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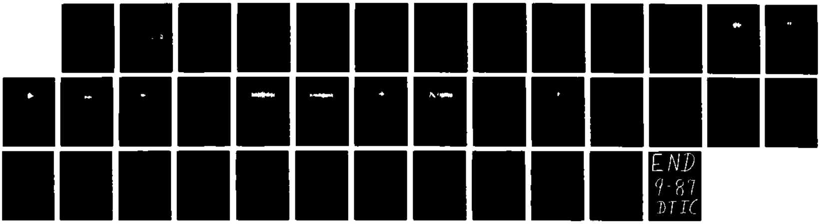

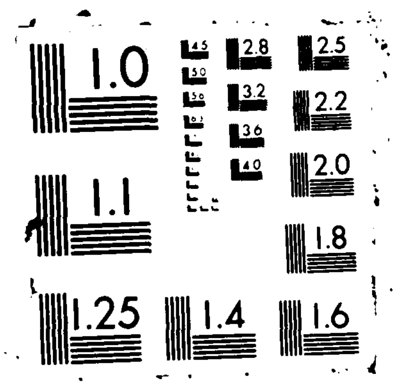
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STRICTLY OSCILLATORY PROCESSES

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by

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Abstract

Empirical evidence shows that the rate of zero-crossings of many stochastic processes tends to increase by repeated differencing. This motivates the definition of a class of processes whose expected oscillation increases monotonically by repeated differencing. The class of strictly stationary processes is a subclass of this class. It is shown that there is a limit to oscillation by proving that the point processes of zero-crossings obtained by repeated differencing converge.

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STRICTLY OSCILLATORY PROCESSES

by

B. Kedem and D. Martin

1. Introduction

In this paper we introduce a class of random processes to which we refer as strictly oscillatory and suggest a method to monitor the oscillation observed in such processes. When a process is second-order stationary, the oscillation observed in the process is described very effectively by the spectrum. When the process is nonstationary, various attempts have been made to extend the notion of the spectrum to model the time varying spectral content of the process. However, a process need not possess moments at all and still appear to be oscillatory. What is needed then is a way to describe oscillation in random phenomena removed from stationarity assumptions and independent of any moment conditions.

In many respects the simplest way to describe the oscillation observed in a stochastic process, stationary or nonstationary, is through the point processes obtained from higher order crossings. The advantages offered by such zero-crossing counts are as follows:

1. The pattern of oscillation changes can be detected and described directly by zero-crossing counts without recourse to any Fourier analysis. Thus we gain simplicity.
2. The zero-crossing counts observed in finite series in discrete time possess all moments regardless of whether the original process has moments or not. Thus we relax the requirement of finite moments.

3. In scientific and engineering applications the amounts of data can be very large and at times forbidding. Zero-crossing counts of filtered series provide effective data reduction.
4. As we shall see, it is possible to construct useful graphical plots from the rates of higher order crossings to monitor spectral changes. In this regard the monotonicity of higher order crossings plays a crucial role.

We shall show that the class of strictly oscillatory processes is quite large, and prove a certain weak convergence associated with such processes. This convergence means that in some sense there is a limit to oscillation.

Sections 2 and 3 motivate our definition of strictly oscillatory processes given in section 4. In section 5 we prove the convergence of point processes defined by higher order crossings, while section 6 discusses briefly the Gaussian case.

2. Evolutionary Spectra Approach to Oscillatory Processes

An attempt to model oscillatory processes has been made in Priestley (1981, ch. 11). It is important to make a distinction between the two approaches and to point to the difficulties one encounters in trying to extend the theory of second-order stationary processes to the nonstationary case. It should be borne in mind that our approach is completely different and is graphical combinatorial in nature.

In classical spectral analysis of time series, it is generally assumed that time series are generated by second-order stationary processes where the autocovariance function and the spectrum form a Fourier pair. When stationarity breaks down, this Fourier relation is no longer true since the autocovariance is time dependent. There were several attempts to define time-varying spectra of nonstationary processes. Perhaps the most complete approach is that of "evolutionary spectra" developed by Priestley (1981). The basic motivation underlying the evolutionary spectra approach stems from the need to model the local behavior of nonstationary processes. Whereas the spectrum of a stationary process describes the power-frequency distribution for the whole process, the evolutionary spectrum is time dependent and describes the local power-frequency distribution at each time instant.

A stationary process $\{Z_t\}$ with spectral distribution function $F(\omega)$ admits the spectral representation

$$Z_t = \int_{-\pi}^{\pi} e^{it\omega}\, d\xi(\omega), \qquad t = 0, \pm 1, \ldots, \tag{1}$$

where $\xi(\omega)$ is a process of orthogonal increments and satisfies

$$E|d\xi(\omega)|^2 = dF(\omega).$$

This representation allows us to identify the component in Z_t which has frequency ω and to determine the contribution of this component to the total power of the process. A way to define a class of nonstationary processes is to replace the exponential function in (1) by a more general class of functions. Priestley (1981, ch. 11) defines an oscillatory process to be a process which admits the representation

$$Z_t = \int_{-\pi}^{\pi} e^{it\omega} A_t(\omega)d\xi(\omega), \qquad t = 0,\pm 1,\ldots \tag{2}$$

where for each ω, $A_t(\omega)$ has a generalized Fourier transform whose modulus has an absolute maximum at the origin, and $\xi(\omega)$ is as in (1). The evolutionary spectrum at time t is defined as

$$dH_t(\omega) = |A_t(\omega)|^2 dF(\omega), \qquad -\pi < \omega \le \pi \tag{3}$$

and it represents the spectral content of the process at time t. From (2) and (3) it is possible to construct a theory of prediction and filtering for nonstationary processes which parallels that for stationary processes.

Though useful in some respect, this theory has its drawbacks when viewed from the point of view of applications. First, the representation (2) is not unique and one is faced with the problem of choosing $A_t(\omega)$. A possible remedy is suggested in Priestley (1981) when $A_t(\omega)$ is a slowly varying function of t for each fixed ω, but is too technical to be described here. Second, it is difficult to prove that a process encountered in practice can in fact be represented by (2). Third, there are processes which appear to be very oscillatory but possess no moments or just first order moments but for which (2) is clearly unsuitable. Fourth, when using the evolutionary spectrum (3) to detect spectral changes in nonstationary time series, we need to consider and compare many different pictures or graphs, one for each time point t. Thus graphical detection of spectral changes can be quite cumbersome using evolutionary spectra. Also, the estimation of numerous spectra can be very time consuming. A less stringent approach, but one which still characterizes the local character of nonstationary processes, can alleviate these difficulties. Such an approach can be developed by higher order crossings of nonstationary processes and is evidently not tied to any moments assumption.

