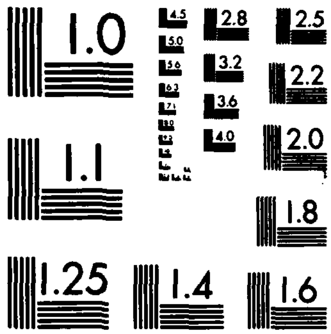
AD-A184 436 A REFINED NONLINEAR ANALYSIS OF LAMINATED COMPOSITE PLATES AND SHELLS(U) VIRGINIA POLYTECHNIC INST AND STATE UNIV BLACKSBURG DEPT OF E J N REDDY AUG 87 1/1

UNCLASSIFIED ARO-21203 7-MA DAAG29-85-K-0007 F/G 20/11 NL




MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

# REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

| | | |
|---|---|---|
| 1. REPORT NUMBER<br>ARO 21203.7-MA | 2. GOVT ACCESSION NO.<br>N/A | 3. RECIPIENT'S CATALOG NUMBER<br>N/A |
| 4. TITLE (and Subtitle)<br>A REFINED NONLINEAR ANALYSIS OF LAMINATED COMPOSITE PLATES AND SHELLS | | 5. TYPE OF REPORT & PERIOD COVERED<br>FINAL TECHNICAL REPORT (1985-1987) |
| | | 6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s)<br>J. N. Reddy | | 8. CONTRACT OR GRANT NUMBER(s)<br>DAAG29-85-K-0007 |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS<br>Department of Engineering Science & Mechanics<br>Virginia Polytechnic Institute and State Univ.<br>Blacksburg, VA 24061 | | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS<br>21203-MA |
| 11. CONTROLLING OFFICE NAME AND ADDRESS<br>U. S. Army Research Office<br>Post Office Box 12211<br>Research Triangle Park, NC 27709 | | 12. REPORT DATE<br>August 1987 |
| | | 13. NUMBER OF PAGES<br>50 |
| 14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) | | 15. SECURITY CLASS. (of this report)<br>Unclassified |
| | | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

DTIC ELECTE OCT 20 1987 S D D

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

NA

18. SUPPLEMENTARY NOTES

The view, opinions, and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Analytical solutions, Finite element analysis, higher-order theories, laminated composite plates and shells, moderate rotation theory, von Karman nonlinearity, shear deformation theories.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This final technical report summarizes in a compact form the results of a two-year research program on the development of refined shear deformation theories of plates and shells, and their analytical solutions. The detailed results are reported in various reports and technical papers during the course of the project. A third-order, nonlinear shear deformation shell theory and finite element model that accounts for parabolic distribution of transverse shear stresses through thickness and the von Karman nonlinear strains was

DD FORM 1473 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

AD-A184 436

DTIC FILE COPY

developed during an investigation sponsored by NASA Langley Research Center during the first year of this research. The theory is further refined and extended and analytical solutions are developed during this research. The most significant contributions of this research are:

1. The development of analytical solutions of the linear theory for various boundary conditions. The space-time concept is used to develop such solutions.

2. The development of moderate rotation theory of laminated composite plates.

3. The development of finite element models for first-ply and post-first-ply failure analysis.

Future research related to refined theories should consider the effects of moderate and large rotations and the development of micro- and macromechanical constitutive models for a better understanding of the structural behavior and prediction of progressive damage and failure in composite structures.

ARO 21203.7-MA

A REFINED NONLINEAR ANALYSIS OF
LAMINATED COMPOSITE PLATES AND SHELLS

FINAL REPORT
Submitted to

ARMY RESEARCH OFFICE
Mathematical Sciences Division
P.O. Box 12211
Research Triangle Park, NC 27709
(Grant DAAG 29-85-K-00017)
ARO Proposal NO: 21203-MA

by

J. N. Reddy
Department of Engineering Science and Mechanics
Virginia Polytechnic Institute and State University
Blacksburg, VA 24061

Report NO. VPI-E-87-19

August 1987

Approved for public release; distribution unlimited.

## TABLE OF CONTENTS

## Preface

The research summarized herein was conducted during the period 1985-87 under Grant DAAG 29-85-K-00017 to Virginia Polytechnic Institute and State University from the Mathematical Sciences Division of the Army Research Office (ARO).

The author gratefully acknowledges the encouragement and support of Dr. Jagdish Chandra of ARO. Also, the author is pleased to acknowledge the technical contributions to this research made by several graduate students, whose names are listed in the Scientific Personnel of this report.

Finally, it is a great pleasure to acknowledge the prompt and skillful typing by Mrs. Vanessa McCoy of many manuscripts prepared under the grant.

# A REFINED NONLINEAR ANALYSIS OF LAMINATED COMPOSITE PLATES AND SHELLS

J. N. Reddy
Department of Engineering Science and Mechanics
Virginia Polytechnic Institute and State University
Blacksburg, VA 24061

## SUMMARY

This final technical report summarizes in a compact form the results of a two-year research program on the development of refined shear deformation theories of plates and shells, and their analytical solutions. The detailed results are reported in various reports and technical papers during the course of the project. A third-order, nonlinear shear deformation shell theory and finite element model that accounts for parabolic distribution of transverse shear stresses through thickness and the von Karman nonlinear strains was developed during an investigation sponsored by NASA Langley Research Center during the first year of this research. The theory is further refined and extended and analytical solutions are developed during this research. The most significant contributions of this research are:

1. The development of analytical solutions of the linear theory for various boundary conditions. The space-time concept is used to develop such solutions.
2. The development of moderate rotation theory of laminated composite plates.
3. The development of finite element models for first-ply and post-first-ply failure analysis.

Future research related to refined theories should consider the effects of moderate and large rotations and the development of micro- and macromechanical constitutive models for a better understanding of the structural behavior and prediction of progressive damage and failure in composite structures.

## 1. INTRODUCTION

The advent of new composite materials and their increasing use in various fields of advanced technology has generated a new interest in the development and solution of consistent refined theories of anisotropic composite plates and shells. This interest is due to the fact that the classical plate theory, in terms of its basic assumptions (i.e. the Kirchhoff hypothesis), comes in conflict with real behavior of these new materials. For example, in contrast to the basic assumption of infinite rigidity in transverse shear in the classical plate theory, the new composite materials exhibit a finite rigidity in transverse shear. This property requires the incorporation of transverse shear deformation effects.

In addition to other shortcomings, the classical plate theory involves a contradiction between the number of boundary conditions physically required to be fulfilled on a free boundary and the number available in theory, which is to be consistent with the order of the associated governing equations (see Stoker [1]). The non-fulfillment of boundary conditions on the bounding surfaces constitutes another feature of the classical theory. In recent years attempts were made to refine the classical theory by: (i) incorporating transverse shear effects, (ii) removing the contradiction which concerns the number of boundary conditions to be prescribed at each edge, and (iii) fulfilling the boundary conditions on the bounding surfaces and, in the case of laminated composite plates and shells, of the continuity conditions at the interfaces between the contiguous layers. In addition, the refined transverse shear deformation theories can be used to model such anisotropic plates and shells whose material exhibits high degree of anisotropy, and are not restricted to the thinness requirement implied by the classical laminate theory. Another feature of refined laminate theories concerns the adequate incorporation of the dynamical effects allowing the evaluation of the lowest and higher natural frequencies.

The shear deformation theories known in the literature can be grouped into two classes: (1) stress-based theories, and (2) displacement-based theories. The first stress-based transverse shear deformable plate theory is due to Reissner [2-4]. The distribution across the thickness of the transverse normal and shear stresses is determined through integration over the thickness of the equilibrium equations of the 3-D elasticity theory. The associated field equations and boundary conditions expressed in terms of 2-D quantities can be determined by using the variational principles of the 3-D elasticity theory, or by considering the moments of $n^{th}$ order of the basic equations of 3-D elasticity theory. Both methods allow the reduction of the 3-D problems to a 2-D equivalent one.

The pioneering work of the displacement-based theories is due to Basset [5]. Based on Basset's representation of displacement field, Hildebrand, Reissner and Thomas [6] developed a variationally consistent first order theory for shells. The field equations were derived using the principle of minimum total potential energy. This results in five equilibrium equations expressed in terms of five displacement quantities.

By using the displacement representation of Basset, Mindlin [7] extended Hencky's theory [8] of isotropic plates to the dynamic case. Historical evidence (from the review of the literature) points out that the basic idea of the displacement-based first-order shear deformation theory came from Basset [5], Hildebrand, Reissner and Thomas [6] and Hencky [8]. Mindlin should be credited with the extension of the theory to the dynamic case. The shear deformation theory of Hencky-Mindlin is referred as the first-order transverse shear deformation theory (see Reddy [9]).

Following these works, many extensions and applications of the two classes of theories were reported in the literature (see e.g. Ambartsumian [10-12], Boal and Reissner [13], Bolle [14], Cheng [15], Gol'denveizer [16-18], Green [19-21], Kromm [22,23], Levinson [24], Librescu [25-30], Lo, Christensen and Wu [31], Medwadowski [32], Murthy [33], Nelson and Lorch [34], Pagano [35,36], Reddy [37-40], Rehfield and Valisetty [41], Reissner [42-47], Schmidt [48], Shirakawa [49], Volterra [50], Whitney and colleqgues [51-54], Wilson and Boresi [55], Yang, Norris and Stavsky [56]).

Extension of the displacement-based theory to the moderately large deflections case is due to Medwadowski [32] and the extension to laminated plates is due to Whitney [51] and Whitney and Pagano [52]. Schmidt [48] presented an extension of Kromm's theory by accounting for moderately large deflections (i.e., in the Von-Karman sense).

The second- and higher-order displacement-based shear deformation theories have been investigated by Nelson and Lorch [34], Librescu [25-28], Lo, Christensen and Wu [31], Levinson [24], Murthy [33] and Reddy [37-39]. Levinson [24] and Murthy [33] presented a third-order theory in which transverse normals are assumed to be inextensibile. The nine displacement functions were reduced to five by requiring that the transverse shear stresses vanish on the bounding planes of the plate. However, both authors (and also Schmidt [48]) used the equilibrium equations of the first-order theory in their analysis and so they are variationally inconsistent. As a consequence, the higher-order terms of the displacement field are accounted for only in the calculation of the strains but not in the governing differential equations or in the boundary conditions. Recently, during the course of the present research, Reddy [37-39] corrected these theories by deriving the governing differential equations by means of the virtual work principle. The theory presented in [37] accounts for the von Karman strains but is limited to orthotropic plates, while that in [38] deals with the small-deflection theory of laminated plates.

Besides the works cited above, there exists numerous other works pertaining to refined theories of elastic plates. Thus, in addition to the works of Basset, Reissner, Hencky and Mindlin that have inspired the elaboration of a great number of works, we mention the symbolic method advanced by Lur'e [57] which has generated a series of works devoted to the refined theory of the isotropic flat plates (see e.g., Westbrook [58,59]). Starting from the three-dimensional field equations of elasticity, methods based on the expansion of all the functions characterizing the plane state of stress into power series over the plate thickness (see Schipper [60]), into Legendre's polynomials (see Cicala [61]), into biharmonic polynomials (see Teodorescu [62]), or that resulting by considering the $n^{th}$ order moments of the equations of three-dimensional elasticity (see Tiffen [63], Tiffen and Lowe [64]) are developed.

Another method which leads to the static and dynamic theory of isotropic and anisotropic plates, is the method of asymptotic integration of the three-dimensional equations of elasticity (see

Gol'denveizer [17,18], Reissner [42], and Green and Naghdi [20]). Other methods are of mixed character which use first, the method of expanding the functions (defining the stress and strain state) into Legendre's polynomials, and then the asymptotic integration method (see, for example, Poniatovskii [65]).

The analysis of laminated composite plates for bending and natural vibration has received widespread attention in recent years. 3-D elasticity solutions for the bending (see Pagano [36]), vibration and buckling of simply supported thick orthotropic rectangular plates and laminates were obtained by Srinivas and his co-workers [66-68]. The Navier solution of simply supported rectangular plates was developed by Whitney and Leissa [52] for classical laminate theory, Pagano [35], Whitney [51], Bert and Chen [69] and Reddy and Chao [39,70] for the first-order transverse shear deformation theory, and by Reddy and his co-workers [37-39,71] for a refined shear deformation theory.

Papers dealing with the response of plates excited by dynamic loads of known time history have been less in number. Sun and Whitney [72,73] have analyzed the response of anisotropic plates in cylindrical bending using Mindlin's [7] theory to account for normal shear stiffness. Sun and Chattopadhyay [74] used the plate equations developed by Whitney and Pagano [53] to analyze a specially orthotropic plate subject to a center impact. Dobyns [75] presented an analysis of simply-supported orthotropic plates subjected to static and dynamic loading conditions. The vibration frequencies and mode shapes were then determined and solutions for plate deflections, bending strains, and normal shear forces due to several transient loads were obtained. Reddy [76,77] obtained the exact form of the spatial variation of the solution to forced motions of rectangular composite plates for two different lamination schemes, under appropriate simply supported boundary conditions. He reduced the problem to the solution of a system of ordinary differential equations in time, which are then integrated numerically using Newmark's direct integration method.

