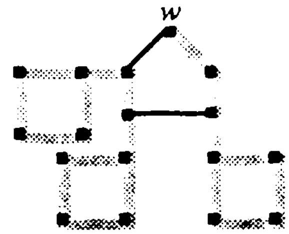
A'_{9}
$\Delta n = 12$ $\Delta m = 15$
$\Delta t \leq 1$ $\Delta f \geq 8$
Figure II.2.13(i)



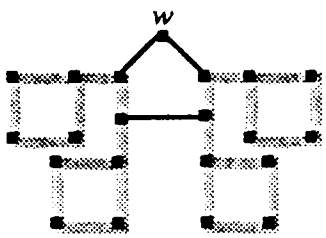
A'_{10}
$\Delta n = 11$ $\Delta m = 14$
$\Delta t \leq 1$ $\Delta f \geq 3$
Figure II.2.13(j)



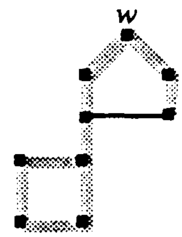
A'_{10}
$\Delta n = 12$ $\Delta m = 15$
$\Delta t \leq 1$ $\Delta f = 8$
Figure II.2.13(k)



A'_9
$\Delta n = 13$ $\Delta m = 17$
$\Delta t \leq 1$ $\Delta f \geq 0$
Figure II.2.13(l)



A'_9
$\Delta n = 20$ $\Delta m = 25$
$\Delta t \leq 2$ $\Delta f \geq 9$
Figure II.2.13(m)



A'_9
$\Delta n = 8$ $\Delta m = 10$
$\Delta t \leq 1$ $\Delta f \geq 1$
Figure II.2.13(n)

Again the tedious and simple verifications of properties (II.2.1) and (II.2.2) are omitted.

For $i = 9$ or 10, let A_i be A_i' followed by applications of A and/or A' until neither A nor A' applies. We state the following lemma without proof. The only assertion in the lemma that needs further verification is the last one.

Lemma II.2.23. For each $i = 1,\ldots,10$, if G is abnormal and $G \in \mathcal{C}$, then $A_i(G)$ is abnormal, $A_i(G) \in \mathcal{C}$, and $n(A_i(G)) < n(G)$. ♠

This completes the proof of Theorem II.2.1' and hence Theorem II.2.1 for triangle-free cacti.

Now suppose that G is a cactus with a triangle $T = (u_1u_2u_3)$. Let G_i be the union of the u_i-components that do not contain T (See Figure II.2.14).

Let $n_i = n(G_i)$. It is clear that $n = n_1 + n_2 + n_3$. For $i,j \in \{1,2,3\}$ and $i \neq j$, let G_{ij} be the graph induced by $V(G_i) \cup V(G_j)$.

Lemma II.2.24. If i, j, and k are distinct, then $I \leq I(G_{ij}) + I(G_k) + 1$.

Proof. Let R_1 and R_2 be optimal representations for G_{ij} and G_k and let $R' = R_i \cup R_j$. Let θ_1

and θ_2 be the u_i-interval and u_j-interval of R_1 whose intersection represents u_iu_j, and let $\theta = \theta_1 \cap \theta_2$. Since u_iu_j is in no cycle within G_{ij}, and hence no triangle, no other interval of R_1 intersects θ. Place a small u_k-interval within θ. ♠

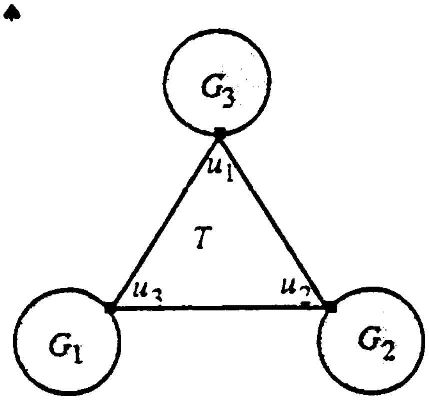


Figure II.2.14

Let $\mathcal{F} = \{G : n(G) \equiv 5 (mod\ 13)\} = \{G : f(G) \equiv 0 (mod\ 13)\}$. If $G \notin \mathcal{F}$, then $f(G) \geq 0$ implies that $f(G) \geq 1$. Since $G \in \Xi$ implies that $f(G)(mod\ 13) \in \{-5,-4,-3,-2,-1\}$, $\mathcal{F} \cap \Xi = \emptyset$.

Now suppose that G is a cactus with at least one triangle T, $G \notin \Xi$, and that Theorem II.2.1' holds for all cacti with fewer vertices or fewer edges than G. Choose $T = (u_1u_2u_3)$ so that G_1 and G_2 are triangle-free. We will be finished proving Theorem II.2.1' if we can prove that $f(G) \geq 0$.

The next two lemmas do not use the fact that G_1 and G_2 are triangle-free.

Lemma II.2.25. For $i \in \{1,2,3\}$, $n_i = 1$ implies that $f \geq 0$.

Proof. Without loss of generality, let $i = 1$. Suppose that $n_1 = 1$. Let $H = G - u_2u_3$. Since no member of Ξ has a cut-edge that is incident to a bivalent vertex, $H \notin \Xi$ and since H is smaller than G, $f(H) \geq 0$. In an optimal representation R' of H, we must have one of the configurations of Figure II.2.15(a).

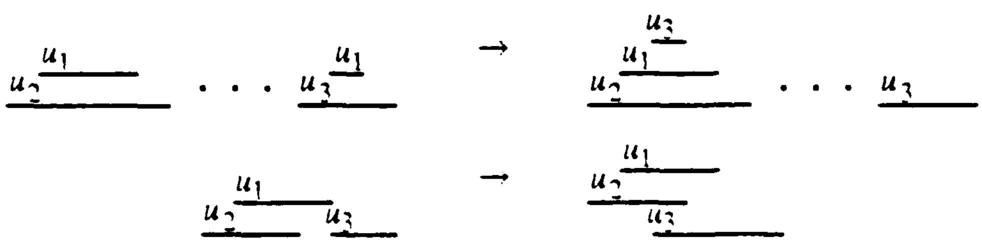


Figure II.2.15(a) Figure II.2.15(b)

We can adjust the representations as shown in Figure II.2.15(b) to obtain a representation for G with the same size. This proves that $l(G) \leq l(H)$ and therefore that $f(G) \geq f(H) \geq 0$. ♠

Lemma II.2.26. For $i \neq j$, $i,j \in \{1,2,3\}$, $f(G_{ij}) < 0$ implies that $f(G) \geq 0$.

Proof. Since G_{ij} is smaller than G, the hypothesis $f(G_{ij}) < 0$ is the same as $G_{ij} \in \Xi$. Note that $u_i u_j$ is a cut-edge of G_{ij} and its deletion leaves the two graphs G_i and G_j. The only members of Ξ that have cut-edges are shown below.

Figure II.2.16

For either of these graphs, the deletion of the (only) cut-edge leaves a graph with two components, at least one of which has only one vertex. Hence if $G_{ij} \in \Xi$, then either $n_i = 1$ or $n_j = 1$. The result now follows from Lemma II.2.25.♠

The following is the most heavily used lemma of this part of the proof.

Lemma II.2.27. Suppose that i, j, and k are distinct, $f(G_{ij}) \geq 0$, $f(G_k) \geq 0$, and $\mathcal{F} \not\supseteq \{G_{ij}, G_k\}$. Then $f(G) \geq 0$.

Proof. By definition, $f(G) = 18(n_i + n_j + n_k) - 13I(G) - 12$. By Lemma II.2.1, $-13I(G) \geq -13(I(G_{ij}) + I(G_k) + 1)$, from which we have $f(G) \geq [18(n_i + n_j) - 13I(G_{ij}) - 12] + [18n_k - 13I(G_k) - 12] - 1 = f(G_{ij}) + f(G_k) - 1$. By the hypotheses, we have $f(G_{ij}) + f(G_k) \geq 1$.♠

We now show that $f(G) \geq 0$. By Lemma II.2.26, we may assume that for all $i,j \in \{1,2,3\}$, $i \neq j$, $f(G_{ij}) \geq 0$. We will consider the following four cases.

i. $G_1, G_2, G_3 \notin \Xi$
ii. $G_1 \in \Xi$, $G_2, G_3 \notin \Xi$
iii. $G_1, G_2 \in \Xi$, $G_3 \notin \Xi$
iv. $G_1, G_2, G_3 \in \Xi$

The arguments of the first three cases do not use the fact that G_1 and G_2 are triangle-free and so any other case can be reduced to one of these four cases by permuting G_1, G_2, and G_3.

Case i. If, for some k, $G_k \notin \mathcal{F}$, then apply Lemma II.2.26. Otherwise, note that $n(G_{12}) \equiv 10 (mod\ 13)$ and so $G_{12} \notin \mathcal{F}$. We can now apply Lemma II.2.27 with $k = 3$.

Case ii. If $G_2 \notin \mathcal{F}$ then apply Lemma II.2.26 with $k = 2$. Hence we may assume that $G_2 \in \mathcal{F}$.

Since $G_1 \in \Xi$ implies that $n_1 \not\equiv 5(\text{mod } 13)$, and $n_2 \equiv 5(\text{mod } 13)$, we have $(n_1 + n_2) \not\equiv 5 \ (\text{mod } 13)$ and $G_{12} \notin \mathcal{F}$. Since $G_3 \notin \Xi$, we can again apply Lemma II.2.27 with $k = 3$.

Case iii. The number of vertices in members of Ξ are 2, 4, 7, 10, and 12. Adding any one of these numbers to itself or any other member of the set does not give something that is $5(\text{mod } 13)$, hence $G_{12} \notin \mathcal{F}$. Now apply Lemma II.2.27 with $k = 3$.

Case iv. It is only for this case that we chose T such that G_1 and G_2 are triangle-free. Let G^* be the graph induced by $V(G_1) \cup V(G_2) \cup \{u_3\}$. If the theorem is false, then $G^* \neq \Gamma_1$. Let $\Delta f = f(G) - f(G_3)$. Note that $\Delta f \geq 18(n(G^*) - 1) - 13I(G^*)$. Moreover, it is easy to use the triangle and the nature of the triangle-free members of Ξ to show that $I(G^*) = m(G^*)$. For example, the candidate for G^* shown in Figure II.2.17(a) has the representation of Figure II.2.17(b).

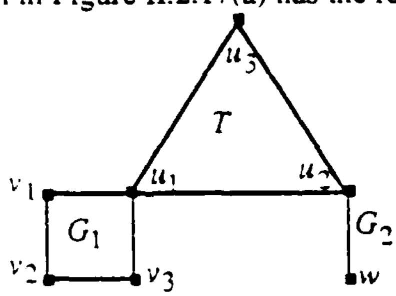


Figure II.2.17(a)

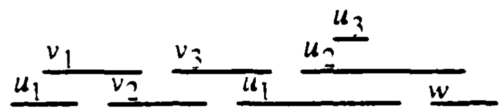


Figure II.2.17(b)

Now for $j = 1,2$, G_j is a triangle-free member of Ξ. Therefore, each G_j is either K_2 or a set of edge-disjoint C_4's.

Because $G_3 \in \Xi$, we know that $f(G_3) \geq -5$. If $G_1 = K_2$ then, unless $G_2 = C_4$, $\Delta f \geq 5$ and $f(G) \geq 0$. If $G_1 = K_2$ and $G_2 = C_4$, then $\Delta f = 4$ and, to avoid $f(G) \geq 0$, G_2 must be C_4. But then $G \in \Xi$ is a type 1 operation applied to K_2 and so is in Ξ.

Hence, each of $\{G_1,G_2\}$ is a set of C_4's. If there are four or more C_4's between G_1 and G_2, then $\Delta f \geq 5$ and $f(G) \geq 0$. If the theorem is false, then it is not true that each of $\{G_1,G_2\}$ consists of one C_4. Hence we may assume that there are exactly three C_4's between $\{G_1,G_2\}$ and we may assume that G^* is one of the three graphs in Figure II.2.18. We then have $\Delta f = 3$. Hence $f(G_1) \leq -4$ and G_3 is one of the graphs of Figure II.2.19.

By inspection, we now can select any of the three graphs of Figure II.2.18 to be G_{12}, either of the graphs of Figure II.2.19 to be G_3, and any vertex of G_3 to be u_3, and the resulting G will be a type 1 operation applied to a member of Ξ. ♠

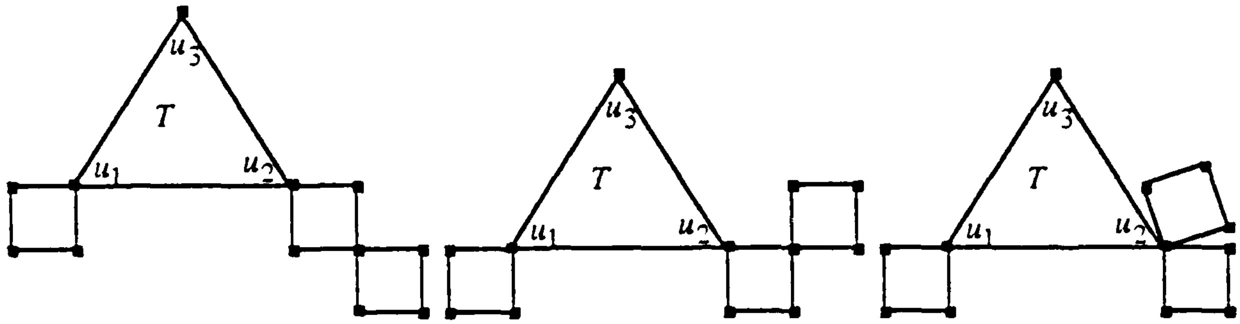


Figure II.2.18

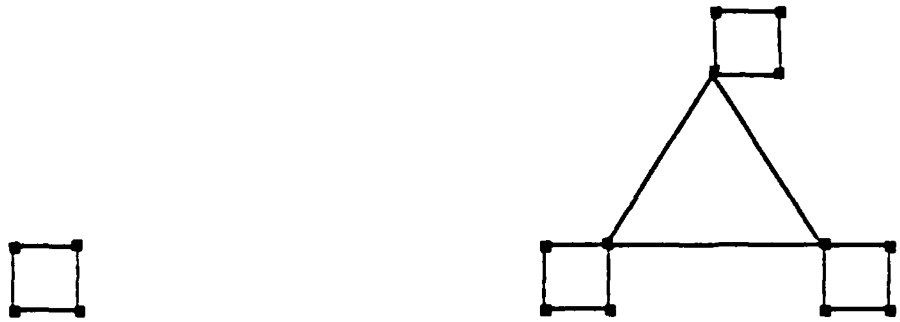


Figure II.2.19

This completes the proof of Theorem II.2.1' and hence of Theorem II.2.1. We now sketch the proof of the analogous result concerning dense cacti.

Theorem II.2.28. If G is a dense cactus that is not in Ξ, then $I(G) \leq \frac{11n(G) - 4}{8}$. Furthermore, for any n, there exists a cactus G_n with n vertices for which $I(G_n) = \lfloor \frac{11n - 4}{8} \rfloor$, and so the result is best possible.

Proof. Note that type 1 operations applied to C_4 show that it is best possible. Furthermore, this time the triangle-free case is simple: a slight modification of the proof of Lemma II.2.13 yields the bound for triangle-free dense cacti since such graphs are Eulerian.

The proof of the general case is by induction on the number of triangles. Select a triangle T such that G_1 and G_2 are triangle-free. Eliminate the possibility of (say) G_1 being trivial by comparing it to the graph with one of the edges to G_1 subdivided: even if depth-3 intervals are allowed, the bivalent vertices of the 4-cycle will force a trail through it and the adjustments of the representation is straightforward.

Since G_1 and G_2 are triangle-free, we may argue as in Lemma II.2.13 to show that both are a collection of 4-cycles. Since there are two or more 4-cycles between G_1 and G_2, it is easy to represent $G_1 \cup G_2 \cup T$ with sufficiently few intervals so that, together with an optimal representation of $G - V(G_1) - V(G_2)$, we have a representation for G with the correct number of intervals. ♠

3. Husimi Trees

The goal of §II.3 is to prove Theorem II.3.1.

Theorem II.3.1. If G is a Husimi tree with $n \geq 4$, then $I \leq \lfloor \frac{3n - 4}{2} \rfloor$. Furthermore, for any $n \geq 4$, there exists a Husimi tree G for which $I(G) = \lfloor \frac{3n - 4}{2} \rfloor$ and so the result is best possible.

The **sun** with n vertices is a graph that consists of a clique $\{u_1,\ldots,u_{n/2}\}$, an independent set $\{v_1,\ldots,v_{n/2}\}$, and the additional edges $\{u_iv_i : i = 1,\ldots,n/2\}$. We denote this S_n. We now show that if $n \geq 4$, then $I(S_n) \geq \frac{3n - 4}{2}$, thereby showing that Theorem II.3.1 is best possible. The basis case, S_4, is trivial.

Now suppose that, for all even $n' < n$, $I(S_{n'}) \geq \frac{3n' - 4}{2}$. Assume that $I(S_n) < \frac{3n - 4}{2}$ and that R is an optimal representation of S_n. Let $V' = V - \{u_1,v_1\}$ and $V'' = \{u_1,v_1\}$. Partition $R(V)$ into $R' \cup R''$, where R' is the set of intervals corresponding to members of V' and R'' is the set of intervals corresponding to members of V''. Note that the graph induced by V' is a sun with $n - 2$ vertices and, by induction, $|R'| \geq \frac{3(n - 2) - 4}{2}$. Therefore $|R''| \leq 2$ and so u_1 and v_1 have just one interval each. By symmetry, we can argue that for any $i = 1,\ldots,n/2$, u_i and v_i have just one interval each. But this contradicts the fact that the graph induced by $\{u_1,u_2,u_3,v_1,v_2,v_3\}$ is not an interval graph (see Figure I.1.1).

To show that the bound of Theorem II.3.1 is best possible for $n = 2k + 1$, subdivide a leaf-edge of S_{2k}.

We now focus on the upper bound. We first show that the bound holds for suns by giving an interval representation of S_n with $\frac{3n - 4}{2}$ intervals. Start with the $k + 2$ intervals listed below.

u_1 - interval $(0,6)$, a u_2 - interval $(3,9)$
v_1 - interval $(1,2)$, a v_2 - interval $(7,8)$
For $i = 3,\ldots,k$, a u_i - interval $(4,5)$

This introduces each edge in the clique and the edges incident to u_1 and u_2. We then use $2(k - 2)$ intervals to represent the remaining k - 2 edges. An example with $n = 8$ is given below.

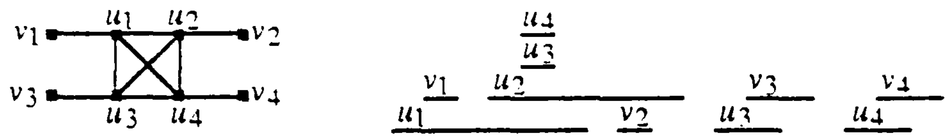


Figure II.3.1

We call this the **standard** representation of the sun.

We now prove the bound by induction on $n(G)$. The basis, $n(G) = 4$, is trivial. Suppose that G is a Husimi tree with n vertices and, if $4 \le n' \le n$ and G' is a Husimi tree with n' vertices, then $I(G') \le \frac{3n' - 4}{2}$. If G is a block, then it is a clique and $I(G) = n$. Hence we may assume that there exists a cut-vertex u.

If u is such that there are three or more u-components, then either $G = K_{1,3}$ or it is possible to group the vertices of G into V' and V'' such that $V' \cap V'' = \{u\}$, $V' \cup V'' = V$, $|V'| \ge |V''| \ge 3$, and there are no edges with one endpoint in V' and the other in V''. Using the fact that any 3-vertex graph has a total interval number of at most three and the induction hypothesis, we can use the union of optimal representations for the graphs induced by V' and V'' to get a representation of G that has at most $\frac{3n - 4}{2}$ intervals. Hence we may assume that there are exactly two u-components. If both u-components have at least four vertices, or if one has three and the other has at least four, then we can again represent them independently within the required number of intervals. If both have three, then we can simply use inspection.

Hence we may assume that, for every cut-vertex u, there are exactly two u-components, one of which has only two vertices. Let u be a cut-vertex and let the two u-components be A and uv. The vertex u is in only one block of A; call it B and recall that B must be a clique. If B contains a cut-vertex u' of G, then the u'-components must be $u'v'$ and $A - v' \cup uv$. This is illustrated in Figure II.3.2, where $B = K_5$.

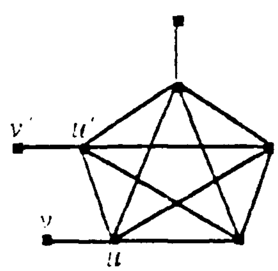


Figure II.3.2

By symmetry, we almost have a sun; some of the independent vertices might be missing. Select u_1 and u_2 so that they *do* have neighboring independent vertices, and follow the standard representation of the sun, omitting the pairs that correspond to edges that are in the sun but not in G; if there are k members of the clique and q members of the independent set, where $2 \le q \le k$, then this representation has $k + 2 + 2(q - 2) \le \frac{3(k+q) - 4}{2}$ intervals.♠

4. Outerplanar Graphs

We now resume our ascending chain of classes. The goal of §II.4 is to prove Theorem II.4.1.

Theorem II.4.1. If G is an outerplanar graph with at least three vertices, then $I(G) \le \lfloor \frac{3n(G) - 2}{2} \rfloor$. Furthermore, for any n, there exists an outerplanar graph for which $I = \lfloor \frac{3n - 2}{2} \rfloor$ and so the result is best possible.

We assume a plane embedding for which the face with n vertices is unbounded. The bound is achieved by any 2-connected outerplanar graph for which the unbounded face is a cycle and either every bounded face is a 4-gon or all but one bounded face is a 4-gon and the remaining bounded face is a 5-gon [2]. For these graphs, $t = 1$ and $m = \lfloor \frac{3n - 2}{2} \rfloor - 1$. Since they are triangle-free, $I = I_2 = 1 + (\lfloor \frac{3n - 2}{2} \rfloor - 1)$ and we have proved that Theorem II.4.1 is best possible.

We now concentrate on 2-connected outerplanar graphs. If such a graph has no 3-gons, then it is intuitive that it has no more edges than the graphs of the previous paragraph. It is easy to use Euler's formula to establish this. We state this as a lemma and omit the proof.

Lemma II.4.2. If G is a 2-connected triangle-free outerplanar graph, then $1 + m \le \lfloor \frac{3n - 2}{2} \rfloor$.♠

Now we establish Theorem II.4.1 for 2-connected outerplanar graphs. We follow Andreae and Aigner's proof. The technique is to find a triangle-free subgraph G' and construct a representation for G that has $1 + m(G')$ intervals.

Lemma II.4.3.[2] If G is a 2-connected outerplanar graph, then we can color the edges red and blue in such a way that the following holds:

- (i) The outside edges are red.
- (ii) Every triangle has at least one blue edge.
- (iii) For every blue edge e, there exists a triangle D_e that consists of e and two red edges. Furthermore, these D_e's are pairwise edge - disjoint.

Proof. We give a slightly different proof than Andreae and Aigner. A **peripheral** face is a face with exactly one chord. If e is an edge that is in two faces, one of which is F, then let $O(e,F)$ be the other face.

Color the outside edges red. If no peripheral face is a triangle, then color the chord e of some peripheral face F red, discard $E(F) - e$, and proceed by induction.

If a peripheral face F is a triangle, then color its chord e blue and consider $F' = O(e,F)$. If an edge of F' is a chord of a peripheral triangle, then color it blue. If not, then color it red. Again proceed by induction, this time applying induction once for each red chord. ♠

We give an example of this coloring in Figure II.4.1. We shade the D_e's and use dark thick lines to indicate blue edges and thick gray lines to indicate red edges. A thin edge is one that is not yet colored.

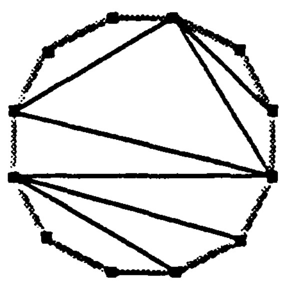


Figure II.4.1(a)

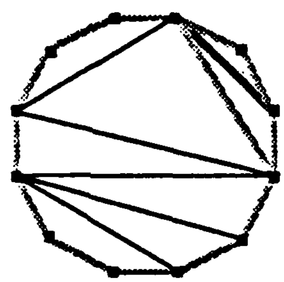


Figure II.4.1(b)

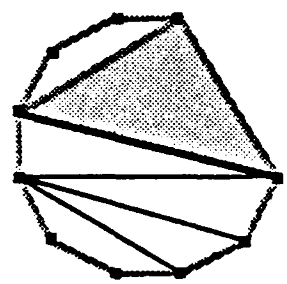


Figure II.4.1(c)

Figure II.4.1(d)

Figure II.4.1(e)

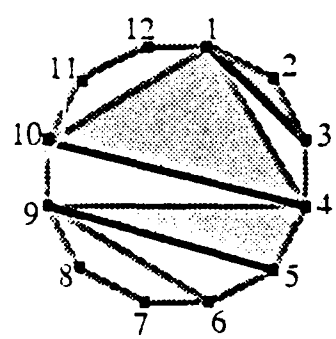


Figure II.4.1(f)

In Figure II.4.1(a), we have the original graph and, because of Lemma II.4.3(i), we color the outside edges "red". In Figure II.4.1(b), the "blue" edge is forced and we can color another edge "red." In Figure II.4.1(c), we first color the "red" chord and then the blue chord.