3. Zero-crossing Rate Processes

Let $\{Z_t\}$, $t = 0,\pm 1,\ldots,$ be a stochastic process defined on some probability space (Ω,A,P). It is assumed the process is not a constant with probability one but that it may be stationary or nonstationary and it may or may not possess moments of any order. Let ∇ be the backward difference operator, $\nabla Z_t = Z_t - Z_{t-1}$, $\nabla^j Z_t = \nabla(\nabla^{j-1} Z_t)$, and define a sequence of binary processes $\{X_t^{(j)}\}$ by

$$X_t^{(j)} \equiv \begin{cases} 1, & \nabla^{j-1} Z_t \geq 0 \\ 0, & \text{otherwise}, \quad t = 0,\pm 1,\ldots . \end{cases}$$

The symbol changes in $\{X_t^{(j)}\}$ correspond to zero-crossings in discrete time in $\{\nabla^{j-1} Z_t\}$. We are interested in the number of such zero-crossings in finite time series. Define

$$d_t^{(j)} = I[X_t^{(j)} \neq X_{t-1}^{(j)}], \quad t = 0,\pm 1,\ldots,$$

where $I[\cdot]$ is the indicator function. When $d_t^{(j)} = 1$ we say that a zero-crossing occurs at time t in $\{\nabla^{j-1} Z_t\}$. The number of zero-crossings in

$$\nabla^{j-1} Z_t, \quad t = 1,2,\ldots,N$$

is denoted by $D_{j,N}$ and is therefore given by

$$D_{j,N} = \sum_{t=1}^{N} d_t^{(j)}, \quad j = 1,2,\ldots .$$

The $D_{j,N}$ are called the (number of) higher order crossings in $\nabla^{j-1} Z_t$, $t = 1,2,\ldots,N$, or HOC for short. Higher order crossings were introduced by Kedem and Slud (1981, 1982) for the purpose of discrimination, while their applications in spectrum analysis are discussed in Kedem (1986a, 1986b, 1987). A review of the theory and application of HOC can be found in Kedem (1986c). This work demonstrates that when it comes to stationary Gaussian processes, HOC possess a surprising amount of spectral information. In fact, in this case the sequence of expected HOC determines uniquely the correlation function $\{\rho_k\}$ and the normalized spectral distribution function $\overline{F}(\omega)$ and we write

$$\{ED_{j,N}\} \Longleftrightarrow \{\rho_k\} \Longleftrightarrow \overline{F}(\omega). \tag{4}$$

Relation (4) emboldens us to believe that higher order crossings may be helpful in tracing spectral changes in nonstationary processes as well.

Observe that in the stationary case the zero-crossing rate $D_{j,N}/N$ converges to a constant in some sense for fixed j as $N \to \infty$. On the other hand for a nonstationary process no such convergence is expected since the spectral content or oscillatory pattern of such a process may be time-varying. This motivates our definition of rate processes for monitoring spectral changes and changes in oscillation patterns.

Definition 1. A (HOC) rate process of order j is defined for each fixed j by the process

$$\overline{D}_j(N) \equiv \frac{D_{j,N}}{N}, \quad N = 1,2,\ldots .$$

It is seen that for each fixed j, $\overline{D}_j(N)$ traces the zero-crossing rate of $\{\nabla^{j-1}Z_t\}$, $t = 1,\ldots,N$, as a function of N. Evidently

$$0 \le \overline{D}_j(N) \le 1$$

and so, any rate process possesses all moments. It is helpful to look at the graphs of rate processes obtained from different time series.

The figures below feature rate processes corresponding to stationary and nonstationary processes. Each figure displays the first eight rate processes $\bar{D}_1,(\cdot),\ldots,\bar{D}_8(\cdot)$, where $\bar{D}_1(\cdot)$ is the lowest curve while $\bar{D}_8(\cdot)$ is the highest. It is seen that changes in the spectral content and in the oscillatory pattern in each case are well captured by the rate processes. Also, the rate processes display a monotone convergence towards a limiting rate process which can be interpreted to mean that there is a limit to the oscillation displayed by a process stationary or not. It is this monotone behavior observed in numerous phenomena which prompted this investigation and consequently our definition of strictly oscillatory processes.

We close this section by noting a useful feature of the figures. As mentioned earlier, if one tries to plot evolutionary spectra on a single coordinate system the graphs may overlap heavily so that separate plots, up to one graph for each time point, are necessary in general. On the other hand our figures provide a compact way to trace the oscillatory history of a process on a single frame and overlapping is prevented due to the displayed monotonicity of rate processes. Now, if one tries to imitate the idea of plotting other quantities as functions of time, overlapping graphs may occur. For example, Figure 12 shows the autocorrelations of the differences $\nabla^j Z_t$ as functions of time where the time series comes from the utterance of "six". It is seen that both, the rate processes and the autocorrelation processes, trace the spectral changes in the time series but in the rate processes no overlapping occurs.

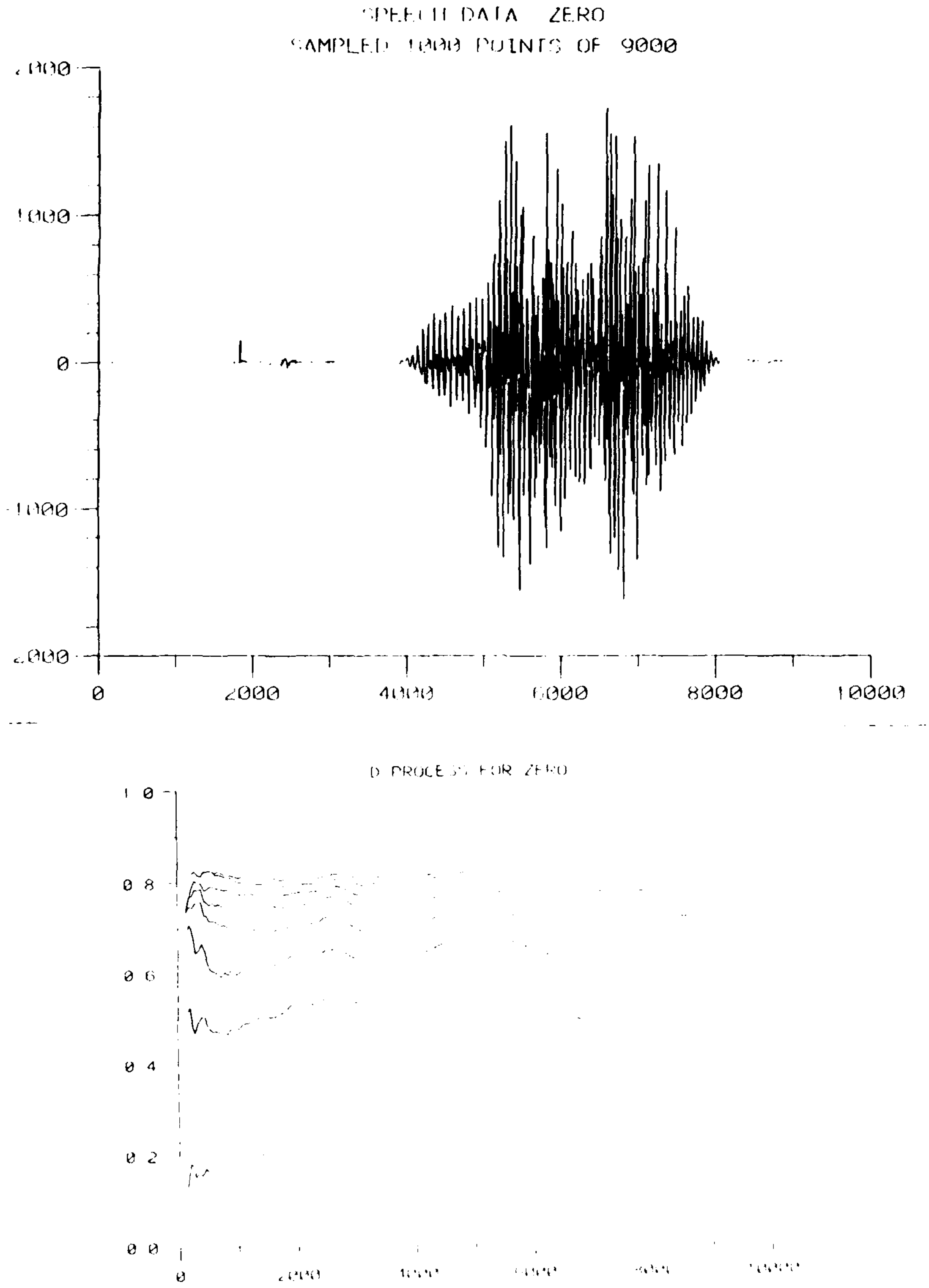


Figure 1. A time series from the utterance of "zero" and the corresponding first eight rate processes.