In geometrically nonlinear theories of elastic anisotropic plates one often assumes that the strains and rotations about the normal to the midplane are infinitesimal, and retains the products and squares of the derivatives of the transverse deflection in the strain-displacement equations (the von Karman assumption; see [[32,37,78-80]). The full geometric nonlinearity (implied by the strain-displacement equations of nonlinear elasticity) in shell theories was considered by Naghdi [81], Librescu [29], Yokas and Matsunaga [82], Habip [83], and Pietraszkiewicz [84], among others. Consideration of full geometric nonlinearity not only results in complex equations, but not warranted in most practical problems. On the other hand, the von Karman nonlinear theory does not account for moderate rotation terms that could be of significance in the analysis (especially in stability problems) of plates while accounting for the transverse normal and shear strains. The small strain and moderate rotation concept was used in the classical theory of plates and shells by Sanders [85], Koiter [86], Reissner [87] and Pietraszkiewicz [88], and in first-order plate and shell theories by Naghdi and Vangsarnpigoon [89], and Librescu and Schmidt [90]. In all these works, no attempt was made to obtain solutions of the theory.

From the review of the literature on various plate and shell theories, one can make the following observations:

1. Analytical solutions of refined theories of anisotropic composite laminates, other than those with simply supported edges, are not available. For example, the Lévy type solutions are not developed for laminated composite plates. This might be possibly due to the difficulty in solving the ordinary differential equations resulting from the Lévy procedure.
2. The most commonly used refined theory is the first-order shear deformation theory, commonly referred to as the Mindlin plate theory. It is well-known that the first-order theory (i) requires the introduction of shear correction factors (to correct for the constant state of transverse shear stresses through thickness), (ii) disregards the effect of transverse and normal stress, and (iii) does not satisfy the stress-free boundary conditions on bounding surfaces. It is also known that the determination of the shear correction factors for anisotropic composite plates is not established, and their values depend on the geometry and lamination scheme of the laminate.
3. Use of refined theories with moderate and large rotations (situations quite commonly encountered in the analysis of helicopter blades, turbine blades, etc.) is not investigated, especially for anisotropic composite laminates.

During the present research analytical solutions based on the space-time concept were obtained and the third shear deformation with moderate rotation terms was developed. A first-ply and post-first-ply analysis capability based on the first-order shear deformation theory was also developed. A brief summary of the first two results is presented in the next section.

## 2. BRIEF OUTLINE OF RESEARCH FINDINGS

### A Third-Order Theory

Consider a laminated plate composed of N orthotropic layers, symmetrically located with respect to the midplane of the laminate. The governing equations of the refined theory are based on the following displacement field [37-39]:

$$u_1 = u + z[\psi_x - \frac{4}{3}(\frac{z}{h})^2(\psi_x + \frac{\partial w}{\partial x})]$$

$$u_2 = v + z[\psi_y - \frac{4}{3}(\frac{z}{h})^2(\psi_y + \frac{\partial w}{\partial y})]$$

$$u_3 = w \tag{1}$$

where $(u_1,u_2,u_3)$ are the displacements along the x, y and z coordinates respectively, (u,v,w) are the corresponding displacements of a point on the midplane of the laminate, and $\psi_x$ and $\psi_y$ are the rotations of a transverse normal about the y- and x-axes, respectively.

The cubic variation of $u_1$ and $u_2$ through laminate thickness introduces higher-order resultants,

$$P_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_i z^3 dz \qquad (i = 1,2,6)$$

$$(R_1,R_2) = \int_{-\frac{h}{2}}^{\frac{h}{2}} z^2(\sigma_5,\sigma_4)dz$$

and laminate stiffnesses,

$$(F_{ij},H_{ij}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij}(z^4,z^6)dz \qquad (i,j = 1,2,6)$$

$$(D_{ij},F_{ij}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij}(z^2,z^4)dz \qquad (i,j = 4,5)$$

For symmetrical cross-ply laminated plates, the following stiffness coefficients vanish [39]:

$$B_{ij} = E_{ij} = 0 \text{ for } i,j = 1,2,4,5,6$$

$$A_{16} = A_{26} = D_{16} = D_{26} = F_{16} = F_{26} = H_{16} = H_{26} = 0$$

$$A_{45} = D_{45} = F_{45} = 0$$

This implies that the effect of coupling between stretching and bending vanishes. For such laminates the governing equations are given by (see [37,38]):

$$\begin{aligned}
&\frac{4}{3h^2}\Big[F_{11}\frac{\partial^3\psi_x}{\partial x^3} + H_{11}(-\frac{4}{3h^2})(\frac{\partial^3\psi_x}{\partial x^3}+\frac{\partial^4 w}{\partial x^4}) + F_{12}\frac{\partial^3\psi_y}{\partial x^2\partial y} + H_{12}(-\frac{4}{3h^2})(\frac{\partial^3\psi_y}{\partial x^2\partial y} \\
&\quad + \frac{\partial^4 w}{\partial x^2\partial y^2}) + F_{12}\frac{\partial^3\psi_x}{\partial y^2\partial x} + H_{12}(-\frac{4}{3h^2})(\frac{\partial^3\psi_x}{\partial y^2\partial x} + \frac{\partial^4 w}{\partial x^2\partial y^2}) + F_{22}\frac{\partial^3\psi_y}{\partial y^3} \\
&\quad + H_{22}(-\frac{4}{3h^2})(\frac{\partial^3\psi_y}{\partial y^3} + \frac{\partial^4 w}{\partial y^4}) + 2F_{66}(\frac{\partial^3\psi_y}{\partial x^2\partial y} + \frac{\partial^3\psi_x}{\partial y^2\partial x}) + 2H_{66}(-\frac{4}{3h^2})(\frac{\partial^3\psi_x}{\partial y^2\partial x} \\
&\quad + \frac{\partial^3\psi_y}{\partial x^2\partial y} + \frac{2\partial^4 w}{\partial x^2\partial y^2})\Big] - \frac{4}{h^2}\Big[D_{55}(\frac{\partial^2 w}{\partial x^2} + \frac{\partial\psi_x}{\partial x}) + F_{55}(-\frac{4}{h^2})(\frac{\partial\psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}) \\
&\quad + D_{44}(\frac{\partial^2 w}{\partial y^2} + \frac{\partial\psi_y}{\partial y}) + F_{44}(-\frac{4}{h^2})(\frac{\partial\psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2})\Big] + \Big[A_{55}(\frac{\partial\psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}) \\
&\quad + D_{55}(-\frac{4}{h^2})(\frac{\partial\psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}) + A_{44}(\frac{\partial\psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}) + D_{44}(-\frac{4}{h^2})(\frac{\partial\psi_y}{\partial y} \\
&\quad + \frac{\partial^2 w}{\partial y^2})\Big] + q = 0 \qquad\qquad (2a)
\end{aligned}$$

$$D_{11}\frac{\partial^2\psi_x}{\partial x^2} + D_{12}\frac{\partial^2\psi_y}{\partial x\partial y} + F_{11}(-\frac{4}{3h^2})(\frac{\partial^2\psi_x}{\partial x^2} + \frac{\partial^3 w}{\partial x^3}) + F_{12}(-\frac{4}{3h^2})(\frac{\partial^2\psi_y}{\partial x\partial y}$$

$$+ \frac{\partial^3 w}{\partial x \partial y^2}) + D_{66}(\frac{\partial^2 \psi_x}{\partial y^2} + \frac{\partial^2 \psi_y}{\partial x \partial y}) + F_{66}(- \frac{4}{3h^2})(\frac{\partial^2 \psi_x}{\partial y^2} + \frac{\partial^2 \psi_y}{\partial x \partial y} + \frac{2\partial^3 w}{\partial x \partial y^2})$$

$$- [A_{55}(\psi_x + \frac{\partial w}{\partial x}) + D_{55}(- \frac{4}{h^2})(\psi_x + \frac{\partial w}{\partial x})] - \frac{4}{3h^2} [F_{11} \frac{\partial^2 \psi_x}{\partial x^2}$$

$$+ H_{11}(- \frac{4}{3h^2})(\frac{\partial^2 \psi_x}{\partial x^2} + \frac{\partial^3 w}{\partial x^3}) + F_{12} \frac{\partial^2 \psi_y}{\partial x \partial y} + H_{12}(- \frac{4}{3h^2})(\frac{\partial^2 \psi_y}{\partial x \partial y} + \frac{\partial^3 w}{\partial x \partial y^2})$$

$$+ F_{66}(\frac{\partial^2 \psi_y}{\partial x \partial y} + \frac{\partial^2 \psi_x}{\partial y^2}) + H_{66}(- \frac{4}{3h^2})(\frac{\partial^2 \psi_x}{\partial y^2} + \frac{\partial^2 \psi_y}{\partial x \partial y} + \frac{2\partial^3 w}{\partial x \partial y^2})]$$

$$+ \frac{4}{h^2} [D_{55}(\frac{\partial w}{\partial x} + \psi_x) + F_{55}(- \frac{4}{h^2})(\psi_x + \frac{\partial w}{\partial x})] = 0 \qquad (2b)$$

$$D_{66}(\frac{\partial^2 \psi_x}{\partial x \partial y} + \frac{\partial^2 \psi_y}{\partial x^2}) + F_{66}(- \frac{4}{3h^2})(\frac{\partial^2 \psi_x}{\partial x \partial y} + \frac{\partial^2 \psi_y}{\partial x^2} + 2 \frac{\partial^3 w}{\partial x^2 \partial y}) + D_{12} \frac{\partial^2 \psi_x}{\partial x \partial y}$$

$$+ D_{22} \frac{\partial^2 \psi_y}{\partial y^2} + F_{12}(- \frac{4}{3h^2})(\frac{\partial^2 \psi_x}{\partial x \partial y} + \frac{\partial^3 w}{\partial x^2 \partial y}) + F_{22}(- \frac{4}{3h^2})(\frac{\partial^2 \psi_y}{\partial y^2} + \frac{\partial^3 w}{\partial y^3})$$

$$- [A_{44}(\psi_y + \frac{\partial w}{\partial y}) + D_{44}(- \frac{4}{h^2})(\psi_y + \frac{\partial w}{\partial y})] - \frac{4}{3h^2} [F_{66}(\frac{\partial^2 \psi_y}{\partial x^2} + \frac{\partial^2 \psi_x}{\partial y \partial x})$$

$$+ H_{66}(- \frac{4}{3h^2})(\frac{\partial^2 \psi_x}{\partial x \partial y} + \frac{\partial^2 \psi_y}{\partial x^2} + \frac{2\partial^3 w}{\partial x^2 \partial y}) + F_{12} \frac{\partial^2 \psi_x}{\partial x \partial y} + H_{12}(- \frac{4}{3h^2})(\frac{\partial^2 \psi_x}{\partial x \partial y}$$

$$+ \frac{\partial^3 w}{\partial x^2 \partial y}) + F_{22} \frac{\partial^2 \psi_y}{\partial y^2} + H_{22}(- \frac{4}{3h^2})(\frac{\partial^2 \psi_y}{\partial y^2} + \frac{\partial^3 w}{\partial y^3})] + \frac{4}{h^2} [D_{44}(\frac{\partial w}{\partial y}$$

$$+ \psi_y) + F_{44}(- \frac{4}{h^2})(\frac{\partial w}{\partial y} + \psi_y)] = 0 \qquad (2c)$$

Here w denotes the transverse displacement, $\psi_x$ and $\psi_y$ are the rotations of the normal to midplane about the y and x axis, respectively, q is the distributed transverse load, and $A_{ij}$, $D_{ij}$, $F_{ij}$, $H_{ij}$ are the plate stiffnesses, defined by

$$(D_{ij},F_{ij},H_{ij}) = \int_{-h/2}^{h/2} Q_{ij}^{(k)}(z^2,z^4,z^6)dz \quad (i,j = 1,2,6)$$

$$(A_{ij},D_{ij},F_{ij}) = \int_{-h/2}^{h/2} Q_{ij}^{(k)}(1,z^2,z^4)dz \quad (i,j = 4,5) \tag{3}$$

Here $Q_{ij}^{(k)}$ denote the reduced orthotropic moduli of the k-th lamina. The boundary conditions of the refined theory are of the form: specify

$$\left.\begin{array}{l} w \text{ or } Q_n \\ \\ \dfrac{\partial w}{\partial n} \text{ or } P_n \\ \\ \psi_n \text{ or } M_n \\ \psi_{ns} \text{ or } M_{ns} \end{array}\right\} \text{on } \Gamma \tag{4}$$

where $\Gamma$ is the boundary of the midplane $\Omega$ of the plate, and

$$\begin{aligned}
M_n &= \hat{M}_1 n_x^2 + \hat{M}_2 n_y^2 + 2\hat{M}_6 n_x n_y \\
M_{ns} &= (\hat{M}_2 - \hat{M}_1) n_x n_y + \hat{M}_6 (n_x^2 - n_y^2) \\
P_n &= P_1 n_x^2 + P_2 n_y^2 + 2P_6 n_x n_y \\
P_{ns} &= (P_2 - P_1) n_x n_y + P_6 (n_x^2 - n_y^2) \\
Q_n &= \hat{Q}_1 n_x + \hat{Q}_2 n_y + \frac{4}{3h^2} \left(\frac{\partial P_{ns}}{\partial s} + \frac{\partial P_n}{\partial n}\right) \\
\hat{M}_i &= M_i - \frac{4}{3h^2} P_i \quad (i = 1,2,6) \\
\hat{Q}_i &= Q_i - \frac{4}{h^2} R_i \quad (i = 1,2) \\
\frac{\partial}{\partial n} &= n_x \frac{\partial}{\partial x} + n_y \frac{\partial}{\partial y} \quad , \quad \frac{\partial}{\partial s} = n_x \frac{\partial}{\partial y} - n_y \frac{\partial}{\partial x}
\end{aligned} \tag{5}$$

