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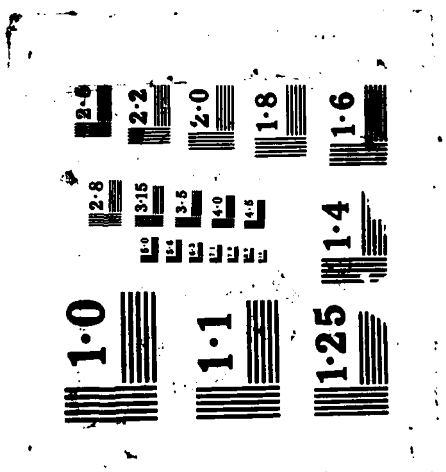

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UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

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We prove an $O(n^2)$ upper bound for the number of separating k-sets in a undirected k-connected graph for fixed k. For fixed k the upper bound is tight up to a constant factor.

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On the Number of Minimum Size Separating Vertex Sets in a Graph

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July 1987

ABSTRACT

We prove an $O(n^2)$ upper bound for the number of separating k-sets in an undirected k-connected graph for fixed k. For fixed k the upper bound is tight up to a constant factor.

1. Introduction

Connectivity is an important graph property and there has been a considerable amount of work on vertex connectivity of graphs ~~[Ev,EvTa,Ga,GiSo,LiLoWi]~~. An undirected graph $G = (V,E)$ is k-connected if for any subset V' of $k-1$ vertices of G the subgraph induced by $V-V'$ is connected [Tu]. A subset V' of k vertices is a separating k-set if the subgraph induced by $V-V'$ is not connected. For k=1 the set V' becomes a single vertex which is called an articulation point, and for k=2,3 the sets V' are called a separating pair and separating triplet, respectively. Efficient algorithms are available for finding all separating k-sets in k-connected undirected graphs for $k \leq 3$. [Ta,HoTa,MiRa,KaRa].

We address the following question: what is the maximum number of separating k-sets in a k-connected undirected graph?

An undirected graph G on n vertices and m edges has a trivial upper bound on the number of separating k-sets of $\binom{n}{k}$ for any k. The graph that achieves it is a graph on n vertices without any edges. For k=1 the maximum number of articulation points in an undirected connected graph is $(n-2)$ and a graph that achieves it is a path on n vertices. For k=2 the maximum number of separating pairs in an undirected biconnected graph is $\frac{n(n-3)}{2}$ and a graph that achieves it is a cycle on n vertices [KaRa2]. For k=3 the maximum number of separating triplets in an

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undirected triconnected graph is $\frac{(n-1)(n-4)}{2}$ and a graph that achieves it is a wheel on n vertices [KaRa2].

We will generalize the result of [KaRa2], to give an $O(3^k n^2)$ upper bound of the number of separating k-sets in an undirected k-connected graphs. The bound is worst-case optimal up to a constant factor for fixed k and we will present a graph (generalizations of cycle and wheel) that achieves it.

2. Upper bound for general k

Let $G=(V,E)$ be an undirected k-connected graph with n vertices and m edges. Let $g(n)$ be the maximum number of separating k-sets for k-connected graphs on n vertices.

Theorem 1 $g(n) = O(3^k n^2)$ for fixed k.

Proof: Let $V' = \{v_1, v_2, \cdots, v_k\}$ be a separating k-set, whose removal separates G into nonempty G_1 and G_2 (see Figure 1).

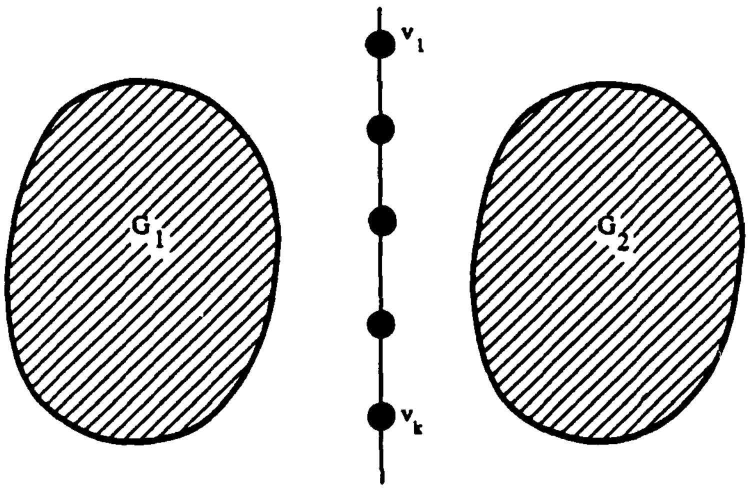

Figure 1.
Separating G into G_1 and G_2 by separating k-set $\{v_1, \cdots, v_k\}$

A separating k-set $\{w_1, w_2, \cdots, w_k\}$ of G is a *cross* separating k-set with respect to V' if $\exists\, i,j: w_i \in G_1$ and $w_j \in G_2$. Let the cardinality of G_1 be l then the cardinality of G_2 is n-l-k. Let the maximum number of cross separating k-sets be $f(l, n-l)$. Then any $g(n)$ that satisfies the recurrence

$$g(n) = \max_l \left[g(l+k) + g(n-l) + f(l, n-l) + 1 \right]$$

is the upper bound on the number of separating k-sets in G.

Lemma 1 $f(l,n-l) \le 3^k(n-l-k)l$

Proof: Let $\{w_1,w_2,\cdots,w_k\}$ be a cross separating k-set with $\{w_1,\cdots,w_s\} \subset G_1$, $\{w_{s+t+1},\cdots,w_k\} \subset G_2$ and $\{w_{s+1},\cdots,w_{s+t}\} \subset \{v_1,\cdots,v_k\}$. The separating k-set $\{w_1,w_2,\cdots,w_k\}$ separates G_1 into G_3 and G_4, separates G_2 into G_5 and G_6, and divides $\{v_1,\cdots,v_k\}$ into $\{v_1,\cdots,v_r\}$, $\{v_{r+t+1},\cdots,v_k\}$ and $v_{r+i} = w_{s+i}$, $i = 1,\ldots,t$. (see Figure 2)

Case 1 None of G_i, $i = 3,4,5,6$ are empty. (see Figure 2)