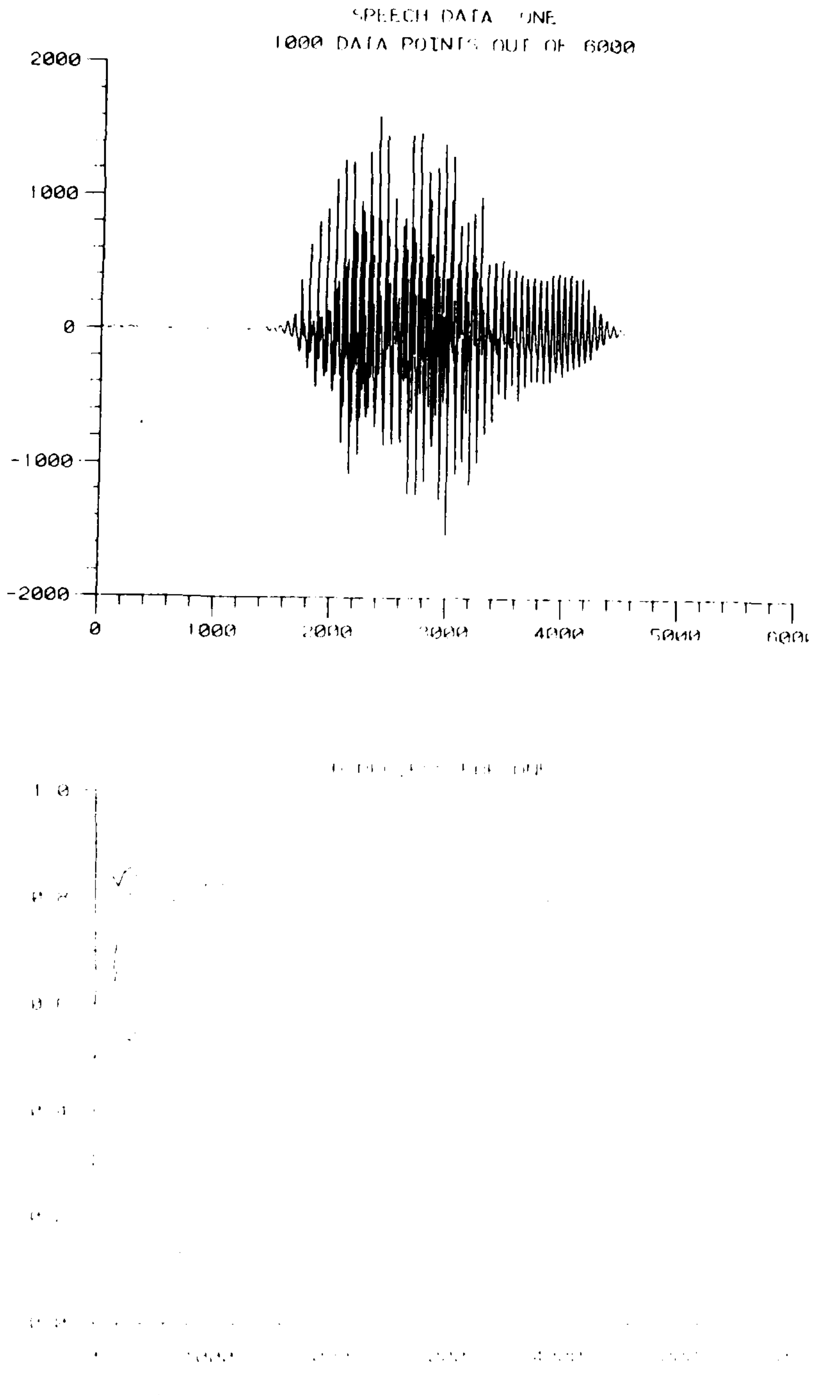


Figure 2. A time series from the utterance of "one" and the corresponding first eight rate processes.

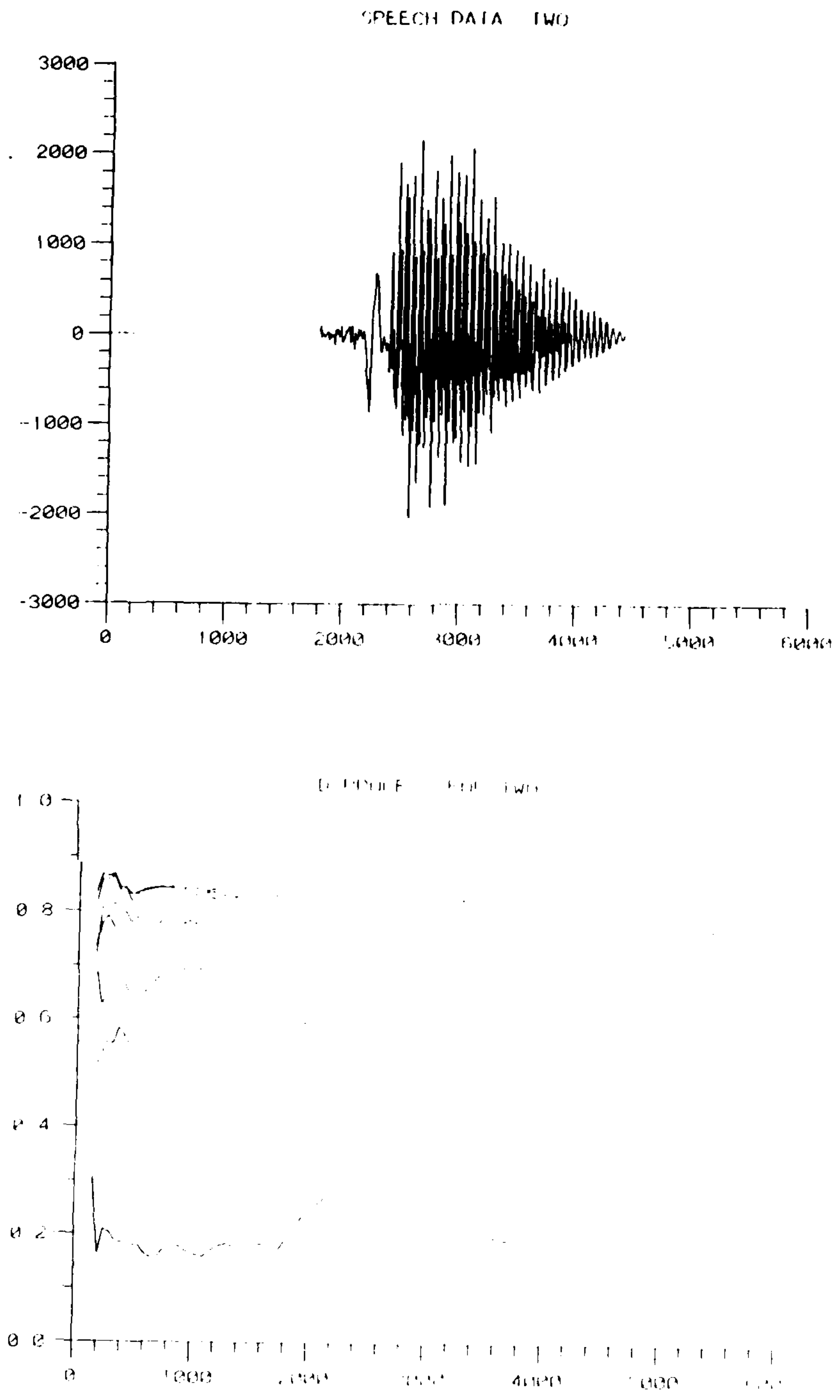


Figure 3. A time series from the utterance of "two" and the corresponding first eight rate processes.