The stress resultants appearing in Eq. (5) can be expressed in terms of the generalized displacements $(w,\psi_x,\psi_y)$ as:

$$M_1 = D_{11}\frac{\partial\psi_x}{\partial x} + D_{12}\frac{\partial\psi_y}{\partial y} + F_{11}(-\frac{4}{3h^2})(\frac{\partial\psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}) + F_{12}(-\frac{4}{3h^2})(\frac{\partial\psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2})$$

$$M_2 = D_{12}\frac{\partial\psi_x}{\partial x} + D_{22}\frac{\partial\psi_y}{\partial y} + F_{12}(-\frac{4}{3h^2})(\frac{\partial\psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}) + F_{22}(-\frac{4}{3h^2})(\frac{\partial\psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2})$$

$$M_6 = D_{66}(\frac{\partial\psi_x}{\partial y} + \frac{\partial\psi_y}{\partial x}) + F_{66}(-\frac{4}{3h^2})(\frac{\partial\psi_x}{\partial y} + \frac{\partial\psi_y}{\partial x} + 2\frac{\partial^2 w}{\partial x\partial y})$$

$$Q_2 = A_{44}(\psi_y + \frac{\partial w}{\partial y}) + D_{44}(-\frac{4}{h^2})(\psi_y + \frac{\partial w}{\partial y})$$

$$Q_1 = A_{55}(\psi_x + \frac{\partial w}{\partial x}) + D_{55}(-\frac{4}{h^2})(\psi_x + \frac{\partial w}{\partial x})$$

$$P_1 = F_{11}\frac{\partial\psi_x}{\partial x} + F_{12}\frac{\partial\psi_y}{\partial y} + H_{11}(-\frac{4}{3h^2})(\frac{\partial\psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}) + H_{12}(-\frac{4}{3h^2})(\frac{\partial\psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2})$$

$$P_2 = F_{12}\frac{\partial\psi_x}{\partial x} + F_{22}\frac{\partial\psi_y}{\partial y} + H_{12}(-\frac{4}{3h^2})(\frac{\partial\psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}) + H_{22}(-\frac{4}{3h^2})(\frac{\partial\psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2})$$

$$P_6 = F_{66}(\frac{\partial\psi_y}{\partial x} + \frac{\partial\psi_x}{\partial y}) + H_{66}(-\frac{4}{3h^2})(\frac{\partial\psi_x}{\partial y} + \frac{\partial\psi_y}{\partial x} + 2\frac{\partial^2 w}{\partial x\partial y})$$

$$R_2 = D_{44}(\frac{\partial w}{\partial y} + \psi_y) + F_{44}(-\frac{4}{3h^2})(\frac{\partial w}{\partial y} + \psi_y)$$

$$R_1 = D_{55}(\frac{\partial w}{\partial x} + \psi_x) + F_{55}(-\frac{4}{3h^2})(\frac{\partial w}{\partial x} + \psi_x) \quad (6)$$

## Analytical Solutions

The present study deals with the development of the Lévy type solution of the refined shear deformation theory of Reddy [12,13] for symmetric rectangular laminates with two opposite edges simply supported and the remaining edges subjected to a combination of free, simply supported and clamped boundary conditions. The state-space concept is used to solve the ordinary differential equations obtained after the application of the Levy solution procedure.

The Lévy method can be used to solve Eqs. (2) for rectangular plates for which two opposite edges are simply supported. The other two edges can each have arbitrary boundary conditions. Here we assume that the edges parallel to the y-axis are simply supported, and the origin of the coordinate system is taken as shown in Fig. 1. The simply supported boundary conditions can be satisfied by trigonometric functions in x. The resulting ordinary differential equations in y can be solved using the state-space concept.

Following the Lévy type procedure, we assume the following representation of the displacements and loading:

$$
\begin{aligned}
w(x,y) &= \sum_{m=1}^{\infty} W_m(y)\sin\alpha x \\
\psi_x(x,y) &= \sum_{m=1}^{\infty} X_m(y)\cos\alpha x \\
\psi_y(x,y) &= \sum_{m=1}^{\infty} Y_m(y)\sin\alpha x \\
q(x,y) &= \sum_{m=1}^{\infty} Q_m(y)\sin\alpha x
\end{aligned} \tag{7}
$$

where $\alpha = \frac{m\pi}{a}$ and $W_m$, $X_m$, $Y_m$ and $Q_m$ denote amplitudes of $w$, $\psi_x$, $\psi_y$ and $q$, respectively. Substituting Eqs. (7) into Eqs. (2), we obtain

$$
\begin{aligned}
e_1 W_m'''' + e_2 W_m'' + e_3 W_m + e_4 X_m'' + e_5 X_m + e_6 Y_m''' + e_7 Y_m' + Q_m &= 0 \\
e_8 W_m'' + e_9 W_m + e_{10} X_m'' + e_{11} X_m + e_{12} Y_m' &= 0 \\
e_{13} W_m''' + e_{14} W_m' + e_{15} X_m' + e_{16} Y_m'' + e_{17} Y_m &= 0
\end{aligned} \tag{8}
$$

Where primes on the variables indicate differentiation with respect to y, and

$$e_1 = -\left(\frac{4}{3h^2}\right)^2 H_{22}$$

$$e_2 = 2\left(\frac{4}{3h^2}\right)^2 \alpha^2 (H_{12} + 2H_{66}) + A_{44} - \frac{8}{h^2} D_{44} + \left(\frac{4}{h^2}\right)^2 F_{44}$$

$$e_3 = -\alpha^2 \left[\left(\frac{4}{3h^2}\right)^2 \alpha^2 H_{11} + \left(\frac{4}{h^2}\right)^2 F_{55} - \frac{8}{h^2} D_{55} + A_{55}\right]$$

$$e_4 = \alpha \frac{4}{3h^2} \left[- F_{12} + \frac{4}{3h^2} H_{12} - 2F_{66} + \frac{8}{3h^2} H_{66}\right]$$

$$e_5 = \alpha^3 \frac{4}{3h^2} \left(F_{11} - \frac{4}{3h^2} H_{11}\right) + \alpha\left[\frac{8}{h^2} D_{55} - \left(\frac{4}{h^2}\right)^2 F_{55} - A_{55}\right]$$

$$e_6 = \frac{4}{3h^2} \left(F_{22} - \frac{4}{3h^2} H_{22}\right)$$

$$e_7 = \alpha^2 \frac{4}{3h^2} \left[-F_{12} - 2F_{66} + \frac{4}{3h^2} (H_{12} + 2H_{66})\right] - \frac{8}{h^2} D_{44} + \left(\frac{4}{h^2}\right)^2 F_{44} + A_{44}$$

$$e_8 = e_4 , \quad e_9 = e_5$$

$$e_{10} = D_{66} - \frac{8}{3h^2} F_{66} + \left(\frac{4}{3h^2}\right)^2 H_{66}$$

$$e_{11} = \alpha^2 \left[-D_{11} + \frac{8}{3h^2} F_{11} - \left(\frac{4}{3h^2}\right)^2 H_{11}\right] + \frac{8}{h^2} D_{55} - \left(\frac{4}{h^2}\right)^2 F_{55} - A_{55}$$

$$e_{12} = \alpha\left[D_{12} + D_{66} - \frac{8}{3h^2} (F_{12} + F_{66}) + \left(\frac{4}{3h^2}\right)^2 (H_{12} + H_{66})\right]$$

$$e_{13} = -e_6 , \quad e_{14} = -e_7 , \quad e_{15} = - e_{12}$$

$$e_{16} = D_{22} - \frac{8}{3h^2} F_{22} + \left(\frac{4}{3h^2}\right) H_{22}$$

$$e_{17} = \alpha^2 \left[-D_{66} + \frac{8}{3h^2} F_{66} - \left(\frac{4}{3h^2}\right)^2 H_{66}\right] + \frac{8}{h^2} D_{44} - \left(\frac{4}{h^2}\right)^2 F_{44} - A_{44}$$

Equations (8) can be written as:

$$W_m'''' = c_1 W_m'' + c_2 W_m + c_3 X_m + c_4 Y_m' + c_o Q_m$$

$$X_m'' = c_5 W_m'' + c_6 W_m + c_7 X_m + c_8 Y_m'$$

$$Y_m'' = c_9 W_m''' + c_{10} W_m' + c_{11} X_m' + c_{12} Y_m \tag{10}$$

where

$$c_1 = \left(\frac{e_4^2}{e_{10}} + \frac{e_4 e_6 e_{12}}{e_{10} e_{16}} - \frac{e_6 e_7}{e_{16}} - e_2\right) / \left(e_1 + \frac{e_6^2}{e_{16}}\right)$$

$$c_2 = \left(\frac{e_4 e_5}{e_{10}} + \frac{e_5 e_6 e_{12}}{e_{10} e_{16}} - e_3\right) / \left(e_1 + \frac{e_6^2}{e_{16}}\right)$$

$$c_3 = \left(\frac{e_{11} e_4}{e_{10}} + \frac{e_{11} e_6 e_{12}}{e_{10} e_{16}} - e_5\right) / \left(e_1 + \frac{e_6^2}{e_{16}}\right)$$

$$c_4 = \left(\frac{e_6 e_{17}}{e_{16}} + \frac{e_4 e_{12}}{e_{10}} + \frac{e_6 e_{12}^2}{e_{10} e_{16}} - e_7\right) / \left(e_1 + \frac{e_6^2}{e_{16}}\right)$$

$$c_o = - \frac{e_{16}}{e_1 e_{16} + e_6^2}$$

$$c_5 = -e_4/e_{10},\ c_6 = -e_5/e_{10},\ c_7 = -e_{11}/e_{10},\ c_8 = -e_{12}/e_{10}$$

$$c_9 = e_6/e_{16},\ c_{10} = e_7/e_{16},\ c_{11} = e_{12}/e_{16},\ c_{12} = -e_{17}/e_{16} \tag{11}$$

The linear system of ordinary differential equations (10) with constant coefficients can be reduced to a single matrix differential equation using the state-space concept (see [91])

$$\underline{x}' = A\underline{x} + \underline{b} \tag{12}$$

This can be done by introducing the variables

$$x_1 = W_m,\ x_2 = W_m',\ x_3 = W_m'',\ x_4 = W_m'''$$

$$x_5 = X_m,\ x_6 = X_m',\ x_7 = Y_m,\ x_8 = Y_m' \tag{13}$$

where

$$\underline{x}' = \begin{Bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \\ x_6' \\ x_7' \\ x_8' \end{Bmatrix}, \quad \underline{b} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ C_0 Q_m \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ c_2 & 0 & c_1 & 0 & c_3 & 0 & 0 & c_4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ c_6 & 0 & c_5 & 0 & c_7 & 0 & 0 & c_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & c_{10} & 0 & c_9 & 0 & c_{11} & c_{12} & 0 \end{bmatrix} \tag{14}$$

The solution of Eq. (12) is given by

$$\underline{x} = e^{Ay}\underline{K} + e^{Ay} \int e^{-Ay}\underline{b}dy \tag{15}$$

where $\underline{K}$ is a constant vector to be determined from the boundary conditions, $e^{Ay}$ denotes the product,

$$e^{Ay} = [c] \begin{bmatrix} e^{\lambda_1 y} & & & \\ & e^{\lambda_2 y} & & \underset{\sim}{0} \\ & & \ddots & \\ \underset{\sim}{0} & & & e^{\lambda_8 y} \end{bmatrix} [c]^{-1} \tag{16}$$

$[c]$ is the matrix of distinct eigenvectors, $\lambda_i (i = 1,2,3,\ldots 8)$ are the eigenvalues associated with matrix A, and $[c]^{-1}$ is the inverse of the eigenvectors matrix $[c]$.

The following boundary conditions are used on the remaining two edges (i.e., the edges parallel to the x-axis) at $y = \mp \frac{b}{2}$.

<u>simply supported</u>: $w = \psi_x = P_2 = M_2 = 0$

<u>clamped</u>: $w = \frac{\partial w}{\partial y} = \psi_x = \psi_y = 0$

<u>free</u>: $P_2 = M_2 = 0$

$$M_6 - \frac{4}{3h^2} P_6 = 0$$

$$Q_2 - \frac{4}{h^2} R_2 + \frac{4}{3h^2} \left(\frac{\partial P_6}{\partial x} + \frac{\partial P_2}{\partial y}\right) = 0 \qquad (17)$$

Numerical results are presented for orthotropic and symmetric cross-ply (0°/90°/0°) plates subjected to three types of loads: uniformly distributed load ($q_0$), triangular distributed load ($2q_0$) and concentrated load P as shown in Fig. 2. The following sets of material properties are used in the calculations:

<u>Material I</u>:

$$\begin{aligned} E_1 &= 20.83 \times 10^6 \text{ psi} \;, \quad E_2 = 10.94 \times 10^6 \text{ psi} \\ G_{12} &= 6.10 \times 10^6 \text{ psi} \;, \quad G_{13} = 3.7 \times 10^6 \text{ psi} \\ G_{23} &= 6.19 \times 10^6 \text{ psi} \;, \quad \nu_{12} = 0.44 \end{aligned} \qquad (18)$$

Material II:

$$E_1 = 19.2 \times 10^6 \text{ psi} \quad , \quad E_2 = 1.56 \times 10^6 \text{ psi}$$
$$G_{12} = G_{13} = 0.82 \times 10^6 \text{ psi} \quad , \quad G_{23} = 0.523 \times 10^6 \text{ psi} \qquad (19)$$
$$\nu_{12} = 0.24$$

The following notation has been used throughout the figures:

SS - simply supported at $y = -b/2$ and at $y = b/2$.

