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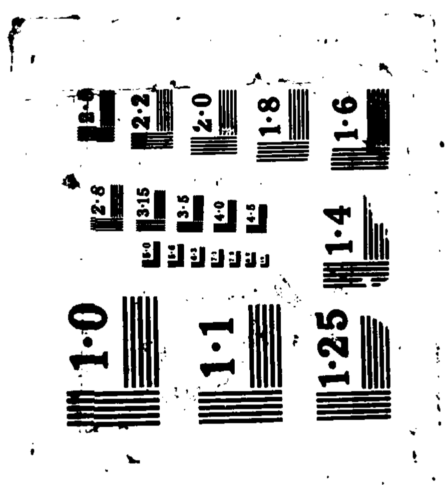

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COORDINATED SCIENCE LABORATORY

College of Engineering
Applied Computation Theory

A CHARACTERIZATION OF SEPARATING PAIRS AND TRIPLETS IN A GRAPH

Arkady Kanevsky
Vijaya Ramachandran

UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

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A Characterization of Separating Pairs and Triplets in a Graph

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ABSTRACT

We obtain tight upper bounds of $\frac{n(n-3)}{2}$ and $\frac{(n-1)(n-4)}{2}$ for the number of separating pairs and triplets in an undirected biconnected and triconnected graph, respectively, where n is the number of vertices in a graph. We present worst-case graphs that exactly achieve our upper bounds. Finally, we give an $O(n)$ characterization for the separating pairs in a biconnected graph.

1. Introduction

Connectivity is an important graph property and there has been a considerable amount of work on algorithms for determining connectivity of graphs [BeX,Ev2,EvTa,Ga,GiSo,LiLoWi]. An undirected graph $G=(V,E)$ is k-*connected* if for any subset V' of $k-1$ vertices of G the subgraph induced by $V-V'$ is connected [Ev]. A subset V' of k vertices is a *separating k-set* if the subgraph induced by $V-V'$ is not connected. For $k=1$ the set V' becomes a single vertex which is called an articulation point, and for $k=2,3$ the set V' is called a separating pair and separating triplet, respectively. Efficient algorithms are available for finding all separating k-sets in k-connected undirected graphs for $k \leq 3$ [Ta,HoTa,MiRa,KaRa].

We address the following question: what is the maximum number of separating pairs and triplets in biconnected and triconnected undirected graphs, respectively?

An undirected graph G on n vertices has a trivial upper bound of $\binom{n}{k}$ on the number of separating k-sets, $k \geq 1$. The graph that achieves this bound for all k is a graph on n vertices without any edges. For $k=1$ the maximum number of articulation points in a *connected* graph is $(n-2)$ and a graph that achieves it is a path on n ver-

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tices.

In this paper we show that for k=2 the maximum number of separating pairs in an undirected biconnected graph is $\frac{n(n-3)}{2}$ and a graph that achieves it is a cycle on n vertices. Further, we observe that there is an $O(n)$ representation for the separating pairs in any biconnected graph (although the number of such pairs could be $\Theta(n^2)$). Finally, we prove that for k=3 the maximum number of separating triplets in a triconnected graph is $\frac{(n-1)(n-4)}{2}$ and we present a graph, namely the *wheel* [Tu], that achieves it.

In a companion paper [Ka1] we prove that the number of separating k-sets in a k-connected graph is $O(c^k n^2)$ and we show that the bound is tight up to the constant c.

2. Graph-theoretic definitions

An *undirected graph* $G=(V,E)$ consists of a *vertex set* V and an *edge set* E containing unordered pairs of distinct elements from V. A *path* P in G is a sequence of vertices $<v_0, \cdots, v_k>$ such that $(v_{i-1}, v_i) \in E, i=1, \cdots, k$. The path P *contains* the vertices $v_0, \cdots, v_k$ and the edges $(v_0, v_1), \cdots, (v_{k-1}, v_k)$ and has *endpoints* v_0, v_k, and *internal vertices* $v_1, \cdots, v_{k-1}$.

We will sometimes specify a graph G structurally without explicitly defining its vertex and edge sets. In such cases, $V(G)$ will denote the vertex set of G and $E(G)$ will denote the edge set of G. Also, if $V' \subseteq V$ and $v \in V$ we will use the notation $V' \cup v$ to represent $V' \cup \{v\}$.

An undirected graph $G=(V,E)$ is connected if there exists a path between every pair of vertices in V. For a graph G that is not connected, a *connected component* of G is an induced subgraph of G which is maximally connected.

A vertex $v \in V$ is an *articulation point* of a connected undirected graph $G=(V,E)$ if the subgraph induced by $V-\{v\}$ is not connected. G is *biconnected* if it contains no articulation point.

Let $G=(V,E)$ be a biconnected undirected graph. A pair of vertices $v_1, v_2 \in V$ is a *separating pair* for G if the induced subgraph on $V-\{v_1, v_2\}$ is not connected. G is *triconnected* if it contains no separating pair.