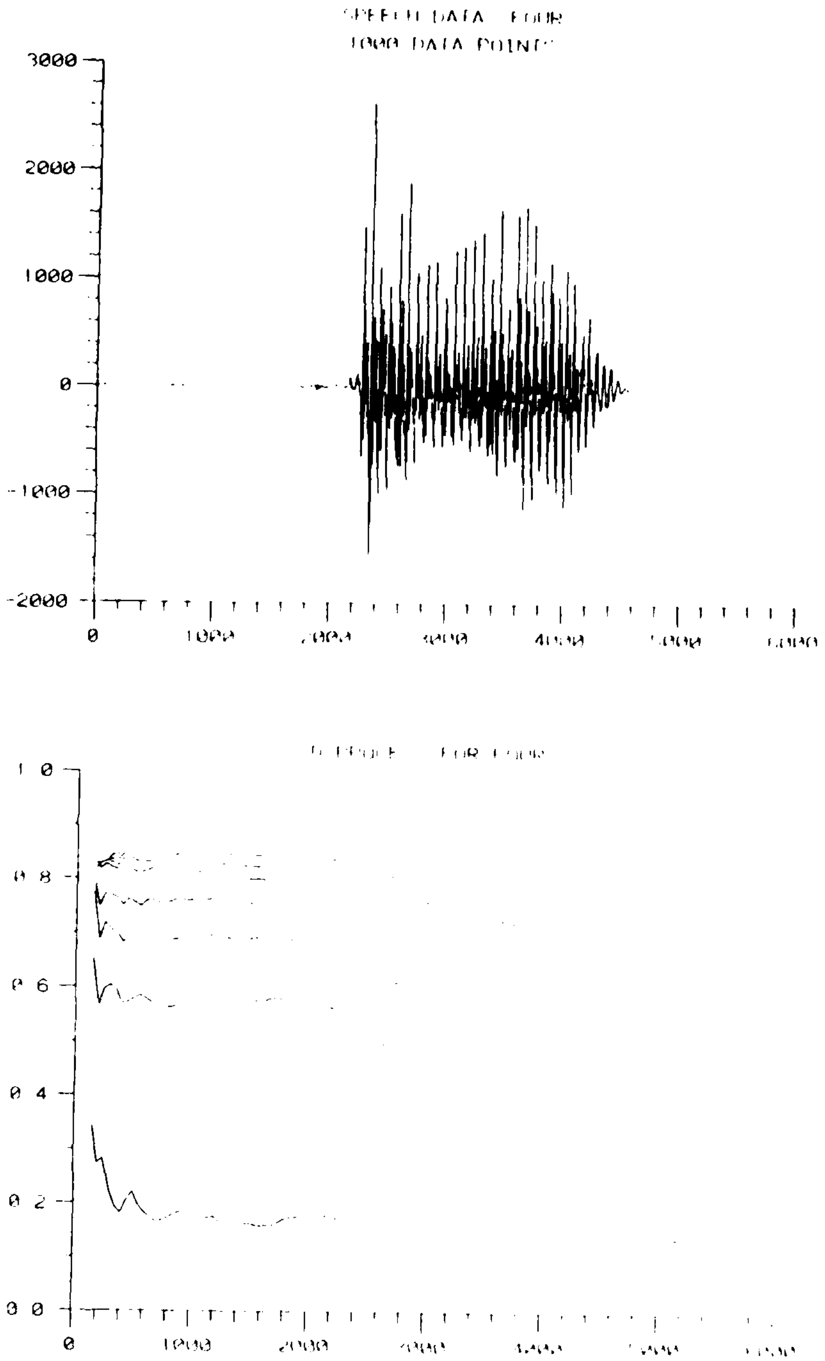


Figure 4. A time series from the utterance of "four" and the corresponding first eight rate processes.

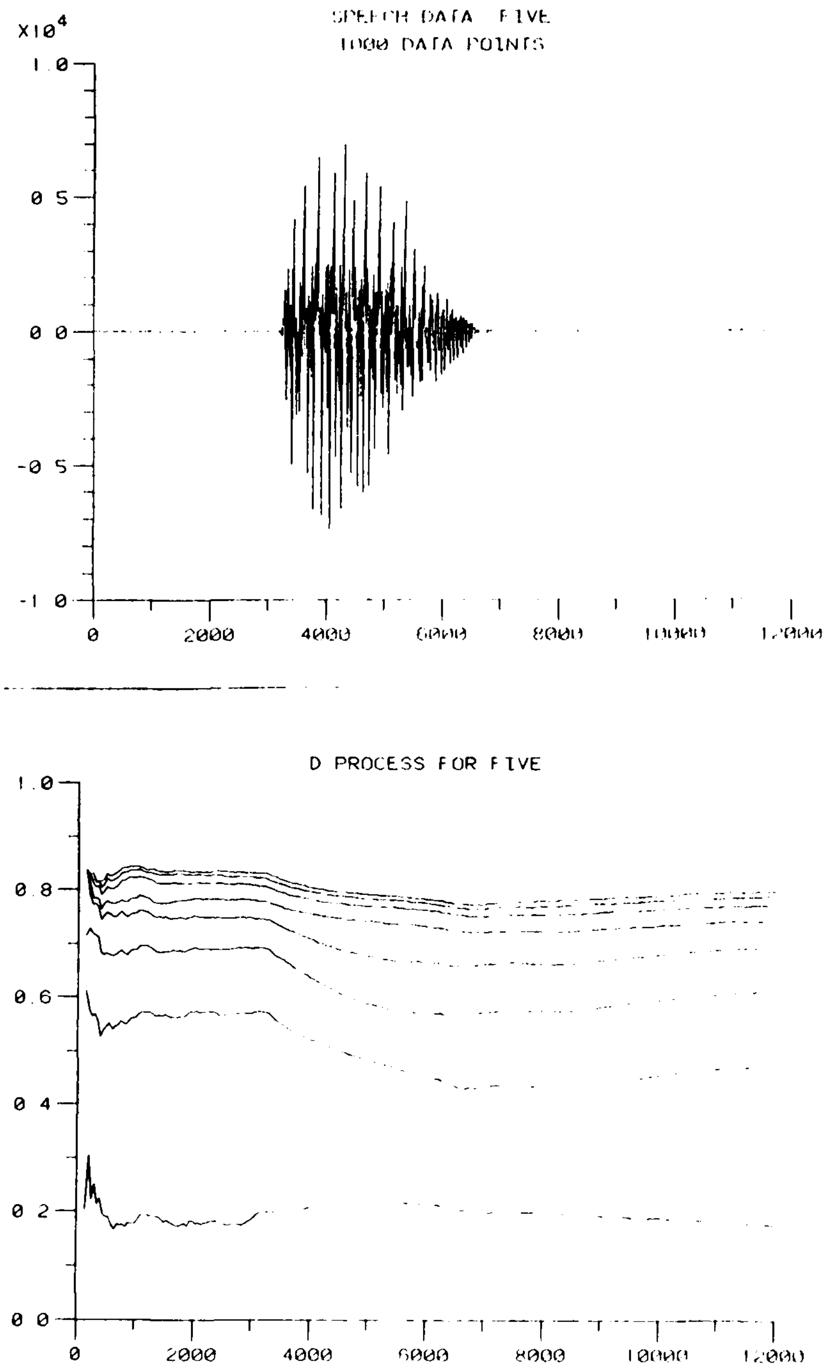
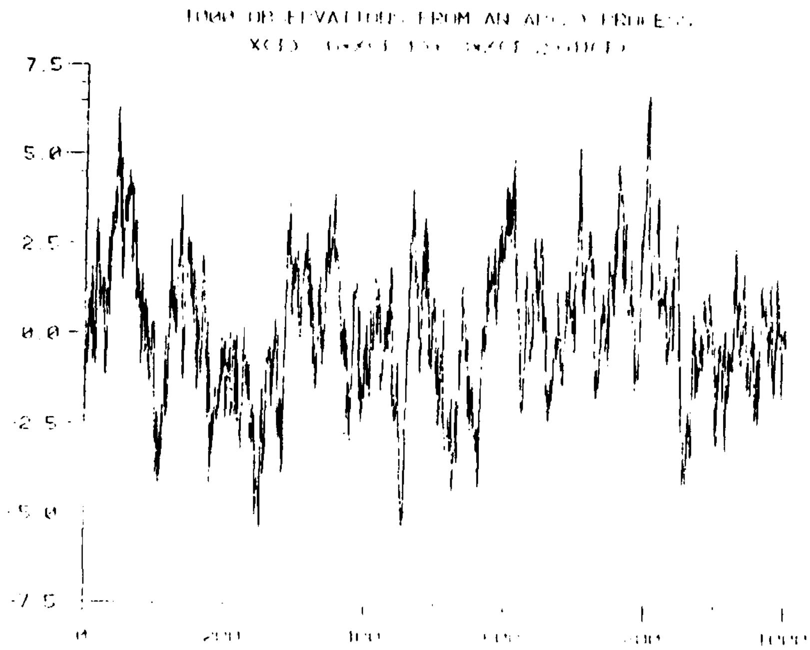