CC - clamped at $y = -b/2$ and at $y = b/2$.

FF - free at $y = -b/2$ and at $y = b/2$.

SC - simply supported at $y = -b/2$ and clamped at $y = b/2$.

SF - simply supported at $y = -b/2$ and free at $y = b/2$.

CF - clamped at $y = -b/2$ and free at $y = b/2$.

UN - uniformly distributed load.

TR - triangular distributed load.

PL - point load at the center of the plate (20)

To show the effect of transverse shear strains on the deflections plots of nondimensionalized center deflection, $\bar{w} = 10^3 w(a/2,0)h^3 E_2/(q_0 a^4)$, versus side to thickness ratio of various plates are presented in Figs. 3-5. The shear deformation effect is more significant in cross-ply plates than in orthotropic plates. Also, the first order shear deformation theory (FSDT) over predicts deflections relative to the higher order theory (HSDT).

Figures 6 and 7 contain plots of the transverse stresses $\sigma_{13}$ through laminate thickness for various boundary conditions. The stresses were computed using lamina constitutive relations. The transverse shear stresses are constant and parabolic, through thickness of each lamina, respectively, for the first- and higher-order theories. The discontinuity at interface of lamina is due to the

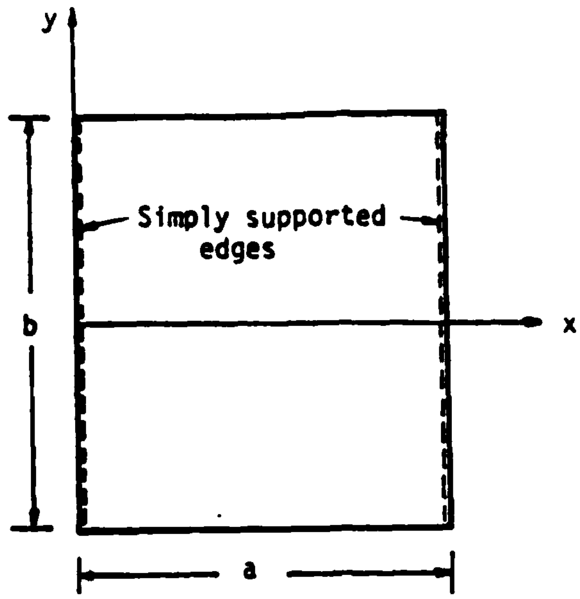



Figure 1. Geometry and coordinate system of rectangular plate

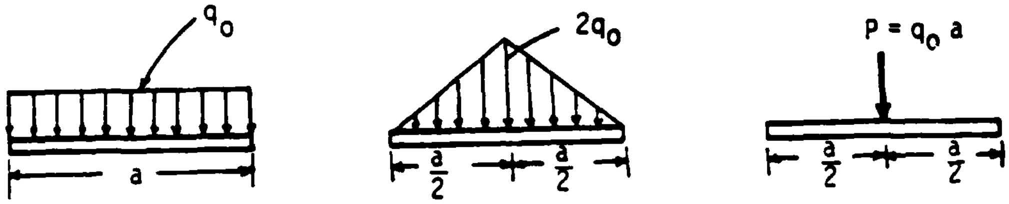



Figure 2. Various types of transverse loads

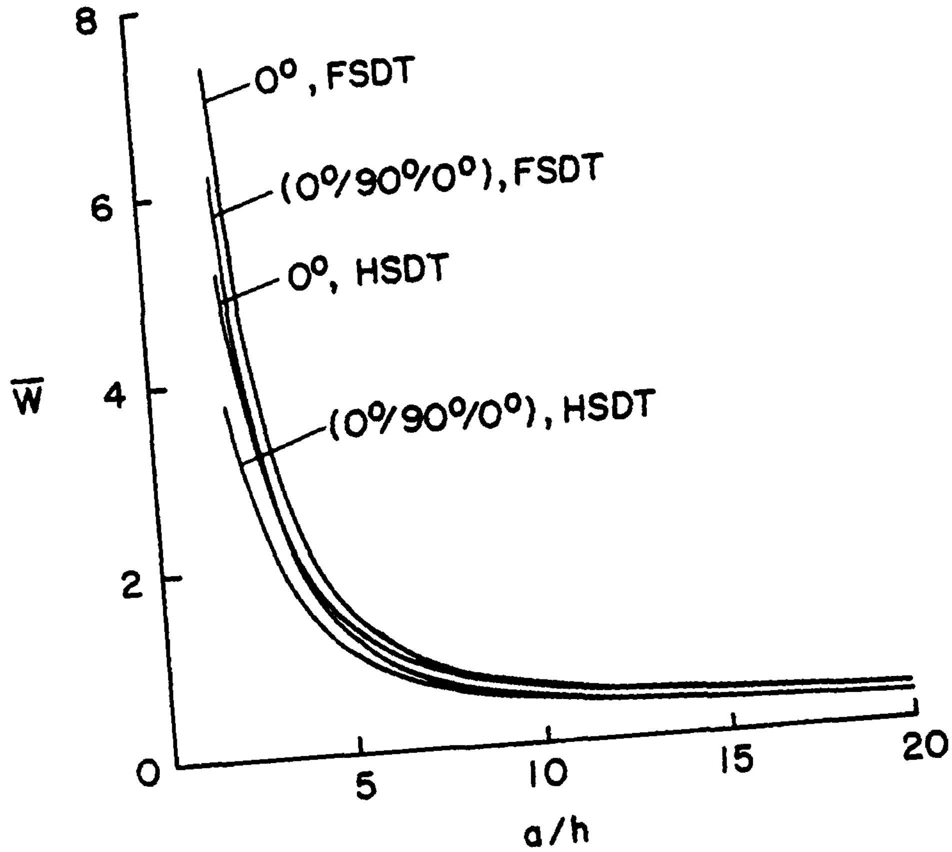



Figure 3. Nondimensionalized center deflection versus side-to-thickness ratio of SSSC plates using the first (FSDT) and higher (HSDT) order shear deformation theories (Material 2, a/b=4, uniform load).

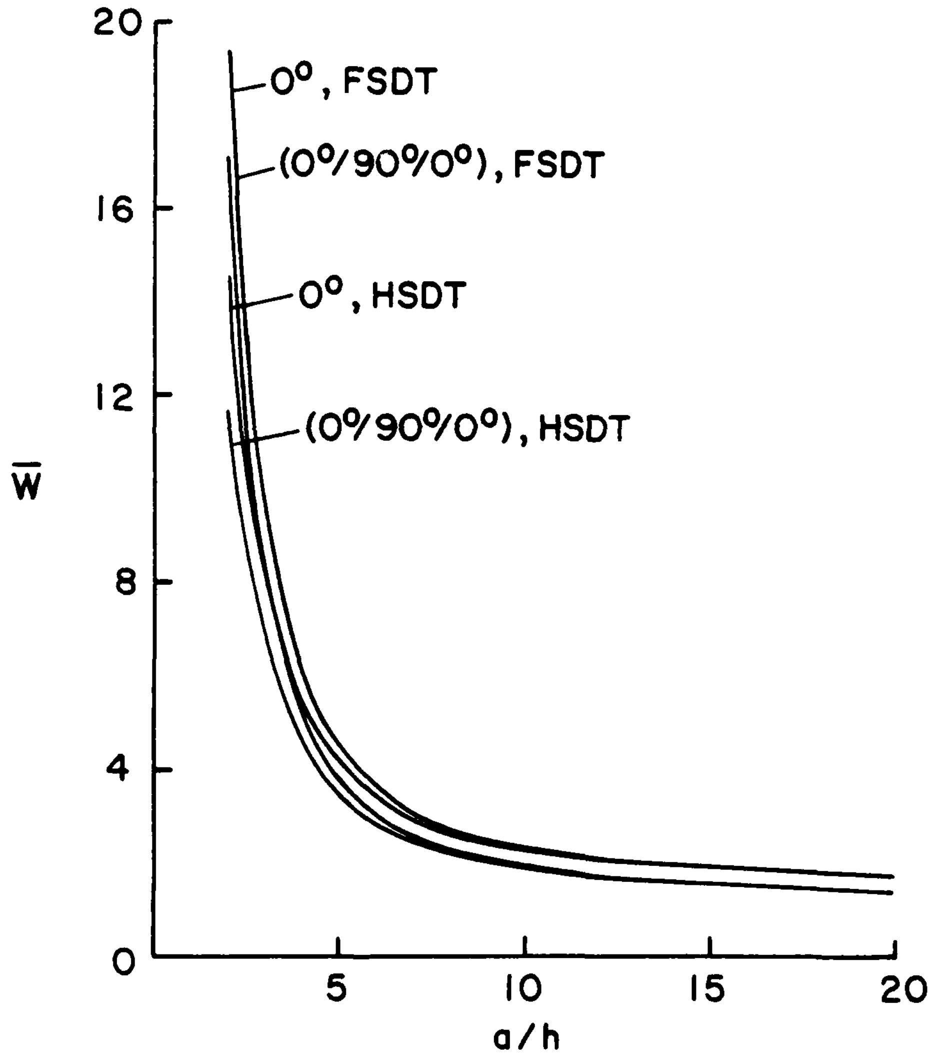



Figure 4. Nondimensionalized center deflection versus side-to-thickness ratio of SSFC plates (Material 2, a/b =4).

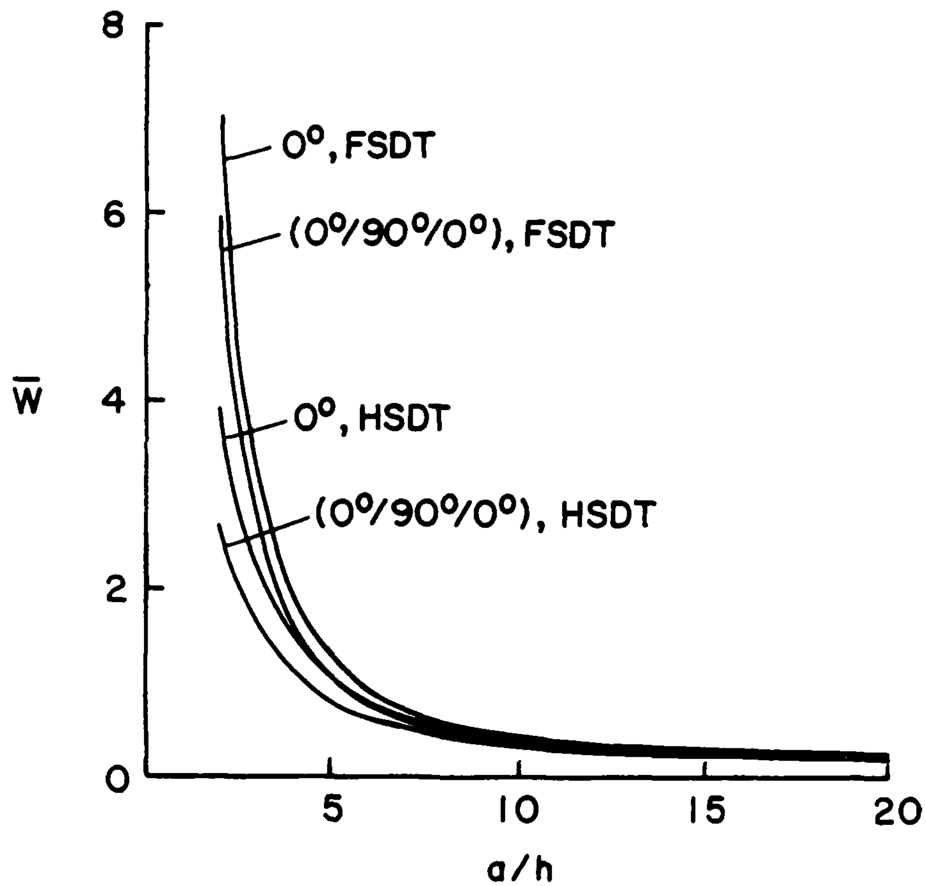



Figure 5. Nondimensionalized center deflection versus side-to-thickness ratio of SSCC plates (Material 2, a/b =4).