A triplet (v_1, v_2, v_3) of distinct vertices in V is a *separating triplet* of a triconnected graph if the subgraph induced by $V - \{v_1, v_2, v_3\}$ is not connected. G is *four-connected* if it contains no separating triplets.

Let $G=(V,E)$ be an undirected graph and let $V'\subseteq V$. A graph $G'=(V',E')$ is a *subgraph* of G if $E'\subseteq E\cap\{(v_i,v_j)\mid v_i,v_j\in V'\}$. The *subgraph of G induced by V'* is the graph $G''=(V',E'')$ where $E''=E\cap \{(v_i,v_j)\mid v_i,v_j\in V'\}$.

3. The tight upper bound for k=2

Theorem 1 The maximum number of separating pairs in an undirected biconnected graph is $\frac{n(n-3)}{2}$.

Proof: Let $\{v_1,v_2\}$ be a separating pair of a biconnected graph G on n vertices and m edges, whose removal separates G into nonempty G_1 and G_2 (see Figure 1).

Let $g(n)$ be the maximum number of separating pairs in a graph on n vertices. Then we can divide all separating pairs into four types:

1). Separating pairs completely inside $G_1\cup\{v_1,v_2\}$,

2). Separating pairs completely inside $G_2\cup\{v_1,v_2\}$,

3). Separating pairs with one vertex from G_1 and one vertex from G_2,

4). The separating pair $\{v_1,v_2\}$.

The number of separating pairs of type one and type two are upper bounded by $g(l+2)$ and $g(n-l)$, respectively, where l is the cardinality of $V(G_1)$ and $n-l-2$ is the cardinality of $V(G_2)$. The number of separating pairs of type three is trivially upper bounded by $l(n-l-2)$. Hence, any function $g(n)$ that satisfies the recurrence

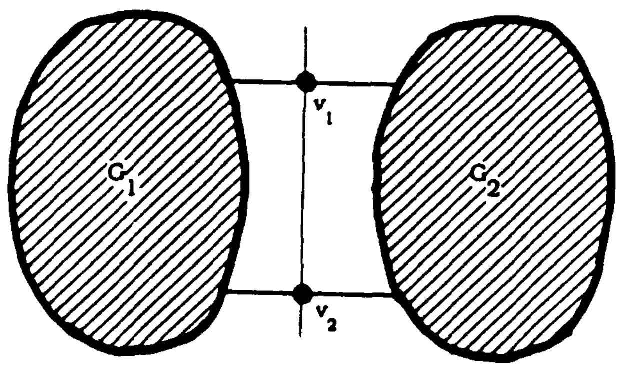

Figure 1.
Separating G into nonempty G_1 and G_2 by separating pair $\{v_1,v_2\}$

$$g(n) = \max_l \left[g(l+2) + g(n-l) + l(n-l-2) + 1 \right].$$

is an upper bound on the number of separating pairs in a graph on n vertices.

We note that $g(n) = \frac{n(n-3)}{2}$ satisfies this recurrence.

□

Graph C_n, the *cycle* on n vertices, has $\frac{n(n-3)}{2}$ separating pairs, so the bound is worst-case optimal.

Even though the number of separating pairs in a biconnected n-node graph $G=(V,E)$ can be as large as $\Theta(n^2)$, we observe that there are more succinct representations for them.

1 The *tree of triconnected components* of a biconnected graph has size $O(m+n)$, where $|E| = m$ [HoTa,MiRa], and this is a representation for all separating pairs together with the triconnected components of the graph.

2 The algorithm in [MiRa] enumerates the separating pairs as a collection $C = \{V_1, \cdots, V_s\}$ of subsets of V, with the interpretation that any pair of vertices within a single V_i is either a separating pair for G or the endpoints of an edge in a specified 'ear' in G, and further, every separating pair for G appears in at least one of the V_i's. It is not difficult to establish that $\sum_{i=1}^{s} |V_i| = O(n)$; thus this gives an $O(n)$ representation for separating pairs. We omit the proof of this result here since it requires extensive background material from [MiRa]. It will appear in [Ka2].

4. The upper bound for k=3

The *wheel* W_n [Tu] is C_{n-1} together with a vertex v and an edge between v and every vertex on C_{n-1}. It is easy to see that W_n is triconnected and has $\frac{(n-1)(n-4)}{2}$ separating triplets. In the following theorem we prove that this is the worst-case for the number of separating triplets in a triconnected graph.

Theorem 3 The number of separating triplets in an undirected triconnected graph is $\le \frac{(n-1)(n-4)}{2}$ for any n.