This brings us to Figure II.4.1(d). We have one more blue edge, shown in Figure II.4.1(e), and we show the entire coloring in Figure II.4.1(f), labelling the vertices for future use.

Theorem II.4.4. If G is a 2-connected outerplanar graph, then $I(G) \le \lfloor \frac{3n-2}{2} \rfloor$.

Proof. Color the edges as in Lemma II.4.3. Let S be the set of red edges. We construct an representation with at most $1 + |S|$ intervals. Since S is triangle-free this will be at most $\lfloor \frac{3n-2}{2} \rfloor$ intervals.

Let the vertex 1 be some vertex with degree at least 3 and number the vertices $1, \ldots, n$ by following the outside face. Start the representation by using the canonical representation of the path $1, \ldots, n$ and insert a small 1-interval into the displayed part of the n-interval. This uses $n + 1$ intervals and represents the n edges on the outside face. We call these the outside intervals.

Each edge (including the red chords) is part of at most one D_e. We will represent each chord by first representing the edges in each D_e and then representing the chords that are not in any D_e. Note that no chord is considered twice (by Lemma II.4.3(iii) of the coloring).

For $i = 1, 2$, or 3, call a D_e type i if it has exactly i edges that are chords of G. Note that type i D_e's have exactly $i - 1$ red chords (by Lemma II.4.3(i),(iii)). We now show how to add $i - 1$ intervals for each type i D_e.

For a type i D_e, extend the outside intervals corresponding to the high-degree vertices until they intersect.

For a type ii D_e, let ij be the outside edge and k be the intersection of the two chords. Take a k-interval and place it within the intersection of the outside i-interval and j-interval.

For a type iii D_e ijk, take an i-interval and a j-interval and place them in the displayed part of the outside k-interval.

Now represent each edge ij that was in no D_e by placing an i-interval inside of the outside j-interval. We have used at most $|S| + 1$ (the " + 1" coming from the $n + 1$ outside intervals representing the n outside edges) intervals. ♠

We give an example of the above construction by using the coloring of the edges that is in Figure II.4.1(f). The representation is given in Figure II.4.2.

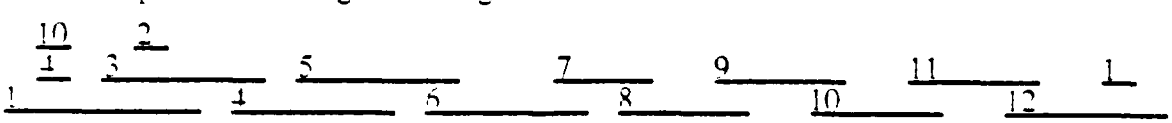


Figure II.4.2

The displayed total interval number (we demand that *every* vertex has a displayed part) can be higher: For example, the total interval number of a 4-cycle with a chord is five, but this graph has no displayed representation with five intervals.

We now present the proof for general outerplanar graphs. It uses the construction of the proof of Theorem II.4.4. Suppose that G is an outerplanar graph. Add edges to the given outerplanar graph until it is 2-connected and then use the representation of Theorem II.4.4. We will be done if we can show that we can remove an arbitrary set of intersections from the representation without increasing the number of intervals.

Consider an outside overlap, say between k and $k+1$ with an interval, say an i-interval placed inside of it. This was placed there because $(k,k+1,i)$ is a type 2 D_e. Note that there cannot also be some j-interval inside of the overlap since the D_e's are edge-disjoint.

To remove the edge $\{k,k+1\}$, shorten k and $k+1$, making sure that i still overlaps the k-interval and the $(k+1)$-interval. To remove the edge $\{i,k\}$, simply move i over to the displayed part of the $(k+1)$-interval and one can similarly remove the edge $\{i,k+1\}$. To remove $\{k,k+1\}$ and $\{i,k\}$, shorten the k-interval until it no longer overlaps the $(k+1)$-interval and make sure that i now overlaps just the $(k+1)$-interval. To remove $\{i,k\}$ and $\{i,k+1\}$, simply remove the i-interval. To remove all three edges, remove the i-interval and shorten the k-interval until it no longer overlaps the $(k+1)$-interval.

Now consider an overlap between a k-interval and a $(k+2)$-interval with a $(k+1)$-interval in the overlap. The same argument shows that any set of adjacencies can be removed. Again we must use the disjointness of the D_e's to ensure that there are not two intervals inside of the overlap.

Now consider a displayed part of an outside interval, say a k-interval. A j-interval placed on the displayed part of the outside k-interval is easy to deal with; remove it. If there are two intervals, say an i-interval and a j-interval on top of each other and the k-interval, then it's harder. Note that these were placed there because of a type 3 D_e and so we can place any 2 of $\{i,j,k\}$ on the third. To remove exactly 1 edge, say (i,j), place k on the displayed parts of i and j. To remove two edges, so that, say only (i,j) remains, place an i-interval on j's outside interval. To remove all 3 intervals, simply remove both i and j. These latter two cases actually save at least one interval.

The final case is if an outside overlap has no interval inside of it and we must remove that edge. Simply shorten one of the intervals until there is no overlap. ♠

5. Planar Graphs

The goal of §II.5 is to prove the following theorem.

Theorem II.5.1. If G is a planar graph and $n(G) \geq 3$, then $l(G) \leq 2n(G) - 3$. Furthermore, for any n, there exists a planar graph G with n vertices for which $l(G) = 2n - 3$ and so the result is best possible.

This is the main result of the thesis. Andreae and Aigner [2] showed that theorem II.5.1 holds for triangle-free graphs and that it is best possible. We enlarge upon their proof that it is best possible.

Given n, let G be a plane graph with n vertices, such that all faces of G are of degree four; we must show that $l(G) \geq 2n - 3$. From $2m(G) = 4\phi(G)$ and Euler's formula, we obtain $m(G) = 2n - 4$. If G has no triangle, then $l_2(G) = l(G)$ and, since $t(G) \geq 1$, it follows that $l(G) \geq 2n - 3$. Hence it is sufficient to show that G has no triangle.

Suppose that G has a triangle C and let H be the graph induced by the vertices of C, together with the vertices that are inside of C. By the definition of G, all bounded faces of H are of degree four and therefore. all but one face of H is of degree four. We then have $2m(H) = 4(\phi(H) - 1) + 3$ and this is impossible since the left side is even and the right side is odd. Hence Theorem II.5.1 is best possible.

Since blocks are graphs, we can apply terms that refer to graphs equally well to blocks. For example, a planar block is a planar graph with no cut-vertex and a plane block is a planar block, together with some planar embedding. We prove Theorem II.5.1 by induction on the number of vertices; the basis step, $n = 3$, is trivial. We now present the induction step if G has a cut-vertex u. If every u-component has two vertices, then G is a star and $l(G) = n(G)$. Otherwise, let A be a u-component with at least three vertices: by induction, $l(A) \leq 2n(A) - 3$. Let B be the graph induced by the non-isolated vertices of $G - E(A)$. Note that $n(G) = n(A) + n(B) - 1$ and $l(G) \leq l(A) + l(B)$. If $n(B) = 2$, then $l(B) = 2$ and $l(G) \leq 2n(A) - 3 + 2 = 2n(G) - 3$. If $n(B) \geq 3$ then, by induction, $l(B) \leq 2n(B) - 3$ and so $l(G) \leq 2n(A) - 3 + 2n(B) - 3 = 2n(G) - 4$. We must still deal with the induction step if G is a block. Hence, it suffices to show:

Lemma II.5.2. If G is a planar block and $n(G) \geq 3$, then $l(G) \leq 2n(G) - 3$.

We first reduce the problem of proving Lemma II.5.2 to that of proving Theorem II.5.5 and then spend all but the last paragraph of §II.5 proving Theorem II.5.5.

For a plane graph G, let $f_i(G)$ be the number of i-gons, and $f_i'(G)$ be the number of bounded i-gons. Let $\lambda(G)$ be the degree of the unbounded face and $\mathcal{B}(G)$ be the set of bounded faces. Note that if $i \neq \lambda(G)$, then $f_i'(G) = f_i(G)$, and that $f'_{\lambda(G)} = f_{\lambda(G)} - 1$.

If e is incident to two bounded faces F and F', then F', the face "opposite" e from F, is denoted $O(e,F)$. If e is incident to the unbounded face and some bounded face F, then $O(e) = F$.

A plane graph G is a **proper supergraph** of a plane graph G' if G is obtained from the embedding of G' by adding a non-empty set of chords of bounded faces without introducing any 3-gon. For example, the graph in Figure II.5.1(b) is a proper supergraph of the graph in Figure II.5.1(a).

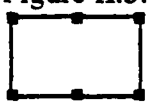


Figure II.5.1(a)

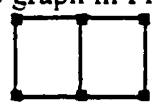


Figure II.5.1(b)

Note that adding a chord to a 4-gon or 5-gon will introduce a 3-gon and therefore will not produce a proper supergraph. It is clear that a plane graph has no proper supergraph if and only each of its bounded faces is of degree at most five.

A **proper** representation R is a representation that has the following properties.

i. R is irredundant.
ii. $k \geq 4 \Rightarrow r_k(R) = 0$
iii. Each depth-3 interval introduces two edges that are incident to the same 3-gon.

If R is a proper representation of G, then, by properties i. and ii. above, $m(G) = r_2(R) + 2r_3(R)$. Let $l'(G)$ be the size of the smallest proper representation of G.

Lemma II.5.3. If G is a proper supergraph of G', then $l'(G') \leq l'(G)$.

Proof. Let $p(G,G') = |E(G) - E(G')|$: we show by induction on p that $E(G) - E(G')$ can be deleted from G without increasing l'. The basis step, $p = 0$, is trivial. Now suppose that G and G' are given and that the lemma holds for all pairs H and H' for which $|E(H) - E(H')| < p(G,G')$ and suppose that $uv \in E(G) - E(G')$. Let $G'' = G' \cup uv$. By induction, $l'(G'') \leq l'(G)$.

Let R be an optimal proper representation of G''. Without loss of generality, we may assume that some v-interval θ_v introduces uv by intersecting an earlier u-interval θ_u.

If the only interval that θ_v intersects is θ_u, then remove θ_v.

Now assume that there exists $w \in V(G) - \{u\}$ and that some w-interval θ_w intersects θ_v. If θ_w also intersects θ_u, then either θ_v or θ_w is a depth-3 interval. By property iii. of proper representations, the two edges introduced by the depth-3 interval must be incident to a 3-gon. Note that uv is not incident to any 3-gon. If θ_v is a depth-3 interval, then vw and vu are two edges of the 3-gon (uvw) and similarly, if θ_w is a depth-3 interval, then wu and wv are two edges of the 3-gon (uvw). But this contradict the fact that uv is in no 3-gon. Hence if θ_v intersects some θ_w, then $\theta_u \cap \theta_w = \emptyset$. To avoid the introduction of the edge uv in R, move the left endpoint of θ_v until it is to the right of the right endpoint of θ_u.

In either case, the resulting representation of G' has no more than $|R|$ intervals and so $I(G') \leq I(G'') \leq I(G)$. ♠

In §II.5, we will deal exclusively with proper representations; we will prove that $I'(G) \leq 2n(G) - 3$. Hence from Lemma II.5.3, we may assume that:

$$i \geq 6 \Rightarrow f_i' = 0 \qquad \text{(II.5.1)}$$

We define the **perimeter** of a plane graph G to be the vertex-set of the unbounded face and denote it $P(G)$. A **contiguous** subset of the perimeter is the set of vertices of some path, all of whose edges are incident to the unbounded face. A **section** of a graph is an induced subgraph with no cut-vertex. For example, a block is a maximal section. In the figures of §II.5, each **straight line segment** indicates a single edge, each **arc** indicates a contiguous subset of the perimeter, each **white area** indicates a single face, and each **shaded area** indicates a section. Each figure represents the class of graphs determined by the lines, curves, white areas, and shaded areas.

We now sketch a proof that Theorem II.5.1 holds for graphs without 3-gons. This case motivates many of the ideas of the proof of the general case.

Let G be a plane block satisfies (II.5.1). Using Euler's formula and $2m(G) = \lambda(G) + f_4(G) + 5f_5(G)$, we obtain $2n(G) - 3 \geq m(G) + 1$. Since $I(G) \leq m(G) + t(G)$, it is sufficient to prove that there is a covering trail T of G. We do this by induction on the number of faces.

We strengthen the induction hypothesis and prove that T can start at any vertex u on the unbounded face and end at any vertex v on the unbounded face. Let $e = uu'$ be an edge on the unbounded face and let $F = O(e)$. If the only vertices of F that are on the unbounded face are u and u', then we can remove e and apply induction. Otherwise, there are several cases. The most difficult case is shown in Figure II.5.2.

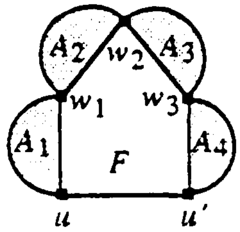


Figure II.5.2

Suppose that $v \in P(A_1)$. Let T begin with $\langle u,u' \rangle$. Since the induction hypothesis applies to each A_i, we can extend T to cover A_4 and end at w_3. We then similarly extend T through A_3, A_2, and A_1, covering these sections, and ending at v.

Similar methods work unless $v \in P(A_3)$. In this case, we consider $F' = O(w_2w_3,F)$ (see Figure II.5.3). We will consider only the case that F' is a 5-gon and all of its vertices are in $P(G)$.

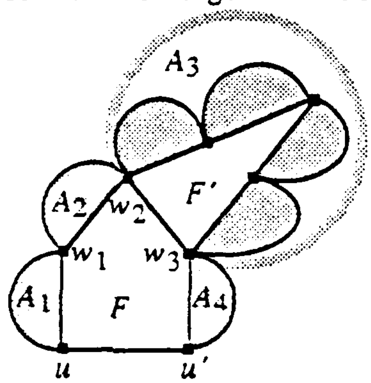


Figure II.5.3

We must treat the possibilities of v lying in each of the gray sections of A_3 as separate cases. We show each case and the corresponding trails in Figure II.5.4.

For graphs with 3-gons, it is possible that $I_2 > 2n - 3$ (e.g., a 4-gon with a chord). Therefore we cannot restrict ourselves to depth-2 representations and we must use depth-3 intervals. We will still use the idea of starting and ending on the unbounded face and, as in Figure II.5.4, we will find "routes" from u to v through some sections.

Let G be a plane graph. From $2m(G) = \lambda(G) + 3f_3^b(G) + 4f_4^b(G) + 5f_5^b(G)$ and $n(G) - m(G) + 1 + f_3^b(G) + f_4^b(G) + f_5^b(G) = 2$, we obtain:

$$2n(G) - 3 = 1 + m(G) - \frac{f_3^i(G)}{2} + \frac{f_5^i(G)}{2} + \frac{\lambda(G) - 3}{2} \qquad \text{(II.5.2)}$$

Our goal is to find a representation R with $2n(G) - 3 \geq |R|$. By (II.5.3), this is equivalent to finding a representation R for which:

$$2 + 2m(G) - f_3^i(G) + f_5^i(G) + (\lambda(G) - 3) - 2|R| \geq 0 \qquad \text{(II.5.3)}$$

We will usually not be interested in the contribution of the unbounded face to the left side of (II.5.4); we define the **profit** $p(R)$ and obtain the following:

$$p(R) = 2 + 2m(G) - f_3^i(G) + f_5^i(G) - 2|R| \qquad \text{(II.5.4a)}$$

$$2((2n(G) - 3) - |R|) = p(R) + (\lambda(G) - 4) \qquad \text{(II.5.4b)}$$

If R is a proper representation, then

$$p(R) = 2 - 2r_1(R) + 2r_3(R) - f_3^i(G) + f_5^i(G) \qquad \text{(II.5.4c)}$$

We call R **profitable** if $p(R) \geq 0$.

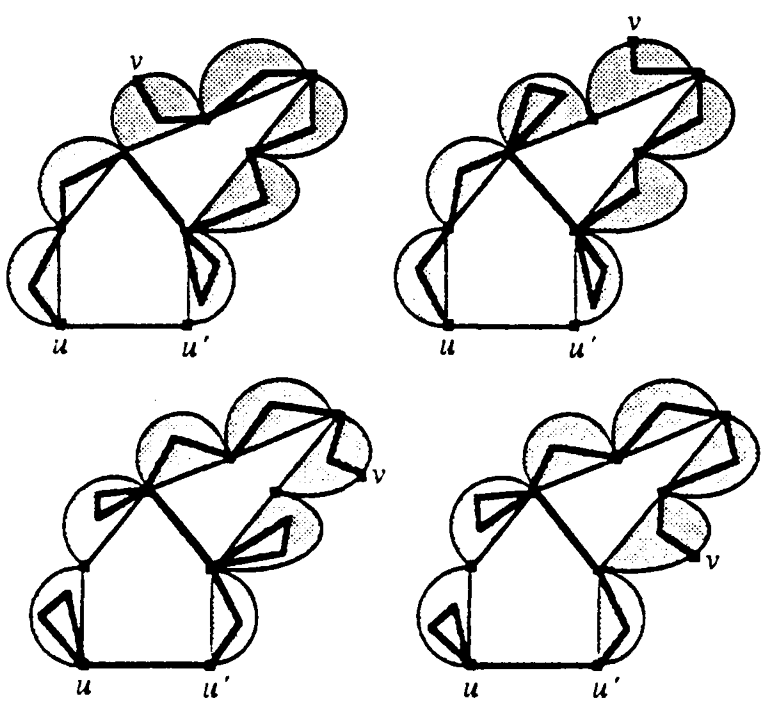


Figure II.5.4

If G is a plane graph and R is a representation of G, then from (II.5.5a) and $2m(G) = 3f_3^i(G) + 4f_4^i(G) + 5f_5^i(G) + \lambda(G)$, we see that the parity of each of the following quantities is the same:

i. The number of odd bounded faces
ii. The profit of R
iii. The degree of the outside face

We define the **parity** $\varepsilon(G)$ of a plane graph G as follows: If the quantities above are even, then

$\varepsilon(G) = 0$ (and G is even), otherwise $\varepsilon(G) = 1$ (and G is odd). Consider the following examples:

Figure II.5.5(a): G_1

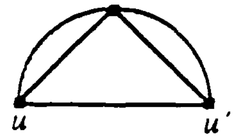


Figure II.5.5(b): G_2

We have $\varepsilon(G_1 - uu') = \varepsilon(G_1)$ and $\varepsilon(G_2 - uu') = 1 - \varepsilon(G_2)$. Note that if G_1 is even, then an even number of blocks of $G_1 - uu'$ are odd, and if G_2 is even, then exactly one block of $G_2 - uu'$ is odd.

Lemma II.5.4. If R is a profitable representation of G and $\lambda(G) \geq 3$, then $|R| \leq 2n(G) - 3$.

Proof. From (II.5.5b), the result is immediate if $\lambda(G) \geq 4$. If $\lambda(G) = 3$, then since $\varepsilon(G) = 1$ and $p(R) \geq 0$, we have $p(R) \geq 1$ and the result again follows.♠

By Lemma II.5.4, in order to prove Lemma II.5.2 (and hence Theorem II.5.1), it suffices to show that there exists a profitable representation for any planar block that satisfies (II.5.1).

We must prove some technical results beyond this and we need several definitions to state them. Let R be a representation of a graph G. If $\theta = [\alpha, \beta]$ is an interval, then $-\theta = [-\beta, -\alpha]$. Let the **reverse** of R be $\{-\theta : \theta \in R(V)\}$ and denote it $\tilde{R}$. If there exists a depth-1 u-interval in R (in $\tilde{R}$), then we say that R **starts** (**ends**) at u. Note that R can start and end at several vertices. If R ($\tilde{R}$) starts at u and the corresponding depth-1 u-interval in R ($\tilde{R}$) is immediately followed in R ($\tilde{R}$) by a depth-2 u'-interval, then R **starts (ends) at the edge** uu'.

Suppose that u and v are vertices and that e and e' are edges. A representation that starts at u and ends at v is a u,v-**representation** and, if profitable, is denoted by $u,G \rightarrow v$. A representation that starts at u and ends at the edge e is a u,e-**representation** and, if profitable, is denoted by $u,G \rightarrow e$. We use analogous definitions and notation for other combinations of u, v, e, and e'.

Suppose that G is a plane block and that $e \in E(G)$. Let R be a proper representation of $G - e$. The e-**profit of** R, $p'(e,R)$, is defined by:

$$p'(e,R) = 2r_3(R) - 2r_1(R) + f_3^{\xi}(G) - f_3^{\xi}(G). \qquad \text{(II.5.6)}$$

If R is a representation of $G - e$ that starts at u and ends at v, and $p'(e,R) \geq 0$, then it is denoted $u,G^e \rightarrow v$.

To see why we include all of $\Xi(G)$ in this definition, and not just $\Xi(G - e)$, consider Figure

II.5.6(a). In Figure II.5.6(b), we look more closely at B. In this example, the face F that is in $\mathcal{B}(B)$ and contains e is a 3-gon.

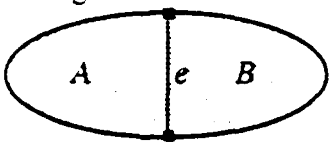


Figure II.5.6(a)

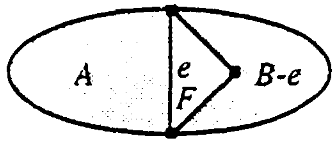


Figure II.5.6(b)

Suppose that we have a representation R of A and a representation R' of $B - e$. Since $R = R \cup R'$ is an irredundant representation of G, we would like to be able to compute the profit of R from the profits of R and R'. Note also that $\mathcal{B}(G)$ is partitioned into $\mathcal{B}(A) \cup \mathcal{B}(B)$. Although $F \notin \mathcal{B}(B - e)$, the profit of a representation of G must take into account the contribution of F. The most convenient way of doing this that allows some combining of R and R' is to assign the contribution of F to the profit of R'. This violates the definition of profit (since R' is a representation of $B - e$ and not B) so we use the term e-profit. Additional details concerning the combining of profits and representations appear in later parts of the proof.

Suppose that G is an odd plane block and not a 3-gon. If $u \in P(G)$ and the only faces that are incident to u are the unbounded face and one bounded 3-gon, then u is called **troublesome**. Let $T(G)$ be the set of troublesome vertices of G. For example, in Figure II.5.7, $T(G) = \{u\}$.

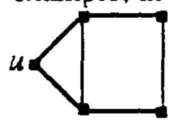


Figure II.5.7: G

Since our graphs have no multiple edges, $\lambda(G) = 3$ implies that $T(G) = \emptyset$. If G is an even block or a 3-gon, then we define $T(G)$ to be the empty set. Note that it is impossible to have two adjacent members of $T(G)$. In particular, if u and u' are one step apart on the unbounded face, then at least one of $\{u,u'\}$ is *not* in $T(G)$.

We have now defined most of the terms in Theorem II.5.5 (below). The terms "u-admissable", "almost profitable", and "difficult triple" are quite technical and we defer their definitions until the proof of the theorem.

Theorem II.5.5. Suppose that G is a plane graph, and that $e = uu'$ and $e' = u'u''$ are edges of the unbounded face. Then we have the following:

(i) If G is an odd block, then G has a profitable u,v-representation unless $u = v \in T(G)$.
(ii) If G is an even block, then G has a profitable e,v-representation.
(iii) If G is an even block, then G has a profitable e,e'-representation.
(iv) If G is u-admissable and $v \neq u$, then G has a profitable u,v-representation.
(v) If G is an odd block and $v \in \{u',u''\}$, then, unless $v = u' \in T(G)$, G has a profitable e,v-representation.
(vi) If G is an even block, $\lambda(G) \geq 4$, and $v \in \{u',u''\}$, then there exists a u,v-representation of $G - e$ for which the e-profit is nonnegative.
(vii) If G is an odd block and (e',e,v) is not a difficult triple, then G has an almost profitable $e,v^{e'}$-representation.

Most of the rest of §II.5 is devoted to proving Theorem II.5.5. The proof is by induction on the number of edges. We will use (e.g.) (i) to refer to Theorem II.5.5(i), or to point out that we are applying the induction hypothesis Theorem II.5.5(i) to a smaller graph; the context will make the meaning of this notation clear.

The critical conclusions of Theorem II.5.5 are (i) and (ii); these show that profitable representations for planar blocks exist. Note that, since an e,v-representation is also a u,v-representation, (ii) implies that if G is an even block, then there exists a profitable u,v-representation of G. We will frequently use this analogue and refer to it simply as (ii).

By considering the reverses of the representations of (ii), (v), and (vi), we immediately obtain:

(ii_r) If G is an even block, then G has a profitable v,e-representation.
(v_r) If G is an odd block and $v \in \{u',u''\}$, then, unless $v = u' \in T(G)$, G has a profitable v,e-representation.
(vi_r) If G is an even block, $\lambda(G) \geq 4$, and $v \in \{u',u''\}$, then there exists a u'',u-representation of $G - e$ for which the e-profit is nonnegative.

We will use these as we would use any of the other induction hypotheses and refer to them as (ii_r), (v_r), and (vi_r).

If R is a representation, then the components can be permuted or reversed without affecting the size of R. Therefore, if R starts or ends at u, then there is a representation R' such that $|R| = |R'|$ (and hence $p(R) = p(R')$) and the first interval of R' is a u-interval. This is useful for combining representations of subgraphs of G to get a representation of G.