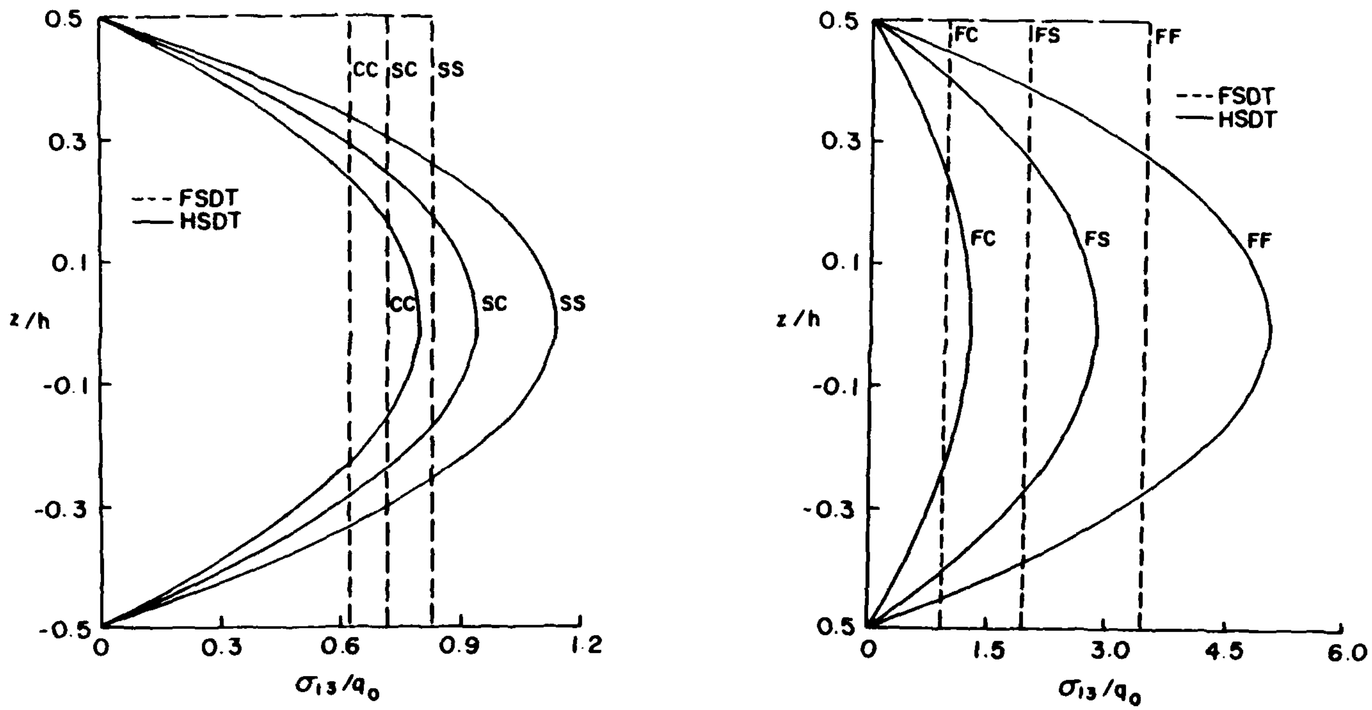



Figure 6. Variation of the transverse shear stress through the thickness of orthotropic plates subjected to uniform load (Material 2, a/b=4, h/a=0.14).

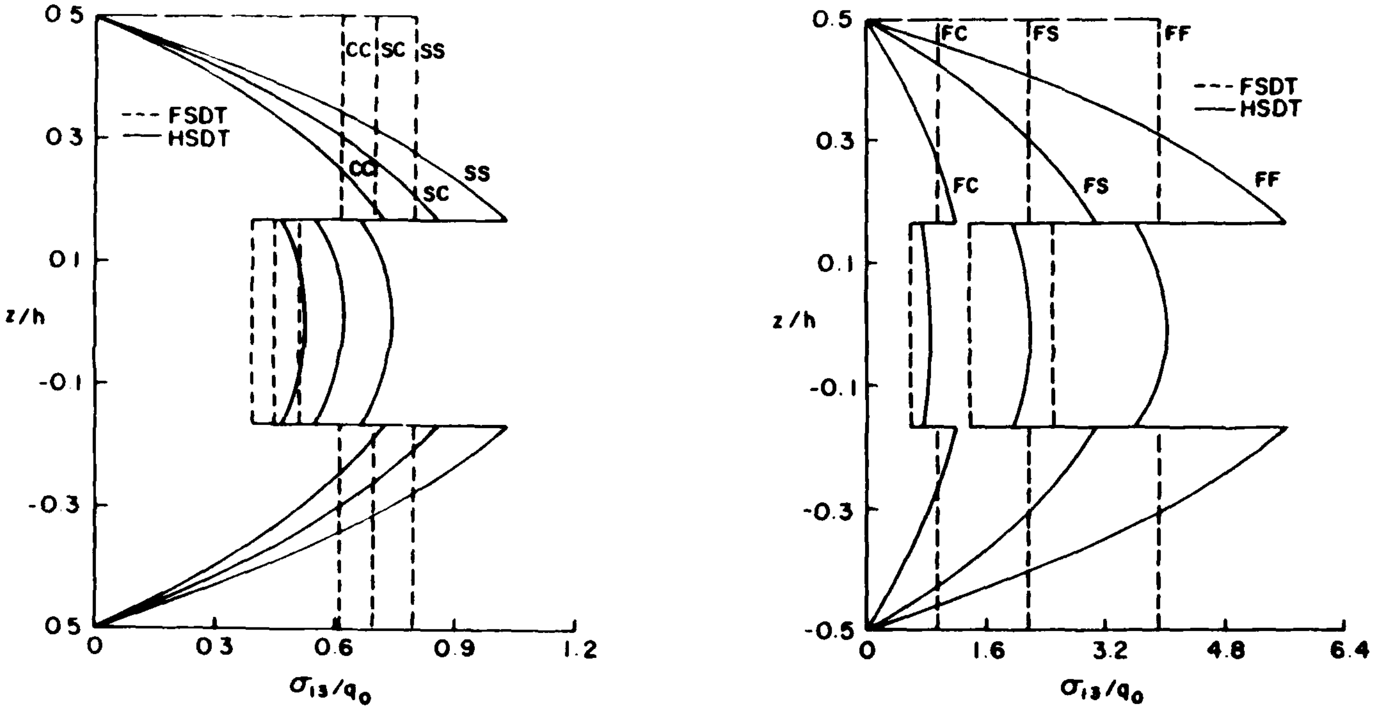



Figure 7. Variation of the transverse shear stress through the thickness of cross-ply (0/90/0) laminates under uniform load (Material 2, a/b=4, h/a=0.14).

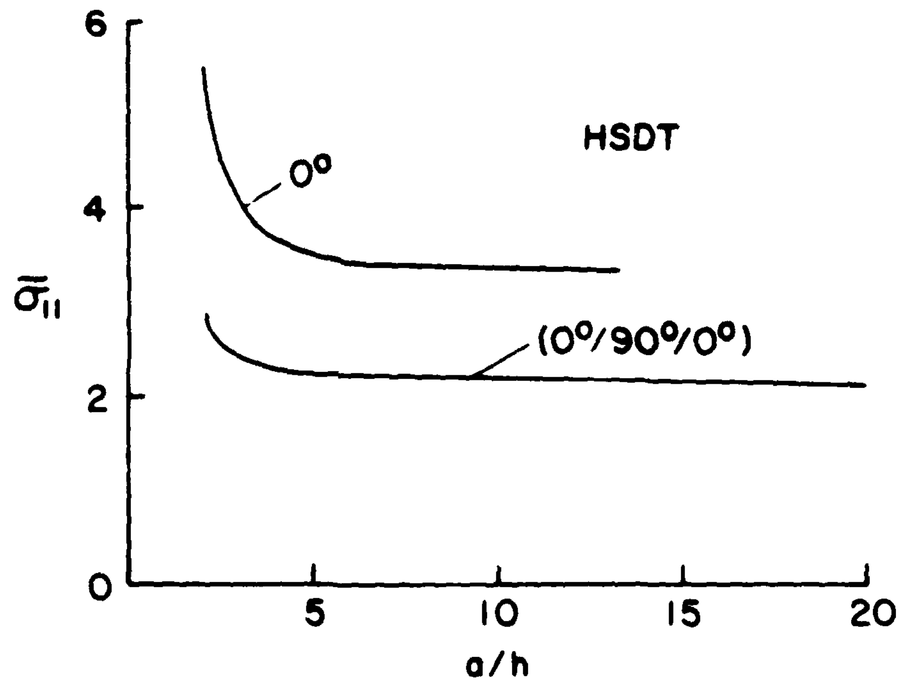



Figure 8. Nondimensionalized center stress versus side-to-thickness ratio for simply supported laminates under uniform load (a/b=4, Material 2).

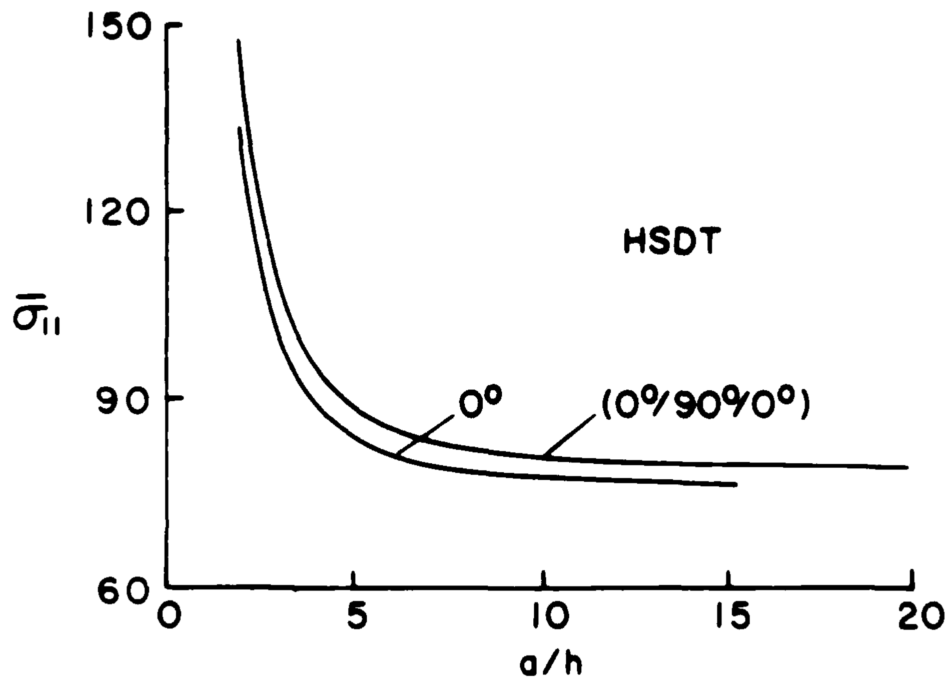



Figure 9. Nondimensionalized center stress versus side-to-thickness ratio for SSFF laminates under uniform load (a/b=4, Material 2).

mismatch of the material properties. When the stresses $(\sigma_x, \sigma_y, \sigma_{xy})$ obtained from the constitutive equations are substituted into the equilibrium equations of elasticity and integrated to determine the transverse shear stresses, the resulting functions will be continuous through the thickness.

Plots of the nondimensionalized center stress, $\bar{\sigma}_{11} = 10^2 \sigma_{11}(\frac{a}{2}, 0, \frac{h}{2})h^2/(q_0 a^2)$, versus side to thickness ratio for simply supported and free-free (SSFF) plates are shown in Figs. 8 and 9. The shear deformation effect is quite significant for a/h ratios smaller than 10.

## A Moderate-Rotation Theory

The theory is a generalization of the classical plate theory, the first-order shear deformation plate theory, and the third-order shear deformation theories of Reddy [37-39]. The theory is based on an assumed displacement field and orders of magnitudes of linear strains and rotations. The associated strain-displacement equations are presented and the equations of motion are derived using the principle of virtual work. Specialization of the equations of motion for various existing theories is presented.

Points of a three dimensional continuum $V$ are denoted by their orthogonal curvilinear coordinates $\underset{\sim}{x} = (x^1, x^2, x^3)$. Covariant and contravariant base vectors at points of the continuum are denoted by $\underset{\sim}{g}_i$ and $\underset{\sim}{g}^i$, respectively. Latin indices are assumed to have values 1, 2, 3, and the Greek indices have values 1, 2. The laminated plate continuum in the undeformed configuration is defined by the Cartesian product of points in the midplane $\Omega$ and the normal $[- h/2, h/2]$:

$$V = \Omega \times [-\frac{h}{2}, \frac{h}{2}]$$

where h denotes the constant thickness of the laminate. Let $x^\alpha$ denote the curvilinear inplane coordinates and $x^3$ be the normal to $\Omega$. The metric tensor components of $\Omega$ are denoted by

$$g_{\alpha\beta} = \underset{\sim}{g}_\alpha \cdot \underset{\sim}{g}_\beta \quad , \quad g^{\alpha\beta} = \underset{\sim}{g}^\alpha \cdot \underset{\sim}{g}^\beta \quad , \quad g^{33} = g_{33} = 1$$
$$\underset{\sim}{g}_\alpha = \frac{\partial \underset{\sim}{r}}{\partial x^\alpha} \quad , \quad \underset{\sim}{g}_\alpha \cdot \underset{\sim}{g}^\beta = \delta_\alpha^\beta \quad , \quad \underset{\sim}{g}_3 = \underset{\sim}{n} \tag{21}$$

where $\underset{\sim}{r}$ is the position vector of a particle $(x^\alpha, x^3)$ at time t, $\delta_\alpha^\beta$ is the Kronecker delta, and $\underset{\sim}{n}$ is the unit normal to the boundary of $\Omega$.