Figure 5. A time series from the utterance of "five" and the corresponding first eight rate processes.



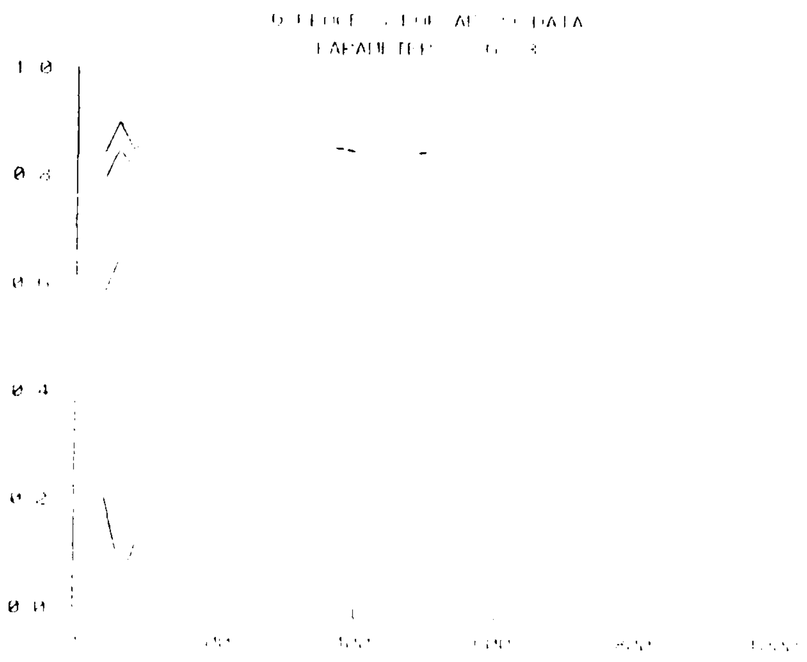


Figure 6. An AR(2) process and its first eight rate processes.

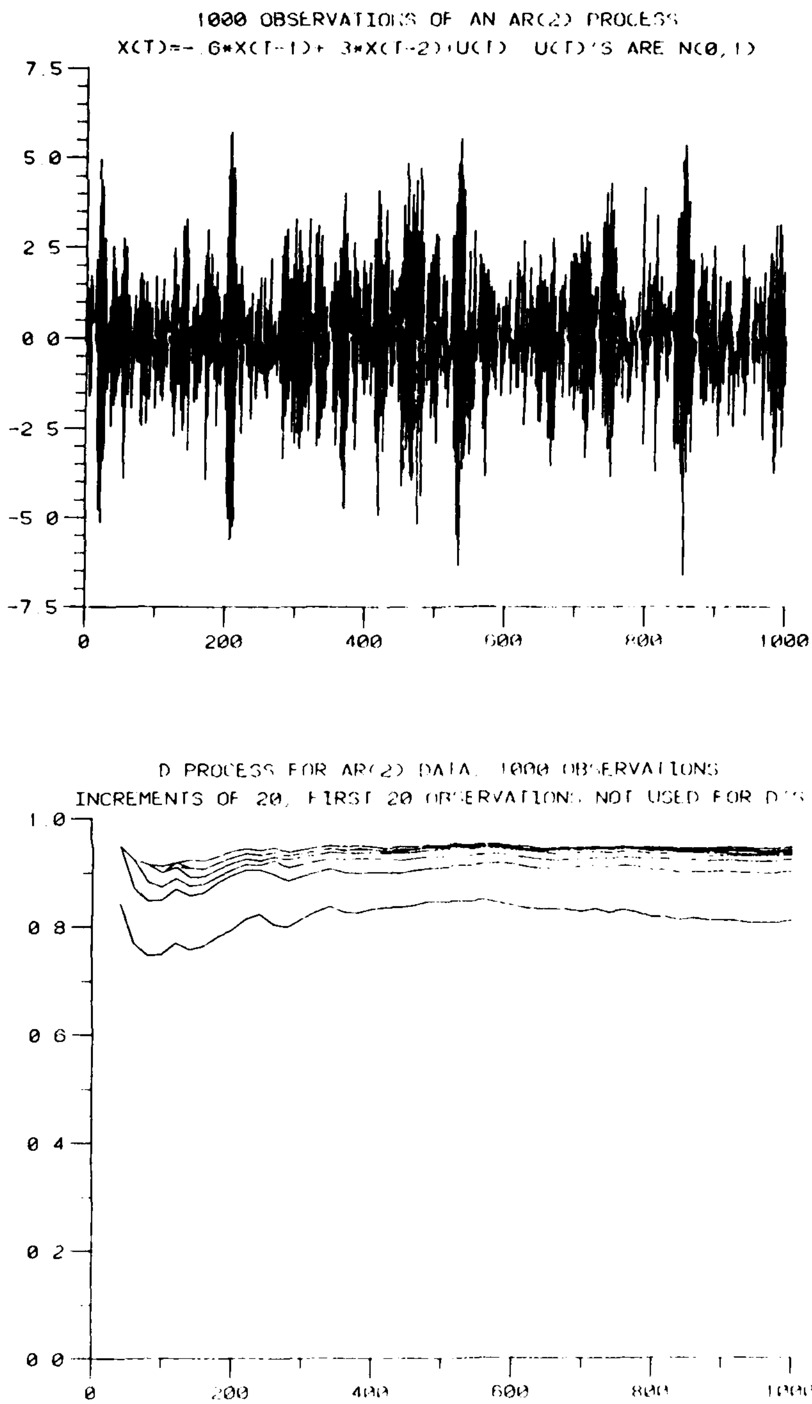


Figure 7. An AR(2) process and its first eight rate processes.