Lemma II.5.6. Suppose that G is a plane graph with the two blocks A and B, $u \in P(A)$, $v \in P(B)$, and $V(A) \cap V(B) = \{w\}$. Suppose also that R_1 is a profitable u,w-representation of A and R_2 is a profitable w,v-representation of B. Then there exists a profitable u,v-representation R of G.

Proof. The hypotheses concerning G are illustrated below.

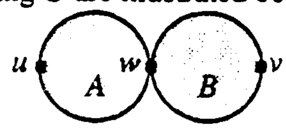


Figure II.5.8

By permuting the components of R_1 and R_2 and shifting all of the intervals of R_2, we may assume that R_1 and R_2 is as below.

Figure II.5.9: R_1 R_2

Let $R' = R_1 \cup R_2$. Note that $f_i'(G) = f_i'(A) + f_i'(B)$ and $r_i(R') = r_i(R_1) + r_i(R_2)$. From the definition of p, we have $p(R') = p(R_1) + p(R_2) - 2$. Since $p(R_1)$ and $p(R_2)$ are both nonnegative, $p(R') \geq -2$.

Now let θ_1 be the w-interval of R_1 that corresponds to R_1 ending at w, and let θ_2 be the w-interval of R_2 that corresponds to R_2 starting at w. Splice θ_1 and θ_2 together to form the new interval θ. Call this representation R; it is irredundant and, from $|R| = |R'| - 1$ and (II.5.4b), we have $p(R) = p(R') + 2 \geq 0$. ♠

Putting the two notations for R_1 and R_2 together and suppressing the second w results in the notation $u.A \rightarrow w.B \rightarrow v$ for describing the profitable u,v-representation of Lemma II.5.6.

A variation of Lemma II.5.6 is the idea of splicing together the two intervals that correspond to an edge. Suppose that A and B are sections of G, $E(G) = E(A) \cup E(B)$ and $E(A) \cap E(B) = \{uu'\}$. Also, suppose that R_1 is a profitable $u.u'u''$-representation of A and that R_2 is a profitable $u'u''.v$-representation of B. As before, we may assume that R_1 and R_2 are as below.

Figure II.5.10: R_1 R_2

Let $R' = R_1 \cup R_2$. Note that R' is *not* a proper representation of G since it is not irredundant. However, if we splice together the u'-intervals *and* the u''-intervals, then the resulting representation is irredundant *and* profitable. We denote it $u,A \to u'u'',B \to v$.

Of course it is possible to use a sequence of splices. In particular, suppose that G has the following characteristics:

i. The blocks of G are $\{A_i : i = 1,\ldots,k\}$.
ii. $u \in A_1$
iii. For $i = 1,\ldots,k-1$, $V(A_i) \cap V(A_{i+1}) = w_i$
iv. $v \in P(A_k)$
v. $\{w_i : i = 1,\ldots,k-1\}$ are distinct.
vi. $j \notin \{i-1, i+1\} \Rightarrow V(A_i) \cap V(A_j) = \emptyset$
vii. $u,A_1 \to w_1$, $\{w_i,A_{i+1} \to w_{i+1} : i = 1,\ldots,k-1\}$, and $w_{k-1},A_k \to v$ exist.

Then it is possible to construct a profitable u,v-representation for G by repeatedly applying Lemma II.5.6. An example that satisfies conditions i. through vi. for $k = 3$ is given below.

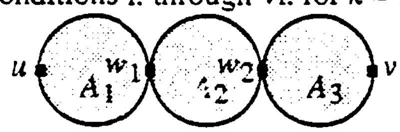


Figure II.5.11

We call a sequence of splices as above a **march**. We also use the term march if we start at an *edge* of the first block or if we end at an edge of the last block.

The representations asserted by Theorem II.5.5 are constructed as follows: A plane block is divided into sections, induction is applied to each section, and the resulting representations are spliced together. From Lemma II.5.6, we can compute the profit of such a representation if we know the profits of the constituent representations.

We use the following abbreviations concerning profitable representations of a section that consists of a single edge $e = uu'$:

$u,uu' \to u$	is abbreviated	$u(u')$
$u,uu' \to u'$	is abbreviated	$u \to u'$
$uu',uu' \to u$	is abbreviated	$uu' \to u$
$u,uu' \to$	is abbreviated	$u \to uu'$

We illustrate the description of a representation that is constructed by splicing together several sections, including some that are single edges. Suppose that we have the following situation:

uu' is a cut-edge of G.
The two components of $G - e$ are A and B.
There exists a profitable v,u-representation of A and a profitable u',w'-representation of B.

Then $v,A \rightarrow u \rightarrow u'B \rightarrow ww' \rightarrow w$ is a profitable v,w-representation of G.

The first interval is always a depth-1 interval and it contributes -2 to the profit. This is offset by the constant in the definition of profit. For any other depth-1 interval, there must be some part of the representation that "creates" a profit of two if the representation is to be profitable. Recall that each 5-gon contributes one to the profit, a profitable representation of an odd block contributes one, and a depth-3 interval contributes two. "Spending" some current or future profit from these sources on a new depth-1 interval is a common tactic and we therefore use the term "**buy at** v" to mean "start a new component of the representation with a depth-1 v-interval." Unless otherwise specified, a **contribution** is understood to mean a contribution of one to the profit and a **negative contribution** is understood to mean a negative contribution of one to the profit.

Consider the following example.

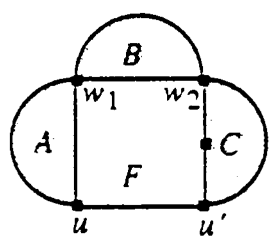


Figure II.5.12: G

Suppose that we wish to show the existence of $u,G \rightarrow v$ and we know only that $v \in P(G)$. Suppose also that $\varepsilon(A) = 1$ and that $u,A \rightarrow w_1$, $w_1,B \rightarrow w_2$, and $w_2,C \rightarrow u'$ all exist. Then the representation $R' = u(u'),A \rightarrow w_1 B \rightarrow w_2,C \rightarrow u'$ has a profit of at least one because of $u,A \rightarrow w_1$. But there is also a contribution to $p(R')$ from the bounded 5-gon F and this has not yet been considered. Hence $p(R') \geq 2$ and we can "afford" to buy at v; let R be the union of R' and a single isolated v-interval. Then $p(R) = p(R') - 2 \geq 0$ and R ends at v. Hence R is a profitable representation that ends at v. The strategy of marching through a sequence of sections to end at any vertex with enough surplus profit to buy at v is called **march and buy**. It occurs in the proof of every induction step of Theorem II.5.5.

The most powerful induction hypotheses of Theorem II.5.5 are (i), (ii), and (v). But (i) and (v) are weakened by the restriction concerning $T(G)$. These restrictions are necessary as shown by the graph of Figure II.5.6.; if there exists a profitable u,u-representation $|R|$, then, from (II.5.4(b)), $|R| \leq$

6 and R starts at u and ends at u'. It is easy to verify that there is no such representation.

Even with this restriction, (i) and (v) are quite powerful because it is often possible to eliminate the possibility of $u \in T(G)$. For example, we have already noted that it is impossible for two consecutive members of the unbounded face to be in $T(G)$. The following lemma is also useful.

Lemma II.5.7. Suppose that G is a plane block, $u, u', u'' \in P(G)$, uu' and $u'u''$ are edges of the unbounded face, $H = G - uu'$, and $v \in \{u', u''\}$. Suppose that H has a cut-vertex. Let A be the block of H that contains the edge $u'u''$ and let w be the cut-vertex of H that is in $P(A)$.

Then $v = w$ implies that $\lambda(A) \leq 4$ and hence $T(A) = \emptyset$.

Before presenting the proof of this lemma, we give a pair of examples that illustrate both the simple nature of the lemma and how it makes the exclusion of 6-gons so convenient. In Figure II.5.13(a), $\lambda(F) = 5$ and $\lambda(A) = 4$. Contrast this with Figure II.5.13(b). In this graph, F is a 6-gon, $\lambda(A) = 5$, and the conclusion of the lemma is false.

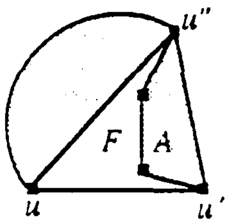


Figure II.5.13(a)

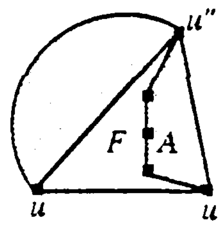


Figure II.5.13(b)

Proof of Lemma II.5.7. Let $F = O(uu')$. Since $v = w$, $V(F) \supseteq P(A)$ and, from $\lambda(F) \leq 5$ and $u \in V(F) - P(A)$, it follows that $\lambda(A) \leq 4$.♠

This lemma is used repeatedly. We note its use the first time we use it and thereafter use it tacitly.

The proof of Theorem II.5.5 is by induction on the number of edges. The basis case for (i), (v), and (vii) is the 3-gon. The basis case for (ii) and (iii) is a single edge. For (iv), the basis case is the 4-gon and for (vi), the basis case is the graph that consists of two incident edges. These are all trivial. Now suppose that G is given and that Theorem II.5.5 holds for all plane graphs with fewer edges than G. We will assume that u' is counterclockwise from u and u'' is counterclockwise from u'. Most steps are justified by (i), (ii), (v), or (vi) and we will not mention these justifications explicitly.

Since nothing is proved until the induction step for each of the parts is established, we use the symbol ♣ instead of ♠ to indicate the end of the induction step for one of the assertions.

We first present the induction step of (vi). Let $F = O(e)$ and $H = G - e$. If $\lambda(F) \geq 4$, then march through the blocks of H to end at v, using Lemma II.5.7. Note that the face F has a nonnegative effect on p.

Now suppose that $\lambda(F) = 3$ so that $\varepsilon(H) = 1$. If H is a block, then use $u,H \rightarrow v$. This provides a contribution to p that offsets the negative contribution of F.

If H is not a block, then define w by $F = (uwu')$. Since H is not 2-connected, $w \in P(G)$ and H has two blocks A and B, where $u \in V(A)$ and $u' \in V(B)$. This is illustrated in Figure II.5.14.

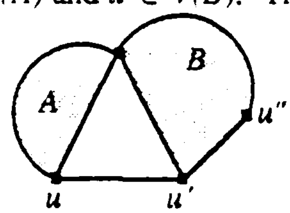


Figure II.5.14

Since $\varepsilon(H) = 1$, exactly one of $\{A,B\}$ is odd. Use $u,A \rightarrow w,B \rightarrow v$. This march yields a contribution as it goes through the odd block (whether A or B) and this offsets the negative contribution of F.♣

We now concentrate on (iv). If G is a plane graph and $u \in P(G)$, then G is **u-admissable** if G has exactly two blocks A and B, $u \in P(A)$, A is even, u is adjacent to the cut-vertex u', and $u' \notin T(B)$. The situation is illustrated below.

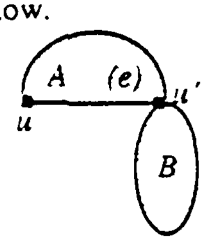


Figure II.5.15: $u' \notin T(B)$

We now present the induction step of (iv). If A is a single edge, then use $u \rightarrow u'$ and then (i) or (ii). Hence $\lambda(A) \geq 4$. If $v \in P(B)$, then use $u,A \rightarrow u',B \rightarrow v$. Hence $v \in P(A) - \{u'\}$. Let $F = O(e)$ and $H = A - e$.

Suppose that $\lambda(F) = 5$. Since A is even, at least one block of H is odd. Start with $u(u')$ and march through the blocks of the remaining graph. Positive contributions to the profit are from F and at least one block of H and we have enough profit to buy at v. This is our first example of march and buy.

Suppose that $\lambda(F) = 4$. If two blocks of H are odd, then we can again march and buy. Hence we may assume that all blocks of H are even. Number the blocks $\{A_i\}$ and cut-vertices $\{w_i\}$ of H by moving clockwise from u around $P(A)$; $u \in P(A_1)$, $V(A_1) \cap V(A_2) = \{w_1\}$, $V(A_2) \cap V(A_3) = \{w_2\}$, etc.. Note that H has at most three blocks.

If $v \in P(A_1)$, then start with $u \to u',B \to u'$ and then march back to v. If H has at least two blocks and $v \in P(A_2)$, then use $u,A_1 \to u \to u',B \to u'$, and then march back to v. If H has three blocks and $v \in P(A_3) - \{w_2\}$, then all four vertices of F are members of $P(A)$. This is illustrated below.

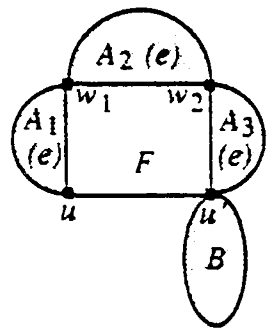


Figure II.5.16

In particular, $w_2 \leftrightarrow u'$. Hence we may use $u(u'),A_1 \to w_1,A_2 \to w_2$ and (iv).

Suppose that $\lambda(F) = 3$: define w by $F = (uwu')$. If $w \notin P(A)$, then H is an odd block and, since $v \neq u'$, we can use $u \to u',B \to u',H \to v$. If $w \in P(A)$, then let A_1 be the block of H that contains u and A_2 be the block of H that contains u'. Since A is even and F is odd, exactly one A_i is odd. If $v \in P(A_1) - \{w\}$, then use $u \to u',B \to u',A_2 \to w,A_1 \to v$. Hence $v \in P(A_2) - \{u'\}$.

If $u \in T(A_1)$, then $w \notin T(A_1)$ and $\varepsilon(A_2) = 0$ so we may use $u \to u',B \to u'$ and (iv). If $u \notin T(A_1)$, then use $u,A_1 \to u \to u',B \to u',A_2 \to v$. ♣

We now concentrate on (ii). Let $F = O(e)$ and $H = G - e$.

If $\lambda(F) = 5$, or $\lambda(F) = 4$ and two blocks of H are odd, then use $e \to u$ and then march and buy. In the former case, one of the blocks of H must be odd since $\varepsilon(H) = 1$.

Now suppose that $\lambda(F) = 4$ and that all blocks are even. Label the blocks $\{A_i\}$ and cut-vertices $\{w_i\}$ of H by moving clockwise from u around $P(A)$; $u \in A_1$, $V(A_1) \cap V(A_2) = w_1$, $V(A_2) \cap V(A_3) = w_2$, etc.. If $v \in P(A_1)$, then start with $e \to u'$ and march from u' back to v. Otherwise, use $e \to u,A_1 \to w_1$ and then (ii) or (iv).

Now suppose that $\lambda(F) = 3$; define w by $F = (uwu')$. If $w \notin P(G)$, then use either $e \to u$ or $e \to u'$, followed by $u,H \to v$ or $u',H \to v$. If $w \in P(G)$, then H has two blocks A and B, where $u \in P(A)$ and $u' \in P(B)$. By symmetry, we may assume that A is even and B is odd. If $v \in P(A)$, then use $e \to u',B \to w,A \to v$ and if $v \in P(B) - \{w\}$, then use $e \to u,A \to w,B \to v$.♣

Before presenting the induction step of the next assertion of Theorem II.5.5, we illustrate (ii) by presenting the induction step of (i) for the case $u \in T(G)$. This will be our first explicit use of a depth-3 interval.

Suppose that $u \in T(G)$. Let $H = G - u$. Start with the depth-1 u-interval and then place depth-2 and depth-3 intervals corresponding to the other vertices of $O(e)$. Extend these latter intervals and then use (ii) on H to finish with $e,H \to v$; since G is odd and $\mathcal{B}(H) = \mathcal{B}(G) - \{F\}$, H is an even block and (ii) is indeed applicable. The depth-3 interval contributes two to p and the 3-gon subtracts one. Hence the representation has a profit of one.♣

We now concentrate on (vii). If R is a representation of $G - e'$ and $p'(e',R) \geq -1$, then R is called **almost profitable**. This term will appear only if there is a missing edge e', R starts at an edge e, and R ends at the vertex v. For such a situation, an almost profitable representation is denoted $e,G^{e'} \rightharpoonup v$.

We say that (e',e,v) is **a difficult triple** if either of the following situations holds:

i. a. $O(e') = (u'wxu'')$, $w \notin P(G)$, and $x \in P(G)$
   b. A is the block of G-e' that contains e, and B is the block that contains u'.
   c. $\varepsilon(A) = 1$ and $\varepsilon(A) = 0$
   d. $v \in P(A) - \{x\}$

ii. a. $O(e') = (u'wu'')$ and $w \in P(G)$
   b. A is the block of $G - e'$ that contains e, and B is the block that contains u'.
   c. $\varepsilon(A) = \varepsilon(B) = 0$
   d. $v \in P(B) - \{w\}$

Therefore, (vii) states that, except for the situations illustrated in Figures II.5.17(a) and II.5.17(b), $e,G^{e'} \rightharpoonup v$ exists.

We now present the induction step of (vii). Let $F = O(e')$ and $H = G - e'$. No matter what F is, start a clockwise march from e through the blocks of H. The hope is that after the march, we will have enough profit to buy at v; note that the statement of (vii) "gives" us some additional profit.

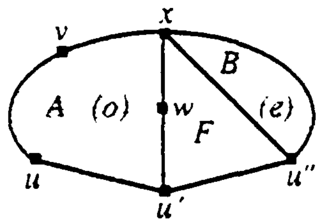


Figure II.5.17(a)

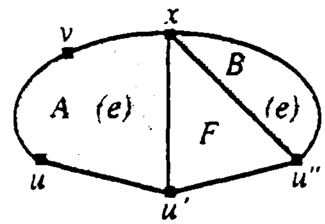


Figure II.5.17(b)

Suppose that the first block of the proposed march is even. Then march through all of the blocks, either ending at v or buying at v after the march. The only time that this tactic is too "expensive" is the second difficult triple.

Therefore we will assume that the first block of the proposed march is odd. In this case, the fact that the representation starts at an *edge* forces us to justify the first step of the march with (v) rather than (i). If the first cut-vertex is only one step away on the unbounded face of the first block of the proposed march, then the march can be completed and there will be enough profit to buy at v.

Now suppose that the first cut-vertex is at least two steps away on the unbounded face of the first section. This cannot happen if $\lambda(F') = 3$.

If $\lambda(F') = 5$, then we can finish the representation of the first block and then buy at some vertex to march through the remaining graph. There is a positive contribution from each of F', the first odd block, and some other odd block (whose existence is inevitable by parity). These allow us enough to buy a *second* depth-1 interval at v.

If $\lambda(F) = 4$, then if H is a block, use $e,H \rightarrow x$ for some $x \in P(H)$ and then buy at v. If v is not in the same block as u', then we have the first difficult triple. Otherwise, finish the first block at some vertex and then buy at the cut-vertex of H or at u' to finish the second block at v.♣

We now develop ideas involving the detailed use of depth-3 intervals. Until now, we have made almost no explicit mention of them. The rest of the induction steps require us to examine them more closely.

Suppose that R is a representation. Each depth-3 interval represents two edges from one 3-gon. The corresponding 3-gons are called **positive** and the other 3-gons **negative**. Each positive 3-gon increases r_3 by one and corresponds to an increase of one in f_3. In this sense, it contributes one to $p(R)$. Each negative 3-gon corresponds to an increase of one in f_3 and therefore contributes negative one to $p(R)$.

If a 3-gon (xyz) is positive, then all three vertices must have intervals that intersect at some point ξ of the real line. Since our representations are irredundant, this is the last place where an x-interval and a y-interval intersect. Suppose that $O(xy,(xyz))$ is also a bounded 3-gon (xyz') and that, prior to ξ, no edge of (xyz') had been introduced. If (xyz') is to be positive, then we must immediately extend the x-interval and the y-interval and place a depth-3 z'-interval that intersects the other two and whose left endpoint is greater than the right endpoint of the z-interval.

One consequence of this is that if $(xy'z)$ is also a bounded 3-gon and, prior to x, no edge of $(xy'z)$ had been introduced, then at least one of $\{(xyz'),(xy'z)\}$ will be negative. This leads us to the concept of paths of positive 3-gons. If, in the plane dual, there is a path of 3-gons, then we can make each of them positive as long as each of the other faces adjacent (in the plane dual) to the 3-gons on the path is of degree at least four or is negative. When we use such a path, we denote it by separating the positive 3-gons by arrows. This is actually just shorthand for considering the 3-gons as sections and explicitly stating the shared edges.

The following two examples illustrate how we combine paths of positive 3-gons with the induction hypotheses of Theorem II.5.5 to establish the existence of profitable representations.

Consider G_1 and G_2 below. For each graph, $\varepsilon(A) = 1$, $\varepsilon(B) = 0$, $v \in P(A)$, and we wish to find a profitable u,v-representation for the entire graph.

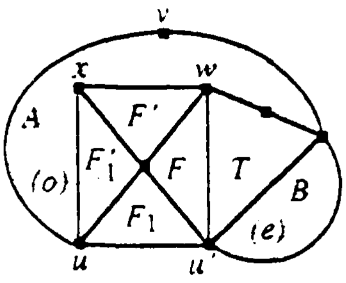


Figure II.5.18(a): G_1

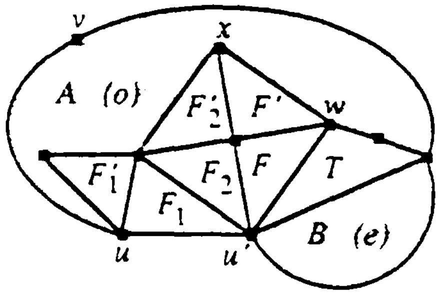


Figure II.5.18(b): G_2

For G_1, start by using (v_r) to justify $uA \to xw$. Since this removes ux, F_1' is negative. Continue with the path of positive 3-gons $F' \to F \to F_1$. Since $\lambda(T) = 4$, T contributes nothing to p and so the use of the edge wu' does not cost anything. Finish by using (ii) to justify $u'B \to u'$. There are four contributions to the profit, one negative contribution, and B is neutral. Hence we can buy at v.

We describe the entire procedure by "$u,A \rightarrow wx,F' \rightarrow F \rightarrow F_1 \rightarrow u',B \rightarrow u'$ and buy at v."

For G_2, it is actually possible to start with the *edge* uu'. Start with $uu' \rightarrow u'$, making F_1 negative. Continue with $u',B \rightarrow u',F_2 \rightarrow F \rightarrow F' \rightarrow wx$. Again, T can be safely ignored but this time F_2' is negative. Since no edge of F_1' has been removed, we can combine F_1' with the odd section A to form an even section and then use (ii) to justify $wx,A \cup F_1' \rightarrow v$. There are three positive contributions, two negative ones, and $A \cup F_1'$ and B are neutral. The entire procedure is described by "$uu' \rightarrow u',B \rightarrow u',F_2 \rightarrow F \rightarrow F' \rightarrow xw,F_1' \cup A \rightarrow v$."

We now present the induction step of (iii). Let $F = O(e)$ and $H = G - e$.

If $\lambda(F) = 5$ (so that H has at least one odd block) or $\lambda(F) = 4$ and H has two odd blocks, then start with $e \rightarrow u$ and march clockwise to the last cut-vertex of H. Note that if H is a block, then this march is vacuous. Then use (ii_r) or (v_r) to determine a vertex at which a depth-1 interval can be bought in order to end at e'. If $\lambda(F) = 4$ and H has no odd blocks, then again start with $e \rightarrow u$ but this time march through the blocks of H to e', using (ii_r) for the last block of H.

Hence we may assume that $\lambda(F) = 3$; define w by $F = (uu'w)$.

Suppose that $w \notin P(G)$ so that H is a block. If $u' \notin T(H)$, then use $e \rightarrow u',H \rightarrow e'$. If $u' \in T(H)$, then $O(e') = (u'wu'')$. Use $e,F \rightarrow uw,G - u' \rightarrow u'' \rightarrow e'$.

Now suppose that $w \in P(G)$ so that H has two blocks. Let A be the block of H that contains u and B be the block that contains e'. Start with $e \rightarrow u,A \rightarrow w$. Then use induction ($ii_r$) or ($v_r$) to finish with $w,B \rightarrow e'$.♣

The induction steps, (i) and (v), are far more difficult to establish. Comparing (i) and (v) with (ii), we see that odd blocks are much more difficult to deal with. Not only are the proofs harder, but the results seem to be weaker. For example, it would be nice if, as in (ii), we could "start at an edge and end anywhere" with odd blocks. But the examples below show that this is not possible.

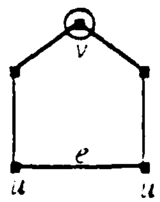


Figure II.5.19(a): G_1

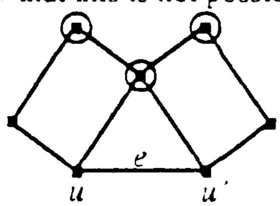


Figure II.5.19(b): G_2

There is no e,v-representation for G_1 and, if v is any of the three circled vertices in G_2, then there is no e,v-representation for G_2.

There is an intuitive basis for suspecting that odd blocks are more difficult to deal with. A profitable representation of an even block gains nothing whereas a profitable representation of an odd block gains one. In essence, both the amount of work and the reward of odd blocks is greater than that of even blocks.

To keep things more compact, we will also reuse labels; it is understood that the definition of a label nullifies any previous use of that label.

We now present the induction step of (v). Let $H = G - e$ and $F = O(e)$.

If $\lambda(F) \geq 4$, then start with $e \rightarrow u$ and march through the blocks of H to v. Hence $\lambda(F) = 3$; define w by $F = (uwu')$.