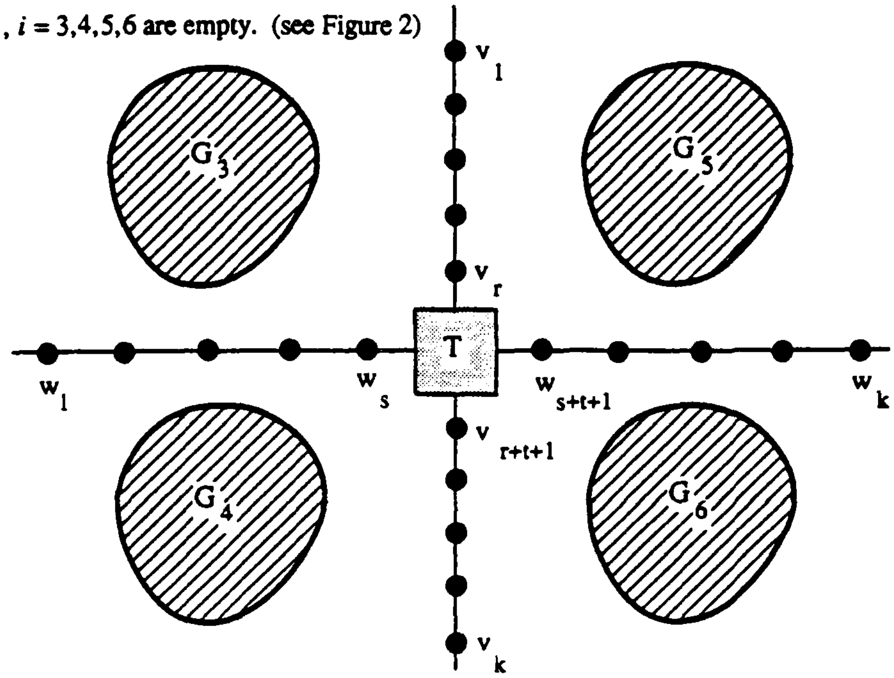

Figure 2.
Separating G into nonempty components by separating k-sets $\{v_1,\cdots,v_k\}$ and $\{w_1,\cdots,w_k\}$.

The sets $\{w_1,w_2,\cdots,w_{s+t},v_1,\cdots,v_r\}$, $\{w_1,w_2,\cdots,w_{s+t},v_{r+t+1},\cdots,v_k\}$, $\{v_1,\cdots,v_{r+t},w_{s+t+1},\cdots,w_k\}$ and $\{v_{r+1},\cdots,v_k,w_{s+t+1},\cdots,w_k\}$ are separating sets of G which separates G_3, G_4, G_5 and G_6 respectively, so their cardinalities are bigger than or equal to k. Then,

$$\begin{cases} s+t+r \ge k \\ r+t+k-s-t \ge k \\ s+t+k-r-t \ge k \\ k-r+k-s-t \ge k \end{cases} \Rightarrow \begin{cases} r+s+t \ge k \\ r \ge s \\ s \ge r \\ k \ge r+s+t \end{cases} \Rightarrow \begin{cases} r = s \\ r+s+t = k \end{cases}$$

We replace the subscript r by s from now on.

Claim 1 $\forall i \ \ i = s+1,\ldots,t \ \exists x_j \in$ nonempty G_j, $j = 3,4,5,6$: $(v_i,x_j) \in E$.

Proof: W.L.O.G. assume $\exists v_i$: $\forall x \in G_3$: $(x, v_i) \notin E$. Then $\{v_1, \cdots, v_{s+t}, w_1, \cdots, w_s\} - \{v_i\}$ is a separating (k-1)-set.

□

Suppose we have a separating k-set $\{w_1, \cdots, w_{s+t}, w_{s+t+1}, \cdots, w_{s+t+a}, u_{s+t+a+1}, \cdots, u_k\}$, where $\{u_{s+t+a+1}, \cdots, u_k\}$ belongs to G_6. This separating k-set separates G_6 into $\hat{G}_6$ and $\overline{G}_6$. (see Figure 3)

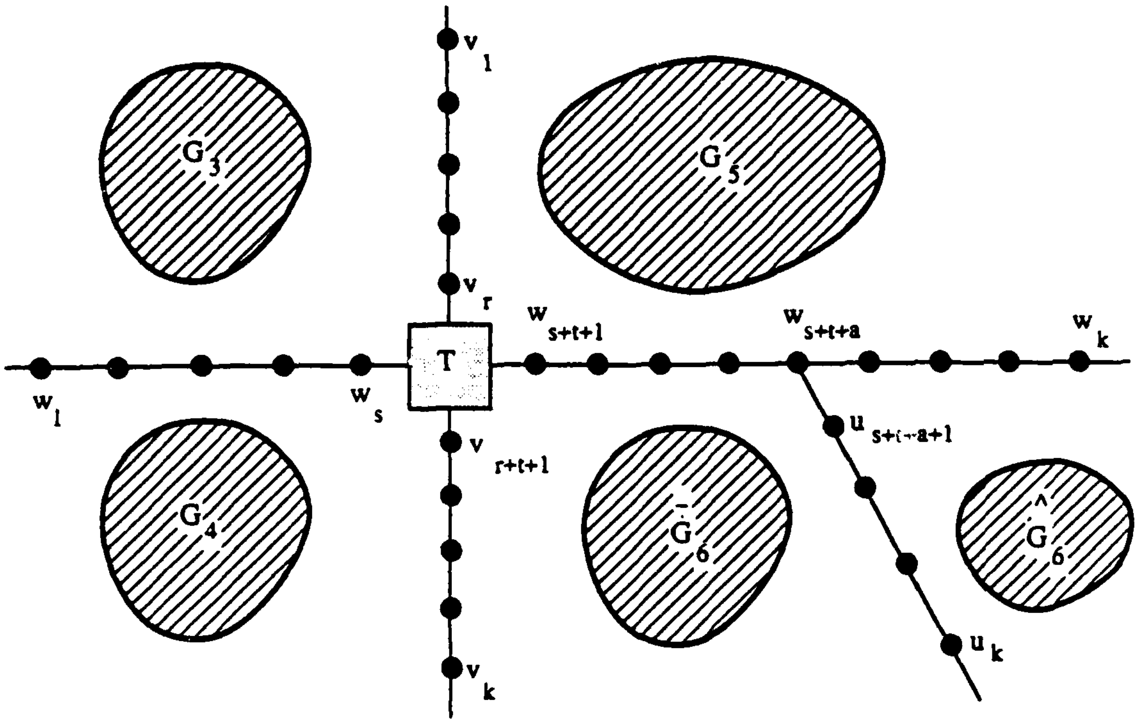

Figure 3.
Illustrating the proof of Claim 2.