Proof: Assume there exists a separating triplet $\{v_1, v_2, v_3\}$ in G, which separates G into nonempty G_1 and G_2 (see Figure 2). Now, we can divide separating triplets in G into 6 distinct types:

1). Separating triplets completely inside $G_1 \cup \{v_1, v_2, v_3\}$,

2). Separating triplets completely inside $G_2 \cup \{v_1, v_2, v_3\}$,

3). Separating triplets with one vertex from G_1, one vertex from G_2 and one vertex from $\{v_1, v_2, v_3\}$,

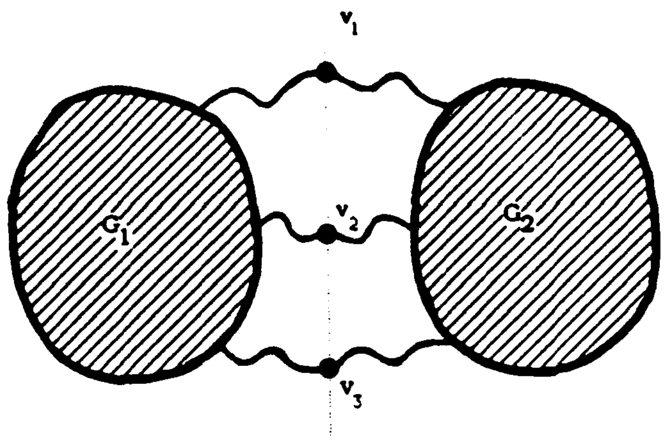

Figure 2.
Separating G into G_1 and G_2 by separating triplet $\{v_1, v_2, v_3\}$

4). Separating triplets with one vertex from G_1 and two vertices from G_2,

5). Separating triplets with two vertices from G_1 and one vertex from G_2,

6). The separating triplet $\{v_1, v_2, v_3\}$.

Let the number of vertices in G_1 be k, then the number of vertices in G_2 is $n-k-3$. Let $g(n)$ be the maximum number of separating triplets in a graph on n vertices, $h(k,n-k)$ be the number of separating triplets of the third type and $f(k,n-k)$ and $f(n-k,k)$ be the number of separating triplets of the fourth and fifth types respectively.

Then any $g(n)$ that satisfies the recurrence

$$g(n) = \max_k (g(k+3) + g(n-k) + h(k,n-k) + f(k,n-k) + f(n-k,k) + 1)$$

is an upper bound on the number of separating triplets in G.

Let us now find the upper bounds for the functions h and f.

Lemma 2: $f(k,n-k) + f(n-k,k) \le \frac{3}{2}(3n-14)$.

Proof: Let $\{w_1, w_2, w_3\}$ be a separating triplet with $w_1 \in G_1$ and $w_2, w_3 \in G_2$. The separating triplet $\{w_1, w_2, w_3\}$ separates G_1 into L_1 and L_2, and separates G_2 into L_3 and L_4 (see Figure 3). Let us see how the original separating triplet $\{v_1, v_2, v_3\}$ is separated by the separating triplet $\{w_1, w_2, w_3\}$.

All $v_i, i=1,2,3$ cannot belong to one separated component of G with respect to the separating triplet $\{w_1, w_2, w_3\}$, otherwise either w_1 would be an articulation point, or $\{w_2, w_3\}$ would be a separating pair, or both. W.L.O.G. assume that v_1 belongs to one separated component and v_2, v_3 to the other.

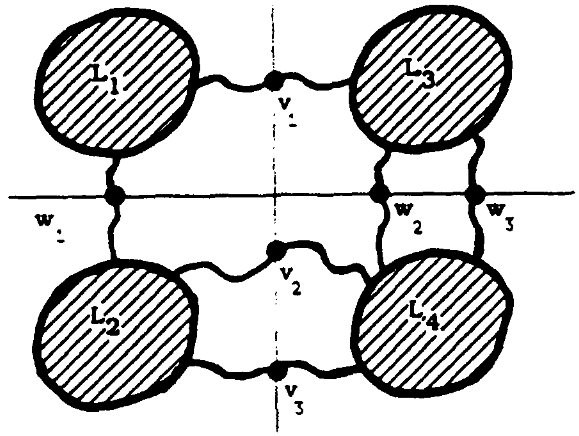

Figure 3.
Separating G_1 into L_1 and L_2 and G_2 into L_3 and L_4 by $\{w_1,w_2,w_3\}$

Subgraph L_1 must be empty, otherwise $\{w_1,v_1\}$ becomes a separating pair. Since the graph is triconnected, $(w_1,v_1)\in E$, $\exists\, x,y\in L_3\cup w_2\cup w_3$: $(x,v_1)\in E$, $(y,v_1)\in E$ and $\forall\, z\in L_2\cup L_4\cup v_2\cup v_3$: $(z,v_1)\notin E$. Hence, vertex w_1 is unique up to a division of the original separating triplet $\{v_1,v_2,v_3\}$ into v_1 and v_2, v_3. So, if there is a separating triplet of the fourth type which separates v_1 from v_2 and v_3 then there is no separating triplet of the fifth type which separates v_1 from v_2 and v_3.

Let us see how many separating triplets of the fourth type there are in G that separate the original separating triplet $\{v_1,v_2,v_3\}$ into v_1 and v_2, v_3. The vertex w_1 must belong to all of them. Let us see the choices for $\{w_2,w_3\}$, such that $\{w_1,w_2,w_3\}$ is a separating triplet of the fourth type.

