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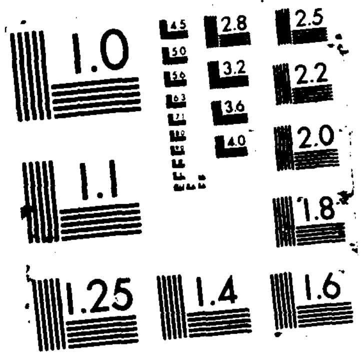
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ABSTRACT

This note extends the Proschan (1965) result on peakedness comparison for convex combinations of i.i.d. random variables from a PF_2 density. Now the underlying random variables are jointly distributed from a Schur-concave density. The result permits a more refined description of convergence in the Law of Large Numbers.

1. Introduction

Proschan (1965) shows that:

1.1 Theorem. Let f be PF_2, $f(t) = f(-t)$ for all t, $X_1, \ldots, X_n$ independently distributed with density f, $\underset{\sim}{p} \overset{m}{\geq} \underset{\sim}{p}'$, $\underset{\sim}{p}$, $\underset{\sim}{p}'$ not identical, $\sum_1^n p_i = 1 = \sum_1^n p_i'$. Then $\sum_1^n p_i' X_i$ is strictly more peaked than $\sum_1^n p_i X_i$.

(Definitions of majorization ($\underset{\sim}{p} \overset{m}{\geq} p'$), PF_2 density, and peakedness are presented in Section 2.)

The Law of Large Numbers asserts that the average of a random sample converges to the population mean under certain conditions. Roughly speaking, Theorem 1.1 states that a weighted average of i.i.d. random variables converges more rapidly in the case in which weights are close together as compared with the case in which the weights are diverse.

In the present note, we extend the basic univariate result to the multivariate situation in which the underlying random variables have a joint Schur-concave density. Theorem 2.4 presents the precise statement of the multivariate extension.

2. Peakedness comparisons

The theory of majorization is exploited in this section to obtain more general versions of the result of Proschan (1965). We begin with some definitions. The definition of peakedness was given by Birnbaum (1948).

Definition 2.1. Let X and Y be real valued random variables and a and b real constants. We say that X is more peaked about a than Y about b if

$$P(|X-a| \geq t) \leq P(|Y-b| \geq t)$$

for all $t \geq 0$. In the case $a = 0 = b$, we simply say that X is more peaked than Y.

Next we define the ordering of majorization among vectors.

Definition 2.2. Let $a_1 \geq \ldots \geq a_n$ and $b_1 \geq \ldots \geq b_n$ be decreasing rearrangements of the components of the vectors a and b. We say that a majorizes b (written $a \overset{m}{\geq} b$) if

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$$

and

$$\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i \qquad \text{for } k = 1, \ldots, n-1.$$

Definition 2.3. A real valued function f defined on R^n is said to be a Schur-concave function if $f(a) \leq f(b)$ whenever $a \geq b$.

A nonnegative function f on $(-\infty,\infty)$ is called a Pólya frequency function of order 2 (PF_2) if log f is concave. If f is a PF_2 function then $\phi(x) = \prod f(x_i)$ is Schur-concave. Thus the random vector $x = (X_1,\ldots,X_n)$ has a Schur-concave density under the conditions of Theorem 1.1. A function f defined on R^n is said to be sign-invariant if $f(x_1,\ldots,x_n) = f(|x_1|,\ldots,|x_n|)$. In the following theorem, we give a peakedness comparison for random variables with a sign-invariant Schur-concave density.

Theorem 2.4. Suppose the random vector $X = (X_1,\ldots,X_n)$ has a sign invariant Schur-concave density. Then for all $t \geq 0$,