If $w \in P(G)$, then H has two blocks A and B, where $u \in P(A)$, $u' \in P(B)$, and $w \in P(A) \cap P(B)$. Since $\lambda(F) = 3$ and G is odd, H is even and so $\varepsilon(A) = \varepsilon(B)$. If $\varepsilon(A) = 1$, then use $e \rightarrow u,A \rightarrow w,B \rightarrow v$. If $\varepsilon(A) = 0$, then use $e,F \rightarrow uw,A \rightarrow w,B^{u'w} \rightarrow v$. The last step is justified by (vi) and it is both unnecessary and unavailable if $u' \in T(G)$ (i.e., $w = u''$).

Hence $w \notin P(G)$. We have the following.

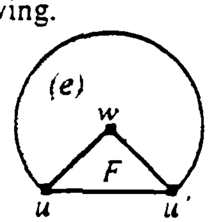


Figure II.5.20

Let $F_1 = O(u'w,F)$ and $H = G - \{e,u'w\}$. Let A be the block of H that contains w and B be the block of H that contains u'.

Suppose that $\lambda(F_1) \geq 4$. Start with $e,F \rightarrow uw$. If $\varepsilon(A) = 1$, then continue with $uw,A \rightarrow x$, for some x, and buy at either v (if H is a block) or the cut-vertex between A and the next block to march to B, ending at v. If $\varepsilon(A) = 0$, then simply march from uw through the blocks of H to v.

Hence $\lambda(F_1) = 3$; define x by $F_1 = (u'wx)$. If $x \in P(G)$, then use $e,F \rightarrow uw,A \rightarrow x,B \rightarrow v$. Hence $x \notin P(G)$. We have the situation below.

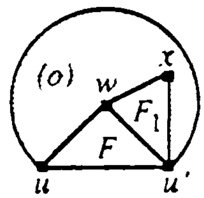


Figure II.5.21

Let $F_2 = O(uw,F)$. Note that if $u' \in F_2$ then, to avoid multiple edges, $\lambda(F_2) = 5$ and u' is two steps away from both u and w on F_2. Let $H = G - \{e,uw\}$. Let A be the block of H that contains u and B be the block of H that contains w. Note that $E(B) \supseteq E(F_1)$ and hence $\lambda(B) \geq 3$. Moreover, the existence of F_1 precludes the possibility of $u' \in T(B)$.

Supppose that $\lambda(F_2) = 5$. If $u' \in F_2$, then start with $e,F \rightarrow u'w,B \rightarrow u'$, march back to u through the blocks of the remaining graph, and buy at v. If $u' \notin F_2$, use $e,F \rightarrow u'w,B \rightarrow v$, and then buy at u to start a march through the blocks of the remaining graph.

Now suppose that $\lambda(F_2) = 4$; define y and z by $F_2 = (uyzw)$. If $x \in F_2$, then $x = y$ or $x = z$. If $x = y$, then $P(B) = \{w,x,z\}$. Use $e,F \rightarrow w,B \rightarrow x,G - E(F) - E(B) \rightarrow v$. If $x = z$, then start with $e,F \rightarrow F_1 \rightarrow xu'$, march through the the blocks of the remaining graph, and then buy at v. Hence $x \notin F_2$. Use $e,F \rightarrow u$ and march through the blocks of the remaining graph to v.

Hence $\lambda(F_2) = 3$; define y by $F_2 = (uyw)$.

If $y = x$, then use $e,F \rightarrow F_1 \rightarrow xu'$ and (ii). If $y \notin P(G)$, then use $e,F \rightarrow u'w,G - \{e,uw\} \rightarrow v$, justifying the last step by (v) and the existence of F_1 (to preclude the possibility of $u' \in T(G - \{e,uw\})$).

Hence $y \in P(G)$. Let A be the block of $G - E(F)$ that contains u and B be the block of $G - E(F)$ that contains w. We have the situation below.

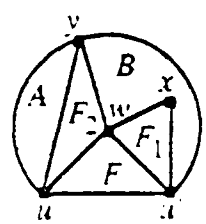


Figure II.5.22

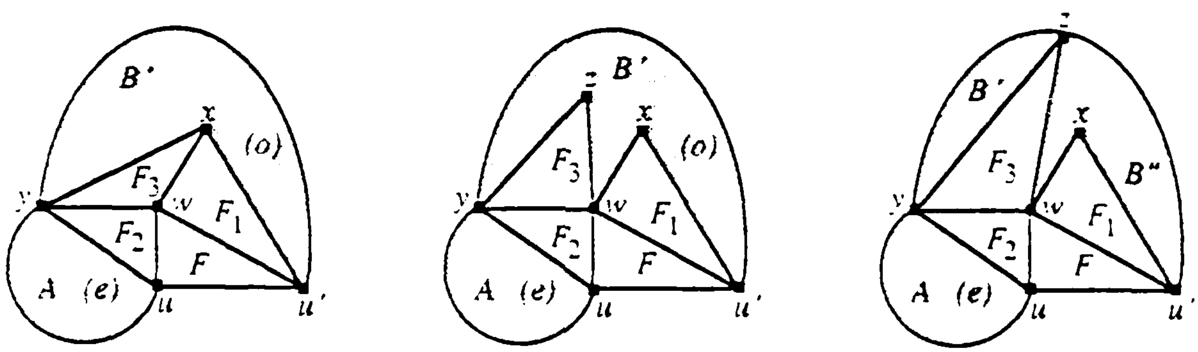
Note that $\varepsilon(A) = \varepsilon(B)$. If $\varepsilon(A) = 1$, then use $e,F \rightarrow u,A \rightarrow y,B \rightarrow v$. Hence $\varepsilon(A) = \varepsilon(B) = 0$. Let $F_3 = O(wy,F_2)$. Let $H = B \cup F_1 - \{wy\}$. Since $\varepsilon(B) = 0$, $\varepsilon(F_1) = 1$ and $\beta(H) = \beta(B) \cup$

$\{F_1\} - \{F_3\}$, it follows that $\varepsilon(H) = 1 - \varepsilon(F_3)$.

Suppose that $\lambda(F_3) \geq 4$. If H is a block, then use $e \to u.A \cup F_2 \to y.H \to v$ unless $y = v = u''$ and $\lambda(F_3) = 4$. In this case $\lambda(H) = 5$ and it is possible for $y = v = u'' \in T(H)$ and we therefore cannot apply (i) for the last step of the preceding procedure. For this case, use $e.F \to u'w.H \cup F_3 \to v.A \to v$. Hence H is not a block. Let C be the block of H that contains w. Because of F_1, $\lambda(C) \geq 3$. Let D be the block of H that shares the cut-vertex z with C.

Recall that $\lambda(F_3) \geq 4$. First assume that $v \in P(C)$ and that either $v \neq z$ or $v \notin T(C)$. Use $e \to u.A \cup F_2 \to y$ and march to v through the blocks of H. Now if $v \notin P(C)$, then $v = u'' \in P(D)$, and $P(F_3) - \{y\} \supseteq P(C)$. Hence $\lambda(C) \leq 4$ and $T(C) = \emptyset$. Use $e.F \to u'w.C \to w \to y.A \to y$ and then march to v. The last possibility for $\lambda(F_3) \geq 4$ is that $v = z \in T(C)$. Since $\lambda(C) \geq 5$, we must have $v = z = u''$ and $P(F_3) - \{y\} \supseteq P(C) - \{u'\}$ and hence $\lambda(C) \leq \lambda(F_3)$. Since $T(C) \neq \emptyset$, $\lambda(C) \geq 5$ and $\lambda(F_3) = 5$. Use $e \to u.A \cup F_2 \to y$, march through the blocks of H and buy at v.

Hence $\lambda(F_3) = 3$; define z by $F_3 = (wyz)$. There are three cases: $z = x$, $z \notin P(G) \cup \{x\}$, and $z \in P(G)$. These are illustrated below.



$z = x$ $\quad$ $z \notin P(G) \cup \{x\}$ $\quad$ $z \in P(G)$

Figure II.5.23

If $z = x$, then start with $e \to u.A \cup F_2 \to yw$. If $v = u'$, then finish with $yw.F_3 \to yx.B' \to u'(w)$. If $v = u''$, then finish with $yw \to w.F_1 \to xu'.B' \to u''$.

If $z \notin P(G) \cup \{x\}$, then, unless $v = u'' = y$ (which implies that $\lambda(B') = 5$), use $e.F \to uw.A \cup F_2 \to y.B' \to v$. If $v = u'' = y$, then use $e.F \to u'w.B' \cup F_1 \cup F_3 \to y.A \to v$.

If $z \in P(G)$, then use $e.F \to uw.A \cup F_2 \to y.B' \to z.B'' \to v$ and march through the blocks of the remaining graph to v. ♣

To present the induction step of (i), we define the **level** of u in Algorithm II.5.8. This algorithm defines the level k, two sets $\{u_i : i = 0,\ldots,k\}$ and $\{u_i' : i = 0,\ldots k\}$ of vertices, two sets $\{F_i : i = 1,\ldots,k\}$ and $\{F_i' : i = 1,\ldots,k\}$ of faces, integers s and t, sets $\{w_j : j = 2,\ldots,s - 1\}$, and $\{w_j : j = 2,\ldots,t - 1\}$ of vertices, faces $F = (u_k w_2 w_3,\ldots,w_{s-1}u')$ and $F' = (u_k x_2 x_3,\ldots,x_{t-1}w_2)$, and a set Q of vertices.

Algorithm II.5.8.

a. Initialize:
   a) $k = 0$, $u_0 = u$, and u_0' is the vertex in P that is *clockwise* from u.
   b) $F = O(e) = (uw_2w_3\ldots w_{s-1}u')$ and $F' = O(uw_2) = (ux_2x_3\ldots x_{t-1}w_2)$
   c) $Q = P(G)$

b. If $\lambda(F) = \lambda(F') = 3, w_2 \notin Q, x_2 \notin Q$, then:
   a) $Q = Q \cup \{w_2,x_2\}$
   b) $k = k + 1$, $u_k = w_2$, $u_k' = x_2$, $F_k = F$, and $F_k' = F'$
   c) $F = O(u_k u', F_k)$; define s, w_2, $w_3,\ldots,$ and w_{t-1} by $F = (u_k w_2 w_3 \ldots w_{s-1}u')$.
   d) $F' = O(u_k w_2)$; define t, x_2, $x_3,\ldots,$ and x_{t-1} by $F = (u_k x_2 x_3 \ldots x_{t-1}w_2)$
   e) Repeat step b..

Three examples appear below.

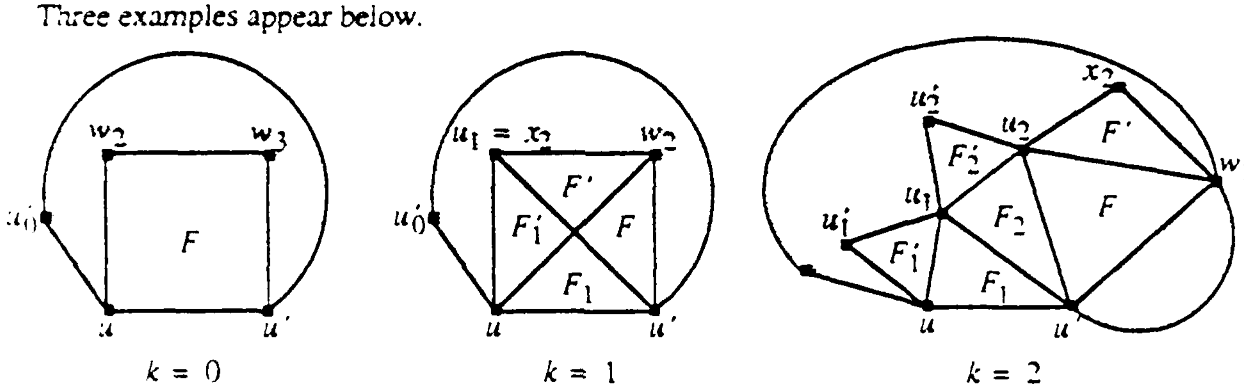


Figure II.5.24

We need to find a profitable u,v-representation for any $v \in P(G)$. We will find a profitable representation for almost any $v \in Q$. The basic tactic is to start with $u \rightarrow e, F_1 \rightarrow \ldots \rightarrow F_k \rightarrow u_k u'$ and continue from there depending upon where v is in Q, relative to F and F'. In this initial part of the proposed procedure, each F_i is positive and each F_i' is negative so when we get to $u_k u'$, we are neither ahead nor behind. Let G' be the graph that remains after this proposed start. After the proposed start, we are starting the representation of G' at $u_k u'$. G' is a block, and $P(G') = Q$. An example of G and G' appears below.

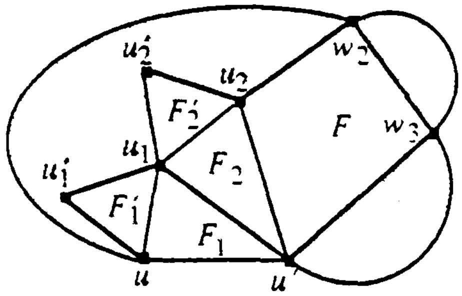


Figure II.5.25(a): G

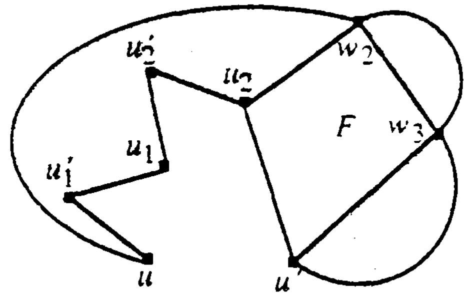


Figure II.5.25(b): G'

By making *all of* the F_i's positive, we have created a situation in which we must start at an *edge* and this, as shown in Figure II.5.20, is not always possible.

Hence we may not always use this tactic. However a case is "mostly" described by the placement of v in Q, relative to F and F'. Suppose that we know every pertinent fact about v, F and F'. We will describe procedures for each of the levels $k = 0$, $k = 1$, etc., until we have a procedure that starts with the *edge* e instead of just the *vertex* u. This is the last level that we need to consider. To see this, recall the graphs G_1 and G_2 of Figures II.5.18(a) and II.5.18(b).

Recall that for G_1, we have a profitable u,v-representation and for G_2, we have a profitable uu',v-representation. Now suppose that the level is $k \geq 2$, we need to find a profitable u,v-representation, and that we have the same situation concerning F and v as in Figure II.5.18: $\lambda(F') = 3$, $x_2 = u'_k$, $\lambda(O(u'w,F)) = 4$, etc.. Then we can start a representation with $e, F_1 \rightarrow F_2 \rightarrow \ldots \rightarrow F_{k-1} \rightarrow u_{k-1}u'$ and conclude with the procedure of G_2. This is possible because the procedure of G_2 starts with the edge $u_{k-1}u'$, and this is where the initial part of the representation currently ends.

Cases are organized by considering the level *last*. Let k be the level. We will asume that $k = 0$ and try to find a profitable u,v-representation that starts at e. If we succeed then, by the above paragraph, we do not have to consider higher values of k. For most of the cases for which this is not possible, we can start at u if $k = 0$ and at e if $k = 1$. For the cases for which we cannot start at e if $k = 1$, then we will be able to start at u if $k \in \{0,1\}$ and at e if $k = 2$.

We must resort to many cases and subcases. We name a subcase by a string of digits. The name of a subcase will describe where in the "tree of cases" the subcase is. For example, Case 213 is the third subcase of the first subcase of the second main case.

Often there are some subcases that are so easily disposed of that they do not merit being considered as separate cases. Following the definition of a subcase, there is often a paragraph or two that disposes of these preliminary cases.

We now present the induction step of (i). Almost all of the rest of §II.5 is spent on this.

The first three main cases correspond to the different ways that Algorithm II.5.8 can terminate because of some "irregularity" of F (i.e., $\lambda(F) = 5$, $\lambda(F) = 4$, and $\lambda(F) = 3$ with $w_2 \in Q$).

Let $H = G - \{u_iu' : i = 0,k\} - \{u_iu_{i+1} : i = 0,k-1\}$. Let A be the block of H that contains u_k and let B be the block of H that contains u'. If there is a block of H that is not B and shares a cut-vertex with A, then call it C. Examples are given in Figures II.5.26, II.5.27, and II.5.28.

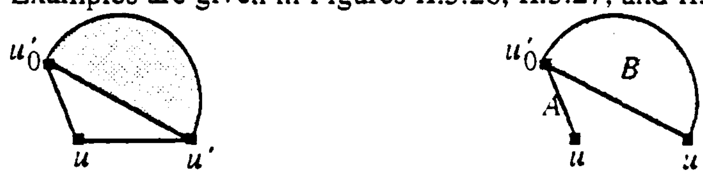


Figure II.5.26: $k = 0$ G H

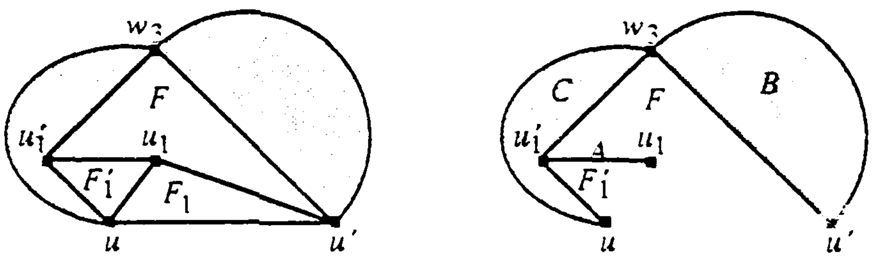


Figure II.5.27: $k = 1$ G H

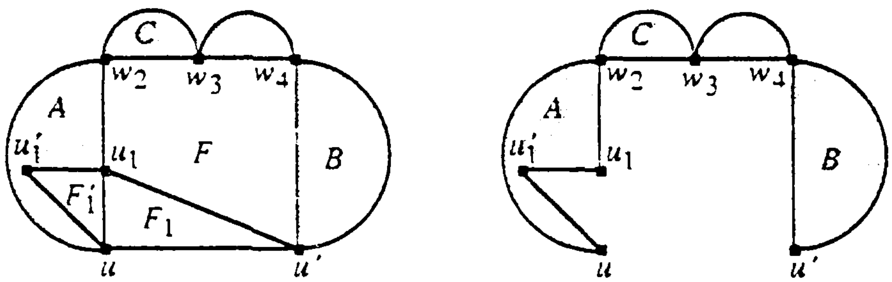


Figure II.5.28: $k = 1$ G H

Case 1 $\lambda(F) = 5$

Suppose that $k = 0$. If H has two odd blocks, then use $e \rightarrow u'$, march and buy. Hence we may [illegible]ume that all blocks of H are even. If H has at most two blocks, then we can start with e and march [illegible]. If H has at least three blocks, then we can start with $e \rightarrow u$ or $e \rightarrow u'$ and use marches or (iv) unless one of the following occurs:

i. The cut-vertices of H are $\{w_2, w_4\}$ and $v \in \{w_2, w_4\}$.
ii. The cut-vertices of H are $\{w_2, w_3, w_4\}$ and $v = w_3$.

For $k = 0$, these are illustrated below. For either case, use $u, A \to u \to u'$ and march to v.

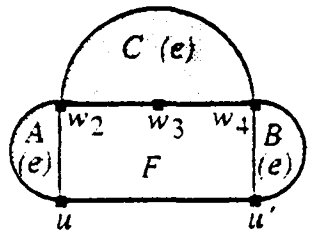


Figure II.5.29(a): $v \in \{w_2, w_4\}$

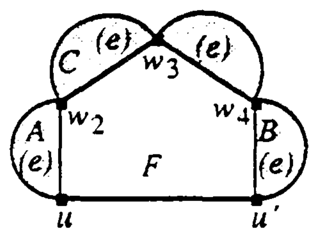


Figure II.5.29(b): $v = w_3$

For $k = 1$, these two situations are illustrated below.

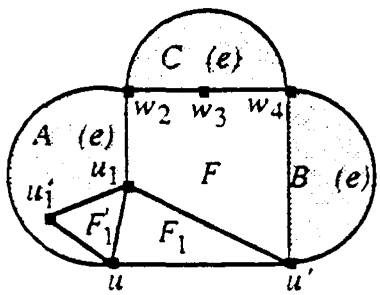


Figure II.5.30(a): $v \in \{w_2, w_4\}$

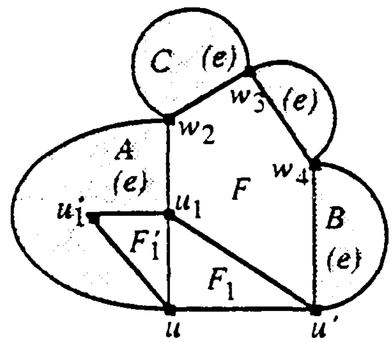


Figure II.5.30(b): $v = w_3$

Start with $e, F_1 \to uu_1$. If $w_2 \neq u_1'$ (as in both figures), then continue with $uu_1, F_1' \cup A \to w_2$, march through all of the blocks, and buy at v. If $w_2 = u_1'$, then $A = u_1u_1'$ (compare with Figure II.5.27). Continue with $uu_1, F_1' \to uu_1'$, march to u', and buy at v.

Case 2 $\lambda(F) = 4$

Suppose that $k = 0$. If H has at most two blocks, then start with either $e \to u$ or $e \to u'$ and march from either u or u' to v. Hence we may assume that H has exactly three blocks. If all three blocks are odd, then start with $e \to u$ and then march and buy. Hence exactly one of the three blocks is odd. It is easy to start at e and march to v if $v \notin P(C)$. Hence we may assume that $v \in P(C)$.

If A is odd, then B and C are even. Use $e \to u, A \to w_2$ and (iv) unless $v = w_2$. If $v = w_2$ and $w_2 \notin T(A)$, then use $e \to u'$, and march to v. The first case for which we must consider $k = 1$ is:

i. $w_2, w_3 \in Q$, $\varepsilon(A) = 1$, $\varepsilon(B) = \varepsilon(C) = 0$, $w_2 \in T(A)$, and $v = w_2$

If B is odd, then A and C are even. Use $e \to u'.B \to w_3$ and (iv) unless $v = w_3$. If $v = w_3$ and $w_3 \notin T(B)$, then use $e \to u$ and march to v. The second case for which we must consider $k = 1$ is:

ii. $w_2, w_3 \in Q$, $\varepsilon(A) = 0$, $\varepsilon(B) = 1$, $\varepsilon(C) = 0$, $w_3 \in T(B)$, and $v = w_3$

If C is odd, then A and B are even. If $v \in \{w_2, w_3\}$, then start with e and march to v, finishing with either $v.A \to v$ or $v.B \to v$. The third case for which we must consider $k = 1$ is:

iii. $w_2, w_3 \in Q$, $\varepsilon(A) = \varepsilon(B) = 0$, $\varepsilon(C) = 1$, and $v \in P(C) - \{w_2, w_3\}$

For $k = 0$, the three "hard" cases are shown below.

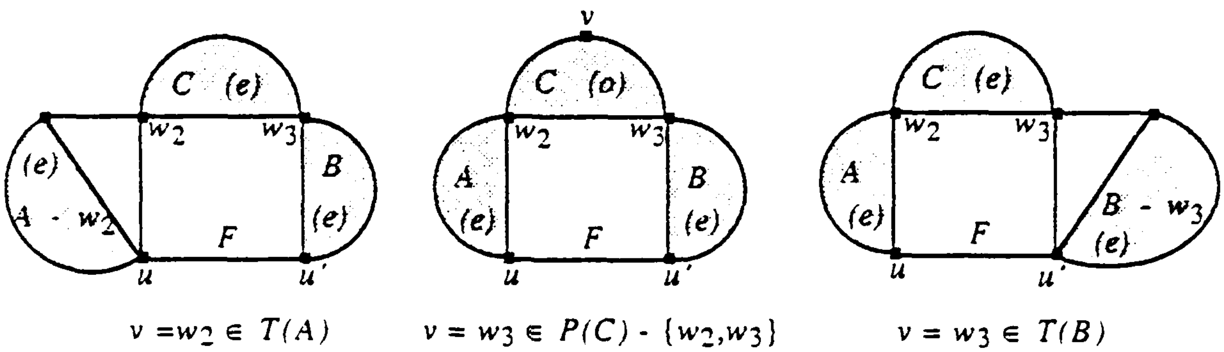


Figure II.5.31

For the first case, note that $u \notin T(A)$. Then for all three hard cases, use $u.A \to u \to u'.B \to w_3, C \to v$.

Now suppose that $k = 1$. For the first case, we have the situation below.

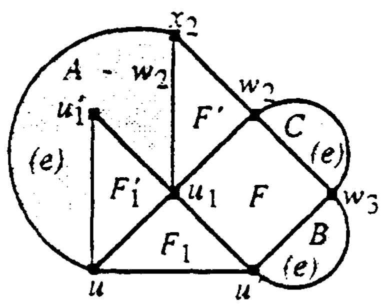


Figure II.5.32

Use $e \to u.(A \cup F_1') - w_2 \to x_2u_1.F' \to u_1 \to u'.B \to w_3, C \to v$.

For the second and third cases, first assume that $w_2 \neq u_1'$. Use $e.F_1 \to uu_1.F_1' \cup A \to w_2.C \to w_3.B \to u'$ and buy at v. If $w_2 = u_1'$, then use $e.F_1 \to F_1' \to uu_1'.C \to w_3.B \to u'$ and buy at v.