Assume there is a separating triplet of the fourth type $\{w_1,u_1,u_2\}$, where $u_1\in L_3$, $u_2\in L_4$. The separating triplet $\{w_1,u_1,u_2\}$ separates L_3 into L'_3 and $\tilde{L}_3$, and separates L_4 into L'_4 and $\tilde{L}_4$ (see Figure 4).

The vertex v_1 is connected by an edge to only one of the $L'_3\cup u_1$ and $\tilde{L}_3$, otherwise $\{w_1,u_1,u_2\}$ is not a separating triplet. If v_1 is not connected to the $L'_3\cup u_1$ and $\tilde{L}_3$ then $\{w_2,w_3\}$ is a separating pair. W.L.O.G. assume $\forall\, x\in\tilde{L}_3$: $(x,v_1)\notin E$. By the symmetry $\{v_2, v_3\}$ is connected to only one of the L'_4 and $\tilde{L}_4$. Let us see how the separating triplet $\{w_1,u_1,u_2\}$ separates $\{w_2,w_3\}$.

If vertices w_2 and w_3 are not separated by $\{w_1,u_1,u_2\}$ then there are four cases to consider.

When w_2 and w_3 belong to the same component as L'_3 and L'_4 with respect to the separating triplet $\{w_1,u_1,u_2\}$ and $\{v_2,v_3\}$ is connected by an edge to $\tilde{L}_4$ then $\{w_1,u_2\}$ is a separating pair which separates $L_2\cup\{v_2,v_3\}\cup\tilde{L}_4$ from $v_1\cup L_3\cup\{w_2,w_3\}\cup L'_4$.

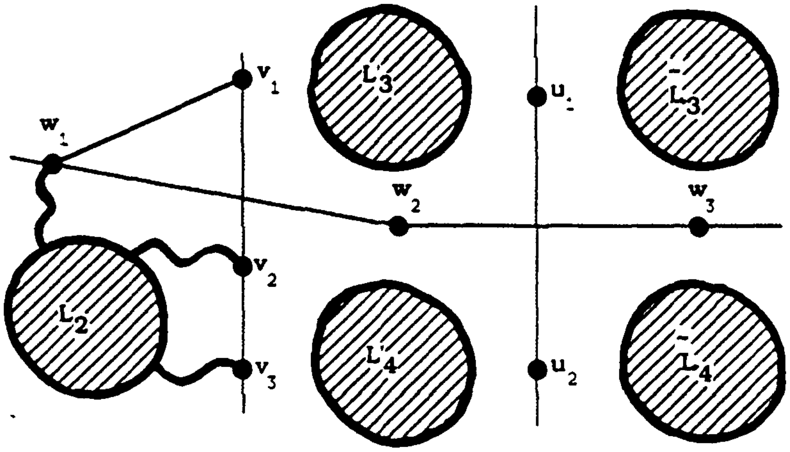

Figure 4.
Separating L_3 into L'_3 and $\tilde{L}_3$ and L_4 into L'_4 and $\tilde{L}_4$ by $\{w_1,u_1,u_2\}$

When w_2 and w_3 belong to the same component as L'_3 and L'_4 with respect to the separating triplet $\{w_1,u_1,u_2\}$ and $\{v_2,v_3\}$ is connected by an edge to L'_4 then $\{u_1,u_2\}$ is a separating pair which separates $\tilde{L}_3 \cup \tilde{L}_4$ from the rest of the graph.

When w_2 and w_3 belong to the same component as $\tilde{L}_3$ and $\tilde{L}_4$ with respect to the separating triplet $\{w_1,u_1,u_2\}$ and $\{v_2,v_3\}$ is connected by an edge to L'_4 then $\{u_1,u_2\}$ is a separating pair which separates $\tilde{L}_3 \cup \{w_2,w_3\} \cup \tilde{L}_4$ from the rest of the graph.

When w_2 and w_3 belong to the same component as $\tilde{L}_3$ and $\tilde{L}_4$ with respect to the separating triplet $\{w_1,u_1,u_2\}$ and $\{v_2,v_3\}$ is connected by an edge to $\tilde{L}_4$ then $\{w_1,u_1\}$ is a separating pair which separates $L'_3 \cup v_1$ from the rest of the graph.

Hence, w_2 and w_3 belong to different components with respect to the separating triplet $\{w_1,u_1,u_2\}$. Subgraph $\tilde{L}_3$ must be empty; otherwise $\{u_1,w_3\}$ becomes a separating pair. Hence, $(u_1,w_3) \in E$, otherwise $\{w_1,w_2\}$ is a separating pair. If $\{v_2,v_3\}$ is connected to L'_4 then $\{u_1,u_2\}$ is a separating pair or $\{w_1,u_1,u_2\}$ is not a separating triplet. So, $\forall\, x \in L'_4$: $(x,v_2) \notin E$, $(x,v_3) \notin E$, $\exists\, y,z \in \tilde{L}_4 \cup \{w_2,w_3\}$: $(y,v_2) \in E$, $(z,v_3) \in E$. Subgraph L'_4 must be empty, otherwise $\{w_2,u_2\}$ is a separating pair or $\{w_1,u_1,u_2\}$ is not a separating triplet. Hence, $(u_2,w_2) \in E$, otherwise $\{w_1,w_3\}$ is a separating pair (see Figure 5).