Claim 2 $\hat{G}_6$ is empty.

Proof: Suppose it is not empty. Then $\{w_{s+1}, \cdots, w_k, u_{s+t+a+1}, \cdots, u_k\}$ is separating set, so its cardinality must be bigger than or equal to k. But the cardinality of this separating set is less than k since $1 \le a \le k-s-t-1$. This contradiction proves the Claim.

□

Claim 3 Number of separating k-sets with $\{w_1, \cdots, w_{s+t}\}$ fixed with at least one vertex from $\{w_{s+t+1}, \cdots, w_k\}$ is $\le 2^s$.

Proof: If there exists such a separating k-set different from $\{w_1, \cdots, w_k\}$ then by Claim 2 its vertices in G_2 different from $\{w_{s+t+1}, \cdots, w_k\}$ belong to the neighbors of $\{w_{s+t+1}, \cdots, w_k\}$. Assume there exists such a separating k-set $\{w_1, \cdots, w_{s+t+a}, u_{s+t+a+1}, \cdots, u_{s+t+a+b}, y_{s+t+a+b+1}, \cdots, y_k\}$, where $\{u_{s+t+a+1}, \cdots, u_{s+t+a+b}\} \subset G_5$ and $\{y_{s+t+a+b+1}, \cdots, y_k\} \subset G_6$. Also, $a, b \geq 1$ and $k-s-t-a-b \geq 1$. (see Figure 4)

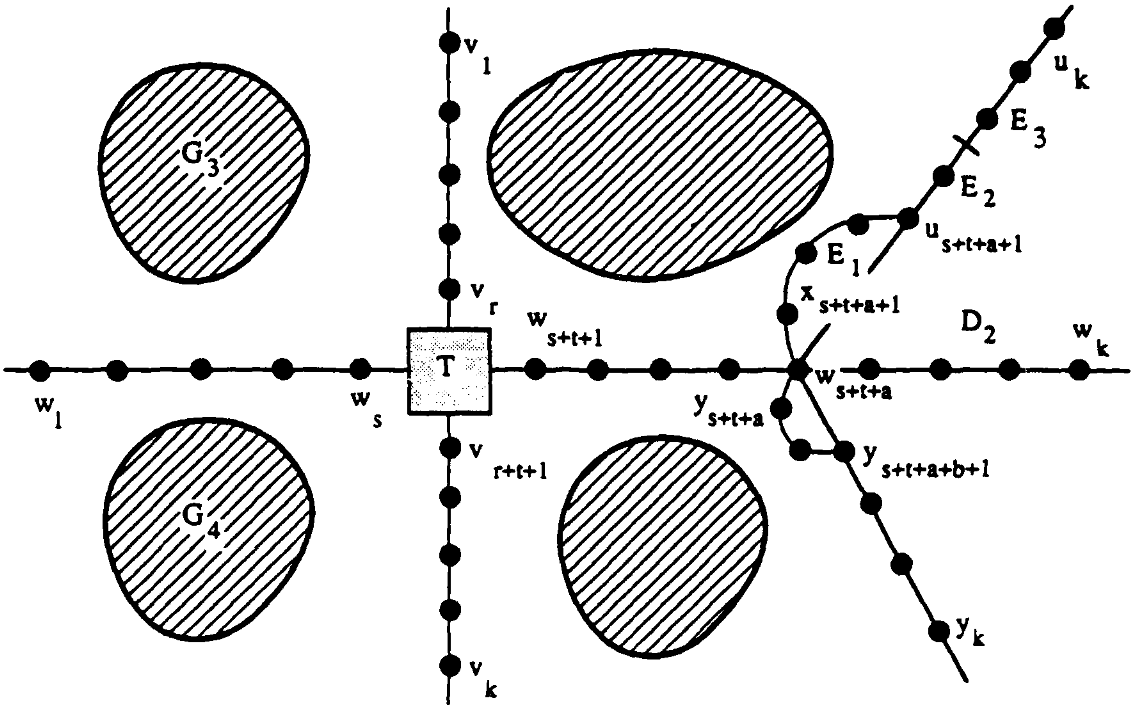

Figure 4.
Illustrating choice for cross separating k-set with $\{w_1, \cdots, w_{s+t}\}$ fixed.

Then $\{w_{s+1}, \cdots, w_{s+t+a}, u_{s+t+a+1}, \cdots, u_{s+t+a+b}, y_{s+t+a+b+1}, \cdots, y_k\}$ separates $\{w_{s+t+a+1}, \cdots, w_k\}$ from the rest of the graph, and cardinality of this separating set is less than k. Hence, if there are separating k-sets $\{w_1, \cdots, w_{s+t+a}, u_{s+t+a+1}, \cdots, u_{s+t+a+b}, y_{s+t+a+b+1}, \cdots, y_k\}$ then either $b = 0$ or $k-s-t-a-b = 0$.

Assume there are two separating k-sets: $\{w_1, \cdots, w_{s+t+a}, u_{s+t+a+1}, \cdots, u_k\}$ and $\{w_1, \cdots, w_{s+t+a}, y_{s+t+a+1}, \cdots, y_k\}$. Then $\{w_{s+1}, \cdots, w_{s+t+a}, u_{s+t+a+1}, \cdots, u_k, y_{s+t+a+1}, \cdots, y_k\}$ separates $\{w_{s+t+a+1}, \cdots, w_k\}$ from the rest of the graph, and cardinality of this separating set is less than k. Hence, if there is separating k-set $\{w_1, \cdots, w_{s+t+a}, u_{s+t+a+1}, \cdots, u_k\}$ then there is no separating k-set $\{w_1, \cdots, w_{s+t+a}, y_{s+t+a+1}, \cdots, y_k\}$.