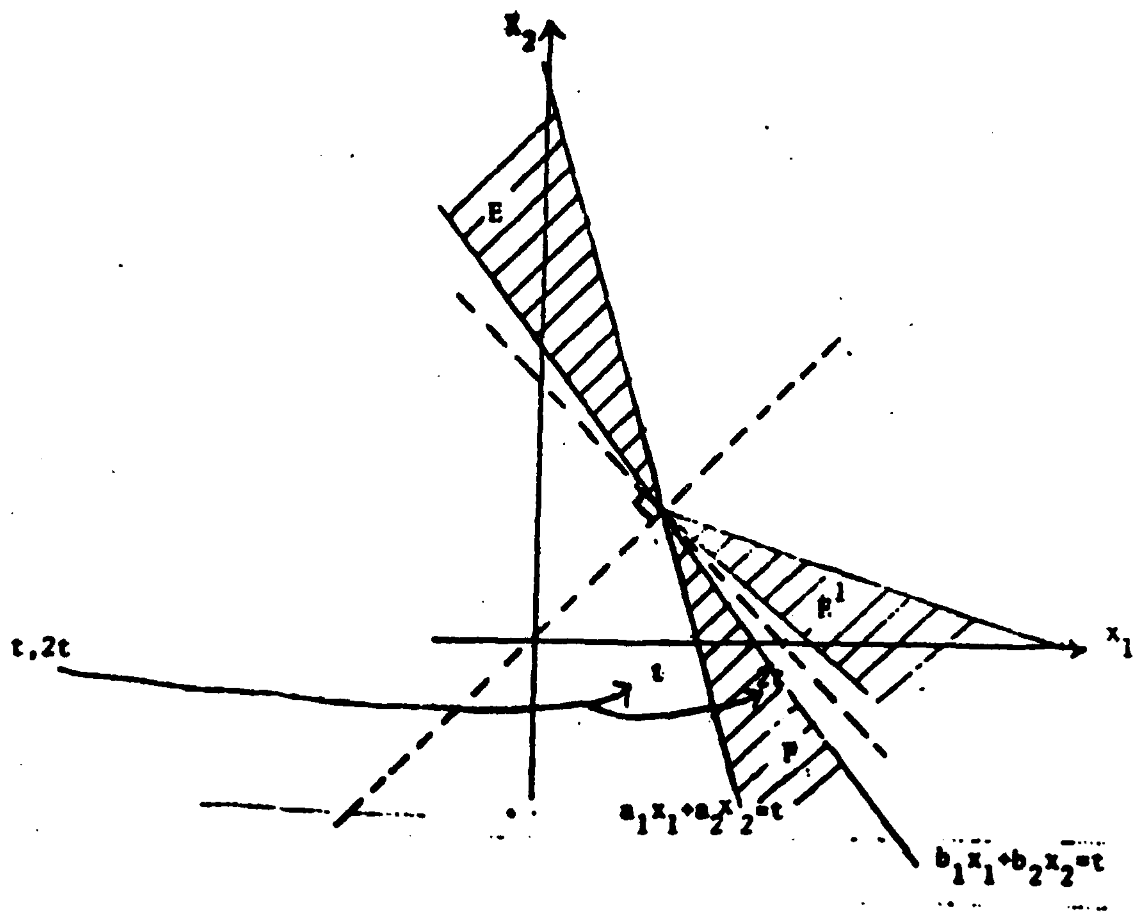
$$\psi(a_1, \ldots, a_n) = P\left(\sum a_i X_i \leq t\right)$$

is a Schur-concave function of $\underset{\sim}{a} = (a_1, \ldots, a_n)$, $a_i \geq 0$ for all i. Equivalently, $\sum b_i X_i$ is more peaked than $\sum a_i X_i$ whenever $\underset{\sim}{a} \overset{m}{\geq} \underset{\sim}{b}$.

Proof.

Without loss of generality, we may assume that $\sum a_i = 1$. We first consider the case $n = 2$. Let $\underset{\sim}{a} = (a_1,a_2)$, $\underset{\sim}{b} = (b_1,b_2)$, $\underset{\sim}{a} \overset{m}{\geq} \underset{\sim}{b}$. Since X_1,X_2 are exchangeable, we may further assume that

$a_1 > b_1 \geq 1/2 \geq b_2 > a_2$. To show that $P(a_1X_1+a_2X_2 \leq t) \leq P(b_1X_1+b_2X_2 \leq t)$ for $t \geq 0$, consider the following diagram:



Since $a_1 > b_1 \geq 1/2$, both lines intersect the x_1-axis in the interval [t,2t] and they intersect the 45 degree line at the point (t,t) ($a_1+a_2 = b_1+b_2 = 1$). We must show that $P(E) \leq P(F)$. Now reflect E across the 45 degree line to form the wedge E´. Then $P(E) = P(E')$ because the joint density f is invariant under permutation. For $k \geq 0$, the line $x_1 - x_2 = k$ intersects E´ at the line segment joining $(t+b_1k, t-b_2k)$ and $(t+a_1k, t-a_2k)$, and it intersects F at the line segment joining $(t+a_2k, t-a_1k)$ and $(t+b_2k, t-b_1k)$. Note that both segments are of equal length. But f sign-invariant and Schur-concave implies that

$$\begin{aligned} f(t+b_1k, t-b_2k) &= f(t+b_1k, b_2k-t) \\ &\leq f(t+b_2k, b_1k-t) \\ &= f(t+b_2k, t-b_1k). \end{aligned}$$

This last fact then clearly implies that $P(E') \leq P(F)$ by conditioning on $X_1 - X_2$.