The above means that for each separating triplet $\{w_1,w_2,w_3\}$ there exists at most one separating triplet $\{w_1,u_1,u_2\}$ such that $u_1 \in L_3$ and $u_2 \in L_4$. So, $\forall\, x \in L'_3$, $\forall\, y \in \tilde{L}_4$ $\{w_1,x,w_3\}$, $\{w_1,x,u_2\}$, $\{w_1,y,w_2\}$, $\{w_1,y,u_1\}$

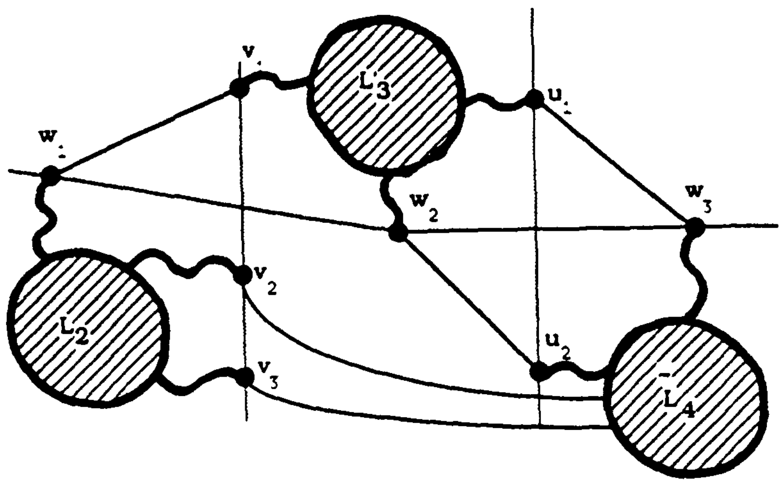

Figure 5.
Illustrating the configuration between separating triplets $\{w_1,w_2,w_3\}$ and $\{w_1,u_1,u_2\}$

and $\{w_1,y,x\}$ are not separating triplets.

Let the number of vertices in L'_3 be l then the number of vertices in $\tilde{L}_4$ will be $(n-k-3-l-4)=(n-k-l-7)$. Then the maximum number of separating triplets that use w_1 is

$$r(n-k-3)=\max_l\left[r(n-k-l-5)-1+r(l+2)-1+4\right]=$$
$$\max_l\left[r(n-k-l-5)+r(l+2)\right]+2,\quad r(2)=1,\ r(1)=0,$$

where $r(n-k-l-5)-1$ counts all separating triplets which use w_1 and two vertices from $\tilde{L}_4\cup u_2\cup w_3$, $r(l+2)-1$ counts all separating triplets which use w_1 and two vertices from $L'_3\cup u_1\cup w_2$ and 4 counts $\{w_1,u_1,u_2\}$, $\{w_1,w_2,w_3\}$, $\{w_1,u_1,w_2\}$ and $\{w_1,u_2,w_3\}$.

The solution for this recurrence is $r(n-k-3)\le\frac{3}{2}(n-k-3)-2$. Since there exists a unique w_1, for every separation of v_i $i=1,2,3$ from the other two v_i's, the upper bound for the separating triplets of the fourth and fifth types together is:

$$f(k,n-k)+f(n-k,k)\le 3\cdot\left(\max_{1\le k\le n-4}\frac{3}{2}\cdot\max((n-k-3),k)-2\right)\le\frac{3}{2}\cdot\left[3(n-4)-2\right]=\frac{3}{2}(3n-14).$$

□

Corollary The maximum number of separating triplets of the fourth type which separate $\{v_i\}$ from $\{v_1,v_2,v_3\}-\{v_i\}$ is $\le\frac{3}{2}(n-k-3)-2$.

Analogously, we can state corollary for the fifth type separating triplet.

Lemma 3 $h(k,n-k) \le k(n-k-3)$.

Proof: Assume there is separating triplet $\{w_1,v_2,w_2\}$ of the third type in G, where $w_1 \in G_1$ and $w_2 \in G_2$. It separates G_1 into K_1 and K_2, and separates G_2 into K_3 and K_4. Vertices v_1 and v_3 must belong to the different components with respect to separating triplet $\{w_1,v_2,w_2\}$, otherwise either $\{w_1,v_2\}$ is a separating pair, or $\{w_2,v_2\}$ is a separating pair, or both.

Claim 1 Vertex v_2 has a direct edge to every nonempty subgraph K_1,K_2,K_3,K_4.

W.L.O.G. assume that K_1 is not empty and $\forall x \in K_1$, $(x,v_2) \notin E$. Then $\{v_1,w_1\}$ is a separating pair of G, which separates K_1 from the rest of the graph.