The displacement vector of a point in the plate at time t is of the form

$$\underset{\sim}{u} = u^\alpha \underset{\sim}{g}_\alpha + u^3 \underset{\sim}{n} = u_\alpha \underset{\sim}{g}^\alpha + u_3 \underset{\sim}{n} \tag{22}$$

where the Einestein summation convention on repeated subscripts is assumed. The covariant components of the Green-Lagrange strain tensor are given by

$$\varepsilon_{ij} = \frac{1}{2} (u_{i|j} + u_{j|i} + u_{m|i} u^{m|j}) \tag{23}$$

where a vertical line denotes covariant differentiation. The strain components $\varepsilon_{ij}$ can be expressed in terms of the linearized strains $e_{ij}$ and rotations $\omega_{ij}$ as

$$\varepsilon_{ij} = e_{ij} + \frac{1}{2} e_{mi} e_j^m + \frac{1}{2} (e_{mi} \omega_j^m + e_{mj} \omega_i^m) + \frac{1}{2} \omega_{mi} \omega_j^m \tag{24}$$

where

$$e_{ij} = \frac{1}{2} (u_{i|j} + u_{j|i}) \quad , \quad \omega_{ij} = \frac{1}{2} (u_{i|j} - u_{j|i}) \tag{25}$$

Following [90], we now assume that the strains $\varepsilon_{ij}$ and rotations $\omega_{ij}$ are of the following magnitude:

$$\epsilon_{ij} = O(\theta^2) \quad , \quad \omega_{\alpha\beta} = O(\theta^2) \quad , \quad \omega_{\alpha 3} = O(\theta) \quad , \quad \theta << 1 \qquad (26)$$

Equation (26) implies that the strains and the rotations about the normal to the midplane are small, and that the rotations of a normal to the midplane are moderate. Such assumptions are justified in view of the large inplane rigidity and transverse flexibility of composite laminates.

Neglecting terms of order ($\theta^4$) and higher in the strain displacement equations (24), we obtain

$$\epsilon_{\alpha\beta} = e_{\alpha\beta} + \underline{\frac{1}{2}\,(e_{3\alpha}\omega^3_\beta + e_{3\beta}\omega^3_\alpha)} + \frac{1}{2}\,\omega_{3\alpha}\omega^3_\beta$$

$$\epsilon_{\alpha 3} = e_{\alpha 3} + \underline{\frac{1}{2}\,(e_{\lambda\alpha}\omega^\lambda_3 + e_{33}\omega^3_\alpha) + \frac{1}{2}\,\omega_{\lambda\alpha}\omega^\lambda_3}$$

$$\epsilon_{33} = e_{33} + \underline{e_{\lambda 3}\omega^\lambda_3} + \frac{1}{2}\,\omega_{\lambda 3}\omega^\lambda_3 \qquad (27)$$

where the underlined terms are of order ($\theta^3$).

The present theory is based on the following assumed variation of the displacement components across the plate thickness:

$$u_\alpha(x^\beta, x^3, t) = u^0_\alpha(x^\beta, t) - x^3 u^0_{3|\alpha} + f(x^3)u^1_\alpha(x^\beta, t)$$

$$u_3(x^\beta, x^3, t) = u^0_3(x^\beta, t) + \hat{u}^0_3(x^\beta, t) \qquad (28)$$

where f is a specified function of the thickness coordinate $x^3$. Note that the transverse deflection is assumed to be independent of $x^3$ and consists of two parts, one due to bending and the other due to transverse shear. The particular form of displacement field is assumed in order to include the displacement fields of the classical plate theory (set $\hat{u}^0_3 = 0$ and $u^1_\alpha = 0$), the first-order shear deformation theory [set $u_3^0 = 0$ and $f(x^3) = x^3$], and the third-order shear deformation theory of Reddy [39] [set $u_3^0 = 0$ and $f(x^3) = x^3[1 - \frac{4}{3}(\frac{x^3}{h})^2]$], among others.

For the displacement field in Eq. (28), the strains for the moderate rotation theory become [consistent with the assumptions in Eq. (26)],

$$
\begin{aligned}
\varepsilon_{\alpha\beta} &= \varepsilon^0_{\alpha\beta} + x^3\varepsilon^1_{\alpha\beta} + f\kappa_{\alpha\beta} \\
\varepsilon_{\alpha 3} &= \varepsilon^0_{\alpha 3} + g\hat{\varepsilon}^0_{\alpha 3} + x^3\kappa^0_{\alpha 3} + gx^3\kappa^1_{\alpha 3} + f\varepsilon^1_{\alpha 3} + fg\tilde{\varepsilon}^1_{\alpha 3} \\
\varepsilon_{33} &= \varepsilon^0_{33} + g\tilde{\varepsilon}^0_{33} + g^2\hat{\varepsilon}^0_{33}
\end{aligned}
\tag{29}
$$

where $g = df/dx^3$, and

$$
\begin{aligned}
&\varepsilon^0_{\alpha\beta} = \frac{1}{2}(u^0_{\alpha|\beta} + u^0_{\beta|\alpha}) + \frac{1}{2}(u^0_{3|\alpha} + \hat{u}^0_{3|\alpha})(u^0_{3|\beta} + \hat{u}^0_{3|\beta}) \\
&\varepsilon^1_{\alpha\beta} = -u^0_{3|\alpha\beta}\ ,\ \kappa_{\alpha\beta} = \frac{1}{2}(u^1_{\alpha|\beta} + u^1_{\beta|\alpha})\ ,\ \varepsilon^0_{\alpha 3} = \frac{1}{2}(\hat{u}^0_{3|\alpha} - u^0_{\beta|\alpha}u^0_{3|\beta}) \\
&\hat{\varepsilon}^0_{\alpha 3} = \frac{1}{2}(u^1_{\alpha} + u^0_{\beta|\alpha}u^1_{\beta})\ ,\ \kappa^0_{\alpha 3} = \frac{1}{2}u^0_{3|\lambda\alpha}u^0_{3|\lambda}\ ,\ \kappa^1_{\alpha 3} = -\frac{1}{2}u^0_{3|\lambda\alpha}u^1_{\lambda} \\
&\varepsilon^1_{\alpha 3} = -\frac{1}{2}u^1_{\lambda|\alpha}u^0_{3|\lambda}\ ,\ \hat{\varepsilon}^1_{\alpha 3} = \frac{1}{2}u^1_{\lambda|\alpha}u^1_{\lambda}\ ,\ \varepsilon^0_{33} = \frac{1}{2}u^0_{3|\alpha}u^0_{3|\alpha} \\
&\tilde{\varepsilon}^0_{33} = -u^0_{3|\alpha}u^1_{\alpha}\ ,\ \hat{\varepsilon}^0_{33} = \frac{1}{2}u^1_{\alpha}u^1_{\alpha}
\end{aligned}
\tag{30}
$$

The dynamic version of the principle of virtual displacements is used to derive variationally consistent equations of motion associated with the displacement field in Eq. (28). The principle can be stated, in the absence of body forces and prescribed tractions, as

$$
0 = \int_0^T [\int_V (\sigma^{ij}\delta\varepsilon_{ij})dV + \int_\Omega q\delta u_3 dA - \int_V \rho(\dot{u}_i\delta\dot{u}_i)dV]dt \tag{31}
$$

where $\sigma^{ij}$ denote the contravariant components of the symmetric stress tensor, $q = q(x^\alpha)$ is the distributed transverse force per unit area, and $\rho$ is the density of the material of the plate. The superposed dot denotes the time derivative, $\dot{u} \equiv \partial u/\partial t$. We introduce the couples and inertias,

$$(N^{\alpha\beta}, M^{\alpha\beta}, P^{\alpha\beta}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma^{\alpha\beta}(1, x^3, f)dx^3$$

$$(Q^{\alpha}, \hat{Q}^{\alpha}, R^{\alpha}, \hat{R}^{\alpha}, S^{\alpha}, \hat{S}^{\alpha}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma^{\alpha 3}(1, g, x^3, x^3 g, f, fg)dx^3$$

$$(N^3, \tilde{N}^3, \hat{N}^3) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma^{33}(1, g, g^2)dx^3 \tag{32}$$

$$I_o = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho dx^3 \ , \ I_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho x^3 dx^3 \ , \ I_1^f = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho f dx^3$$

$$I_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho (x^3)^2 dx^3 \ , \ \hat{I}_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho x^3 f dx^3 \ , \ I_2^f = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho f^2 dx^3 \tag{33}$$

The equations of motion of the theory are obtained by substituting Eq. (30) for the strains in terms of the displacements $(u_\alpha^o, u_3^o, \hat{u}_3^o, u_\alpha^1)$ into Eq. (31), integrating by parts to transfer differentiation from the displacements to the stress resultants and couples, collecting the coefficients of the various virtual displacements, and invoking the fundamental lemma of the calculus of variations. We obtain the following six equations:

$$\delta u_\alpha^o: \quad N^{\alpha\beta}|_\beta - \underline{(Q^\beta u^o_{3|\alpha})}|_\beta + \underline{(\hat{Q}^\beta u^1_\alpha)}|_\beta = I_o\ddot{u}^o_\alpha - I_1\ddot{u}^o_{3|\alpha} + I_1^f\ddot{u}^1_\alpha$$

$$\delta u_3^o: \quad M^{\alpha\beta}|_{\alpha\beta} + [N^{\alpha\beta}(u^o_{3|\beta} + \hat{u}^o_{3|\beta})]|_\alpha - \underline{(Q_\alpha u^o_{\beta|\alpha})}|_\beta$$

$$- \underline{(R^\alpha|_\alpha u^o_{3|\beta})}|_\beta + \underline{(\hat{R}^\alpha u^1_\beta)}|_{\beta\alpha} - \underline{(S^\alpha u^1_{\beta|\alpha})}|_\beta + \underline{(N^3 u^o_{3|\alpha})}|_\alpha$$

$$- \underline{(\tilde{N}^3 u^1_\alpha)}|_\alpha = q + I_o(\ddot{u}^o_3 + \ddot{\hat{u}}^o_3) + I_1\ddot{u}^o_{\alpha|\alpha} + \hat{I}_2\ddot{u}^1_{\alpha|\alpha} + I_2\ddot{u}^o_{3|\alpha\alpha}$$

$$\delta\hat{u}_3^o: \quad [N^{\alpha\beta}(u^o_{3|\beta} + \hat{u}^o_{3|\beta})]|_\alpha + Q^\alpha|_\alpha = q + I_o(\ddot{u}^o_3 + \ddot{\hat{u}}^o_3)$$

$$\delta u_\alpha^1: \quad P^{\alpha\beta}|_\beta - \hat{Q}^\beta(\delta_{\alpha\beta} + \underline{u^o_{\alpha|3}}) + \underline{\hat{R}^\beta u^o_{3|\alpha\beta}} + \underline{\hat{S}^\beta|_\beta u^1_\alpha}$$

$$- \underline{(S^\beta u^o_{3|\alpha})}|_\beta - \underline{\hat{N}^3 u^1_\alpha} + \underline{\tilde{N}^3 u^o_{3|\alpha}} = I_1^f\ddot{u}^o_\alpha - \hat{I}_2\ddot{u}^o_{3|\alpha} + I_2^f\ddot{u}^1_\alpha \qquad (34)$$

where the underlined terms are entirely due to the inclusion of moderate rotations (i.e., over and above the von Karman nonlinear terms).

Equations (34) can be specialized to the three different theories discussed earlier. The equations are summarized below:

(i) <u>Classical Plate Theory</u> ($\hat{u}^o_3 = 0, \;\; u^1_\alpha = 0$)

$$N^{\alpha\beta}|_\beta - \underline{(Q^\beta u^o_{3|\alpha})}|_\beta = I_o\ddot{u}^o_\alpha$$

$$M^{\alpha\beta}|_{\alpha\beta} + (N^{\alpha\beta}u^o_{3|\beta})|_\alpha - \underline{(Q^\alpha u^o_{\beta|\alpha})}|_\beta - \underline{(R^\alpha|_\alpha u^o_{3|\beta})}|_\beta + \underline{(N^3 u^o_{3|\alpha})}|_\alpha$$

$$= q + I_o\ddot{u}^o_3 + I_1\ddot{u}^o_{\alpha|\alpha} - I_2\ddot{u}^o_{3|\alpha\alpha} \qquad (35)$$

(ii) <u>First-Order Shear Deformation Plate Theory</u>

($u^o_3 = 0, \;\; f = x^3$)

$$N^{\alpha\beta}|_\beta + \underline{(Q^\beta u^1_\alpha)}|_\beta = I_o\ddot{u}^o_\alpha + I_1\ddot{u}^1_\alpha$$

$$Q^\alpha|_\alpha + N^{\alpha\beta}\hat{u}^o_{3|\beta})|_\alpha = q + I_o\ddot{\hat{u}}^o 3$$

$$M^{\alpha\beta}|_\beta - Q^\beta(\delta_{\alpha\beta} + \underline{u^o_{\alpha|\beta}}) - \underline{N^3 u^1_\alpha} + \underline{R^\beta|_\beta u^1_\alpha} = I_1\ddot{u}^o_\alpha + I_2\ddot{u}^1_\alpha \qquad (36)$$

(iii) Third-Order Shear Deformation Plate Theory

$$(u_3^0 = 0, \quad f = x^3[1 - \frac{4}{3}(x^3|h)^2])$$

$$N^{\alpha\beta}|_\beta + \underline{(\hat{Q}^\beta u_\alpha^1)}|_\beta = I_0\ddot{u}_\alpha^0 + I_1^f\ddot{u}_\alpha^1$$

$$Q^\alpha|_\alpha + (N^{\alpha\beta}\hat{u}^0_{3|\beta})|_\alpha = q + I_0\ddot{\hat{u}}_3^0$$

$$P^{\alpha\beta}|_\beta - \hat{Q}^\beta(\delta_{\alpha\beta} + \underline{u^0_{\alpha|\beta}}) + \underline{\hat{S}^\beta|_\beta u^1_\alpha} - \underline{\hat{N}^3 u^1_\alpha} = I_1^f\ddot{u}^0_\alpha + I_2^f\ddot{u}^1_\alpha \qquad (37)$$

Note that several other theories can be obtained from Eq. (35) as special cases. For example, the refined theory of Kromm [22,23], along with that of Basset [5], can be obtained by setting $\hat{u}_3^0 = 0$.