Assume there are two separating k-sets $\{w_1, \cdots, w_{s+t+a}, u_{s+t+a+1}, \cdots, u_k\}$ and $\{w_1, \cdots, w_{s+t+a}, x_{s+t+a+1}, \cdots, x_k\}$ where x's and u's belong to G_5. Let $\{u_{s+t+a+1}, \cdots, u_k\} \cap \{x_{s+t+a+1}, \cdots, x_k\} = E_2$. Let $\{u_{s+t+a+1}, \cdots, u_k\} - E_2 = E_1$ and $\{x_{s+t+a+1}, \cdots, x_k\} - E_2 = E_3$. Let $\{w_{s+t+a+1}, \cdots, w_k\} = D_2$. If D_2 is not connected to $E_1 \cup E_3$ then $\{w_1, \cdots, w_{s+t+a}\} \cup E_2$ is a separating set with cardinality less than k. So, D_2 is connected to $E_1 \cup E_3$. W.L.O.G. assume $\exists x \in E_1 : (x,z) \in E, y \in D_2$. By Claim 2 E_1 must be in the same set where $G_5 - E_2 - E_3$ with respect to the separating set $\{w_1, \cdots, w_{s+t+a}, x_{s+t+a+1}, \cdots, x_k\}$. But E_1 and D_2 are connected to each other in $G - \{w_1, \cdots, w_{s+t+a}, x_{s+t+a+1}, \cdots, x_k\}$ and do not belong to the same separating component of $G - \{w_1, \cdots, w_{s+t+a}, x_{s+t+a+1}, \cdots, x_k\}$. Hence, there is a unique $\{u_{s+t+a+1}, \cdots, u_k\}$ for each D_2 such that $\{w_1, \cdots, w_{s+t+a}, u_{s+t+a+1}, \cdots, u_k\}$ is a separating k-set.

The number of different D_2 is

$$\le \binom{s}{0} + \binom{s}{1} + \cdots + \binom{s}{s} \le 2^s.$$

Hence, the number of separating k-sets with $\{w_1, \cdots, w_{s+t}\}$ fixed and with at least one vertex from $\{w_{s+t+1}, \cdots, w_k\}$ is $\le 2^s$.

□

The cardinality of the set of neighbors of $\{w_{s+t+1}, \cdots, w_k\}$ in G_5 is bigger than or equal to s. Divide the vertices in G_2 into a collection C of nonintersecting sets $V_1, V_2, \cdots$ of s vertices such that $\{w_1, \cdots, w_{s+t}\} \cup V_i$ is a separating k-set. The cardinality of collection C is $\le \frac{n-l-k}{s}$. Hence, the number of choices for $\{w_{s+t+1}, \cdots, w_k\}$ is $\le 2^s \, \frac{n-l-k}{s}$.

The number of ways $\{w_1, \cdots, w_s\}$ can be chosen for fixed $\{w_{s+1}, \cdots, w_k\}$ is $\le \frac{l}{s} 2^s$, by the symmetry. Hence, for the Case 1

$$f(l, n-l) \le \sum_{t=0}^{k-2} \binom{k}{t} 2^s \, \frac{l}{s} \, 2^{(k-s-t)} \, \frac{n-l-k}{s},$$

$$f(l, n-l) \le l\,(n-l-k) \sum_{t=0}^{k-2} \binom{k}{t} 2^{k-t} \le l\,(n-l-k)\, 3^k.$$

□

Case 2

For the cases, when at least one of the G_i, $i=3,4,5,6$ is empty and $l \ge k$, we will do the following observation. W.L.O.G. assume G_3 is empty. The sets $\{w_1, w_2, \cdots, w_{s+t}, v_{r+t+1}, \cdots, v_k\}$, $\{v_1, \cdots, v_{r+t}, w_{s+t+1}, \cdots, w_k\}$ and $\{v_{r+1}, \cdots, v_k, w_{s+t+1}, \cdots, w_k\}$ are separating sets of G which separates G_4, G_5 and G_6 respectively, so their cardinalities are $\ge k$. Then,

$$\begin{cases} r+t+k-s-t \geq k \\ s+t+k-r-t \geq k \\ k-r+k-s-t \geq k \end{cases} \Rightarrow \begin{cases} r \geq s \\ s \geq r \\ k \geq r+s+t \end{cases} \Rightarrow \begin{cases} r = s \\ r+s+t \leq k \end{cases}$$

By taking $g(l+2k)$ instead of $g(l+k)$ in the recurrence , by taking $g(n-l+k)$ instead of $g(n-l)$ in the recurrence and choosing $\{w_{s+t+1}, \cdots, w_k\}$ from $n-l-2k$ and $\{w_1, w_2, \cdots, w_s\}$ from $l-k$ in the recurrence for $n-2k \geq l \geq k$ we cover all other cases.

□

The solution to the recurrence

$$g(n) = \max_{n-2k \geq l \geq k} \left[g(l+2k) + g(n-l+k) + 3^k (l-k)(n-l-2k) \right]$$

is $g(n) \leq \frac{1}{2} 3^k n^2 - \frac{9}{2} 3^k k^2$. (For more details see Appendix 1).

□

Case 3 $l \leq k$

For the case when $l \leq k$ we will do the analysis in a different way. First, among all separating k-sets choose one which maximizes the cardinality l of G_1, where $l \leq k$ (see Figure 1). Assume that we have *cross* separating k-sets. By Case 1 the number of *cross* separating k-sets when G_i, $i=3,4,5,6$ are nonempty is $\leq 3^k l (n-l-k)$. If one of G_i, $i=3,4,5,6$ (see Figure 2) is empty we have two cases: case i when G_3 or G_4 is empty, and case ii when G_5 or G_6 is empty.

For case i W.L.O.G. assume G_3 is empty. Among all *cross* separating k-sets choose the one which maximizes $\bar{l} \leq k$ for the separating k-set $\{w_1, \cdots, w_k\}$. So, either $\{v_1, \cdots, v_s\} \cup G_5$ has cardinality $\bar{l} \leq k$ or $G_4 \cup \{v_{s+t+1}, \cdots, v_k\} \cup G_6$ has cardinality $\bar{l} \leq k$. Next we will upper bound the number of these *cross* separating k-sets.