□

Now, we will prove that there are no separating triplets of the third type which use v_1 or v_3. We will prove this by contradiction. W.L.O.G. assume there is a separating triplet $\{u_1,v_1,u_2\}$, where $u_1 \in G_1$ and $u_2 \in G_2$ (u_1 may be equal to w_1 and u_2 may be equal to w_2).

Case 1: $u_1 \in K_2$, if K_2 is not empty (see Figure 6).

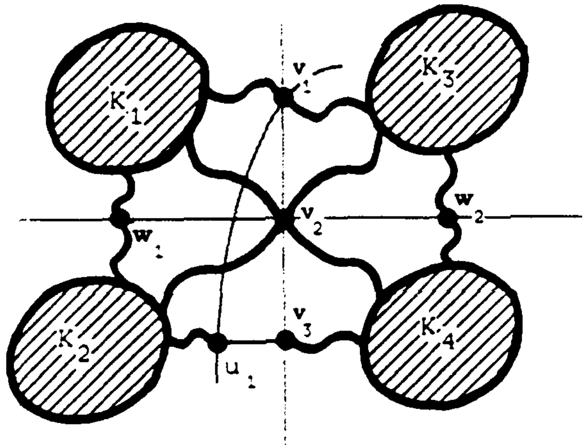

Figure 6.
Illustrating Case 1 in the proof of Lemma 3

By Claim 1 for v_1 and the existence of separating triplet $\{u_1,v_1,u_2\}$, K_1, w_1, K_2-u_1 belong to the same connected component with respect to separating triplet $\{u_1,v_1,u_2\}$. If v_2 belongs to the same component then $\{v_1,u_1\}$ is a separating pair which separates $K_3 \cup w_2 \cup K_4 \cup v_3$ from the rest of the graph. If v_2 does not belong to the same component then $\{v_1,u_1\}$ is a separating pair which separates $K_1 \cup w_1 \cup K_2 - u_1$ from the rest of the graph.

Analogously, $u_2 \notin K_4$.

Case 2: $u_1 = w_1$.

Since $\{u_1, v_1, u_2\}$ is a separating triplet then v_2 does not have any edges to K_1 and hence, K_1 is empty by Claim 1. But then $\{v_1, u_2\}$ is a separating pair, if $\{u_1, v_1, u_2\}$ is a separating triplet.

Analogously, $u_2 \neq w_2$.

Case 3: $u_1 \in K_1$ and $u_2 \in K_3$.

If $\{u_1, v_1, u_2\}$ is a separating triplet then either $\{u_1, u_2\}$, or $\{u_1, v_1\}$, or $\{v_1, u_2\}$ is a separating pair.

That means that if there is a separating triplet of the third type which uses one of the $v_i, i=1,2,3$ then there are no separating triplets of the third type that use the other $v_j, j=1,2,3, j \neq i$.

Since the number of choices for w_1 is $|V(G_1)| = k$ and the number of choices for w_2 is $|V(G_2)| = (n-k-3)$, the number of separating triplets of the third type is $h(k, n-k) \leq k(n-k-3)$.

$\square$

Let us now tighten the upper bound for the number of separating triplets in the triconnected graph G. Assume that $\{v_1, v_2, v_3\}$ divides the graph such that the ratio $\frac{|V(G_1)|}{|V(G_2)|}$ is as close to one as possible over all separating triplets in G. From Lemma 3 we know that there is a unique vertex among $\{v_1, v_2, v_3\}$ that participates in the separating triplets of the third type. W.L.O.G., let this vertex be v_2.

Lemma 4: If there is a separating triplet of the fourth type or the fifth type that separates v_2 from v_1 and v_3 then there are no separating triplet of the third type.

Proof : W.L.O.G., assume there exists a separating triplet of the fourth type $\{w_1, w_2, w_3\}$, with $w_1 \in G_1$ and $w_2, w_3 \in G_2$, which separates v_2 from v_1 and v_3. It separates G_1 into K_1 and K_2, and separates G_2 into K_3 and K_4. From the proof of Lemma 2, K_1 is empty, $(w_1, v_2) \in E$ and $(x, v_2) \notin E$, $\forall x \in G_1 \cup v_1 \cup v_3 - w_1$ (see Figure 7).