Case 3 $\lambda(F) = 3$ and $w_2 \in Q$

Note that $\varepsilon(A) = \varepsilon(B)$. If $\varepsilon(A) = 1$ and $k = 0$, then start with $e \to u$ or $e \to u'$ and march to v unless $v = w_2 \in T(A) \cap T(B)$. In this case, use $u,A \to uw_2,F \to w_2u',B \to u'$ and buy at v. If $\varepsilon(A) = 1$, $k = 1$, and $v = w_2 \in T(A) \cap T(B)$, then define w by $w \leftrightarrow w_2$, $w \in P(A)$, and $w \neq u_1$. Use $e \to u,(A \cup F_1') - w_2 \to u_1w,(ww_2u_1) \to F \to w_2u',B \to u'$ and buy at v.

If $\varepsilon(A) = 0$, $v \in P(B)$, and $k = 0$, then use $u,A \cup F \to w_2u',B \to v$. If $\varepsilon(A) = 0, v \in P(B)$, $k = 1$, and $w_2 = u_1'$, then use $e,F_1 \to F_1' \to uu_1',B \to v$. If $\varepsilon(A) = 0, v \in P(B)$, $k = 1$, and $w_2 \neq u_1'$, then use $e,F_1 \to uu_1',F_1' \cup A \to w_2,B \to v$.

Hence $\varepsilon(A) = \varepsilon(B) = 0$ and $v \in P(A) - \{w_2\}$. Let $A' = A - u_kw_2$. Let A_1 be the block of A' that contains u_k and A_2 be the block of A' that contains w_2. If there is a block of A' that is not A_2 and shares a cut-vertex with A_1, then call it A_3. We give an example of these definitions in Figure II.5.33. Note that in our example, $k = 1$ and $x_2 = u_1'$ so that A_1 is a single edge.

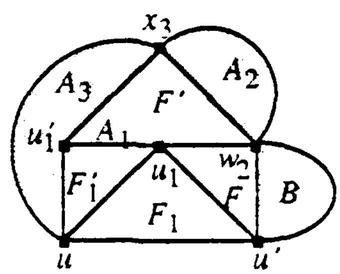


Figure II.5.33

Case 31 $\lambda(F') \geq 4$

Suppose that $k = 0$. If $\lambda(F') = 5$ or if $\lambda(F') = 4$ and A' has two odd blocks, then use $e,F \cup B \to w_2$, march and buy. If $v \in P(A_1)$, then use $e,F \cup B \to w_2$ and march back to v. If $\lambda(F') = 4$, all blocks of A' are even, $v \notin P(A_1)$, and $k = 0$, then use $u,A_1 \to u,F \cup B \to w_2$, and then (ii) or (iv).

Hence $\lambda(F') = 4$, all blocks of A' are even, $v \notin P(A_1)$, and $k > 0$. We will start the remaining procedures of Case 31 with e and therefore we may assume that $k = 1$.

Case 311 $x_2 = u_1'$ (See Figure II.5.33)

If $x_3 \in P(G)$, then A_3 exists. Then if $v \in P(A_2)$, use $e \to u,A_3 \cup F_1' \to u_1,F \cup B \to w_2,A_2 \to v$. If $v \in P(A_3) - \{x_3\}$, then use $e \to u',F \cup B \to w_2,A_2 \to x_3,A_3 \cup F_1' \to v$.

If $x_3 \notin P(G)$, then use $e \to u', F \cup B \to w_2, A_2 \cup F_1' \to v$.

Case 312 $x_2 \neq u_1'$

Note that since $\varepsilon(A_1) = 0$, we cannot have $x_3 = u_1'$. Hence we may use $e, F_1 \to uu_1, A_1 \cup F_1' \to u_1 \to w_2, B \to w_2$ and then (ii) or (iv).

Case 32 $\lambda(F') = 3$

Suppose that $k = 0$. If $x_2 \notin P(G)$ or if $x_2 \in P(G)$ and $v \in P(A_1) - \{x_2\}$, then use $e.F \cup B \to w_2$ and march to v. Hence we may assume that $x_2 \in P(G)$ and $v \in P(A_2)$. Note that $A_1 \neq A_2$ and $\varepsilon(A_1) = 1 - \varepsilon(A_2)$.

If $u \in T(A_1)$, then $x_2 \notin T(A_1)$ and $\varepsilon(A_2) = 0$ and we can therefore use $e,F \cup B \to w_2$ and (iv). If $u \notin T(A_1)$ and $k = 0$, then use $u,A_1 \to u,F \cup B \to w_2,A_2 \to v$. Hence $x_2 \in Q$, $v \in P(A_2), u_k \notin T(A_1)$, and $k > 0$. We will start the remaining procedures of Case 32 with e and so we may assume that $k = 1$. Therefore this summary reduces to:

$x_2 \in P(G) \cup \{u_1'\}$, $v \in P(A_2)$, $u_1 \notin T(A_1)$, and $k = 1$.

Case 321 $x_2 = u_1'$

If $w_2 \in T(A_2)$, then define w by $w \leftrightarrow w_2$, $w \neq u_1'$, and $w \in P(A_2)$. This is illustrated below.

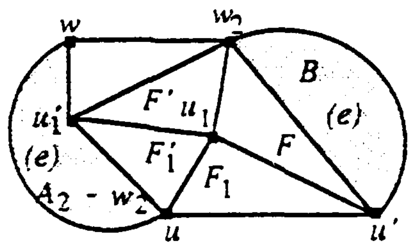


Figure II.5.34

If $v \neq w_2$, then use $e,F_1 \to F \to F' \to (u_1'ww_2) \to u_1'w,A_2 - w_2 \to v$ and then buy at u' to finish with $u'.B^{uw_2} \to w_2$. If $v = w_2$, then use the above procedure except that it is immaterial where the first sequence ends and, if $\lambda(B) = 2$, then the second sequence is replaced by "buy at v."

If $w_2 \notin T(A_2)$, then use $e.F_1 \to F \to F' \to u_1'w_2,A_2 \to w_2.B^{uw_2} \to u'$ and buy at v. The last step before the purchase at v is unnecessary and unavailable if $\lambda(B) = 2$.

Case 322 $x_2 \neq u_1'$

First assume that $\varepsilon(A_1) = 1$. If $x_2 \in T(A_1)$, then define x by $x \leftrightarrow x_2$, $x \neq u_1$, and $x \in P(A_1)$.

Use $e \to u, A_1 \cup F_1' - x_2 \to xu_1,(x_2u_1x) \to x_2u_1, B \cup F \cup A_2 \cup F' \to v$. If $x_2 \notin T(A_1)$ then, since $\varepsilon(A_2) = 0$, we can use $e.F_1 \to u_1u', F \cup B \to w_2$ and (iv).

If $\varepsilon(A_1) = 0$, then use $e.F_1 \to uu_1.F_1' \cup A_1 \to u_1 \to w_2,B \to w_2,A_2 \to v$.

The remaining main cases correspond to terminating Algorithm II.5.8 because F' is irregular. The ways for this to happen are $\lambda(F') = 5$, $\lambda(F') = 4$ or $\lambda(F') = 3$ and $x_2 \in Q$. We will split the last one into two cases and these will be defined later. Since F' is the reason that Algorithm II.5.8 terminated, $\lambda(F) = 3$ and that $w_2 \notin Q$.

It is convenient at this time to let H, A, B and C denote graphs or sections other than what they did during the first three cases. These graphs will play essentially the same role as their namesakes above. Let $H = G - \{u_iu' : i = 0,k\} - \{u_iu_{i+1} : i = 0,k-1\} - u_kw_2$. Let A be the block of H that contains u_k and B be the block of H that contains w_2. We give an example of G and H below. In this example, $k = 1$.

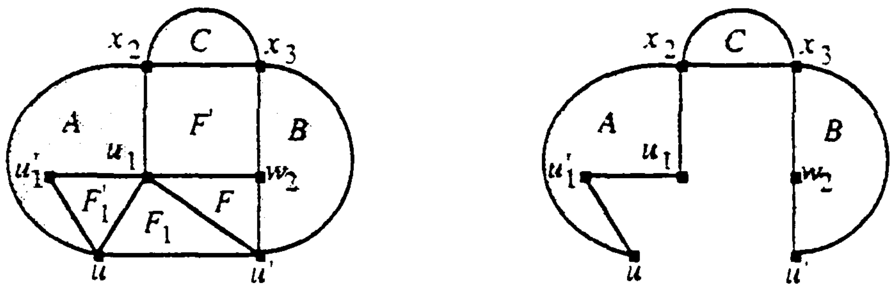


Figure II.5.35 G H

Case 4 $\lambda(F') = 5$

Suppose that $k = 0$. If no x_i is in $P(G)$, then use $e.F \to w_2u',H \to y$ for some $y \in P(G)$ and then buy at v. Hence some $x_i \in P(G)$. Let x be the cut-vertex in H that is in B. If $x = x_4$, then use $e.F \to w_2u'.B \to x$, march to u, and buy at v. Hence $x_4 \notin P(G)$. If $v \in P(B) - \{x\}$, then start with $e \to u(w_2)$ and march to v. Otherwise use $e.F \to u'w_2.B \to y$ for some $y \in P(B)$, buy at u or x, and march to v.

Case 5 $\lambda(F') = 4$

Suppose that $k = 0$. If no x_i is in $P(G)$, then use $e.F \to w_2u'.H \to v$. Hence we may assume that some $x_i \in P(G)$. Note that H has an even number of odd blocks.

Case 5i $x_3 \in Q$

Suppose that $k = 0$. If H has an odd block, then use $e,F \rightarrow w_2u',B \rightarrow x_3$, march and buy. Hence H has no odd blocks. Note that this means that $x_3 \neq u_0'$, for that would force $\lambda(A) = 3$.

Case 511 $x_2 \notin Q$

If $v \in P(A)$ and $k = 0$, then use $e,F \rightarrow w_2u',B \rightarrow x_3,A \rightarrow v$. If $v \in P(B)$, and $k = 0$, then use $u,A \rightarrow u,B \cup F \rightarrow v$. If $v \in P(B)$, and $k = 1$, then use $e,F_1 \rightarrow uu_1,F_1' \cup A \rightarrow u_1 \rightarrow w_2,B \rightarrow v$.

Case 512 $x_2 \in Q$

Suppose that $k = 0$. If $v \notin P(B)$, then use $e,F \rightarrow w_2u',B \rightarrow x_3$ and (iv). We must consider separate levels if $v \in P(B)$. If $k = 0$ and $v \neq x_3$, then use $u,A \rightarrow x_2,C \rightarrow x_3,B \cup F \rightarrow v$. If $k = 0$ and $v = x_3$, then use $u,A \rightarrow u,B \cup F \rightarrow x_3,C \rightarrow x_3$.

Now suppose that $k = 1$. All remaining procedures of Case 512 will start with e and therefore we do not need to consider more levels. If $x_2 = u_1'$, then unless $v = x_3 \in T(B \cup F)$, use $e \rightarrow u,A \cup F_1' \rightarrow x_3,B \cup F \rightarrow v$. For this exceptional case, define x by $x \leftrightarrow x_3$, $x \neq w_2$, and $x \in P(B)$ and then use $e \rightarrow u'(u_1),B - x_3 \rightarrow w_2x,(w_2xx_3) \rightarrow w_2 \rightarrow u_1,F_1' \rightarrow uu_1',A \rightarrow v$.

If $x_2 \neq u_1'$, then use $e,F_1 \rightarrow uu_1,F_1' \cup A \rightarrow u_1 \rightarrow w_2$ and then (iv).

Case 52 $x_3 \notin Q$

If $x_2 \in Q$, then H has just the two blocks A and B and they have the same parity. Suppose that $\varepsilon(A) = 1$ and $k = 0$. If $v \in P(A)$, then use $e,F \rightarrow w_2u',B \rightarrow y$ for some $y \in B$. Then buy at u or x_2 to end at v. If $v \notin P(A)$, then use $e \rightarrow u(w_2),A \rightarrow x_2,B \rightarrow v$.

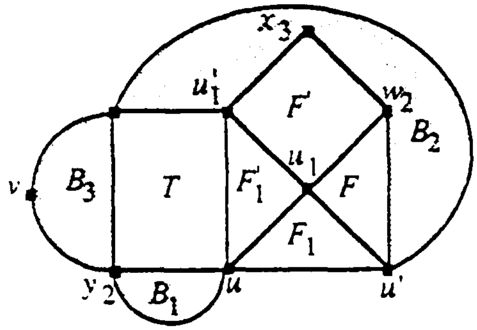
Hence $\varepsilon(A) = \varepsilon(B) = 0$. If $k = 0$ and $v \in P(A)$, then use $e,F \rightarrow w_2u',B \rightarrow x_2,A \rightarrow v$.

The only remaining subcase of Case 52 (and Case 5) is $x_2 \in Q$, $x_3 \notin Q$, $\varepsilon(A) = \varepsilon(B) = 0$, and $v \in P(B)-\{w_2\}$. If $k = 0$, then use $u,A \rightarrow u,B \cup F \rightarrow v$. We will now need to consider $k = 1$ and, for some subcases, $k = 2$.

Suppose that $k = 1$. If $x_2 \neq u_1'$, then use $e,F_1 \rightarrow uu_1,A \cup F_1' \rightarrow u_1 \rightarrow w_2,B \rightarrow v$. Hence we may assume that $x_2 = u_k'$.

Note that $A = u_kx_2$ and that B is "most of the graph." Let $T = O(u_{k-1}u_k',F_k')$. Define $y_2, y_3, \ldots$ by $T = (u_{k-1}y_2y_3\ldots u_k')$. Let $J = G-\{u_i: i. = 1,k\}-\{e,u_{k-1}u_k'\}$. Let B_1 be the block of J that contains

u_{k-1}, B_2 be the block of J that contains u_k', and if there is a block of J that is not equal to B_1 and shares a cut-vertex with B_1, call it B_3. We use the notations B_1, B_2, and B_3 since we are breaking up B. These ideas are illustrated below. For this example, $\lambda(T) = 4$.



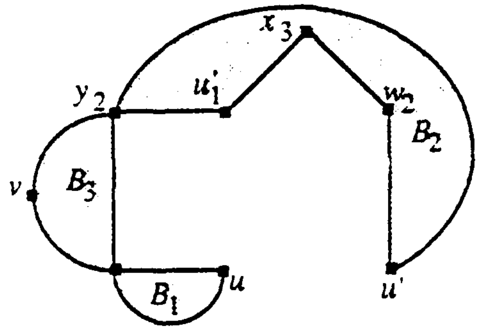


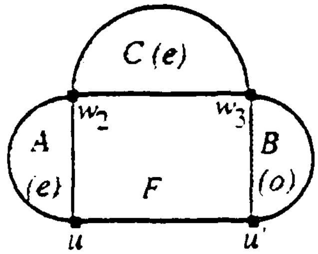
Figure II.5.36 $\quad G \quad\quad J$

Case 521 $\lambda(T) \geq 4$, $k \geq 1$

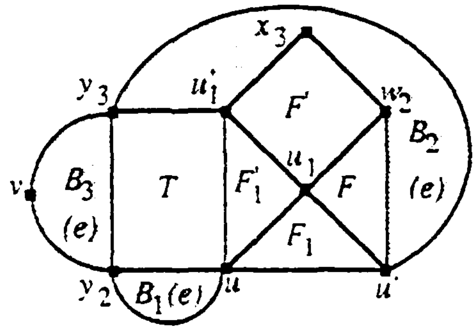
All algorithms of Case 521 start with e and so we do not need to consider higher values of k.

If we start with $e \to u.F_1' \to uu_1'$, we have a current profit of zero, the remaining graph is a block, and we are starting at an edge of its unbounded face. Hence we need only consider the subcases of Cases 1 and 2 for which we needed to consider values of k other than 0. For $k = 0$, all of these subcases' solutions started with "$u.A \to u \to u'$." If we replace that by $e \to u.B_1 \to u.F_1' \to u_1'$, we have the same situation. Hence we can always start with e.

We give an example of these parallel problems below.



$v = w_4 \in T(B)$



$v = y_3 \in T(B_2 \cup F)$

Figure II.5.37

Recall that the procedure for the first graph of Figure II.5.37 is $u.A \to u \to u'.B \to w_4.C \to v$. In its place, we use $e \to u.B_1 \to u.F_1' \to u_1'.B_2 \cup F \to y_3.B_3 \to v$ for the second graph of Figure II.5.37.

The reader might wonder why this approach does not work if $\lambda(T) = 3$ (Case 522 below). The reason is that in the procedures of the first two main cases, the edge uu' is not part of a 3-gon and so is accounted for by $u \to u'$ or $u(u')$. These can be replaced by $u,F_1' \to u_1'$ or $u,F_1' \to u$. Comparing these two algorithms, one can say that uu' of the first two main cases is "similar" to F_1' of Case 521. But if $\lambda(F_1) = 3$, then uu' might be represented as part of a 3-gon and there is no convenient way of transforming such an procedure to one for Case 522 and gaining a unit of profit from F_1'.

Case 522 $\lambda(T) = 3$, $k \geq 1$

Suppose that $k = 1$. If $y_2 = u_0'$ and $v \neq u$, then start with $e \to u,F_1' \to T \to u_0'u_1'$. The remaining graph is an even block and so we can start from the edge $u_0'u_1'$ and end at v. If $y_2 = u_0'$ and $v = u$, then use $e,F_1 \to F_1' \to u_1 \to w_2,J\text{-}u \to u_0' \to u$. If $y_2 = u'$, then uu' is a multiple edge. If $y_2 = x_3$, then use $e,F_1 \to F_1' \to T \to ux_3,J\text{-}u_1' \to w_2 \to u_1$ and buy at v. If $y_2 = w_2$, then the perimeter of B_2 is $(u_1'x_3w_2)$. We illustrate this below; note that we had to contort our diagram extensively to maintain the straight lines for edges.

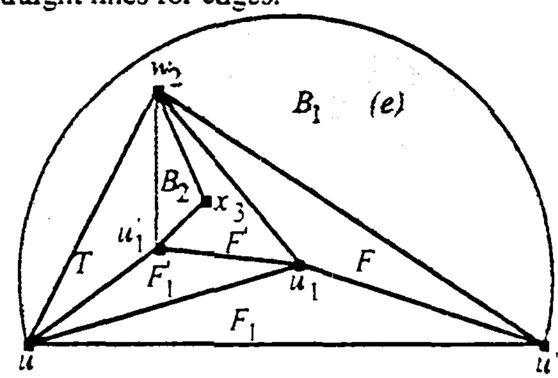


Figure II.5.38: $y_2 = w_2$

Use $e,F_1 \to F_1' \to u_1'u_1,B_2 \cup F' \to w_2,B_1 \to v$.

If y_2 is some "interior" point that has not been mentioned (i.e., $B_1 = B_2$), then use $e,F_1 \to F_1' \to u_1 \to w_2,B_2 \to v$. Hence $y_2 \in Q - \{u_k'\}$. This is illustrated below for $k = 1$ and $k = 2$.

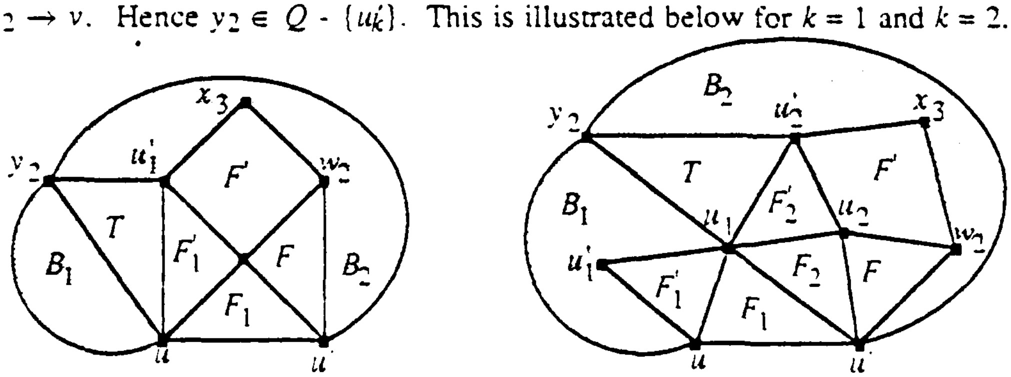


Figure II.5.39 $k = 1$ $k = 2$

We now resume the assumption that $k = 1$. If $v \in P(B_1)$, then unless $v = y_2 \in T(B_1)$, use $e,F_1 \to F_1' \to u_1 \to w_2,B_2 \to y_2,B_1 \to v$. If $y_2 \in T(B_1)$, then we can use $e \to u,B_1 \to u,F_1' \to u_1,B_2 \cup F \to v$ for *any* $v \in P(B_2)$.

Hence $v \in P(B_2) - \{y_2\}$.

If $\varepsilon(B_1) = 1$, then use $e \to u,F_1' \to u,B_1 \to y_2$. What's left is an odd block and we are starting at the vertex y_2, a vertex that we do not need to worry about ending at.

Hence $\varepsilon(B_1) = 0$.

If $k = 1$, then start with $u,B_1^{uy_2} \to y_2$. Again we are left with an odd block and we are starting at y_2. Note that the first step of the algorithm is justified since $\lambda(B_1) \geq 4$ and this is known because $y_2 \neq u_0'$. If $k = 2$, note that $y_2 \neq u_1'$. Start with $e,F_1 \to uu_1,B_1 \cup F_1' \to y_2$. Again we are left with an odd block and we are starting at y_2.

We are left to consider $\lambda(F') = 3$. Recall that $H = G - \{u_iu' : i = 0,k\} - \{u_iu_{i+1} : i = 0,k - 1\} - u_kw_2$, A is the block of H that contains u_k, and that B is the block of H that contains w_2. It is convenient to immediately split this case into the two cases $x_2 \neq u_k'$ and $x_2 = u_k'$.

Case 6 $\quad \lambda(F') = 3$ and $x_2 \neq u_k'$

Suppose that $k = 0$. Note that $x_2 \neq u'$ since we have no multiple edges. Hence $\lambda(B) \geq 3$. If $v \in P(A)$, then unless $v = x_2 \in T(A)$, use $e,F \to w_2u',B \to x_2,A \to v$.

For this exceptional case, if $k = 0$, then use $u,A \to u,F \cup B \to v$. If $k = 1$, then use $e,F_1 \to uu_1,F_1' \cup A \to x_2u_1,F' \cup B_2 \to v$. Note that the step "$uu_1,F_1' \cup A \to x_2u_1$" is our first use of (iii).

Hence $v \in P(B) - \{x_2\}$. Note that $\varepsilon(A) = 1 - \varepsilon(B)$.

If $k = 0$ and $\varepsilon(A) = 0$, then start with $u,A^{ux_2}, \to x_2$. The remaining graph is a block and we are now starting at the vertex x_2. If $k = 0$ and $\varepsilon(A) = 1$, then start with $u,A \to x_2$. Exactly the same situation as before now exists.

If $k = 1$ and $\varepsilon(A) = 0$, then start with $e,F_1 \to uu_1,F_1' \cup A \to u_1 \to w_2$. The remaining graph is an odd block but we are now starting at a vertex that we do not need to be able to end at. If $k = 1$ and $\varepsilon(A) = 1$, then start with $e,F_1 \to uu_1,F_1' \cup A \to x_2u_1,F' \to x_2w_2$. We are now starting at the *edge* x_2w_2 but the remaining graph is even.

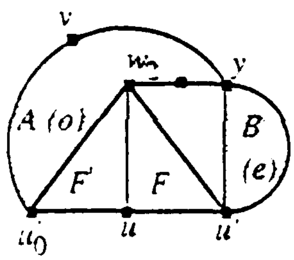
Case 7 $\lambda(F') = 3$ and $x_2 = u_k'$

Suppose that $k = 0$. If $v = u$, then use $e,F \to w_2u',G - u \to u_0' \to u$. Otherwise, we apply (vii) after $e,F \to F' \to u_0'w_2$ unless $(u'w_2,w_2u_0',v)$ is a difficult triple of $G - u$.

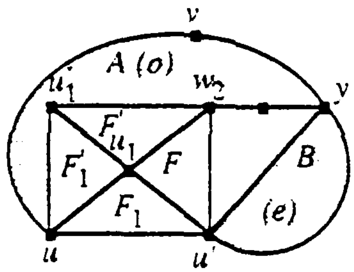
Hence we have a difficult triple. This is the most difficult part of the proof. We will now frequently redefine H, A, B, and C. We classify difficult triples as type i. or type ii.. Type i. difficult triples are the ones that have a 4-gon (see the definition of difficult triple).

Case 71 Difficult triple type i.

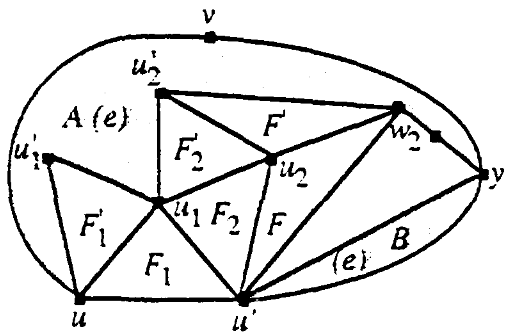
We must resort to considering levels $k = 0$, 1, and 2. Define sections A and B, as well as the vertex y that they have in common as in Figures II.5.40(a), II.5.40(b), and II.5.40(c).