Assume we have another *cross* separating k-set which uses $\{w_1, \cdots, w_{s+t}\}$. If it lies completely inside either $\{v_1, \cdots, v_s\} \cup G_5$ whose cardinality $\bar{l} \leq k$ or $G_4 \cup \{v_{s+t+1}, \cdots, v_k\} \cup G_6$ whose cardinality $\bar{l} \leq k$ then we can upper bound all *cross* separating k-sets for G_3 empty and all separating k-sets completely inside $G_1 \cup \{v_1, \cdots, v_k\}$ by $g(\bar{l}+2k)$. Since G_4 can also be empty, to cover all separating k-sets outside $G_2 \cup \{v_1, \cdots, v_k\}$ we can upper bound them by $g(\bar{l}+3k)$. Note, that it also covers all other cases: case ii and cases when more than one of G_i, $i=3,4,5,6$ is empty. So, we need to show that all *cross* separating k-sets lies within $\bar{l}+2k$ area. Recall that we choose

$\{w_1, \cdots, w_k\}$ to maximize $\bar{l}$ among all *cross* separating k-sets.

Suppose $\exists \{u_{s+t+1}, \cdots, u_k\} \subset G_2$ such that $\{w_1, \cdots, w_{s+t}, u_{s+t+1}, \cdots, u_k\}$ is a cross separating k-set with $\{u_{s+t+1}, \cdots, u_{s+t+d}\} \subset G_5$, $\{u_{s+t+d+h+1}, \cdots, u_k\} \subset G_6$, and $\{u_{s+t+d+1}, \cdots, u_{s+t+d+h}\} \subset \{w_{s+t+1}, \cdots, w_k\}$. The separating k-set $\{w_1, \cdots, w_{s+t}, u_{s+t+1}, \cdots, u_k\}$ separates G_5 into G_7 and G_8, separates G_6 into G_9 and G_{10}, and divides $\{w_{s+t+1}, \cdots, w_k\}$ into $\{w_{s+t+1}, \cdots, u_{s+t+e}\}$, $\{u_{s+t+e+h+1}, \cdots, u_k\}$ and $w_{s+t+e+i} = u_{s+t+d+i}$, $i = 1, \ldots, h$. Sets $\{u_{s+t+1}, \cdots, u_{s+t+d}\}$ and $\{u_{s+t+d+h+1}, \cdots, u_k\}$ are nonempty. (see Figure 5)

Since $\{v_1, v_2, \cdots, v_k\}$ is a separating k-set, for $\forall x \in G_1$ and $\forall y \in G_2$: $(x,y) \notin E$. Analogously, $\{w_1, w_2, \cdots, w_k\}$ is a separating k-set, so for $\forall x \in \{v_1, v_2, \cdots, v_s\} \cup G_5$ and $\forall y \in G_4 \cup \{v_{s+t+1}, \cdots, v_k\} \cup G_6$: $(x,y) \notin E$. For $\forall v_i$ $i = 1, \ldots, s$ $\exists x \in \{w_1, \cdots, w_s\}$, $\exists y \in G_4$:$(v_i,x) \in E$, $(v_i,y) \in E$ otherwise either $\{v_1, v_2, \cdots, v_{s+t}, w_1, w_2, \cdots, w_s\} - v_i$ or $\{v_1, v_2, \cdots, v_{s+t}, w_{s+t+1}, \cdots, w_s\} - v_i$ is a separating (k-1)-set. Analogously, $\forall v_i$ $i = s+t+1, \ldots, k$ $\exists x \in G_4$, $\exists y \in G_6$: $(v_i,x) \in E$, $(v_i,y) \in E$; $\forall w_i$ $i = 1, \ldots, k$ $\exists x \in \{v_1, \cdots, v_s\}$, $\exists y \in G_5$: $(w_i,x) \in E$, $(w_i,y) \in E$; $\forall w_i$ $i = s+t+1, \ldots, k$ $\exists x \in G_5$, $\exists y \in G_6$: $(w_i,x) \in E$, $(w_i,y) \in E$.

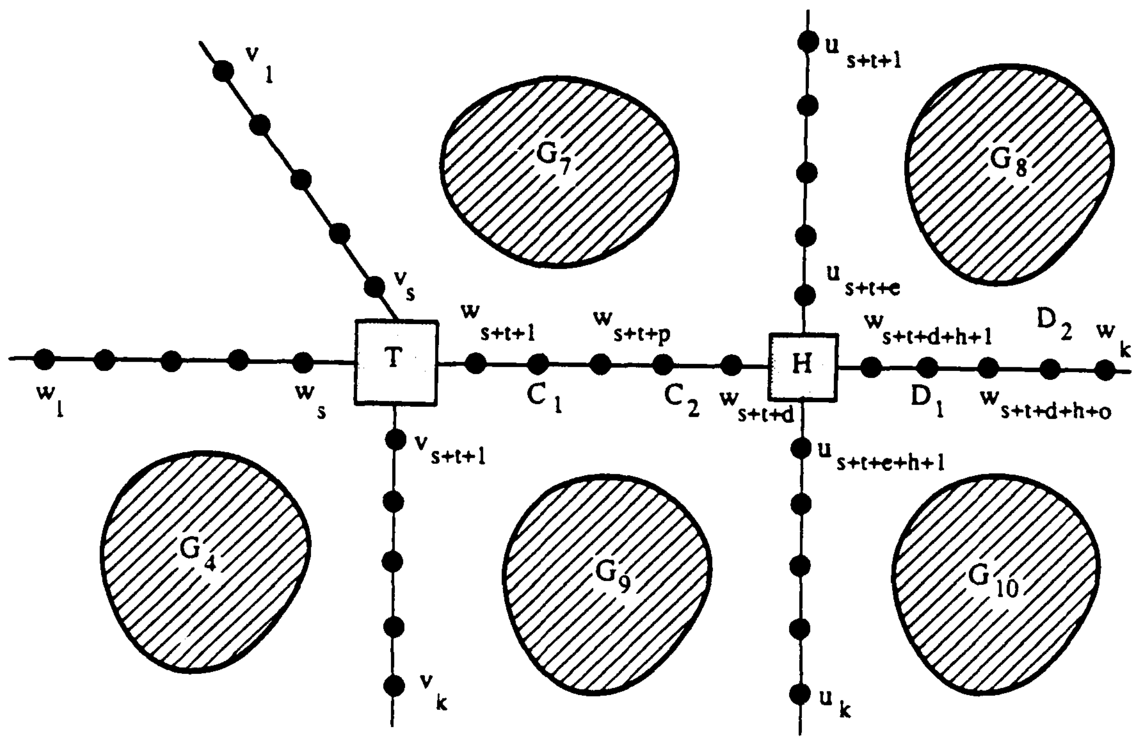

Figure 5.
Separating G into components by separating k-sets
$\{v_1, \cdots, v_k\}$, $\{w_1, \cdots, w_k\}$ and $\{w_1, \cdots, w_{s+t}, u_{s+t+1}, \cdots, u_k\}$.