Assume there is a separating triplet of the third type $\{u_1, v_2, u_2\}$, where $u_1 \in G_1$ and $u_2 \in G_2$. By Claim 1 v_2 must be connected by an edge to every nonempty component of G_1, G_2 which is created by the separator $\{u_1, v_2, u_2\}$. By the proof of Lemma 3 $u_1 = w_1$. If v_1 and v_3 are separated by $\{w_1, w_2, w_3\}$ then $(v_2, w_2) \in E$, $(v_2, w_3) \in E$ and $(x, v_2) \notin E, \forall x \in G_2 - w_2 - w_3$. Furthermore, by Claim 1, no separating triplet of the third type exists. If v_1 and v_3 are not separated by $\{w_1, w_2, w_3\}$ then $\{v_2, u_2\}$ is a separating pair. These two contradictions prove the

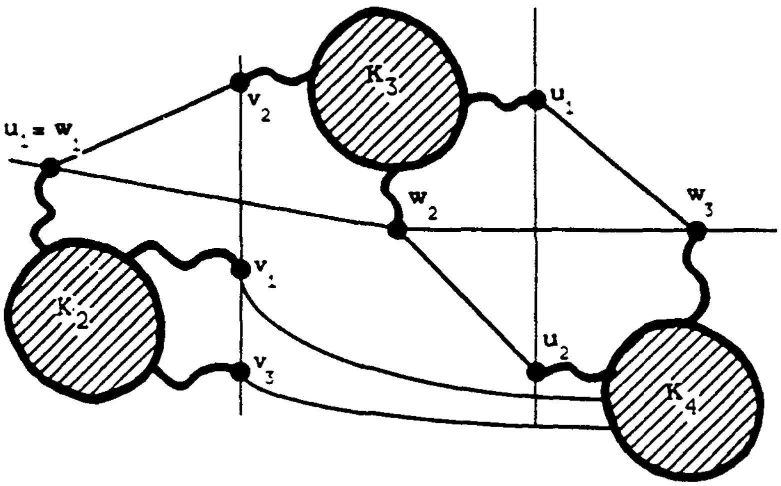

Figure 7.
Illustrating the proof of Lemma 4

lemma.

□

Now we will do a case by case analysis of trade-offs between separating triplets of the third type and the separating triplets of the fourth type and the fifth type.

Case 1: There are no separating triplets of the fourth type or the fifth type.

Let $g(n)$ be the maximum number of separating triplets of G on n vertices. Then, using Lemma 3 we obtain the following recurrence relation

$$g(n) = \max_{1 \le k \le n-4} (g(k+3) + g(n-k) + k(n-k-3) + 1)$$

The smallest function satisfying this recurrence is $g(n) = \frac{1}{2}n^2 - \frac{5}{2}n + 2$. Note that, with this solution, equality holds since the wheel W_n has this number of separating triplets.

□

By Lemma 2, if there exists a separating triplet of the fourth type that separates v_1 from v_2 and v_3, then no separating triplet of the fifth type exists which separates v_1 from v_2 and v_3. Since the separating triplets of the fourth type and the fifth type are analogous, we need only consider one of them in the case analysis.

Case 2: There is a separating triplet of the fourth type that separates v_1 from v_2 and v_3.

Let $\{w_1, w_2, w_3\}$ be such a separating triplet, with $w_1 \in G_1$ and $w_2, w_3 \in G_2$. It separates G_2 into G'_2 and $\tilde{G}_2$ and $G_1 = \{w_1\} \cup \tilde{G}_1$. Furthermore, suppose $\{w_1, w_2, w_3\}$ maximizes $|V(G'_2)|$, where G'_2 is the part of G_2

separated by $\{v_1, w_2, w_3\}$. Define $\tilde{G}_2 = G_2 - G'_2 - w_2 - w_3$ and let $|V(G'_2)| = l$. Now we will consider three cases depending on whether separating triplets of the fourth and fifth types exist, which separate v_3 from v_1, v_2. We do not restrict separating triplets which involve v_2.

Case A: There are no separating triplets of the fourth type or the fifth type that separate v_3 from v_1 and v_2.

If there is a separating triplet $\{u_1, v_2, u_2\}$, of the third type where $u_1 \in G_1$ and $u_2 \in G_2$, then $u_2 \in \tilde{G}_2$ by Claim 1. Hence, the following recurrence relation is obtained using the corollary to lemma 2:

$$g(n) = \max_{1 \le k \le n-5}\left(g(k+3) + g(n-k) + \max_{0 \le l \le n-k-5}\left(k(n-k-l-5) + \frac{3}{2}(l+2) - 2\right) + 1\right).$$

Since the function to be maximized is linear in l, the maximum is reached at one of the endpoints of the interval for l. If $k \le 1$ then the maximum is reached when $l = n-k-6$. But in this case $\{v_1, w_2, w_3\}$ would be chosen instead of $\{v_1, v_2, v_3\}$. If $k > 1$ then the maximum is reached when $l = 0$ and the recurrence becomes

$$g(n) = \max_{1 \le k \le n-5}\left(g(k+3) + g(n-k) + k(n-k-5) + 2\right),$$

whose solution is no greater than the bound of Case 1.

Case B: There is a separating triplet of the fourth type which separates v_3 from v_1 and v_2.

Let $\{x_1, x_2, x_3\}$ be such a separating triplet, with $x_1 \in G_1$ and $x_2, x_3 \in G_2$. Furthermore, suppose $\{x_1, x_2, x_3\}$ maximizes $|V(\bar{G}_2)|$, where $\bar{G}_2$ is the part of G_2 separated by $\{v_3, x_2, x_3\}$.