$k = 0$
Figure II.5.40(a)



$k = 1$
Figure II.5.40(b)



$k = 2$
Figure II.5.40(c)

If $k = 0$, then use $u(u_0').F \to u'.B \to y,A \to v$. If $k = 1$, then use $u,A \to x_2w_2F_1' \to F \to F_1 \to u',B \to y$ and buy at v: If $k = 2$, then use $e \to u',B \to u',F \to F_2 \to F_2' \to u_1u_2',A \cup F_1' \to v$.

Case 72 Difficult triple type ii.

We immediately separate into cases for levels 0, 1, and 2 and we define H, A, and B for each of these. In fact we make very sporadic use of the fact that $(u'w_2,w_2u_k',v)$ is a difficult triple of $G - \{u_0,\ldots,u_k\}$.

Case 721 $k = 0$

Let $T = (w_2u'y) = O(w_2u'.F)$. Because F and F' are symmetric with respect to u, we may assume that $T' = O(u_0'w_2,F')$ is a 3-gon $(u_0'w_2y')$. Let $H = G - E(F) - E(F')$ and let A, B, and C be the blocks of H that contain $u_0'y'$, $y'w_2$, and yu' respectively. Except H, each of these concepts is illustrated below.

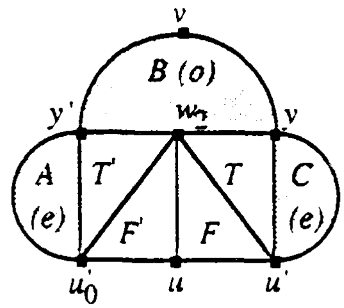


Figure II.5.41

Since both $(u'w_2, w_2u_0', v)$ and $(u_0'w_2, w_2u', v)$ are difficult triples of $G - u$, A and C are even and $v \in P(B) - \{y, y'\}$. Let $T'' = O(y'w_2, T')$.

If $\lambda(T'') \geq 4$, then start with $e, F \rightarrow F' \rightarrow u_0'w_2, A \cup T' \rightarrow y'$. Then march through the blocks of $B - y'w_2$ to y. This is possible since y is in the last block of the march and either y is not equal to the cut-vertex between the last two blocks or the outside face of the last block has length at most four by Lemma II.5.7. The march, together with T'' will gain one more. Finish with $y, C \rightarrow u'$ and buy at v.

If $\lambda(T'') = 3$, then define y'' by $T'' = (y'w_2y'')$. If $y'' \notin P(B)$, then start with $e \rightarrow u, F' \rightarrow u_0'w_2, A \cup T' \rightarrow y'$. What's left is an odd block and, since $v \neq y'$, we can finish at v. If $y'' = y$, then use $e \rightarrow u'. T \cup C \rightarrow w_2, F' \rightarrow u_0'w_2, A \cup T' \rightarrow y'. B - w_2 \rightarrow v$.

Hence we may assume that $y'' \in P(B) - \{y'\}$. Let B_1 be the block of $B - w_2y'$ that contains y' and B_2 be the block of $B - w_2y'$ that contains y. Note that $\varepsilon(B_1) = \varepsilon(B_2)$. This is illustrated below:

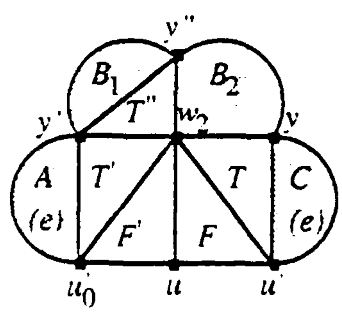


Figure II.5.42

If $\varepsilon(B_1) = 1$, then use $e \rightarrow u'. T \cup C \rightarrow w_2y. B_2y''. B_1 \rightarrow y'. A \cup T' \rightarrow u_0'w_2, F' \rightarrow u$ and buy at v.

If $\varepsilon(B_1) = 0$ and $A \neq u_0'y'$, then use $e \to u',C \cup T \to w_2y,B_2 \to w_2y'',T'' \to T' \to F' \to u_0',A^{u_0'y'} \to y',B_1^{y'y''} \to y''$ and buy at v. The last step is unnecessary and unavailable if $\lambda(B_1) = 2$.

If $\varepsilon(B_1) = 0$ and $A = u_0'y'$, then use $e,F \to F' \to T' \to w_2y',T'' \cup B_1 \to y'',B_2^{y''w_2} \to y,C \to u'$ and buy at v

Case 722 $k = 1$

For this case, we ignore almost everything that we now know about the graph and pursue an entirely different line that involves a different difficult triple.

Use $e,F_1 \to F_1' \to F' \to u_1'w_2$ and then (vii) unless $(uu_1',u_1'w_2,v)$ is a difficult triple of $G - u_1 - e$. Let $T = O(uu_1',F_1')$. Let $H = G - u_1 - \{uu_1',e\}$,$A$ be the block of H that contains u, and B be the block of H that contains u_1'. Let y be the cut-vertex between A and B. Note that $v \in P(B) - \{y\}$.

If $\lambda(T) = 4$, then start with $u,A \to y$. The rest of the graph is a block and we are starting at the vertex y.

If $\lambda(T) = 3$, then use $u,A \to u,F_1' \to F_1 \to F \to u'w_2,B \to v$.

Case 723 $k = 2$

For this case, we will use a sequence of up to three difficult triples to "box in" v until we have identified enough of the graph to write down procedures for the remaining cases.

We start by using $e,F_1 \to F_2 \to F_2' \to F' \to w_2u_2'$ and (vii) unless $(u_1u_2',u_2'w_2,v)$ is a difficult triple of $G - E(F_1) - u_2$. Let $T = O(u_1u_2',F_2')$. Let $H = G - E(F_1) - u_2 - u_1u_2'$, A be the block of H that contains u_1, and B be the block of H that contains u_2'. Let y be the cut-vertex between A and B. Note that $v \in P(B) - \{y\}$. Examples corresponding to the types of difficult triples are shown below.

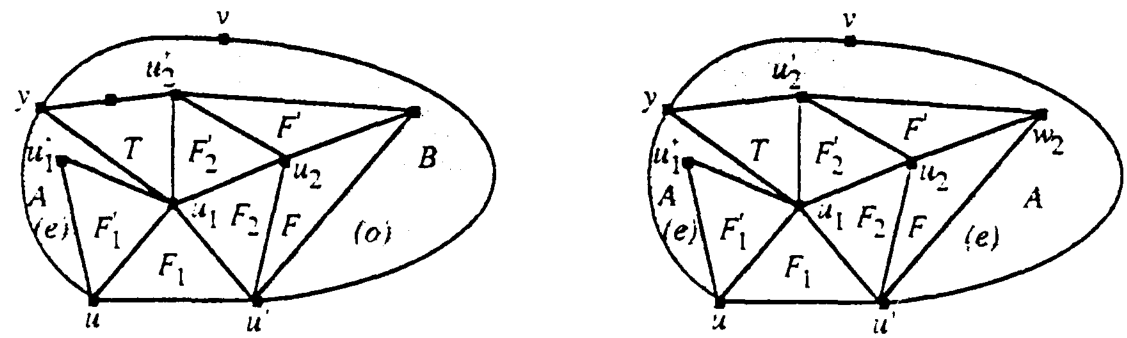


Figure II.5.43

If $\lambda(T) = 4$ and $y = u_1'$, then use $e,F_1 \to F_1' \to uu_1',B \cup F_2' \cup F' \cup F \to v$. If $\lambda(T) = 4$ and

$y \neq u'_1$, then use $e, F_1 \rightarrow uu'_1, A \cup F'_1 \rightarrow y, B \cup F'_2 \cup F' \cup F \rightarrow v$. If $\lambda(T) = 3$ and $y \neq u'_1$, then use the same procedure. Hence we may assume that $\lambda(T) = 3$ and $y = u'_1$. This is shown below.

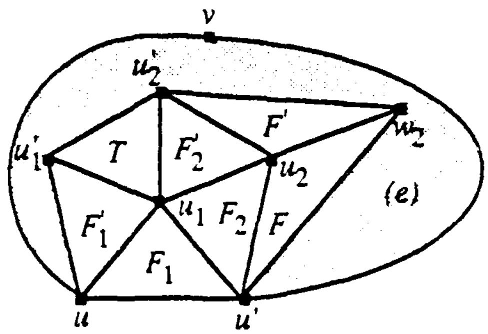


Figure II.5.44

We know from the definition of case 72 that $(u'w_2, w_2u'_2, v)$ is a type ii. difficult triple of $G - u_2 - E(F_1)$. Let $T' = O(u'w_2, F)$. Again we redefine A and B. At this point, it is simpler to appeal to diagrams for the definitions of the face T' and the sections A and B. These definitions, together with the definition of the vertex z are given in Figure II.5.45.

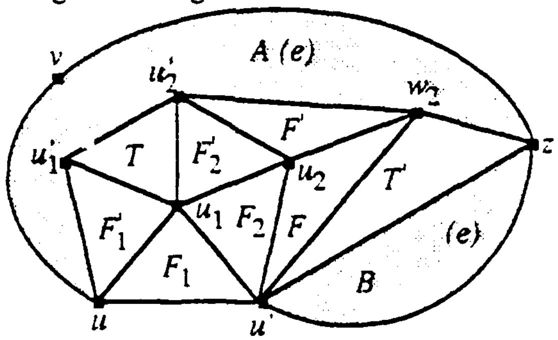


Figure II.5.45

Note that $v \in P(A)$, as indicated in Figure II.5.45 and $\lambda(T') = 3$. Use $e, F_1 \rightarrow u', B' \rightarrow u', F \rightarrow F' \rightarrow F'_2 \rightarrow T \rightarrow u'_1u'_2$ and (vii) unless $(w_2u'_2, u'_2u'_1, v)$ is a difficult triple of A. Therefore, we have either Figure II.5.46(a) or Figure II.5.46(b), again using the diagrams to define A_1, A_2, and z'.

At this point, v has exhausted all of its hiding places. Note that z' must be in $P(G)$ because [illegible] relative positions of v and z' and the fact that $v \in P(G)$. For either case, start with e, F [illegible] $z, A_2 \rightarrow w_2, F \rightarrow F' \rightarrow F'_2 \rightarrow T \rightarrow u'_2u'_1$. For Figure II.5.46(a), finish with [illegible] z' and buy at v. For Figure II.5.46(b), finish with $u'_2u'_1 \rightarrow v$. ♣

AD-A189 648 THE TOTAL INTERVAL OF A GRAPH(U) ILLINOIS UNIV AT URBANA COORDINATED SCIENCE LAB T M KRATZKE JAN 88 UILU-ENG-88-2202 N00014-85-K-0570 2/2

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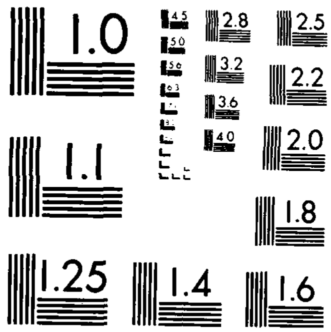

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

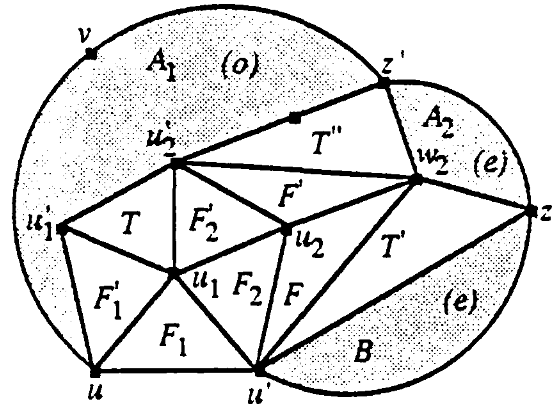

Figure II.5.46(a)

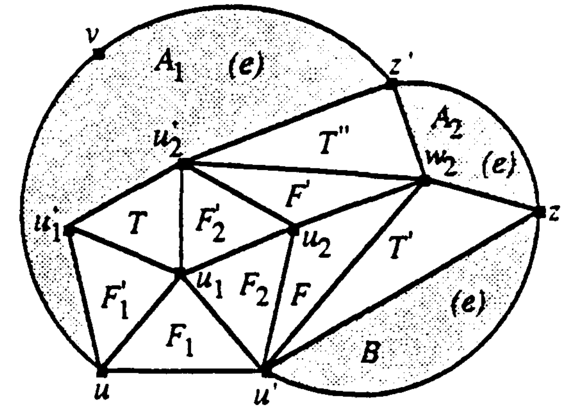

Figure II.5.46(b)

We have now established each of the induction steps of Theorem II.5.5 and have therefore completed the proofs of Theorem II.5.5, Lemma II.5.2, and Theorem II.5.1. ♠

As another application of profitable representations, we now show how they can be used for proving the bound for 2-connected outerplanar graphs (See Theorem II.4.4). Suppose that G is 2-connected and outerplanar and let R be a profitable representation. From (II.5.5b) and $\lambda(G) = n(G)$, we have $2((2n(G) - 3) - |R|) = p(R) + n - 4$ and, because n is odd implies that $p(R) \geq 1$, we also have $2n(G) - 3 - |R| \geq \lfloor \frac{n(G) - 3}{2} \rfloor$ or $|R| \leq 2n(G) - 3 - (\lceil n(G)/2 \rceil - 2) \leq \lfloor 3n(G)/2 \rfloor - 1$.

6. Connected Graphs

In §II.6, we establish bounds on I and I_3 for connected graphs in terms of n. In particular, we prove the following theorem.

Theorem II.6.1. For any connected graph G with n vertices, where $n \geq 4$, we have the following:

(i) If $G = K_{\lfloor n/2 \rfloor \lceil n/2 \rceil}$, then $I(G) = \lfloor n^2/4 \rfloor + 1$. If $G \in \{K_4, K_5\}$, then $I_3(G) = \lfloor n^2/4 \rfloor + 1$.

(ii) If $G \notin \{K_4, K_5, K_{\lfloor n/2 \rfloor \lceil n/2 \rceil}\}$, then $I_3(G) \leq \lfloor n^2/4 \rfloor$.

Moreover, for any $n \geq 4$, there exists a graph G' with n vertices for which $I(G') = \lfloor n^2/4 \rfloor$ and so (ii) is best possible.

Proof. For the first assertion of (i), it is clear that $t(K_{\lfloor n/2 \rfloor \lceil n/2 \rceil}) = 1$. Since $K_{\lfloor n/2 \rfloor \lceil n/2 \rceil}$ is triangle-free, $I(K_{\lfloor n/2 \rfloor \lceil n/2 \rceil}) = I_2(K_{\lfloor n/2 \rfloor \lceil n/2 \rceil}) = m + 1 = \lfloor n/2 \rfloor \lceil n/2 \rceil + 1 = \lfloor n^2/4) \rfloor + 1$. The second assertion of (i) be verified directly.

By removing one edge from $K_{\lfloor n/2 \rfloor \lceil n/2 \rceil}$, we obtain a graph that demonstrates the last statement in the theorem.

For (ii), we use induction on n. Because of the nature of the proof, we must verify $n = 4$ and $n = 5$ for our basis case and we do this by finding representations for each graph with four or five vertices.

Now suppose that $G \neq K_{\lfloor n/2 \rfloor \lceil n/2 \rceil}$ and that $n \geq 6$. If $G \in \{K_6, K_7\}$, then we use one of the representations in Figure II.6.2

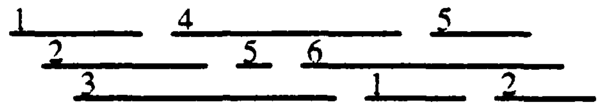

Figure II.6.2(a)

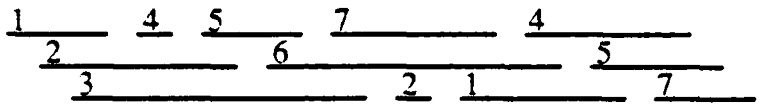

Figure II.6.2(b)

If $G \notin \{K_4, K_5, K_6, K_7, K_{\lfloor n/2 \rfloor \lceil n/2 \rceil}\}$, then, there is an edge uv such that $G - \{u,v\} \notin \{K_4, K_5, K_{\lfloor (n-2)/2 \rfloor \lceil (n-2)/2 \rceil}\}$ and, by induction, $\{I_3(G - \{u,v\}) \leq \lfloor \frac{(n-2)^2}{4} \rfloor = \lfloor n^2/4 \rfloor - (n-1)$. Let $H = G - \{u,v\}$. We will be done if we can prove:

$$I_3(G) \leq I_3(H) + (n-1) \tag{II.6.1}$$

Let R' be an optimal depth-3 representation for H and let w and x be the vertices that correspond to the first two intervals of R'. The situation is depicted below.

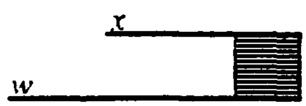

Figure II.6.3: R'

We adjust R' to form a representation R'' of $H \cup \{uz : z \in N_G(u)\} \cup \{vz : z \in N_G(v)\}$. Seven cases and the corresponding adjustments are shown in Figure II.6.4. Any other case can be reduced to one of the above seven by exchanging the roles of u and v and/or of w and x.

For each possible representation R'', $|R''| \leq |R'| + 3 = I_3(H) + 3$. Moreover, there is a displayed u-interval θ_u, a displayed v-interval θ_v, and an intersection θ_{uv} of some u-interval and some v-interal that intersects no other interval. For each $y \notin \{u,v,w,x\}$, place a small y-interval inside θ_u, θ_v, or θ_{uv}, depending on whether y is adjacent to u but not v, y is adjacent to v but not u, or y is adjacent to both u and v. If y is not adjacent to either u or v, then do not add any y-interval. Call the resulting representation R. By this construction, $|R| \leq (I_3(H) + 3) + n - 4 = I_3(H) + (n - 1)$. Hence, we have established (II.6.1).♠

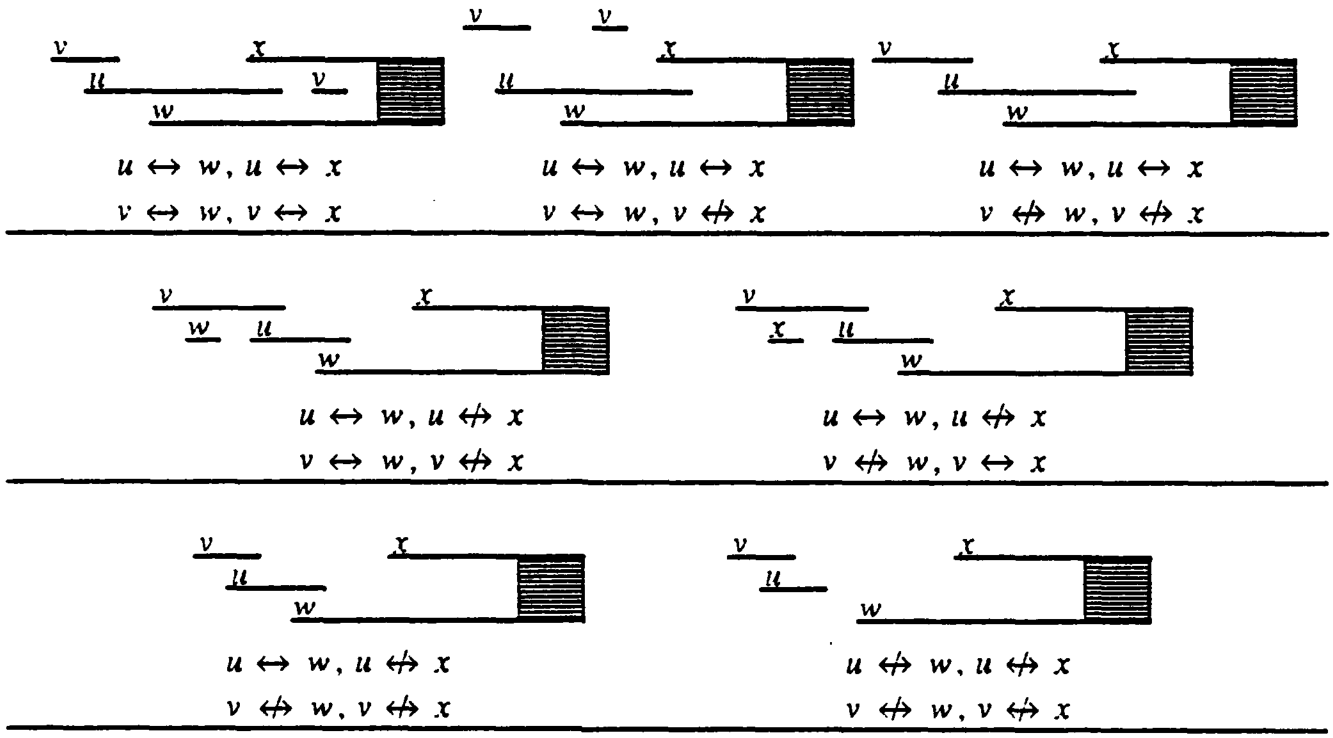

Figure II.6.4

III. THE TOTAL INTERVAL NUMBER AND THE NUMBER OF EDGES

1. Preliminary Results

In §III, we consider bounds on the total interval number in terms of the number of edges of a graph. For all classes in §III for which we establish a best possible upper bound, there exist triangle-free extremal graphs. Therefore, I_2 is very important and we will rely heavily on trail covers.

When there are no restrictions on the graphs, it is easy to see that, given m, the only graph with m edges that requires one trail per edge is m copies of K_2. Hence, for all graphs, $I \leq I_2 \leq 2m$ and $I_2 = 2m$ only for m copies of K_2.

We now establish the bound for connected graphs. Recall the set $\mathcal{P}$ of graphs of §II.1. It is clear that if $G \in \mathcal{P}$, then $I(G) = \dfrac{5m(G)+2}{4}$.

Theorem III.1.1. If G is connected, $m(G) > 1$, and $G \notin \mathcal{P}$, then $I_2(G) < \dfrac{5m(G)+2}{4}$.

Proof. Let G be a connected graph that is not in $\mathcal{P}$. If G is a tree, then the result follows from Corollary II.1.12 and the remarks following it. Hence we may assume that G has a cycle. If it has two or more cycles, then choose any edge that is in a cycle and snip (Lemma I.6.2(iii)) it. The number of cycles decreases, the number of edges remains the same, and the trail cover number does not decrease. Hence we may assume that G has exactly one cycle C and that if we snip any edge of the cycle, we obtain a member of $\mathcal{P}$. For example, if $m(G) = 10$, then G must be one of the graphs below.

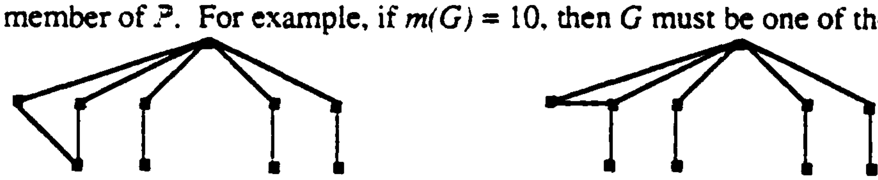

Figure III.1.1

For such graphs, we have $t = 1 + \dfrac{m-6}{4}$ and hence $I_2 = \dfrac{5m-2}{4} < \dfrac{5m+2}{4}$. ♠

We can modify the proofs of Corollary II.1.12 and Theorem III.1.1 to show that, given m, the only graphs with m edges for which $I = \dfrac{5m+1}{4}$ are trees that arise from subdividing one edge of the member of $\mathcal{P}$ that has $m - 1$ edges, or are trees formed by two augmentations at black vertices and $\dfrac{m-11}{4}$ augmentations at white vertices.

2. Classes Defined by Minimum Degree

In §III.2, we show how placing a lower bound on δ tends to decrease the maximum value of I/m. The proof of the best possible bound seems quite difficult for $\delta \geq d$ for $d \geq 3$. We will concentrate on requiring $\delta \geq 2$, where we have a complete solution.

Theorem III.2.1. If G is not a 4-cycle, a 5-cycle, or a 6-cycle, and $\delta(G) \geq 2$, then $I(G) \leq \frac{9m(G)+1}{8}$. Furthermore, for any $m \geq 8$, there exists a graph G with m edges for which $I(G) = \lfloor \frac{9m(G)+1}{8} \rfloor$ and so the result is best possible.

Let $f(G) = 9m(G) - 8I(G) + 1$. We will prove an equivalent version of Theorem III.2.1 that we call Theorem III.2.1'.

Theorem III.2.1'. If G is not a 4-cycle, a 5-cycle, or a 6-cycle, and $\delta(G) \geq 2$, then $f(G) \geq 0$. Furthermore, for any $m \geq 8$, there exists a graph G with m edges for which $0 \leq f(G) < 8$ and so the result is best possible.

We first give the extremal graphs that show that Theorem III.1.1' is best possible. A $(1,3)$-tree is a tree for which each vertex is of degree 1 or 3. Let G' be a $(1,3)$-tree, n_1 be the number of leaves, and n_3 be the number of branchpoints. Since $m(G') = n_1(G) + n_3(G) - 1$, there are n_1 leaf-edges and $n_3 - 1$ edges between branchpoints. Hence if we add isolated 4-cycles at each leaf and twice subdivide the edges between the vertices of degree 3, we get a graph G that satisfies:

$$m(G) = 3(n_3(G) - 1) + 5n_1(G) \qquad \text{(III.2.1)}$$

The trail cover number of G is the same as the trail partition number and this is half the number of odd vertices. Hence we have:

$$t(G) = \frac{n_1(G) + n_3(G)}{2} \qquad \text{(III.2.2)}$$

Counting the edges of G' by $m(G) = \frac{n_1 + 3n_3}{2}$ and also $m(G) = n_1(G) + n_3(G) - 1$, we get:

$$n_1(G) - n_3(G) = 2 \qquad \text{(III.2.3)}$$

Combining (III.2.1), (III.2.2), and (III.2.3), we see that $8t(G) - 1 = m(G)$. Subdividing any edge up to seven times gives graphs in the other residue classes of eight for which the bound of Theorem III.2.1' is sharp.