Let us see how the separating k-set $\{w_1, \cdots, w_{s+t}, u_{s+t+1}, \cdots, u_k\}$ separates G.

The sets $\{v_{s+t+1}, \cdots, v_k\}$ and G_4 always belong to the same connected component of G with respect to the separating k-set $\{w_1, \cdots, w_{s+t}, u_{s+t+1}, \cdots, u_k\}$. Let $\{v_1, v_2, \cdots, v_s\} = A$, $\{v_{s+t+1}, \cdots, v_k\} \cup G_4 = B$, $\{w_{s+t+1}, \cdots, w_{s+t+e}\} = C$, $\{w_{s+t+e+h}, \cdots, w_k\} = D$. Note that C and D can be split into two component $C_1 = \{w_{s+t+1}, \cdots, w_{s+t+p}\}$ and $C_2 = \{w_{s+t+p+1}, \cdots, w_{s+t+e}\}$, $D_1 = \{w_{s+t+e+h+1}, \cdots, w_{s+t+e+h+o}\}$ and $D_2 = \{w_{s+t+e+h+o+1}, \cdots, w_k\}$ respectively, by separating k-set $\{w_1, \cdots, w_{s+t}, u_{s+t+1}, \cdots, u_k\}$. (see Figure 5). We allow C_1, C_2, D_1 and D_2 to be empty. Then, the components of $G - \{w_1, \cdots, w_{s+t}, u_{s+t+1}, \cdots, u_k\}$ are:

Case a) A and $B \cup G_9 \cup G_{10} \cup G_7 \cup G_8 \cup C \cup D$,

Case b) B and $A \cup G_9 \cup G_{10} \cup G_7 \cup G_8 \cup C \cup D$,

Case c) $A \cup G_7$ and $B \cup G_9 \cup G_{10} \cup G_8 \cup C \cup D$,

Case d) $A \cup G_8$ and $B \cup G_9 \cup G_{10} \cup G_7 \cup C \cup D$,

Case e) $B \cup G_9$ and $A \cup G_7 \cup G_{10} \cup G_8 \cup C \cup D$,

Case f) $B \cup G_{10}$ and $A \cup G_9 \cup G_7 \cup G_7 \cup C \cup D$,

Case g) $A \cup G_7 \cup C_1$ and $B \cup G_9 \cup G_{10} \cup G_8 \cup D \cup C_2$,

Case h) $A \cup G_8 \cup D_1$ and $B \cup G_9 \cup G_{10} \cup G_7 \cup C \cup D_2$,

Case i) $B \cup G_9 \cup C_1$ and $A \cup G_7 \cup G_{10} \cup G_8 \cup D \cup C_2$,

Case j) $B \cup G_{10} \cup D_1$ and $A \cup G_9 \cup G_7 \cup G_8 \cup C \cup D_2$,

Case k) $A \cup G_7 \cup G_8 \cup C_1 \cup D_1$ and $B \cup G_9 \cup G_{10} \cup C_2 \cup D_2$,

Case l) $A \cup G_7 \cup G_9 \cup C$ and $B \cup G_8 \cup G_{10} \cup D$,

Case m) $B \cup G_7 \cup G_9 \cup C$ and $A \cup G_8 \cup G_{10} \cup D$.

Cases *a)* and *b)* are analogous, so we will look only at the case *a)*. Since $\{v_1, \cdots, v_s\}$ separated from $G_7 \cup G_8 \cup C \cup D$ by $\{w_1, \cdots, w_{s+t}, u_{s+t+1}, \cdots, u_k\}$, so $\{w_1, \cdots, w_{s+t}, u_{s+t+1}, \cdots, u_{s+t+d+r}\}$ separates A from the rest of the graph G. The cardinality of $\{w_1, \cdots, w_{s+t}, u_{s+t+1}, \cdots, u_{s+t+d+r}\}$ is $\geq k$, if and only if $k - s - t - d - r \leq 0$, in other words $\{u_{s+t+d+r+1}, \cdots, u_k\}$ is empty. This contradict the assumption of the choice for $\{u_{s+t+1}, \cdots, u_k\}$.

Cases *c)*, *d)*, *e)* and *f)* are symmetric, so we will look only at the case *c)*. Since $\{w_1, \cdots, w_{s+t}, u_{s+t+1}, \cdots, u_k\}$ separates $G_3 \cup \{v_1, \cdots, v_s\} \cup G_7$ from the rest of the graph G, $\forall x \in G_7 \cup \{v_1, \cdots, v_s\}$ and $\forall y \in C \cup D$, $(x,y) \notin E$. So, $\{w_1, \cdots, w_{s+t}, u_{s+t+1}, \cdots, u_{s+t+d+r}\}$ separates $A \cup G_7$ from the rest of the graph G.

and $\{u_{s+t+d+r+1}, \cdots, u_k\}$ must be empty in order for the cardinality of this set be $\geq k$. This again contradict the assumption of the choice for $\{u_{s+t+1}, \cdots, u_k\}$.

Cases *g), h), i)* and *j)* are symmetric, so we will look only at the case *g)*. First, $\{w_1, \cdots, w_{s+t+p}, w_{s+t+e}, \cdots, w_{s+t+e+h}, u_{s+t+1}, \cdots, u_{s+t+d}\}$ is a separating set, so its cardinality is $\geq k$. Hence, $|C_1| \geq$ the cardinality of $\{u_{s+t+d+h+1}, \cdots, u_k\}$. Second, $\{w_1, \cdots, w_{s+t}, w_{s+t+p+1}, \cdots, w_{s+t+e+h}, u_{s+t+d+h+1}, \cdots, u_k\}$ is another separating set , so its cardinality is $\geq k$. Hence, $|C_2| \geq$ the cardinality of $\{u_{s+t+1}, \cdots, u_{s+t+d}\}$. Combining these two inequalities we get that D is empty and $|C_1|$ = the cardinality of $\{u_{s+t+d+h+1}, \cdots, u_k\}$, $|C_2|$ = the cardinality of $\{u_{s+t+1}, \cdots, u_{s+t+d}\}$. Also, G_8 and G_{10} are empty. But then we would choose either $\{w_1, \cdots, w_{s+t+p}, u_{s+t+1}, \cdots, u_{s+t+d}\}$ or $\{w_1, \cdots, w_{s+t}, w_{s+t+p+1}, \cdots, w_{s+t+e+h}, u_{s+t+d+h+1}, \cdots, u_k\}$ to maximize $\bar{l}$ instead of $\{w_1, \cdots, w_k\}$. Hence, no $\{w_1, \cdots, w_{s+t}, u_{s+t+1}, \cdots, u_k\}$ exist.