The result for $n \geq 3$ now follows since

$$\begin{aligned} &P(\Sigma a_i X_i \leq t) \\ &= E\,[P(a_1X_1 + a_2X_2 \leq t - \sum_3^n a_iX_i \mid X_3, \ldots, X_n)] \end{aligned}$$

and the conditional density $f(x_1, x_2 \mid x_3, \ldots, x_n)$ is also Schur-concave and sign invariant.

For an example of a Schur-concave density function that is also sign-invariant, consider the multivariate Cauchy density:

$$f(x_1, \ldots, x_n) = \pi^{-(n+1)/2}\Gamma((n+1)/2)(1+\sum_{i=1}^n x_i^2)^{-(n+1)/2}$$

The following result is an immediate consequence of Theorem 2.4.

Corollary 2.5. Let $X_1, \ldots, X_n$ be random variables with joint Schur-concave sign-invariant density f. Then $\frac{1}{k}\sum_{i=1}^{k} X_i$ is increasing in peakedness as k increases from l to n.

Proof.

Let $\underset{\sim}{a}_1 = (1,0, \ldots 0)$, $\underset{\sim}{a}_2 = (\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0), \ldots,$ and $\underset{\sim}{a}_n = (\frac{1}{n}, \ldots, \frac{1}{n})$, where each vector contains n components. Then $\underset{\sim}{a}_1 \overset{m}{\geq} \underset{\sim}{a}_2 \overset{m}{\geq} \ldots \overset{m}{\geq} \underset{\sim}{a}_n$. The result follows from Theorem 2.4.

Suppose $\underset{\sim}{X} = (X_1,\ldots,X_n)$ and $\underset{\sim}{Y} = (Y_1,\ldots,Y_n)$ are independently distributed with respective densities f and g where both f and g are Schur-concave and sign invariant. Then Theorem 2.4 implies that $\sum b_i (X_i + Y_i)$ is more peaked than $\sum a_i (X_i + Y_i)$ whenever $\underset{\sim}{a} \overset{m}{\geq} \underset{\sim}{b}$. This is true because the convolution of Schur-concave functions is Schur-concave. However, if $Y_1, \ldots, Y_n$ are i.i.d. Cauchy, then the joint density g given by

$$(2.1) \qquad g(y_1,\ldots,y_n) = (\frac{a}{\pi})^n \prod_{i=1}^{n} (1+ a^2 y_i^2)^{-1}, a > 0,$$

is not Schur-concave. In Theorem 2.6 below, we show that $\sum b_i (X_i + Y_i)$ is more peaked than $\sum a_i (X_i + Y_i)$ whenever $\underset{\sim}{a} \overset{m}{\geq} b$. This result identifies a different class of densities for which the conclusion of Theorem 2.4 holds.

Theorem 2.6. Suppose that the random vector $\underset{\sim}{X} = (X_1,...,X_n)$ has a sign-invariant Schur-concave density f. Let $Y = Y_1, ..., Y_n$ be i.i.d. Cauchy with joint density g as given in (2.1). Let $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ be independent, and $\underset{\sim}{a} \overset{m}{\geq} \underset{\sim}{b}$ where $a_i \geq 0$, $b_i \geq 0$ for all i and $1 = \sum_1^n a_i = \sum_1^n b_i$. Then $\sum_1^n b_i (X_i + Y_i)$ is more peaked than $\sum_1^n a_i (X_i + Y_i)$.

Proof.

Since f is sign invariant both $\sum_1^n a_i X_i$ and $\sum_1^n b_i X_i$ are symmetric random variables. We use the fact that $\sum_1^n a_i Y_i$, $\sum_1^n b_i Y_i$ have the same distribution as does Y_1. The result now follows Theorem 2.4 and the Lemma of Birnbaum (1948) by noting that Y_1 has a symmetric and unimodal density.

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[1] Birnbaum, Z.W. (1948). On random variables with comparable peakedness. Ann. Math Statist. 19 76-81.

[2] Proschan, F. (1965). Peakedness of distributions of convex combinations. Ann. Math. Statist. 36 1703-1706.

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