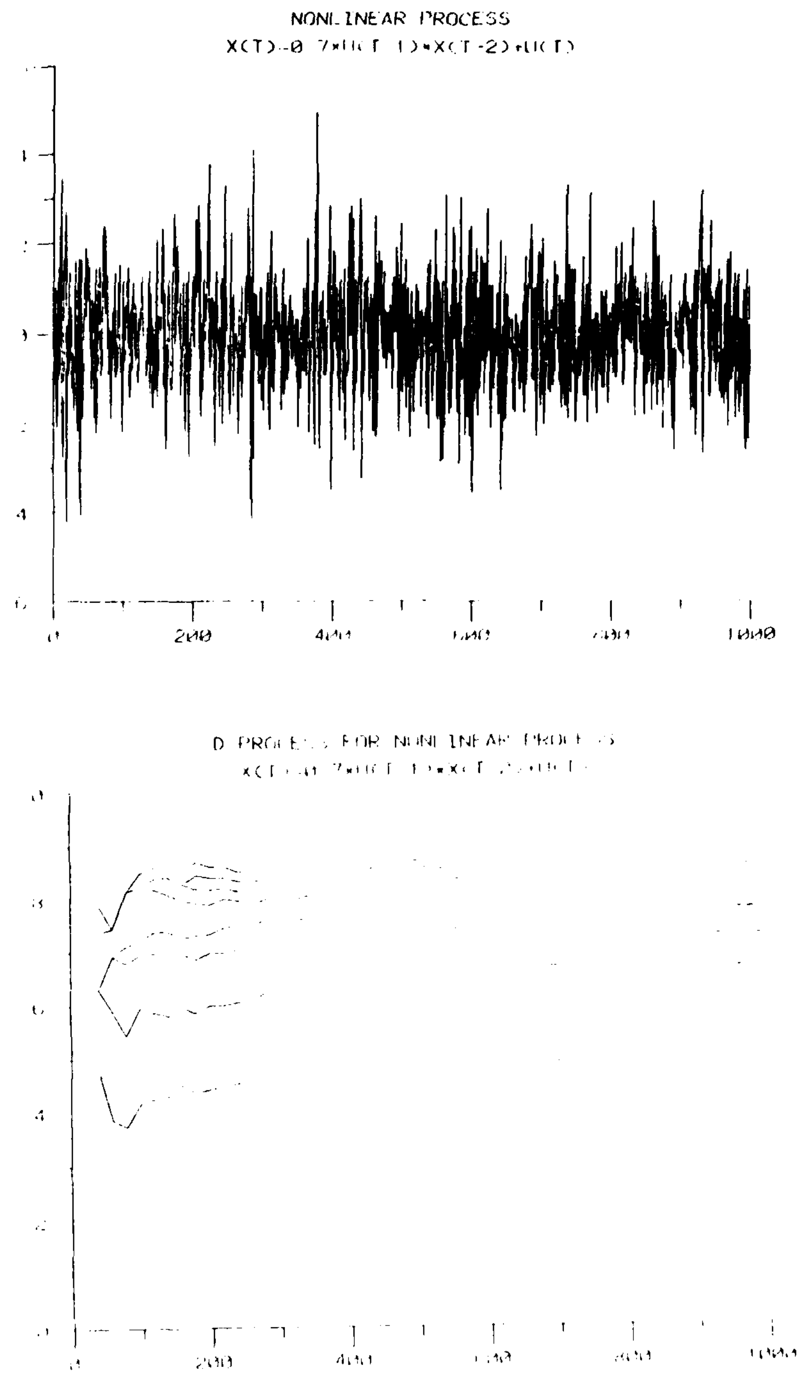


Figure 8. A nonlinear process and its first eight rate processes.

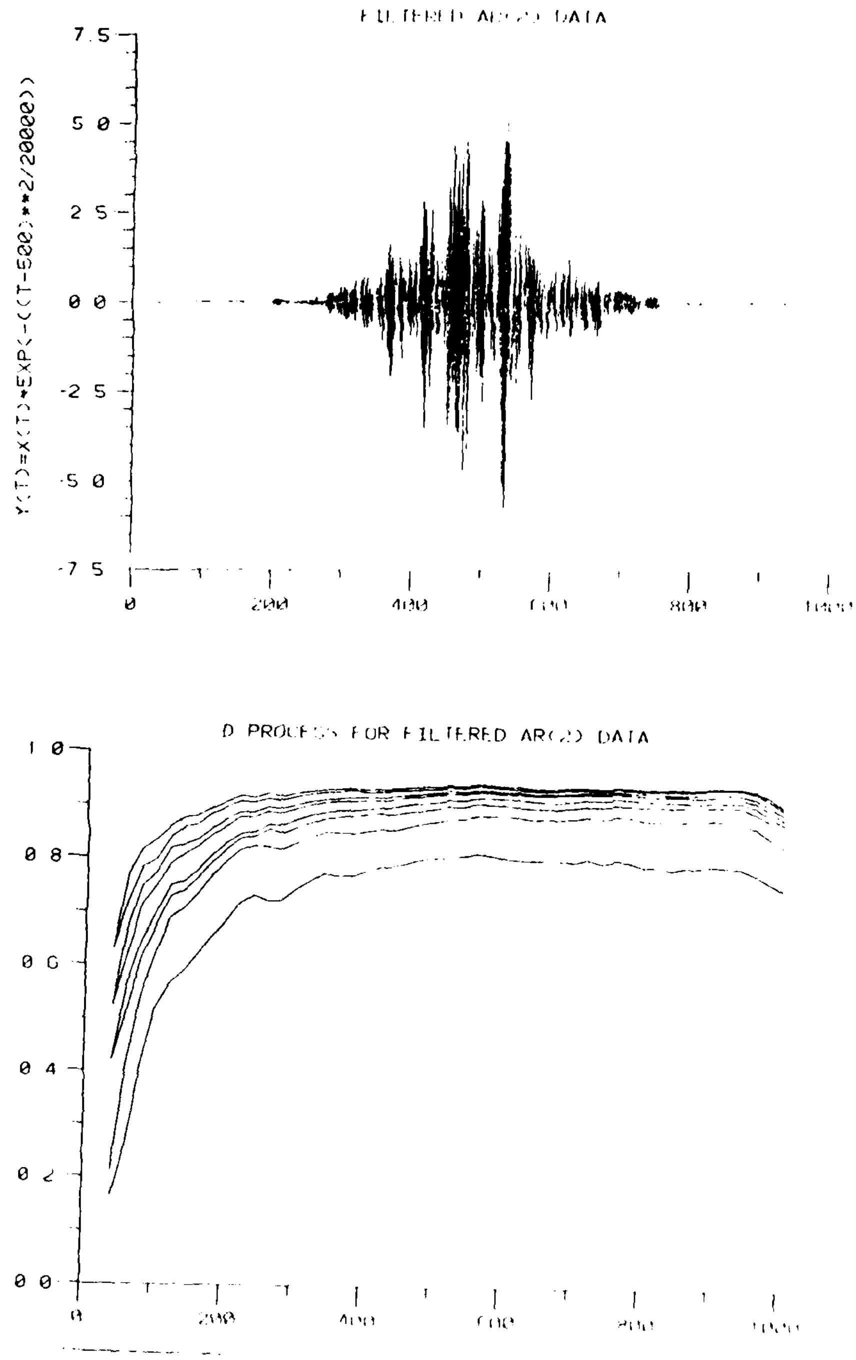


Figure 9. A uniformly modulated AR(2) process and its first eight rate processes.