Consider case *k)*. First, $\{w_1, \cdots, w_{s+t+p}, w_{s+t+e}, \cdots, w_{s+t+e+h+o}, u_{s+t+1}, \cdots, u_{s+t+d}\}$ is a separating set, so its cardinality is $\geq k$. Hence, $|C_1| + |D_1| \geq$ the cardinality of $\{u_{s+t+d+h+1}, \cdots, u_k\}$. Second, $\{w_1, \cdots, w_{s+t}, w_{s+t+p+1}, \cdots, w_{s+t+e+h}, w_{s+t+e+h+o+1}, \cdots, w_k, u_{s+t+d+h+1}, \cdots, u_k\}$ is another separating set , so its cardinality is bigger than or equal to k. Hence, $|C_2| + |D_2| \geq$ the cardinality of $\{u_{s+t+1}, \cdots, u_{s+t+d}\}$. Combining these two inequalities we get that $|C_1| + |D_1|$ = the cardinality of $\{u_{s+t+d+h+1}, \ldots, u_k\}$ and $|C_2| + |D_2|$ = the cardinality of $\{u_{s+t+1}, \ldots, u_{s+t+d}\}$. But then we would choose either $\{w_1, \cdots, w_{s+t+p}, w_{s+t+e}, \cdots, w_{s+t+e+h+o}, u_{s+t+1}, \cdots, u_{s+t+d}\}$ or $\{w_1, \cdots, w_{s+t}, w_{s+t+p+1}, \cdots, w_{s+t+e+h}, w_{s+t+e+h+o+1}, \cdots, w_k, u_{s+t+d+h+1}, \cdots, u_k\}$ to maximize $\bar{l}$ instead of $\{w_1, \cdots, w_k\}$. Hence, no $\{w_1, \cdots, w_{s+t}, u_{s+t+1}, \cdots, u_k\}$ exist.

Cases *l)* and *m)* are symmetric, so we will look only at the case *l)*. First, $\{w_1, \cdots, w_{s+t+e+h}, u_{s+t+1}, \cdots, u_{s+t+d}\}$ is a separating set, so its cardinality is bigger than or equal than k. Hence, the cardinality of C is bigger than or equal to the cardinality of $\{u_{s+t+d+h+1}, \cdots, u_k\}$ and the cardinality of $\{u_{s+t+1}, \cdots, u_{s+t+d}\}$ is bigger than or equal to the cardinality of D. Second, $\{w_1, \cdots, w_{s+t}, w_{s+t+e+h+1}, \cdots, w_k, u_{s+t+d+h+1}, \cdots, u_k\}$ is another separating set , so its cardinality is bigger than or equal to k. Hence, the cardinality of C is less or equal to the cardinality of $\{u_{s+t+d+h+1}, \cdots, u_k\}$ and the cardinality of $\{u_{s+t+1}, \cdots, u_{s+t+d}\}$ is less or equal to the cardinality of D. Combining these four inequalities we get that $|C|$ = the cardinality of $\{u_{s+t+d+h+1}, \cdots, u_k\}$ and the cardinality of $\{u_{s+t+1}, \cdots, u_{s+t+d}\}$ is $= |D|$. But then we would choose either $\{w_1, \cdots, w_{s+t+e+h}, u_{s+t+1}, \cdots, u_{s+t+d}\}$ or

$\{w_1, \cdots, w_{s+t}, w_{s+t+e+h+1}, \cdots, w_k, u_{s+t+d+h+1}, \cdots, u_k\}$ to maximize $\bar{l}$ instead of $\{w_1, \cdots, w_k\}$.

So, no separating k-set $\{w_1, \cdots, w_{s+t}, u_{s+t+1}, \cdots, u_k\}$ exists. Hence, the maximum number of *cross* separating k-sets and separating k-sets in $G_1 \cup \{v_1, \cdots, v_k\}$ is $\leq g(\bar{l}+3k)$. Hence, for Case 2 we get the following recurrence

$$g(n) = \max_{l \leq k} \left[g(\tilde{l}+3k) + g(n-\tilde{l}) + 3^k \tilde{l}\,(n-\tilde{l}-k) \right],$$

where $\tilde{l} = \max(l, \bar{l})$. The solution to this recurrence is $g(n) \leq \frac{1}{2} 3^k n^2 - \frac{13}{2} 3^k k^2$. (For more details see Appendix 2).

□

Combining the results of Cases 1, 2 and 3 we get the upper bound on the number of separating k-sets in a k-connected undirected graph

$$g(n) = O(3^k n^2),$$

and for fixed k this bound becomes $O(n^2)$.

□

3. Lower bound

Let us now generalize the *wheel* graph and *cycle* graph [KaRa2] for the lower bounds for odd and even k respectively. (see figure 6)

Take $\frac{n}{k}$ complete graphs on k vertices, arranged in a cycle. Two adjacent complete graphs are connected via $\frac{k}{2}$ edges, one edge per vertex. Removal of these $\frac{k}{2}$ vertices and analogous $\frac{k}{2}$ vertices in nonadjacent complete graph will separated the graph. Each of these $\frac{k}{2}$ vertices has only one edge to one of the connected components of the graph after separation, so we can replace each vertex by its unique neighbor not in its complete graph. That gives rise to 2^k factor. Since a cycle has $\frac{n(n-3)}{2}$ separating pairs we get $\frac{n^2}{k^2}$ factor. Hence, the number of separating k-sets for the both graphs is $\Omega\left(2^k \frac{n^2}{k^2}\right)$.