## References


1. Stoker, J. J., "Mathematical Problems Connected with the Bending and Buckling of Elastic Plates," Bull. Amer. Math. Soc., Vol. 48, pp. 247-261, 1942.

2. Reissner, E., "On the Theory of Bending of Elastic Plates," J. Math. Phy., Vol. 23, pp. 184-191, 1944.

3. Reissner, E., "The Effect of Transferse Shear Deformation on the Bending of Elastic Plates," J. Appl. Mech., Vol. 12, No. 1, pp. A-69 to A-77, 1945.

4. Reissner, E., "On Bending of Elastic Plates," Q. Appl. Math., Vol. 5, pp. 55-68, 1947.

5. Basset, A. B., "On the Extension and Flexure of Cylindrical and Spherical Thin Elastic Shells," Phil. Trans. Royal Soc., (London), Ser. A, Vol. 181, No. 6, pp. 433-480, 1890.

6. Hildebrand, F. B., Reissner, E. and Thomas, G. B., "Notes on the Foundations of the Theory of Small Displacements of Orthotropic Shells," NACA Technical Note No. 1833, March 1949.

7. Mindlin, R. D., "Influence of Rotatory Inertia and Shear on Flexural Motions of Isotropic, Elastic Plates," J. Appl. Mech., Vol. 18, pp. 31-38, 1951.

8. Hencky, H., "Über Die Berücksichtigung Der Schubverzerrung in Ebenen Platten," Ing. Arch., Vol. 16, pp. 72-76, 1947.

9. Reddy, J. N., "A Review of the Literature on Finite-Element Modeling of Laminated Composite Plates," The Shock and Vibration Digest, Vol. 17, No. 4, pp. 3-8, April 1985.

10. Ambartsumian, S. A., "On the Theory of Anisotripic Shells and Plates," IUTAM Symp. on non-homogenity in elasticity and plasticity, Warsaw, Pergamon Press, 1958.

11. Ambartsumian, S. A., Theory of Anisotropic Shells, NASA Report TT F-118, 1964.

12. Ambartsumian, S. A., "Theory of Anisotropic Plates," Izdat. Nauka, Moskva (1967), English transl. by Technomic Publishing Co. (1969).

13. Boal, J. L. and Reissner, E., "Three-dimensional Theory of Elastic Plates with Transverse Inextensibility," J. Math. Phys., Vol. 39, pp. 161-181, 1960.

14. Bolle, L.. "Contribution an Probleme Lineaire de Flexion Oune Plaque Elastique," Bull. Technique de la Suisse Romande, Parts 1 and 2, Vol. 73, pp. 281-285 and 293-298, 1947.

15. Cheng, S., "Elasticity Theory of Plates and a Refined Theory," J. Appl. Mech., Vol. 60, pp. 644-650, 1979.

16. Gol'denveizer, A. L., "O Teorii Izgiba Plastinok Reissnera (On Reissner's Theory of the Bending of Plates)," Izvestiya AN SSSR, OTN, No. 4, pp. 102-109, 1958 (translation of the paper available as NASA Technical Translation F-27, May 1960).

17. Gol'denveizer, A. L., "Derivation of an Approximate Theory of Bending of a Plate by the Method of Asymptotic Integration of the Equations of the Theory of Elasticity," Priklad matem. i mechan, Vol. 26, p. 4, 1962.

18. Gol'denveizer, A. L., "Derivation of an Approximate Theory of Bending of a Plate by the Method of Asymptotic Integration of the Equations of the Theory of Elasticity," Priklad, matem. i mechan, Vol. 27, p. 4, 1963.

19. Green, A. E., "On Reissner's Theory of Bending of Elastic Plates," Q. Appl. Math., Vol. 7, No. 2, pp. 223-228, 1949.

20. Green, A. E. and Naghdi, P. M., "Some Remarks on the Linear Theory of Shells," Quart. Journ. Mech. and Appl. Math., Vol. 18, No. 3, pp. 257-276, 1965.

21. Green, A. E. and Naghdi, P. M., "A Theory of Laminated Composite Plates," IMA J. Appl. Mathem., Vol. 29, pp. 1-23, 1982.

22. Kromm, A., "Verallgeneinerte Theorie der Plattenstatik," Ing. Arch., Vol. 21, pp. 266-286, 1953.

23. Kromm, A., "Über die RandquerKrafte bei gestutzten Platten," ZAMM, Vol. 35, pp. 231-242, 1955.

24. Levinson, M., "An Accurate Simple Theory of the Statics and Dynamics of Elastic Plates," Mech. Res. Commun., Vol. 7, pp. 343-350, 1980.

25. Librescu, L., "On the Theory of Elastic Anisotropic Shells and Plates," Inzh. Zhurnal, Vol. 4, p. 3, 1964.

26. Librescu, L., "On the Thermoelastic Problem of Shells Treated by Eliminating the Love-Kirchhoff Hypothesis," in Non-Classical Shell-Problems, North-Holland Publ. Co., Amsterdam, P.W.N., Warsaw, 1964.

27. Librescu, L., "The Thermoelastic Problem of Shells and Plates Approached by Eliminating the Love-Kirchhoff Hypothesis," St. Cer. Mec. Apl. Vol. 21, No. 2, pp. 351-365, 1966.

28. Librescu, L., "On the Theory of Anisotropic Elastic Shells and Plates," Int. J. Solids Struct., Vol. 3, pp. 53-68, 1967.

29. Librescu, L., Elastostatics and Kinetics of Anisotropic and Heterogeneous Shell-Type Structures, Noordhoff, Leyden, The Nethrlands, 1975.

30. Librescu, L., "On the Refined Theory of Multilayered Anisotropic Shells," Mechanika Polimerov, Part I, 6, 1974, Part II, 1, 1975 (English translation by Plenum Press).

31. Lo, K. H., Christensen, R. M. and Wu, E. M., "A High-Order Theory of Plate Deformation," Parts 1 and 2, J. Appl Mech., Vol. 44, pp. 663-676, 1977.

32. Medwadowski, S. J., "A Refined Theory of Elastic, Orthotropic Plates," J. Appl. Mech., Vol. 25, pp. 437-443, 1958.

33. Murthy, M. V. V., "An Improved Transverse Shear Deformation Theory for Laminated Anisotropic Plates," NASA TP-1903, Nov. 1981.

34. Nelson, R. B. and Lorch, D. R., "A Refined Theory for Laminated Orthotropic Plates," J. Appl. Mech., Vol. 41, pp. 177-183, 1974.

35. Pagano, N. J., "Exact Solutions for Composite Laminates in Cylindrical Bending," J. Composite Materials, Vol. 3, No. 3, pp. 398-411, 1969.

36. Pagano, N. J., "Exact Solutions for Rectangular Bidirectional Composites and Sandwich Plates," J. Composite Materials, Vol. 4, pp. 20-34, 1970.

37. Reddy, J. N., "A Refined Nonlinear Theory of Plates with Transverse Shear Deformation," Int. J. Solids & Struct., Vol. 20, No. 9/10, pp. 881-896, 1984.

38. Reddy, J. N., "A Simple Higher-Order Theory for Laminated Composite Plates," J. Appl. Mech., Vol. 51, pp. 745-752, 1984.

39. Reddy, J. N., Energy and Variational Methods in Applied Mechanics, John Wiley, New York, 1984.

40. Reddy, J. N., "On Kinematic and Constitutive Models of Laminated Composite Plates," Proc. ARO Workshop on Constitutive Models, Virginia Polytechnic Institute and State University, Blacksburg, VA, March 24-26, 1986, SIAM Publications, Philadelphia.

41. Rehfield, W. L., and Valisetty, R. R., "A Simple Refined Theory for Bending and Stretching of Homogeneous Plates," AIAA Journal, Vol. 22, pp. 90-95, Jan. 1984.

42. Reissner, E., "On the Foundations of the Theory of Elastic Shells," Proc. 11th Intl. Congr. Appl. Mech., Münich, 1964, H. Görtler (ed.) pp. 20-30, Springer 1966.

43. Reissner, E., "A Consistent Treatment of Transverse Shear Deformations in Laminated Anisotropic Plates," AIAA J., Vol. 10, No. 5, pp. 716-718, 1972.

44. Reissner, E., "On Transverse Bending Including the Effect of Transverse Shear Deformation," Int. J. Solids & Struct., Vol. 11, pp. 569-573, 1975.

45. Reissner, E., "On the Theory of Transverse Bending of Elastic Plates," Int. J. Solids & Struct., Vol. 12, pp. 545-554, 1976.

46. Reissner, E., "Note on the Effect of Transverse Shear Deformation in Laminated Anisotropic Plates," Comput. Meth. Appl. Mech. Engng., Vol. 20, No. 2, pp. 203-209, 1979.

47. Reissner, E., "Reflections on the Theory of Elastic Plates," Appl. Mech. Rev., Vol. 38, No. 11, pp. 1453-1464, 1985.

48. Schmidt, R., "A Refined Nonlinear Theory of Plates with Transverse Shear Deformations," J. Industrial Mathematics Society, Vol. 27, Part 1, pp. 23-28, 1977.

49. Shirakawa, K., "Bending of Plates Based on Improved Theory," Mech. Res. Commun., Vol. 10, pp. 205-211, 1983.

50. Volterra, E., "Effect of Shear Deformations on Bending of Rectangular Plates," J. Appl. Mech., Vol. 27, No. 3, pp. 594-596, 1960.

51. Whitney, J. M., "The Effect of Transverse Shear Deformation on the Bending of Laminated Plates," J. Comp. Materials, Vol. 3, No. 3, pp. 534-547, 1969.

52. Whitney, J. M. and Leissa, A. W., "Analysis of Heterogeneous Anisotropic Plates," J. Appl. Mech., Vol. 36, pp. 261-266, 1969.

53. Whitney, J. M. and Pagano, N. J., "Shear Deformation in Hetergeneous Anisotropic Plates," J. Appl. Mech., Vol. 37, pp. 1031-1036, 1970.

54. Whitney, J. M. and Sun, C. T., "Transient Response of Laminated Composite Plates Subjected to Transverse Dynamic Loading," J. Acoust. Soc. Am., Vol. 61, pp. 101-104, 1977.

55. Wilson, P. E. and Boresi, A. P., "Annotated Bibliography of Selected References on the Theory of Elastic Plates," T.A.M., Report 158, University of Illinois, 1960.

56. Yang, P. C., Norris, C. H. and Stavsky, Y., "Elastic Wave Propagation in Heterogeneous Plates," Int. J. Solids & Struct., Vol. 2, No. 4, pp. 665-684, 1966.

58. Westbrook, D. R., "A Symbolic Approach to Dynamical Problems in Plates," Research Paper No. 22, June 1967, Univ. of Calgary, Canada.

59. Westbrook, D. R., "Symbolic Approach to Dynamical Problems in Plates," J. Acoustical Soc. of America, Vol. 44, No. 4, pp. 1083-1092, 1968.

60. Schipper, J. F., "An Exact Formulation of the Linear Equation for Thick Orthotropic Shells with Arbitrary, Imposed Temperature and Force Fields and Temperature-Dependent Parameters," Developments in Theoretical Mechanics, Vol. 3, Edited by W. A. Shaw, pp. 255-278, 1967.

61. Cicala, P., "Consistent Approximations in Shell Theory," J. Engng. Mech. Div. Proc. Amer. Soc. Civil Eng., Vol. 88, p. 4, 1962.

62. Teodorescu, P. P., "Spatial Problems in the Theory of Elasticity," Edit. Acad. RSR, Bucuresti, pp. 349-360, 1970.

63. Tiffen, R., "An Investigation of the Transverse Displacement Equations of Elastic Plate Theory," Quart. J. Mech. & Appl. Math., Vol. 14, p. 59, 1961.