Most of the rest of §III.2 is devoted to establishing the upper bound of Theorem III.2.1'. For any graph G we color the edges and define the **cut-graph** $K(G)$ as follows. Color all cut-edges red and the other edges blue. Let G_r (G_b) be the subgraph of G that is induced by the red (blue) edges. Let $K(G)$ be the intersection graph of the vertex sets of the components of G_r and G_b. We will soon see that a red component and a blue component intersect in at most one vertex. Therefore it doesn't matter if we define $K(G)$ as a simple graph (as we have) or if we define an edge for each member of the intersection between sets.

For $X \in V(K(G))$, we will use "X" to refer to either the vertex in $K(G)$ or the subgraph of G that is induced by the vertices of X. If $X \leftrightarrow Y$ in $K(G)$, then select $q(X,Y) \in V(X) \cap V(Y)$. If we want to distinguish which graph we are considering when applying q, we will use a subscript (e.g.$q_{G'}$).

For the reader that is familiar with other intersection graphs (e.g. the block cut-vertex tree), the proof that $K(G)$ is a tree is routine. We write it out here.

If X corresponds to a red (blue) component, then color X red (blue). This coloring of the cut-graph is called the **canonical** coloring. A coloring of any graph is called **proper** if $X \leftrightarrow Y$ imples that X and Y are different colors.

Lemma III.2.2. For any graph G, $K(G)$ is a tree.

Proof. It is clear that $K(G)$ is connected. We must show that it has no cycle. We first show that the canonical coloring is proper. If $X \leftrightarrow Y$ and X and Y are the same color, then there exists $v \in V(X) \cap V(Y)$. Since components of the same color do not intersect, X and Y must have different colors

Next, we show that if X is red, then X is a tree. Red edges are precisely those that are in no cycle of G, and deleting the blue edges cannot introduce any cycles. Hence X has no cycles and, since it is connected, X must be a tree.

Now suppose that $K(G)$ has a cycle. Here we specifically allow multiple edges to correspond to some intersection with at least two vertices. Suppose that $C = (X_0, \ldots X_{k-1})$ is the smallest cycle in $K(G)$. Since the canonical coloring is proper, we may assume that X_0 is red and both X_1 and X_{k-1} are blue. Note that, since we have not eliminated the possibility of multiple edges, it is possible that $X_1 =$

X_{k-1} (i.e., $k = 1$).

For what follows, take all subscripts modulo k. Choose $x_i \in q(X_iX_{i+1})$. If $x_i = x_j$, and $j > i$, then $(X_0,\ldots,X_iX_{j+1},\ldots,X_{k-1})$ is a smaller cycle than C, and so we may assume that the x_i's are distinct.

Let P_i be a path within X_i between x_i and x_{i+1}. It is clear that $P_0P_1,\ldots,P_{k-1}$ is a cycle within G. But since $x_0 \neq x_1$, P_0 is a set of red edges, each of which is in a cycle. Each of these edges is therefore not a cut-edge, contradicting the definition of a red edge.♠

We assume the canonical coloring of the edges for the rest of §III.2. A cycle C with exactly one branchpoint u is called a **pendant** cycle and u is called the **base** of C. Let $\mathcal{F}$ be the set of graphs G that satisfy:

i. $\delta(G) \geq 2$
ii. G is connected.
iii. G has no pendant 3-cycle whose base is of degree three.
iv. $m(G) \geq 7$

Let $f'(G) = 9m(G) - 8I_2(G) + 1$. Note that $f'(G) = m(G) - 8t(G) + 1$ and that $f'(G) > 0$ if and only if $8t(G) > m(G) + 1$.

For a graph G, let $\eta(G) = 2m(G) - n(G)$. Most of the proof of the upper bound is an inductive argument on η that shows $G \in \mathcal{F}$ implies that $f'(G) \geq 0$.

Lemma III.2.3. Let G be a graph with minimum value of $\eta(G)$ such that $G \in \mathcal{F}$ and $8t(G) > m(G) + 1$. Then G has the following properties:

(i) If $uv \in E(G)$, and u and v are branchpoints, then uv is red.
(ii) If $uv, vw \in E(G)$, $d(v) = 2$, and both u and w are branchpoints, then uv and vw are both red.
(iii) If u, v, and w are bivalent and $u \leftrightarrow v \leftrightarrow w$, then u, v, and w are the bivalent members of an isolated 4-cycle whose base is of degree 3.
(iv) If C is a pendant cycle with base u, then C is a 4-cycle and $d(u) = 3$.

Proof. (i) and (ii): From Lemma I.6.2(iii),(iv), snipping and double snipping do not decrease the trail cover number. Both operations leave m unchanged, increase n and hence decrease η. Therefore, if the result of either operation is a member of $\mathcal{F}$, then that would contradict the minimality of G. By the hypotheses, both operations do not create a leaf or a pendant 3-cycle and hence they must disconnect the graph. Hence, uv (in (i)) and both uv and vw (in (ii)) must be cut-edges.

(iii): Let $H = G \bullet uv$; by Lemma I.6.2(ii), $t(H) = t(G)$ and simple arithmetic shows that $\eta(H) = \eta(G) - 1$. Hence $H \notin \mathcal{F}$. If $m(H) < 7$, then G is a 7-cycle and $f(G) = 0$. The only other property of $\mathcal{F}$ that H can violate is iii. and, if this is the case, then u, v, and w are as asserted.

(iv): By (iii), we may assume that C is a pendant 3-cycle (uvw) and, by property iii. of $\mathcal{F}$, $d(u) \geq 4$. Let H be the graph that results from removing w and then snipping uv. Note that $t(H) = t(G)$ and $\eta(H) < \eta(G)$. It is clear that H satisfies the first three properties of $\mathcal{F}$ and, if $m(H) < 7$, that $f(G) = 0$. Hence $H \in \mathcal{F}$, contradicting the minimality of G.♠.

Let $\mathcal{F}'$ be the set of graphs G in $\mathcal{F}$ with minimal $\eta(G)$ such that $f(G) < 0$; we will show that $\mathcal{F}' = \emptyset$. If $G \in \mathcal{F}'$, then, by Lemma II.2.3(iv), every pendant cycle is a 4-cycle whose base is of degree three. We strip the pendant cycles (leaving the bases) from such a graph G to obtain the **reduced graph** G'. By the leaflessness of graphs in $\mathcal{F}$, it is easy to retrieve G from G'; G has a pendant 4-cycle at a vertex u of G' if and only if u is a leaf of G'. Let $\Xi' = \{G' : G \in \mathcal{F}'\}$. To prove Theorem III.2.1', we will show that Ξ' is empty.

For any $G' \in \Xi'$, the canonical coloring is inherited from the corresponding $G \in \mathcal{F}'$; the edge incident to a pendant cycle is red and stripping the pendant cycles merely deletes a component of G_b that is a blue leaf of $K(G)$.

A **staple graph** is a graph that is obtained from some multigraph H, called an **underlying multigraph** by twice subdividing each edge. It is easy to show that, unless $G = C_{3k}$ for some k, there is a unique underlying multigraph. For a staple graph G with underlying multigraph H, we call the vertices created during the subdivisions of the members of $E(H)$ **new** vertices and the other vertices **old**. We will not need to refer explicitly to the underlying multigraph; we will simply refer to new and old vertices. By considering the edges incident to new vertices, it is easy to see that the trail cover problem of a staple graph is the same as the problem of partitioning the edges into trails and the answer to this problem is half the number of vertices with odd degree.

Lemma II.2.4 Let Y be a component of G_b'. Then

(i) Y is a staple graph.
(ii) The only vertices of Y that are in some component of G_r' are old vertices.
(iii) $\delta(Y) \geq 2$

Proof. From Lemma III.2.3, we see that the only blue edges of G' are paths $\langle u,v,w,x \rangle$ where u and x are branchpoints and v and w are bivalent. Hence Y is a staple graph and the only vertices of Y that are in some component of G_r' are branchpoints of G'. Hence (i) and (ii) are proved.

We now prove (iii). Suppose that $x \in V(X) \cap V(Y)$, (and hence X is red), and $d_Y(x) = 1$. Define y by $xy \in E(Y)$. Then xy is in some cycle of G' but that cycle must include at least one edge of X since, except for y, all neighbors (in G') of x are in $V(X)$. This contradicts the definition of a red edge. ♠

Lemma III.2.5. Suppose that $G' \in \Xi'$ and, in the canonical coloring of $K(G')$, X is a red leaf. Then X is a single edge.

Proof. In $K(G')$, X has exactly one neighbor Y, or X is the only vertex of $K(G')$. In the first case, let $w = q(XY)$ and in the second case, let w be a peripheral vertex of X. In either case, let u be at a maximum distance (in X) from w. Note that u has only one neighbor v and, if the lemma is false, then $v \neq w$. Furthermore, v has only one non-leaf neighbor v'.

Suppose that $d_G(v) = 2$. Let G'' be the graph obtained from G by contracting the edge uv. Since G'' is connected and the contraction of uv does not introduce any leaves or pendant 3-cycle, it follows that if $G'' \notin \mathcal{F}$, then $m(G'') < 7$ and so $m(G) = 7$. But then G has a leaf and hence $G'' \in \mathcal{F}$. Because of the pendant 4-cycle containing u, it is clear that $t(G'') = t(G)$. Moreover, it is easy to verify that $f'(G'') = f'(G) - 1$ and $\eta(G'') = \eta(G) - 1$, contradicting the minimality of G.

Now suppose that $d_G(v) > 3$ so that v has at least three leaf neighbors u,u', and u'' in G'. Let G'' be the graph obtained from G by removing u, u', and the pendant 4-cycles containing u and u'. It is easy to verify that $G'' \in \mathcal{F}$. By using one trail to cover $E(G) - E(G'')$, we see that $t(G) < t(G'') + 1$ and so $f'(G'') < f'(G) - 2$. Then $\eta(G'') = \eta(G) - 1$ contradicts the minimality of G.

Finally suppose that $d_G(v) = 3$ so that $N_G(v) = \{u,u',v'\}$, where u and u' are leaves in G'. If $d_G(v') \geq 2$, then we can mimic the above argument to get the result, this time using the trail to cover

vv'. Hence we may assume that $N_G''(v') = \{v,v''\}$ and we have the situation illustrated below.

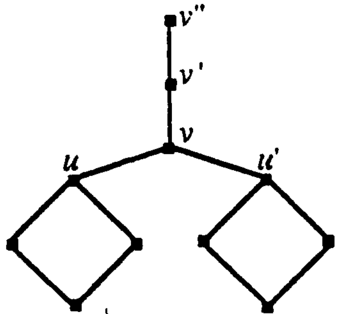

Figure III.2.1

Now let G^* be the graph obtained from G'' by adding vertices x and x', and edges so that $(v'v''xx')$ is a 4-cycle. Note that it is a pendant cycle of G^* and that its base is v''. This is illustrated below.

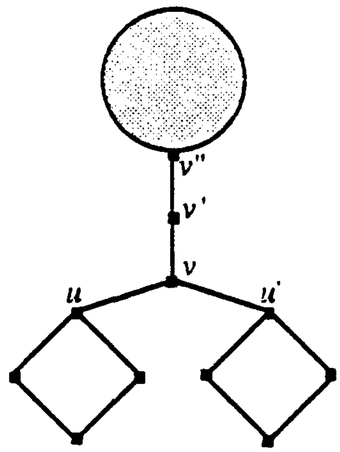

Figure III 2.2(a): G

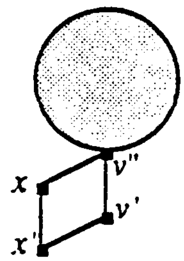

Figure III 2.2(b): G^*

It is again easy to verify that $G^* \in \mathcal{F}$. Since $\eta(G) = \eta(G) - 9$, the minimality of G shows that there exists a trail cover $\mathcal{T}^*$ of G^* such that $8|\mathcal{T}^*| - 1 \leq m(G^*)$. Moreover, it is clear that we may assume some trail $T^* \in \mathcal{T}^*$ traverses the new 4-cycle. If we remove this 4-cycle from T^*, then we do not increase the number of trails and the edge $v''v'$ is covered. Hence we have a trail cover $\mathcal{T}''$ of G'' and $|\mathcal{T}''| = |\mathcal{T}^*|$. Then $\mathcal{T} = \mathcal{T}'' \cup \{T\}$ is a trail cover of G, it follows that $f'(G^*) < f'(G)$ and so $f'(G) > 0$.♠

Theorem III.2.6 If $G \in \mathcal{F}$, then $f'(G) \geq 0$.

Proof. Suppose not: let $G \in \mathcal{F}'$ and the corresponding $G' \in \Xi'$ be as constructed earlier. By Lemma III.2.5, we may assume that each red leaf of $K(G')$ corresponds to a single edge. This eliminates the possibility that $K(G')$ is a single red vertex. If $K(G')$ has a peripheral red leaf, then let Y be its neighbor; otherwise let Y be a peripheral blue leaf. Y has at most one non-leaf neighbor. If Y has a non-leaf neighbor, then call it W, let $w = q(YW)$, and let Q be the component of $G - E(W)$ that contains Y. If Y has no non-leaf neighbor, then choose $w \in V(Y)$ arbitrarily and let $Q = G$. Let R be the collection of red edgs that correspond to the leaf neighbors of Y (R may be empty). Note that Q contains Y, together with R and the pendant 4-cycles. In Figure II.2.3, we illustrate the situation. In that figure, we have put a leaf-edge of G' incident to w to emphasize that such edges and their corresponding pendant 4-cycles do not belong to Q.

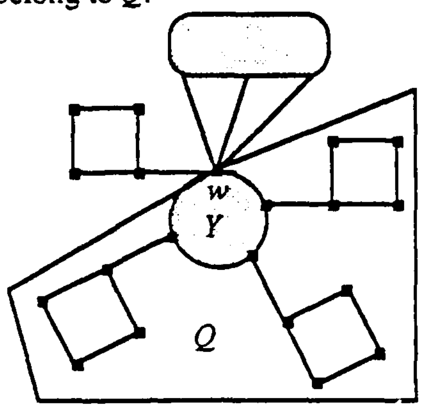

Figure III.2.3: G (Note: Y is a staple graph.)

Now fix some underlying multigraph H of Y. Let n_1 be the number of odd vertices in H and n_2 be the number of even vertices in H. Let $n_1 = n_{11} + n_{12}$, where n_{11} is the number of odd vertices in H that are incident to members of R. Let $n_2 = n_{21} + n_{22}$, where n_{21} is the number of even vertices in H that are incident to members of R. Since H has no vertices of degree one, and $m(Y) = 3m(H)$, $m(Y) \geq 3\,\frac{3n_1 + 2n_2}{2}$. Since each red edge and corresponding pendant 4-cycle has five edges, we have:

$$m(Q) \geq 3\,\frac{3n_1 + 2n_2}{2} + 5(n_{11} + n_{21}) \qquad \text{(III.2.4)}$$

Now we count the odd vertices of Q. Each member of R provides one from the base of the corresponding pendant 4-cycle, each even vertex of Y that is incident to an edge of R provides one, and each odd vertex of $Q - R$ that is not incident to any edge of R provides one. Dividing by two to compute the number of trails necessary to partition the edges gives:

$$\begin{aligned}&\text{If } n_1 - n_{21} = 0, \text{ then } t(Q) = 1.\\ &\text{If } n_1 - n_{21} > 0, \text{ then } t(Q) \leq n_1/2 + n_{21}.\end{aligned} \qquad \text{(III.2.5)}$$

Note that Q cannot be a 6-cycle since G has no pendant 6-cycle and hence $m(Y) \geq 9$. Let T be a trail partition of Q. From $m(Q) \geq 9$, (III.2.4), (III.2.5), and $n_2 \geq n_{21}$, we have $t(Q) \leq 8m(Q)$. Let Q' be the graph induced by the vertices of the non-trivial component of $G - E(H)$.

If $f'(Q') \geq 0$, then we can cover Q' and Q separately and show that $f'(G) \geq 0$. Hence we have $f'(Q') < 0$ and, by the minimality of G, that $Q' \notin \mathcal{F}$. It then follows that either $m(Q') < 7$ or else $\delta(Q') < 2$.

Suppose that $m(Q') < 7$. For some $T \in \mathcal{T}$, T visits w, we can route T through Q' and have T cover $E(Q')$ before continuing. Therefore $t(G) = t(Q)$. But it then quickly follows that $f'(G) > 0$, contradicting $G \in \mathcal{F}'$.

Hence $\delta(Q') = 1$ and $N_{Q'}(w) = \{w'\}$. First assume that $d_Q(w)$ is odd. Then in $\mathcal{T}$, a trail T ends at w and so can be extended to w' and, if necessary, beyond to a branch point; call the resulting set of trails $\mathcal{T}'$. Note that $|\mathcal{T}| = |\mathcal{T}'|$. Remove the edges of $\mathcal{T}'$ from Q' to form Q''. If $f'(Q'') \geq 0$, then we can take a trail cover $\mathcal{T}''$ of Q'', together with $\mathcal{T}'$ and get a trail cover of G with sufficiently few trails. If $f'(Q'') < 0$, then, by the minimality of G, $Q'' \notin \mathcal{F}$ and it is easy to show that this implies that $m(Q'') < 7$. But then T can be extended to cover Q'' and again we have a trail cover of G with sufficiently few trails.

Hence we may assume that $d_Q(w)$ is even.

We will again cover Q and Q' separately. Define Q'' to be the graph obtained from Q' by adding two vertices and edges so that w and w' are part of a 4-cycle. It is clear that $t(Q') = t(Q'')$. Furthermore, it is again easy to eliminate the possibility of $Q'' \notin \mathcal{F}$ and so we may assume that $8t(Q'') - 1 \leq m(Q'')$. From $t(Q) \leq n_1/2 + n_{21}$, $m(Q) \geq 9n_1/2 + 8n_{21} + 3n_{22} + 5n_{11}$, and $n_{22} \geq 1$, we have $8t(Q) \leq m(Q) - 3$. Since $8t(Q') = 8t(Q'') \leq m(Q'') + 1 = m(Q') + 3$, we have $8t(G) \leq 8t(Q) + 8t(Q') \leq m(Q') + 3 + m(Q) - 3 = m(G) < m(G) + 1$, again contradicting $f'(G) < 0$.♠

We are now ready to prove Theorem III.2.1'. We verify directly that if $m(G) < 7$ and $G \notin \{C_4, C_5, C_6\}$, then $f(G) \geq 0$. From Theorem III.2.6, we may assume that G has a pendant 3-cycle whose base is of degree three. Replace each of these 3-cycles with a pendant 4-cycle with the same base to obtain the graph H. By Theorem III.2.6, H has a depth-2 representation R such that $|R| \leq \frac{9m(H) - 1}{8}$. For every pendant 4-cycle $(uvwx)$ with base u, we may assume that R contains a configuration as in Figure III.2.4(a). For each such cycle that must be replaced by a 3-cycle (uvw) to return to G, we modify R as in Figure III.2.4(b).

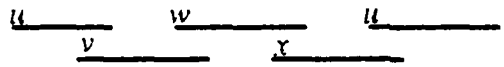

Figure III.2.4(a)

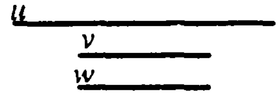

Figure III.2.4(b)

This saves one edge and two intervals. Hence, if there are k such 3-cycles, then $I(G) \leq \frac{9(m(G)+k)+1}{8} - 2k = \frac{9m(G)+1}{8} - 7k/8$. This completes the proofs of Theorem III.2.1' and Theorem III.2.1.

Theorem III.2.7. If G is a graph, and $\delta(G) \geq 2$, then $I(G) \leq 5m(G)/4$. Furthermore, for any $m \geq 8$, there exists a graph G with m edges for which $I(G) = 5m(G)/4$ and so the result is best possible.

Proof. We establish best possible first. If $m \equiv 0 (mod\ 4)$, then let G be $m/4$ copies of C_4. For other residue classes, subdivide edges of one of the C_4's.

Let $\alpha(G) = I(G)/m(G)$; we must prove that $\alpha \leq 5/4$. Let G be a graph that maximizes I/m. Let $\beta(G) = max\{\alpha(H) : H \text{ is a component of } G\}$. We have $\alpha(G) \leq \beta(G)$, with equality if and only if, for every component H of G, $\alpha(H) = \alpha(G)$. Hence we may assume that the components of G are identical. Let H be a component of G.

From Theorem III.2.1, if $m(H) > 6$, then $\alpha(H) \leq 8/7$. If $m(H) \leq 6$, then we can verify directly that the maximum value of α is achieved uniquely by C_4.♠

We finish §III.2 by showing that for any positive integer d, there exists $\varepsilon_d > 0$ and an infinite sequence of graphs with $\delta = d$ for which $I > (1 + \varepsilon_d)m$. This is of interest because it is not true for the two other connectivity parameters; if $\kappa' \geq 4$, then $I \leq m + 1$.

Given d and r, let $G_{d,r}$ be the graph formed by the following procedure. Start with r disjoint copies of $K_{d,d}$, each with one distinguished vertex. Add a new vertex u and edges from u to each of the distinguished vertices. We have $t(G_{d,r}) = \lfloor \frac{r+1}{2} \rfloor$ and $m(G_{d,r}) = r(d^2 + 1)$. Simple arithmetic then shows that, for any r, we can use $\varepsilon_d = \frac{1}{2(d^2+1)}$. For this construction, we can say that it takes $2(d^2 + 1)$ edges to force an additional trail. We do not claim that this is the minimum but we do believe that the number of edges necessary to force new trails grows quadratically with d.

3. Classes Defined by Connectivity Parameters

In §III.3, we resolve the extremal problem for $I(G)$ in terms of $m(G)$ for some connectivity classes. We consider edge-connectivity κ and the vertex-connectivity κ. In particular, we prove the following.

Theorem III.3.1. If G is a graph with $\kappa'(G) \geq 2$ and $m(G) \geq 9$, then $I_2(G) \leq \lfloor 10m(G)/9 \rfloor$. Furthermore, for any $m \geq 9$, there exists a 2-edge-connected graph with m edges for which $I = \lfloor 10m/9 \rfloor$ and so the result is best possible.

The examples that we will use to show that Theorem III.3.1 is best possible are 2-connected, which yields:

Corollary III.3.2. If G is a graph with $\kappa(G) \geq 2$ and $m(G) \geq 9$, then $I_2(G) \leq \lfloor 10m(G)/9 \rfloor$. Furthermore, for any $m \geq 9$, there exists a 2-connected graph with m edges for which $I = \lfloor 10m/9 \rfloor$ and so the result is best possible.

In Theorem III.3.1 and Corollary III.3.2, the upper bound is for the parameter I_2 whereas the "best possible" clause is in terms of I. Therefore the statements are slightly stronger than if they were completely in terms of I or I_2. It is easy to show that if G is a graph for which $m(G) < 9$ but G has a triangle, then $I(G) \leq 10m(G)/9$.

As mentioned earlier, Andreae and Aigner showed that if $\kappa' \geq 4$, then $I \leq m + 1$. We will contrast that with $\kappa = 3$ by showing that there exists $\varepsilon > 0$ and an infinite set of 3-connected graphs for which $I > (1 + \varepsilon)m$.

We now begin the discussion of Theorem III.1.1. Phrasing it in terms of trail covers, we want to show that $t(G) \leq m(G)/9$ for 2-edge-connected graphs, and we want to construct triangle-free 2-connected graphs for which $t = \lfloor m/9 \rfloor$.

We first establish that Theorem III.1.1 is best possible and at the same time eliminate staple graphs from consideration for violating the upper bound.

Let G be a 2-connected staple graph and let G' be the underlying multigraph of G. Let n_i be the number of vertices of G' that have degree i. Then $m(G) = \frac{3}{2}\sum_{i=1}^{n} in_i$ and $t(G) = \frac{1}{2}\sum_{i \text{ is odd}} n_i$. Since G is 2-connected, $n_1 = 0$ and it quickly follows that t/m is maximized when $n_3 = n(G')$. In this case,

$t = m/9$.

Therefore, the staple graph of any 3-regular 2-connected multigraph shows that Theorem III.3.1 is best possible; these are the only staple graphs achieving the bound and none exceeds it. An example appears below.

Figure III.3.1

These graphs are all 2-connected, and therefore Corollary III.3.2 follows from Theorem III.3.1. Subdividing any edge of such a graph gives the best possible result for graphs of other residue classes of nine. Furthermore, we may now exclude staple graphs when considering the upper bound.

A **unit** is a maximal induced 2-edge-connected subgraph. A **trivial** unit is one with no edge, a **non-trivial** unit is one with at least one edge, a **small** unit is a non-trivial unit with at most eight edges, and a **large** unit is one with at least nine edges. An **interunit edge** is an edge whose endpoints are in different units.

Note that units are vertex disjoint. Moreover, there is at most one interunit edge between any two units.