□

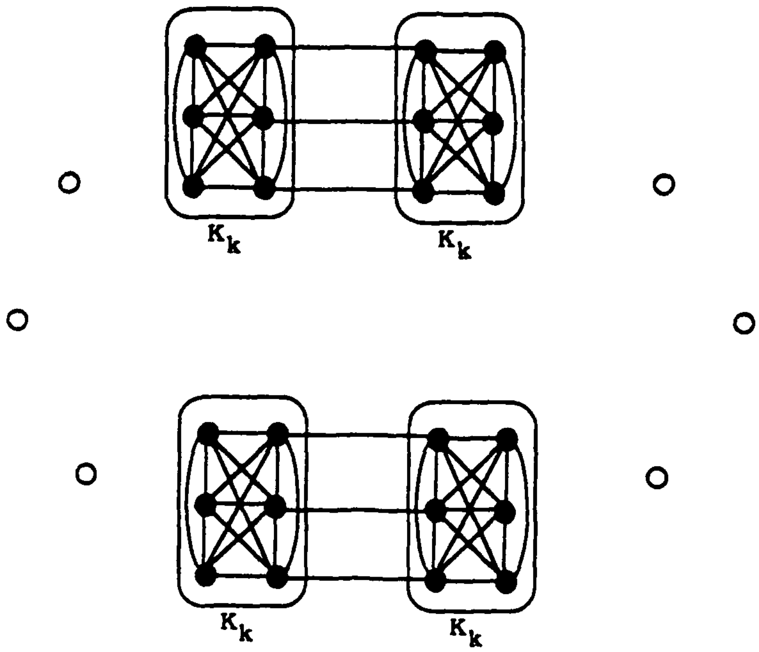

Figure 6
Generalization of cycle for even k.

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REFERENCES

[Ev] S.Even, "An algorithm for determining whether the connectivity of a graph is at least *k*," *SIAM J. on Computing*, vol. 4, 1975, pp. 393-396.

[EvTa] S.Even,R.E.Tarjan, "Network flow and testing graph connectivity," *SIAM J. on Computing*, vol. 4, 1975, pp.507-518.

[Ga] Z.Galil, "Finding the vertex connectivity of graphs," *SIAM J. on Computing*, vol. 9, 1980, pp. 197-199.

[GiSo] M.Girkar,M.Sohoni, "On finding the vertex connectivity of graphs," *Tech. Report ACT-77*, Coordinated Science Laboratory, University of Illinois, Urbana, Il, May, 1987

[HoTa] J.E.Hopcroft,R.E.Tarjan, "Dividing a graph into triconnected components," *SIAM J. on Computing*, vol. 2, 1973, pp. 135-158.

[KaRa] A.Kanevsky,V.Ramachandran, "Improved algorithms for graph four-connectivity," *Proc. 28th IEEE Ann. Symp. on Foundation of Computer Science*, Oct., 1987, to appear.

[KaRa2] A.Kanevsky,V.Ramachandran, "A characterization of separating pairs and triplets in a graph," *Tech. Report ACT-79*, Coordinated Science Laboratory, University of Illinois, Urbana, IL, July, 1987.

[LiLoWi] N.Linial,L.Lovasz,A.Wigderson, "A physical interpretation of graph connectivity, and its algorithmic applications," *Proc. 27th IEEE Ann. Symp. on Foundations of Comp. Sci.*, 1986, pp. 39-48.

[MiRa] G.L.Miller,V.Ramachandran, "A new graph triconnectivity algorithm and its parallelization," *Proc. 19th ACM Ann. Symp. on Theory of Computing*, New York, NY, May 1987, pp. 335-344.

[Ta] R.E.Tarjan, "Depth-first search and linear graph algorithms," *SIAM J. on Computing*, vol. 1, 1972, pp. 146-159.

[Tu] W.T.Tutte, *Connectivity in Graphs*, University of Toronto Press, 1966.

Appendix 1.

Assume $g(n) = c_1 3^k n^2 + c_2 3^k k^2$. Then the recurrence

$$g(n) = \max_{l \ge k}\left[g(l+2k) + g(n-l+k) + 3^k (l-k)(n-l-2k) \right]$$

becomes

$$c_1 3^k n^2 + c_2 3^k k^2 = c_1 3^k (l+2k)^2 + c_2 3^k k^2 + c_1 3^k (n-l+k)^2 + c_2 3^k k^2 + 3^k (l-k)(n-l-2k).$$

$$c_1 n^2 + c_2 k^2 = c_1 l^2 + 4c_1 lk + 4c_1 k^2 + 2c_2 k^2 + c_1 n^2 + c_1 l^2 + c_1 k^2 - 2c_1 nl + 2c_1 nk - 2c_1 lk + nl - kn - l^2 + kl - 2lk + 2k^2,$$

$$0 = l^2 (2c_1 - 1) + \ln (1 - 2c_1) + lk (2c_1 - 1) + nk (2c_1 - 1) + k^2 (5c_1 + c_2 + 2).$$

Setting $c_1 = \frac{1}{2}$ we get

$$0 = k^2 (\frac{9}{2} + c_2).$$

So, $g(n) \le \frac{1}{2} 3^k n^2 - \frac{9}{2} 3^k k^2$.

□

Appendix 2.

Assume $g(n) = c_1 3^k n^2 + c_2 3^k k^2$. Then the recurrence

$$g(n) = \max_{l \le k}\left[g(l+3k) + g(n-l) + 3^k l (n-l-k) \right]$$

becomes

$$c_1 3^k n^2 + c_2 3^k k^2 = c_1 3^k (l+3k)^2 + c_2 3^k k^2 + c_1 3^k (n-l)^2 + c_2 3^k k^2 + 3^k l (n-l-k).$$

$$c_1 n^2 + c_2 k^2 = c_1 l^2 + 6c_1 lk + 9c_1 k^2 + 2c_2 k^2 + c_1 n^2 + c_1 l^2 - 2_1 nl + nl - l^2 - lk,$$

$$0 = l^2(2c_1 - 1) + nl(1 - 2_1) + lk(6c_1 - 1) + k^2(9c_1 + c_2).$$

Since $l \le k$ and setting $c_1 = \frac{1}{2}$, we get

$$0 = k^2(\frac{13}{2} + c_2)$$

So, $g(n) \le \frac{1}{2} 3^k n^2 - \frac{13}{2} 3^k k^2$.

□

END

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