64. Tiffen, R. and Lowe, P. G., "An Exact Theory of Plane Stress," J. London Math. Soc., Vol. 40, p. 72, 1962.

65. Poniatovskii, V. V., "On the Bending Theory of Anisotropic Plates," Prikladimatem, i mechan, Vol. 28, p. 6, 1964.

66. Srinivas, C. V., Rao, J. and Rao, A. K., "An Exact Analysis for Vibration of Simply Supported Homogeneous and Laminated Thick Rectangular Plates," J. Sound & Vibration, Vol. 12, pp. 187-199, 1970.

67. Srinivas, S. and Rao, A. K., "Bending, Vibration and Buckling of Simply Supported Thick Orthotropic Rectangular Plates and Laminates," Int. J. Solids & Struct., Vol. 6, pp. 1463-1481, 1970.

68. Hussainy, S. A. and Srinivas, S., "Flexure of Rectangular Composite Plates," Fibre Science and Technology, Vol. 8, pp. 59-76, 1975.

69. Bert, C. W. and Chen, T. L. C., "Effect of Shear Deformation of Antisymmetric Angle-Ply Laminated Rectangular Plates," Int. J. Solids Struct., Vol. 14, pp. 465-473, 1978.

70. Reddy, J. N. and Chao, C. W., "A Comparison of Closed Form and Finite Element Solutions of Thick Laminated Anisotropic Rectangular Plates," Nuclear Engng. Design, Vol. 64, pp. 153-167, 1981.

71. Reddy, J. N. and Phan, N. D., "Stability and Vibration of Isotropic, Orthotropic and Laminated Plates According to a Higher-Order Shear Deformation Theory," J. Sound & Vibration, Vol. 98, No. 2, pp. 157-170, 1985.

72. Sun, C. T. and Whitney, J. M., "Forced Vibrations of Laminated Composite Plates in Cylindrical Bending," J. Acoust. Soc. Am., Vol. 55, pp. 1003-1008, 1974.

73. Sun, C. T. and Whitney, J. M., "Dynamic Response of Laminated Composite Plates Under Initial Stress," AIAA Journal, Vol. 14, pp. 268-270, 1976.

74. Sun, C. T. and Chattopadhyay, S., "Dynamic Response of Anisotropic Plates Under Initial Stress Due to Impact of a Mass," J. Appl. Mech., Vol. 42, pp. 693-698, 1975.

75. Dobyns, A. I., "Analysis of Simply-Supported Orthotropic Plates Subject to Static and Dynamic Loads," AIAA J., Vol. 19, No. 5, pp. 642-650, 1981.

76. Reddy, J. N., "On the Solutions to Forced Motions of Rectangular Composite Plates," J. Appl. Mech., Vol. 49, pp. 403-408, 1982.

77. Reddy, J. N., "A Note on Symmetry Considerations in the Transient Response of Symmetrically Laminated Anisotropic Plates," Int. J. Num. Engng., Vol. 20, pp. 175-181, 1984.

78. Ebcioglu, I. K., "A Large Deflection Theory of Anisotropic Plates," Ingenieur-Archiv, Vol. 33, pp. 396-403, 1964.

79. Chia, C. Y., Nonlinear Analysis of Plates, McGraw-Hill, New York, 1980.

80. Reddy, J. N., "Geometrically Nonlinear Transient Analysis of Laminated Composite Plates," AIAA Journal, Vol. 21, No. 4, pp. 621-629, 1983.

81. Naghdi, P. M ., "The Theory of Shells and Plates," in Handbuch der Physik, Vol. VIa/2, S. Flugge (ed.), pp. 425-640, Springer-Verlag, New York, 1972.

82. Yokoo, Y. and Matsunaga, H., "A General Theory of Elastic Shells," Int. J. Solids & Struct., Vol. 10, pp. 261-274, 1974.

83. Habip, L. M., "Theory of Elastic Plates in the Reference State," Int. J. Solids & Struct., Vol. 2, pp. 157-166, 1966.

84. Pietraszkiewicz, W., Finite Rotations and Lagrangian Description in the Non-Linear Theory of Shells, Polish Academy of Sciences, Institute of Fluid-Flow Machinary, Polish Scientific Publishers, Warsaw, Poland, 1979.

85. Sanders, J. L., "Nonlinear Theories for Thin Shells," Quart. Appl. Math., Vol. 21, pp. 21-36, 1963

86. Koiter, W. T., "On the Nonlinear Theory of Thin Elastic Shells," Proc. Kon. Ned. Ak. Wet., Series B., Vol. 69, pp. 1-54, 1966.

87. Reissner, E., "Rotationally Symmetric Problems in the Theory of Thin Elastic Shells," Proc. 3rd U.S. Nat. Congr. of Appl. Mech., pp. 51-69, 1958.

88. Pietraszkiewicz, W., "Lagrangian Desription and Incremental Formulation in the Nonlinear Theory of Thin Shells," Int. J. Non-Linear Mechanics, Vol. 19, pp. 115-140, 1984.

89. Naghdi, P. M. and Vongsarnpigoon, L., "A Theory of Shells with Small Strains Accompanied by Moderate Rotations," Arch. for Rational Mechanics and Analysis, Vol. 83, pp. 245-283, 1983.

90. Librescu, L. and Schmidt, R., "Higher-Order Moderate Rotation Theories for Elastic Anisotropic Plates," Finite Rotations in Structural Mech., (Proc. of Euromech Symposium), No. 197, Jablonna, Poland, 1985, W. Pietraszkiewicz (ed.), Springer-Verlag, Berlin, pp. 158-174, 1986.

91. Franklin, J. N., Matrix Theory, Prentice-Hall, Englewood Cliffs, NJ, 1968.

## 3. PUBLICATIONS RESULTING FROM THE RESEARCH


1. J. N. Reddy and N. D. Phan, "Stability and Vibration of Isotropic, Orthotropic and Laminated Plates According to a Higher-Order Deformation Theory," J. Sound and Vibration, Vol. 98, No. 2, pp. 157-170, 1985.

2. J. N. Reddy and C. F. Liu, "A Higher-Order Shear Deformation Theory of Laminated Elastic Shells," Int. J. Engng. Sci., Vol. 23, No. 3, pp. 319-330, 1985.

3. J. N. Reddy, "A Refined Shear Deformation Theory for Laminated Anisotropic Plates," ARO Report 85-1, Trans. Second Army Conf. on Appl. Math. and Computing, pp. 331-342, 1985.

4. C. F. Liu, "Geometrically Nonlinear Analysis of Composite Laminates Using a Refined Shear Deformation Shell Theorey," Ph.D. Dissertation, Virginia Polytechnic Institute and State University, Blacksburg, VA, Dec. 1985.

5. N. S. Putcha and J. N. Reddy, "A Refined Mixed Shear Flexible Finite Element for the Nonlinear Analysis of Laminated Plates," Computers & Structures, Vol. 22, No. 2, pp. 529-538, 1986.

6. N. S. Putcha and J. N. Reddy, "Stability and Natural Vibration Analysis of Laminated Plates by Using a Mixed Element Based on a Refined Plate Theory," J. Sound and Vibration, Vol. 104, No. 2, pp. 285-300, 1986.

7. J. N. Reddy, "On Mixed Finite-Element Formulations of a Higher-Order Theory of Composite Laminates," Finite Element Methods for Plate and Shell Structures, T. J. R. Hughes and E. Hinton (eds.), Pineridge Press, U.K., pp. 31-57, 1986.

8. J. N. Reddy and E. Barbero, "A Refined Nonlinear Theory of Laminated Composite Plates," Proc. of The Fourth Annual Review of Virginia Tech Center for Composite Materials and Structures, May 11-14, 1987.

9. A. K. Pandey and J. N. Reddy, "Failure Analysis of Composite Plates and Shells Using a Macroscopic Failure Criteria," Proc. of the Fourth Annual Review of Virginia Tech Center for Composite Materials and Structures, May 11-14, 1987.

10. J. N. Reddy, "A General Two-Dimensional Theories of Laminated Composite Plates," Communi. Appl. Numer. Meth., Vol. 3, pp. 173-180, 1987.

11. J. N. Reddy and D. Sandidge, "Mixed Finite Element Models for Laminated Composite Plates," J. Engng. for Industry, Vol. 109, pp. 39-45, 1987.

12. J. N. Reddy, "On a Third-Order Shear Deformation Theory of Laminated Composite Plates," Proc. of XII Southeastern Conference on Theoretical and Applied Mechanics, (Selected papers), to appear.

13. J. N. Reddy, A. A. Khdeir and L. Librescu, "The Levy Type Solutions for Symmetric Rectangular Composite Plates Using the First-Order Shear Deformation Theory," J. Appl. Mech., to appear.

14. L. Librescu, A. A. Khdeir and J. N. Reddy, "A Comprehensive Analysis of the State of Stress of Elastic Anisotropic Flat Plates Using Refined Theories," Acta Mechanica, to appear.

15. J. N. Reddy, "A Small Strain and Moderate Rotation Theory of Elastic Anisotropic Plates," J. Appl. Mech., to appear.

16. A. A. Khdeir, J. N. Reddy and L. Librescu, "Analytical Solution of a Refined Shear Deformation Theory for Rectangular Composite Plates," Int. J. Solids and Structures, to appear.

17. L. Librescu, A. A. Khdeir and J. N. Reddy, "Further Results Concerning the Dynamic Response of Elastic Orthotropic Plates," (with L. Librescu and A. A. Khdeir), AIAA Journal, in review.

18. L. Librescu and J. N. Reddy, "A Critical Review and Generalization of Transverse Shear Deformable Anisotropic Plate Theories," Int. J. Engineering Science, in review.

# 4. SCIENTIFIC PERSONNEL

Faculty

J. N. Reddy (Principal investigator), Professor
L. Librescu,* Professor

Graduate Students

| | |
|---|---|
| R. Arenburg | (Ph.D.) |
| E. Barbero | (Ph.D.) |
| A. Khdeir | (Ph.D.) |
| C. L. Liao | (Ph.D.) |
| C. F. Liu | (Ph.D.) |
| A. K. Pandey | (Ph.D.) |
| N. D. Phan* | (Ph.D.) |
| S. N. Putcha* | (Ph.D.) |
| D. Rourk | (Ph.D.) |

*not supported by the project.

## 5. HONORS AND AWARDS

J. N. Reddy

- Appointed as the Clifton C. Garvin Professor of Engineering at Virginia Tech, 1985
- Received the 1985 Alumni Award for Research Excellence at Virginia Tech.
- Elected as Fellow of the American Academy of Mechanics, 1986
- Received the von Humboldt Foundation Research Award, 1986

N. D. Phan Awarded M.S. degree, 1984

S. N. Putcha Awarded Ph.D. degree, 1984

C. F. Liu Awarded Ph.D. degree, 1985

A. A. Khdeir Awarded Ph.D. degree, 1986

D. Rourk Awarded Ph.D. degree, 1986

C. L. Liao Awarded Ph.D. degree, 1987

A. J. Pandey Awarded Ph.D. degree, 1987
AIAA Jefferson Gablett Award, 1987

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

# REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

| | | |
|---|---|---|
| 1. REPORT NUMBER | 2. GOVT ACCESSION NO. N/A | 3. RECIPIENT'S CATALOG NUMBER N/A |
| 4. TITLE (and Subtitle) A REFINED NONLINEAR ANALYSIS OF LAMINATED COMPOSITE PLATES AND SHELLS | | 5. TYPE OF REPORT & PERIOD COVERED FINAL TECHNICAL REPORT (1985-1987) |
| | | 6. PERFORMING ORG. REPORT NUMBER Report No. VPI-E-87.19 |
| 7. AUTHOR(s) J. N. Reddy | | 8. CONTRACT OR GRANT NUMBER(s) Proposal No. 21203-MA Grant DAAG 29-85-K-00017 |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Engineering Science & Mechanics Virginia Polytechnic Institute and State Univ. Blacksburg, VA 24061 | | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 21203-MA |
| 11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office Post Office Box 12211 Research Triangle Park, NC 27709 | | 12. REPORT DATE August 1987 |
| | | 13. NUMBER OF PAGES 50 |
| 14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) | | 15. SECURITY CLASS. (of this report) Unclassified |
| | | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

NA

18. SUPPLEMENTARY NOTES

The view, opinions, and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Analytical solutions, Finite element analysis, higher-order theories, laminated composite plates and shells, moderate rotation theory, von Karman nonlinearity, shear deformation theories.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This final technical report summarizes in a compact form the results of a two-year research program on the development of refined shear deformation theories of plates and shells, and their analytical solutions. The detailed results are reported in various reports and technical papers during the course of the project. A third-order, nonlinear shear deformation shell theory and finite element model that accounts for parabolic distribution of transverse shear stresses through thickness and the von Karman nonlinear strains was

DD FORM 1473 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) developed during an investigation sponsored by NASA Langley Research Center during the first year of this research. The theory is further refined and extended and analytical solutions are developed during this research. The most significant contributions of this research are:

1. The development of analytical solutions of the linear theory for various boundary conditions. The space-time concept is used to develop such solutions.

2. The development of moderate rotation theory of laminated composite plates.

3. The development of finite element models for first-ply and post-first-ply failure analysis.

Future research related to refined theories should consider the effects of moderate and large rotations and the development of micro- and macromechanical constitutive models for a better understanding of the structural behavior and prediction of progressive damage and failure in composite structures.

END

10-87

DTIC