Vertices $x_2, x_3 \in \tilde{G}_2 \cup w_2 \cup w_3$, otherwise G is not triconnected. Define $\hat{G}_2 = \tilde{G}_2 - \bar{G}_2 - x_2 - x_3$ and let $|V(\bar{G}_2)| = \bar{l}$. If there is a separating triplet of the third type $\{u_1, v_2, u_2\}$, where $u_1 \in G_1$ and $u_2 \in G_2$, then by Claim 1 $u_2 \in \hat{G}_2$. Hence, the following recurrence relation is obtained using the corollary to lemma 2:

$$g(n) = \max_{1 \le k \le n-5}\left(g(k+3) + g(n-k) + \max_{\substack{0 \le l \le n-k-5 \\ 0 \le \bar{l} \le n-k-l-5}}\left(k(n-k-l-\bar{l}-5) + \frac{3}{2}(l+\bar{l}+4) - 4\right) + 1\right).$$

As in Case A, the maximum is reached when $l = \bar{l} = 0$, if $k > 1$. Hence, the equality becomes

$$g(n) = \max_{1 \le k \le n-5}\left(g(k+3) + g(n-k) + k(n-k-5) + 3\right),$$

which again gives a worse upper bound than the bound of Case 1. If $k=1$ then the maximum is reached when either $l = n-k-5$ and $\bar{l} = 0$ or $\bar{l} = n-k-5$ and $l = 0$. But in this case either $\{v_1, w_2, w_3\}$ or $\{v_3, x_2, x_3\}$ would be chosen instead of $\{v_1, v_2, v_3\}$.

Case C: There is a separating triplet of the fifth type which separates v_3 from v_1 and v_2.

Let $\{x_1, x_2, x_3\}$ be such a separating triplet, with $x_1 \in G_2$ and $x_2, x_3 \in G_1$. Furthermore, suppose $\{x_1, x_2, x_3\}$ maximizes $|V(\bar{G}_1)|$, where $\bar{G}_1$ is the part of G_1 separated by $\{v_3, x_2, x_3\}$. Define $G'_1 = G_1 - \bar{G}_1 - x_2 - x_3 - w_1$ and let $|V(\tilde{G}_1)| = \tilde{l}$. Since $\{v_1, v_2, v_3\}$ was chosen as the initial separating triplet instead of $\{v_1, v_2, x_1\}$ or $\{w_1, v_2, v_3\}$, $| \, |V(G_1)| - |V(G_2)| \, | \le 1$. Therefore, $k = \lfloor \frac{n-3}{2} \rfloor$ or $\lceil \frac{n-3}{2} \rceil$. Since these two cases are analogous, assume $k = \lfloor \frac{n-3}{2} \rfloor$.

If there is a separating triplet of the third type $\{u_1, v_2, u_2\}$, where $u_1 \in G_1$ and $u_2 \in G_2$, then by Claim 1 $u_1 \in G'_1 \cup w_1$ and $u_2 \in \tilde{G}_2 \cup x_1$. Hence, the recurrence relation obtained is using the corollary to lemma 2:

$$g(n) = g(\lfloor \frac{n+3}{2} \rfloor) + g(\lceil \frac{n+3}{2} \rceil) + \max_{\substack{0 \le l \le \lceil \frac{n-3}{2} \rceil - 2 \\ 0 \le \tilde{l} \le \lfloor \frac{n-3}{2} \rfloor - 2}} \left(\left(\lceil \frac{n-3}{2} \rceil - l - 1 \right)\left(\lfloor \frac{n-3}{2} \rfloor - \tilde{l} - 1 \right) + \frac{3}{2}(l + \tilde{l} + 4) - 3 \right) \right).$$

The right hand side is bilinear in l and $\tilde{l}$, hence the maximum is reached at the endpoints of the intervals. If l or $\tilde{l}$ is equal to 0 then we get a degenerate case that is equal to case A. If $l = \lceil \frac{n-3}{2} \rceil - 2$ and $\tilde{l} = \lfloor \frac{n-3}{2} \rfloor - 2$ then the equality becomes

$$g(n) = g(\lfloor \frac{n+3}{2} \rfloor) + g(\lceil \frac{n+3}{2} \rceil) + \frac{3}{2}(n-3) - 2).$$

The solution to this recurrence is $\le \frac{3}{2} n \log_2 n + \frac{13}{2}$. For any $n \ge 19$ this solution gives an upper bound smaller than $\frac{(n-1)(n-4)}{2}$. All triconnected graphs on $5 \le n \le 18$ vertices with constraints of Case C have less number of separating triplets than the wheel on n vertices. Hence, for case 2

$$g(n) \le \frac{(n-1)(n-4)}{2}$$

for all n.

Note: Case 2 includes the case when no separating triplet of the third type exists.

This concludes the case by case analysis of the trade-offs between separating triplets of G of the third type and the separating triplets of the fourth and fifth types.

The established upper bound on the number of separating triplets of G for all n is

$$g(n) \le \frac{(n-1)(n-4)}{2}.$$

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