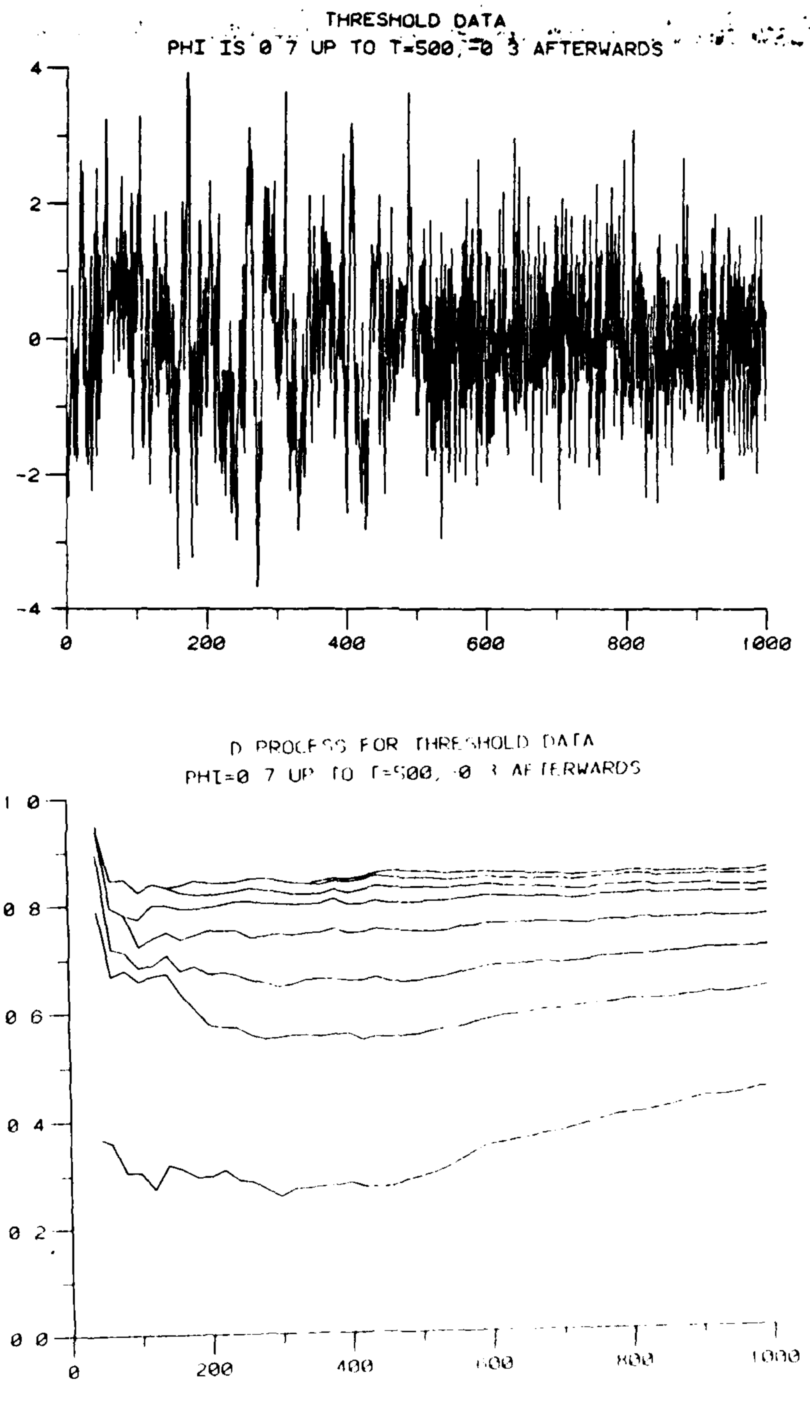


Figure 10. The rate processes of a first order threshold process.

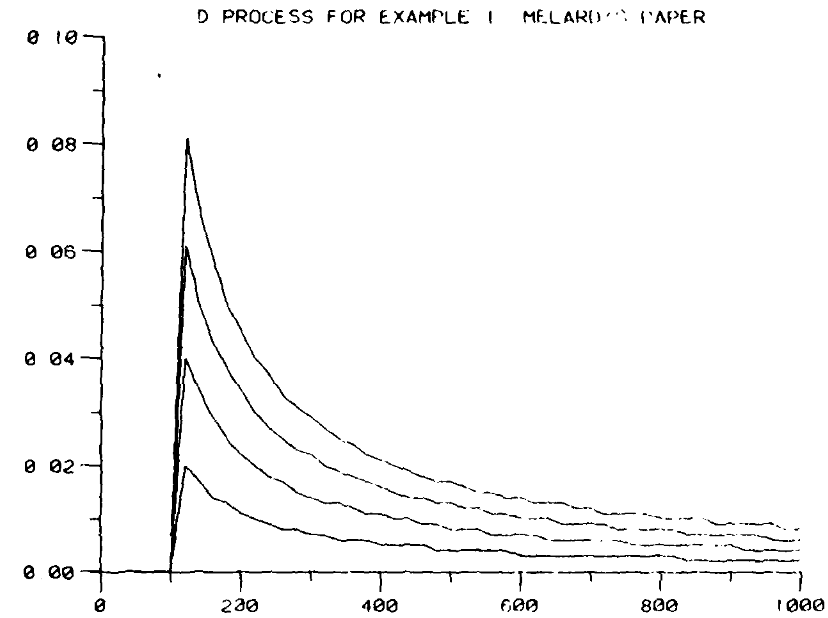


Figure 11. The rate processes of the nonstationary oscillatory process (Mélard (1985)),

$$Z_t = \begin{cases} Y_1, & t = 1 \\ Y_2, & t = 2 \\ 0, & \text{otherwise} \end{cases}$$