For a graph G, the **unit graph** $U(G)$ is defined by $V(U(G))$ is the set of units and, for $X,Y \in V(U)$, $X \leftrightarrow Y$ if and only if there is an interunit edge between X and Y.

Lemma III.3.3. For a connected graph G, $U(G)$ is a tree.

Proof. Since G is connected, U is connected. If $XY \in E(U(G))$ is in some cycle, then the vertex-sets corresponding to the units of this cycle form an induced 2-edge-connected graph, contradicting the fact that each of these vertex-sets induces a maximal 2-edge-connected subgraph. ♠

We will prove Theorem III.3.1 inductively. As in §III.2, we let $\eta(G) = 2m(G) - n(G)$. Let $\mathcal{F}$ be the family of graphs with minimal $\eta(G)$ among the 2-edge-connected graphs for which $t(G) > m(G)/9$, i.e., $\mathcal{F}$ is the set of "η-minimal" counterexamples. We will establish properties of graphs in $\mathcal{F}$ and eventually show that $\mathcal{F} = \emptyset$.

Note that if H is a 2-edge-connected graph with at most eight edges, then $t(H) = 1$. Hence, if H is 2-edge-connected, then $m(H) < m(G)$, and $t(H) = t(G)$, then G cannot be a minimal member of $\mathcal{F}$.

An edge uv is a **link** if both of its vertices are bivalent. In a staple graph, every edge is either a link or it is incident to a link.

Lemma III.3.4. If $G \in \mathcal{F}$, then G has the following properties:

(i) G has no incident links.
(ii) G has no pendant cycle.
(iii) For any $e \in E(G)$, $G - e$ is not 2-edge-connected.
(iv) If $u \in V$ and $d(u) = 2$, then $G - u$ is not 2-edge-connected.
(v) G is not a staple graph.
(vi) G has an edge that is neither a link nor is incident to a link.

Proof. For (i) through (iv), we modify G to obtain a graph that contradicts the minimality of G. For (i), if e and e' are incident links, then contract e. This decreases m but not t. For (ii), if G has a pendant cycle C, then subdivide one of the edges that is incident to the base but not in C and remove C, leaving its base. This decreases m without decreasing t. For (iii), snip e, and for (iv), double snip the edges incident to u.

We have already shown (v) and (vi) is just a restatement of the fact that G is not a staple graph.♠

We stated before that in a staple graph, every edge is either a link or it is incident to a link. It is easy to see that the reverse is almost true; if a graph has this property and satisfies Lemma III.3.4(i) then the branchpoints on any trail must occur exactly three apart, and G is a staple graph.

Lemma III.3.5. If $G \in \mathcal{F}$ and $e \in E(G)$, then $U(G - e)$ is a path whose ends are units containing the endpoint of e.

Proof. Let $G' = G - e$. By Theorem III.3.4(iii), G is not 2-edge-connected and so $U(G')$ is non-trivial. Since G is 2-edge-connected, G' is connected and, by Lemma III.3.3, $U(G')$ is a tree. If we add edges to G' to make it 2-edge-connected, then we will need an endpoint of one edge to be in each unit that corresponds to a leaf of G'. Since $G' \cup e$ is 2-edge-connected, e has an endpoint in each leaf of $U(G')$. Hence there must be only two leaves of $U(G')$ and $U(G')$ is a path whose ends correspond to units containing the endpoints of e.♠

Lemma III.3.6. If $G \in \mathcal{F}$, then there exists $e \in E(G)$ such that $G - e$ has at least two non-trivial units.

Proof. We first show that if $G - e$ has at most one non-trivial unit, then e is a link or is incident to a link. Let $G' = G - e$ and $e = uv$. Since $G - e$ is not 2-edge-connected, $A \neq B$. By hypothesis, at most one of A and B is non-trivial. Let A be the unit of G' that contains u and B be the unit of G' that contains v. If both A and B are non-trivial, then we are done. If they are both trivial, then uv is a link and again we are done. Hence we may assume that A is trivial and B is not.

Since $U(G')$ is a path with one of the endpoints $A = \{u\}$, u has only two neighbors in G, one of which is v; call the other one w. Let C be the unit of G' that contains w. If $C = B$, then $d(u) = 2$ and $G - u$ is 2-edge-connected, violating Lemma III.3.4(iv). Hence $C \neq B$. If C is non-trivial, then B and C are two non-trivial units of G. If C is trivial, then e is incident to the link uw.

Now if the lemma is false then, for every e, $G - e$ has at most one non-trivial unit and, by the above, every edge is a link or is incident to a link. But this violates Lemma III.3.4(vi). ♠

Now fix $G \in \mathcal{F}$, and let e be an edge of G such that $G - e$ has at least two non-trivial units. From Lemma III.3.5, G looks like a "cycle" of non-trivial units of $G - e$, where the "edges" are paths. We illustrate this below; each gray circle represents a non-trivial unit.

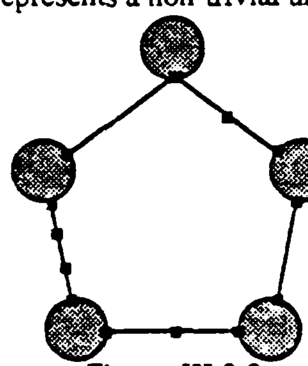

Figure III.3.2

We call the paths between the non-trivial units **bridges**. From Lemma III.3.4(i), bridges can have length one, two, or three; we call these **short**, **medium**, and **long** bridges respectively. We call the endpoints of the bridges **terminals**.

Let k be the number of non-trivial units of $G - e$. For what follows, do all arithmetic modulo k. Label the units $A_0, \ldots A_{k-1}$ and bridges $B_0, \ldots B_{k-1}$ so that, for any i, the bridge B_i is between units A_i

and A_{i+1}. Suppose that B_i has p_i vertices and label these $u_{i,j}$, where $j = 0,\ldots,p_i$ so that $u_{i,0} \in V(A_i)$, $u_{i,j} \leftrightarrow u_{i,j+1}$, and $u_{i,p_i} \in V(A_{i+1})$. Note that $1 \le p_i \le 3$.

From Lemma III.3.4(iii), no cycle has a chord. In particular, no unit of G - e has a cycle that has a chord. With this restriction, we illustrate all possible small units in Figures III.3.3.

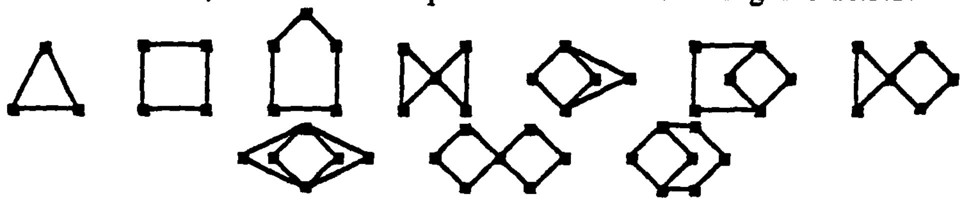

Figure III.3.3(a)

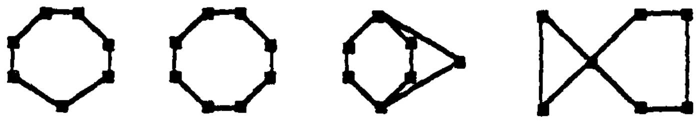

Figure III.3.3(b)

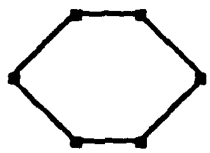

Figure III.3.3(c)

Lemma III.3.7. If A is a unit of G - e, then either A is large or $A = C_6$ with terminals at opposite vertices.

Proof. If A appears in Figure III.3.3(a), then no matter what the terminals of A are, it is possible to cover all edges of A with a trail that starts at one terminal and ends at the other. Form G' by replacing the bridges on either side of the A, together with A itself by a long bridge. This does not decrease t and it does decrease m. By the minimality of G, we can eliminate all of the graphs of Figure III.3.3(a) from consideration as units of G. We can use the same argument if $A = C_6$ and the terminals are not opposite.

If $A = C_6$ and the terminals are opposite, then every trail cover of G ends within A; i.e., it is impossible to cover the edges with a single trail that starts at u and ends at v. Now suppose that A appears in Figure III.3.3(b). Replace A by C_6 with opposite terminals and call the resulting graph G'. By the minimality of G, $t(G') \le m(G')/9$. Replace the trail T (in G') that starts within the 6-cycle by a trail (in G) that starts at an appropriate vertex within A, visits every vertex of A and leaves at the same terminal that T does. This demonstrates that $t(G') \ge t(G)$, contradicting the minimality of G. ▲

If A_i is a small unit of $G - e$, and A_{i-1} and A_{i+1} are both large units of $G - e$, then we say that A_i is an **isolated small** unit of $G - e$.

We now group edges into **packets.** The edges of some of the long bridges will be in no packet but each of the other edges will be in exactly one packet. Furthermore, no packet will contain edges from two different large units of $G - e$.

We start by describing the **cores** of the packets. The cores are subsets of packets that determine the packets and there is one core for each packet. Each large unit is the core of one packet. If A_{i-1} is large, $A_i, \ldots, A_{i+\upsilon}$ are small, and $A_{i+\upsilon+1}$ is large, then $E(A_i) \cup E(B_i) \cup E(A_{i+1})$ is the core of one packet, $E(A_{i+2}) \cup E(B_{i+2}) \cup E(A_{i+3})$ is the core of one packet, etc.. The last core from these $\upsilon + 1$ units contains three units and two bridges if $\upsilon + 1$ is odd and two units and one bridge if $\upsilon + 1$ is even. If a small unit is isolated but one of the incident bridges is long, then the unit and the long bridge form the core of a packet.

One characteristic of packets that will be needed is that they contain at least nine edges. The packets that have been defined so far comfortably satisfy this requirement and we call them **ample.** The remaining packets are created from some of the isolated small units and we must work harder to ensure that they contain nine edges. We call these packets **scant.**

If A_i is an isolated small unit, neither B_{i-1} nor B_i is long, and at least one of $\{B_{i-1}, B_i\}$ is medium, then we make a scant packet from the medium bridge, A_i, and the edge incident to A_i from the other bridge.

We now assign most of the remaining edges to ample packets. These edges are called **supplementary.** If A_i is a unit whose edges are not assigned to any packet, then it is an isolated small unit, both B_{i-1} and B_i are short, and both A_{i-1} and A_i are large. Assign $E(B_{i-1}) \cup E(A_i) \cup E(B_i)$ to the packet that contains A_{i-1}. If uv is part of a short or medium bridge, and uv has not been assigned to any packet, then at least one of $\{u,v\}$ is in an ample packet. If u is in an ample packet A, then assign uv to A. Otherwise, assign uv to the packet containing v.

This completes the assignment of edges to packets and hence the definition of packets. The definition is not precise since there is more than one way of assigning edges to packets: we will just assume that some assignment that agrees with the above rules is given. The **boundary points** of a packet are the vertices of the packet that are farthest counterclockwise and clockwise in the cycle of units. Suppose that A is some packet. For some i, A contains all of the edges of A_i and none of the edges of A_{i-1}. We define the **low** boundary point to be $u_{i-1,j}$ where $j = min\{q : u_{i-1,q} \in V(A)\}$. For some i, A contains all of the edges of A_i and none of the edges of A_{i+1}. We define the **high** boundary point to be $u_{i,j}$ where $j = max\{q : u_{i,q} \in V(A)\}$. The **spans** are the paths (some of them have no edges and some of them have three edges) between the packets. We call the spans of length three **great**.

In Figure III.3.4, we give an example of a cycle of units. The large circles represent large units and the small circles represent small units. Our choice of packets is designated by the polygons. The polygon corresponding to the only scant packet is white and the polygons corresponding to the ample packets are light gray. The boundary points of the packets are designated with large dots.

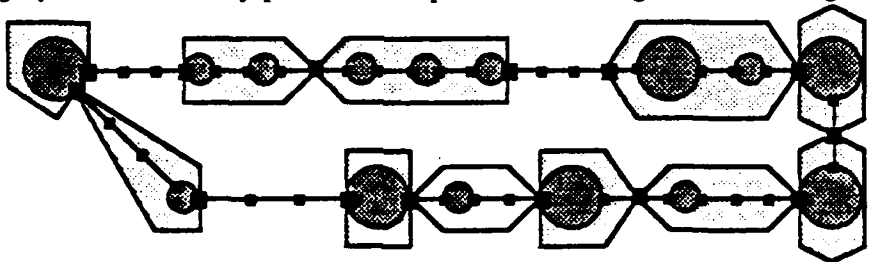

Figure III.3.5

Theorem III.3.8. If A is an ample packet and $m(A) \geq 9$, then $t(A) \leq m(A)/9$.

Proof. If the core of A consists of one (resp. two, three) small units, then it has at least nine (resp. thirteen, twenty) edges and can be covered with one (resp. one, two) trails in such a way that these trails contain both boundary points of the packets. Therefore, if A has any supplementary edges, then they are immediately covered.

If A contains a large unit A, then there are two possible sources of supplementary edges. Suppose that we have eight supplementary edges from a small unit and two short bridges. This is depicted in Figure III.3.5(a).

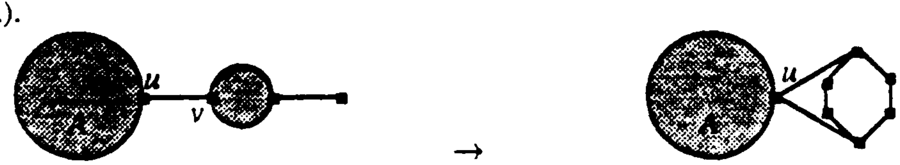

Figure III.3.5(a): A Figure III.3.5(b): A^*

Note that $m(A^*) = m(A)$ and A^* is 2-edge-connected. By the minimality of G, A^* can be covered with $m(A^*)/9$ trails. Moreover, since it is impossible to cover the new edges with a trail that starts and ends at u, we are forced to start a trail T within the graph induced by the new edges. We may assume that T continues to u and then (if necessary) into A. Replace T by a trail in A that starts at v, goes around the 6-cycle and then follows the subtrail of T that is in A. Thus we have a trail cover of A that has at most $m(A)/9$ trails.

Suppose now that A has a supplementary edge uv from a short or medium bridge, and that uv is incident to a large unit A, as in Figure III.3.6(a).

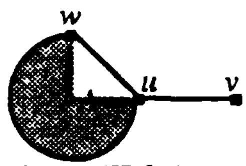

Figure III.3.6(a): A

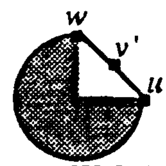

Figure III.3.6(b): A^*

We subdivide an edge uw where $w \neq v$ and remove uv to obtain A^*. By the minimality of G and the 2-edge-connectedness of A^*, $t(A^*) \leq m(A^*)/9 = m(A)/9$ and, from the "snipping lemma" (Lemma I.6.2(iii)) we have $t(A) \leq t(A^*)$.

We can combine these two techniques to get a trail cover of A with the required number of trails ♠.

If there are no great spans, then every edge is in some packet and we can apply Theorem III.3.8 to each packet, take the union of the resulting trail covers, and obtain $t(G) \leq m(G)/9$. We need a lemma to deal with great spans.

Lemma III.3.9. Suppose that $G_0,\ldots,G_{j-1}$ are graphs with disjoint vertex-sets, and that, for each j, $\langle u_j,v_j,w_j,x_j\rangle$ is a trail within G_j and v_j and w_j are bivalent. Let $H = \cup_{j=0}^{r-1} G_i$ and $H' = H - \{u_jv_j : j = 0,\ldots,r-1\} \cup \{v_ju_{j+1} : j = 0,\ldots,j-1\}$. Then $t(H') \leq t(H)$.

Proof. Note that $t(H) = \sum_{j=0}^{r-1} t(G_j)$. From Lemma I.6.2(ii), for each j, there is an optimal trail cover $\mathcal{T}_j$ of G_j that contains a trail T_j that has a subtrail $\langle u_j,v_j,w_j,x_j\rangle$. When we remove each u_jv_j, the total number of trails increases by at most r (fewer if some T_j is closed).

Now if we can take two ends u and v of different trails and make the trails into one trail by adding the edge uv to one trail and then concatenate these trails, we decrease the number of trails by one. We do this $r - 1$ times by adding the edges $\{v_ju_{j+1} : j = 0,\ldots,r-2\}$. The proof will be done if adding the edge $v_{r-1}u_0$ also decreases the number of trails by one.

If it does not, then v_{i-1} and u_0 are in the same trail so that adding the edge $v_{r-1}u_0$ simply makes a closed trail from an open trail. But if this is the case, then each T_j was a closed trail and removing the edges u_jv_j did not increase the number of trails at all. Then in this case, $t(H') \leq t(H) - (r - 1)$.♠

Suppose that there are r packets. For what follows, do all arithmetic modulo r. Label the packets $A_0, \ldots A_{r-1}$ so that, for any j, there is a span C_j between packets A_j and A_{j+1}. Let u_j be the high boundary point of A_j and x_j be the low boundary point of A_j so that C_j is a path between u_j and x_{j+1}.

If there are at least two long bridges, then let them be C_1 and C_s. Form the graph G' as follows. Delete the edges of C_1 and C_s. Insert a path of length three between u_1 and x_{s+1}, and another path of length three between x_2 and u_s.

This graph has the same number of edges as G and it has two components G'_1 and G'_2, each of which has at least nine edges and is 2-edge-connected. By the minimality of G, for $l = 1,2$, there exists a trail cover T_l of G'_l with $m(G'_l)/9$ trails. We now apply Lemma III.3.9 to G'_1 and G'_2 to obtain $t(G) \leq m(G'_1)/9 + m(G'_2)/9 = m(G)/9$.

If there is exactly one great span, then we must look more closely at the individual packets. Let $\rho_j \equiv m(A_j)(mod\ 9)$. By Theorem III.3.8, we can cover the packets with $\frac{m(G) - 3 - \sum_{i=1}^{r}\rho_j}{9}$ trails so that if $3 + \sum_{i=1}^{k}\rho_j \geq 9$, we can use a single trail for the great span and still have at most $\frac{m(G)}{9}$ trails.

If $3 + \sum_{j=1}^{k}\rho_j < 9$, then, for each j, $\rho_j \leq 5$ and we can add three edges to A_j without increasing the bound for covering A_j. We can then discard the great span and insert paths of length three between the low and high terminals of each packet and apply Lemma III.3.9. Then contract the newly created great spans and we have a sufficiently small trail cover. This procedure is illustrated in Figure III.3.7.This completes the proof of Theorem III.3.1.♠

Two-edge-connected 3-regular staple graphs show that there exists $\varepsilon > 0$ and an infinite set of graphs such that $l/m \geq (1 + \varepsilon)$. Since 4-edge-connected graphs have a Hamiltonian path, $t = 1$ for these graphs and the same cannot be said for them. There are two intermediate classes. These are 3-edge-connected and 3-connected. We now describe examples to show that these are more like the 2-connected graphs.

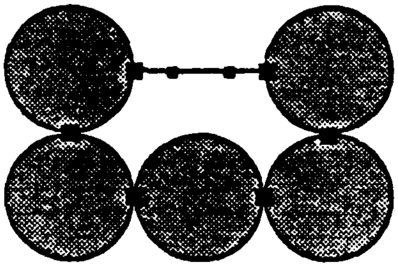

Figure III.3.7(a)

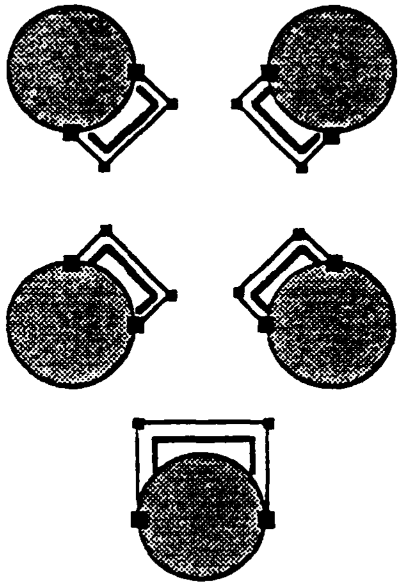

Figure III.3.7(b).

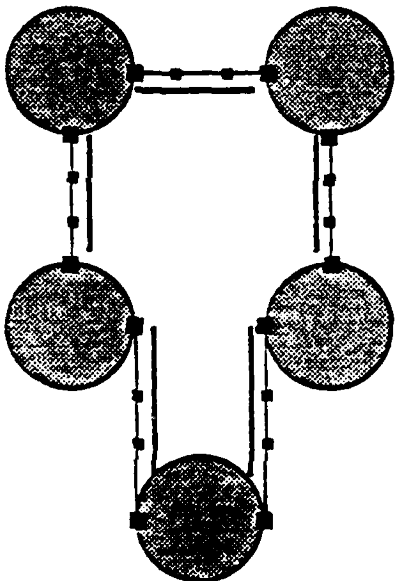

Figure III.3.7(c)

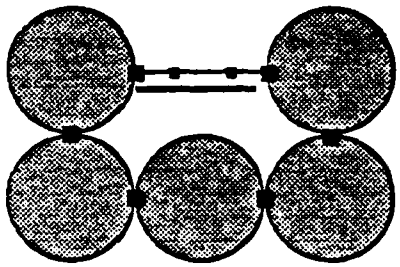

Figure III.3.7(d)

Since our ultimate focus is the total interval number and not just the trail cover number, we must select sequences such that the graphs are triangle-free and so $I = m + t$.

Recall that the Petersen graph, shown in Figure III.3.8(a), is 3-connected, 3-regular, symmetric, and not Hamiltonian. Select one vertex u. Replace each vertex $v \neq u$ with the graph H as in Figure I.6.2. and call the resulting graph G'.

Now suppose that T covers G' and starts and ends at u. Since there are only three edges from any copy of H to the rest of the graph, T cannot enter and leave a copy of H more than once. Because of the edges within the copy, T must enter it. By contracting all of the edges within each copy of H, and contracting the corresponding edges of T, we get a Hamiltonian cycle of the Petersen graph and this is impossible.

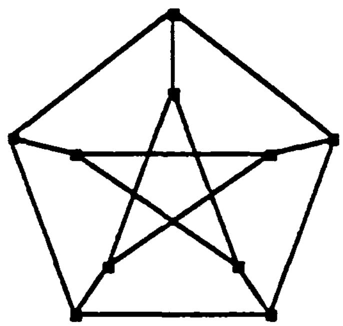

Figure III.3.8(a)

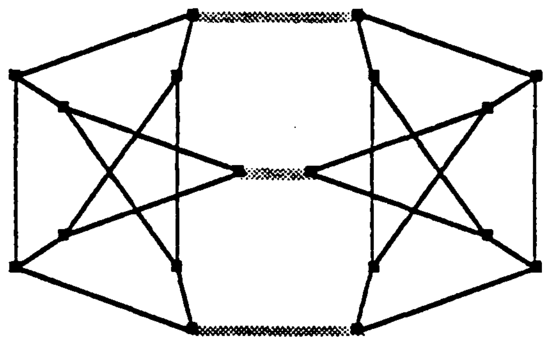

Figure III.3.8(b)

Let G_k be the graph that consists of $2k - 1$ copies of G' with the vertices u identified. From the above, no trail can enter a copy of G', cover it, and leave it. If we try to use two trails to cover a copy of G', then the second trail uses up the last edge in G' that is incident to u on its way in and therefore it must end inside the copy. We have now shown that there is at least one endpoint of some trail in each copy and so we need at least k trails to cover G_k.

Since $m(H) = 9$, $m(G') = 15 + 9 \cdot 9$. Then since $m(G_k) = (2k - 1)m(G')$, $m(G_k) = 192k - 96$ and we have a sequence of graphs for which $l/m = \frac{193k - 96}{192k - 96} \geq 1 + 1/192$. We say that, in this construction, it takes 192 edges to force another trail. We do not claim that this is the most efficient use of edges for forcing additional trails.

We now discuss a construction of 3-connected graphs. Consider the Thomassen graph, shown in Figure III.3.8(b). Replace *each* vertex by a copy of H, subdivide the three edges that are in no 5-cycle (drawn with thick gray lines), and call the resulting graph G'. Take $2k$ copies of G' and call them $H_0,\ldots,H_{2k-1}$. For each $i = 0,\ldots,2k - 1$, let u_i, v_i, and w_i be the three bivalent vertices. Taking all subscripts modulo $2k$, add edges $\{u_i v_{i+1} : i = 0,\ldots,2k - 1\}$ and $\{w_i w_{i+k} : i = 0,\ldots,k - 1\}$. Call this graph G_k; it is 3-connected. Moreover, it is 3-regular so $m = 3n/2$. Since $n = 2k[18 \cdot 7 + 3]$, we have $m = 387k$.

We now show that each H_i has *two* endpoints of a trails from any trail cover. If not, then we may assume that, for some trail cover, the left side of G_0 has no endpoint. Since there are only three edges into this half, it is impossible for two trails to enter *and* leave and therefore some trail T enters the left half, covers all of its edges and leaves. If we take the portion of T that is in the left half, contract all of the edges of any copy of H, and then identify the three vertices marked x, x', and x'', then we have a Hamiltonian cycle of the Petersen graph and this is impossible. Therefore, we have at least two end-points in each H_i and so we have at least $2k$ trails. Since G_k has $387k$ edges, we have a sequence of graphs for which $l/m = 389k/387k \geq 1 + 1/193.5$. Here it takes 193.5 edges to force each trail, just slightly more than it took for our 3-edge-connected construction.

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