$$EY_1 = EY_2 = 0$$

$$\mathrm{Var}\ Y_1 = \mathrm{Var}\ Y_2 = 1$$

$$EY_1Y_2 = 0$$

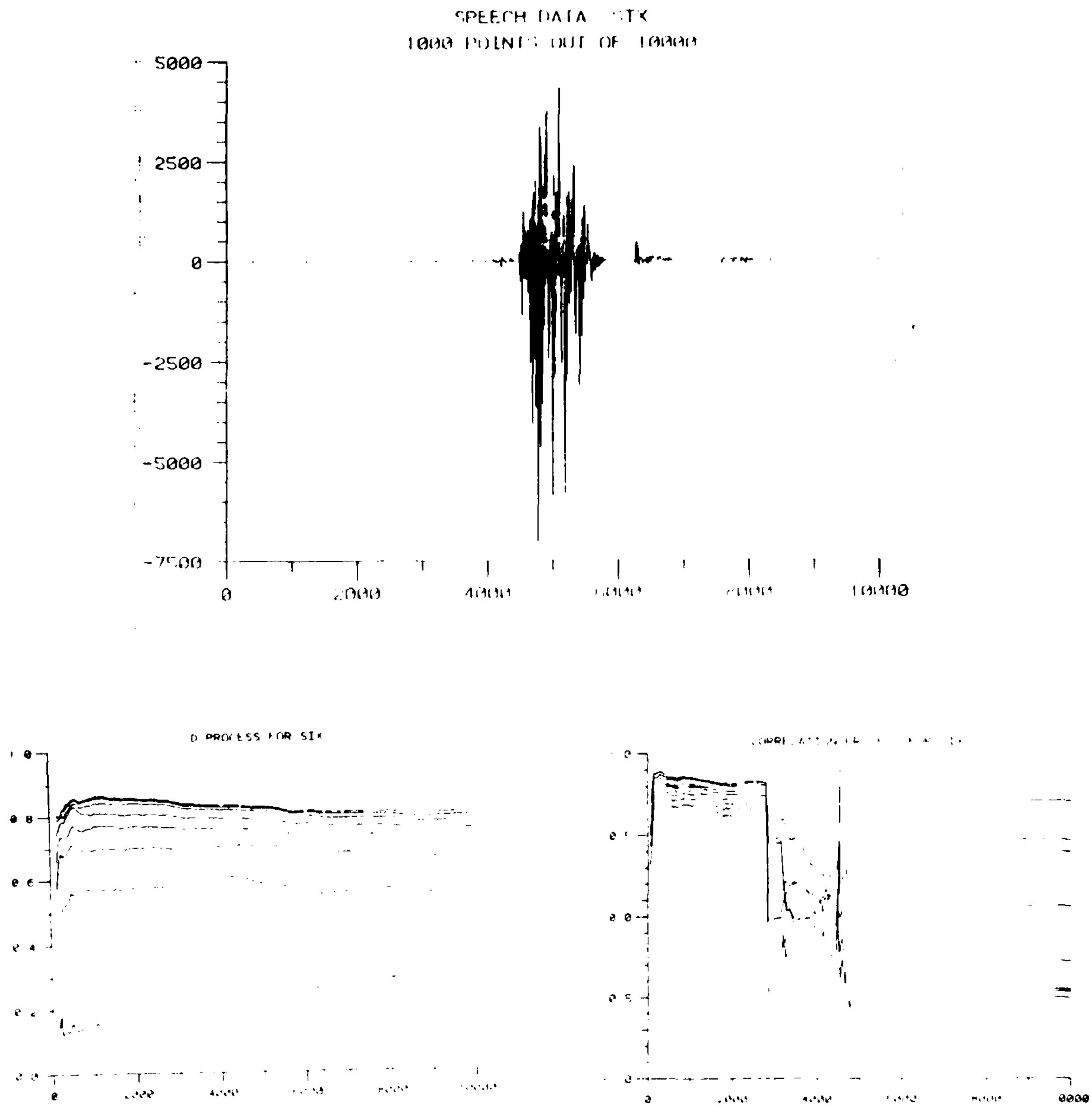


Figure 12. Rate processes and correlation process from an utterance of "six". In the correlation processes overlapping occurs.

4. Definition of Strictly Oscillatory Processes

The monotone behavior of rate processes observed in the figures is quite a general phenomenon and can be attributed to the difference operation. For example, in the second-order stationary case, a difference operation acts as a high pass filter which gives more power to higher frequencies. This amplification causes the process to be more and more oscillatory with each application of ∇. In general, differencing causes values at neighboring points to be negatively correlated and thus increase the number of symbol changes in $\{X_t^{(j)}\}$. It seems therefore that the requirement that $E\overline{D}_j(N)$ increases as $j \to \infty$ for each fixed N is quite reasonable due to this observed generality. This leads to our definition of strictly oscillatory processes.

To define the class of strictly oscillatory processes, and at the same time describe the monotone convergence observed in the figures we appeal to the theory of point processes as developed by Kallenberg (1976).

Recall that $\{Z_t\}$, $t = 0,\pm1,\ldots,$ is a process defined on the probability space (Ω,A,P) and let $\mathcal{S} = [0,\infty)$ be equipped with the Borel field $\mathcal{G}$. Further let $\mathcal{B}$ be the class consisting of all bounded sets in $\mathcal{G}$. The set of all locally finite measures μ that are nonnegative and integer valued is denoted by $\mathcal{N}$. Thus $\mu \in \mathcal{N}$ if for $B \in \mathcal{B}$

$$\mu(B) \in \mathbb{Z}^+ \equiv \{0,1,2,\ldots\}.$$

In $\mathcal{N}$ we introduce the σ-algebra $\mathcal{F}$ generated by the mappings $\mu \to \mu(B)$, $B \in \mathcal{B}$. This is the smallest σ-algebra that makes these mappings measurable. By a point process on $\mathcal{S}$ we mean any measurable mapping of (Ω,A,P) into $(\mathcal{N},\mathcal{F})$.

With an obvious extension of the previous notation, let $D_j(B)$, $B \in \mathcal{B}$, be the number of zero-crossings generated by $\nabla^{j-1}Z_t$ in B. Then for each realization of $\{Z_t\}$, $D_j(\cdot)$ defines a measure in N and so for each fixed j, $D_j(\cdot)$ is a point process. In other words, the values of $D_j(\cdot)$ when ranging over Ω are measures in N. The sequence $\{D_j(\cdot)\}$ can now be used in defining the class of oscillatory processes.

The following definition reflects our experience that by differencing any finite section of a process the corresponding higher order crossings tend to increase. Note that since t is defined over all the integers, theoretically we do not have any problem with repeated differencing applied to finite sections. On the other hand, in practice only finite records are available and we lose one observation with each difference. However this does not pose any serious problem since the amount of differencing needed in practice is very small as is well indicated by the above figures.

Definition 2. A stochastic process $\{Z_t\}$, $t = 0,\pm 1,\ldots,$ is called strictly oscillatory if and only if

$$ED_j(B) \le ED_{j+1}(B)$$

for every $B \in \mathcal{B}$ and $j = 1,2,\ldots$.

The class of strictly oscillatory processes is quite large. Strictly stationary processes are members of this class and second-order stationary processes become strictly oscillatory by undergoing a sufficient amount of differencing. However, oscillatory processes in the sense of Priestley (1981) need not be strictly oscillatory. Our first example concerns stationary Gaussian processes.

Theorem 1. A zero mean stationary Gaussian process is strictly oscillatory.

Proof. Let ρ_1, $\rho_\nabla(1)$ be the first autocorrelations in Z and ∇Z respectively. Then from Kedem (1984) $\rho_\nabla(1) \le \rho_1$. By stationarity, it is sufficient to consider the sequence of higher order crossings. Consider $D_{1,N}$ and $D_{2,N}$. Then the Gaussian assumption entails

$$ED_{1,N} = N(\frac{1}{2} - \frac{1}{\pi}\sin^{-1}\rho_1) \le N(\frac{1}{2} - \frac{1}{\pi}\sin^{-1}\rho_\nabla(1)) = ED_{2,N}.$$

But since the differences $\{\nabla^j Z_t\}$ are again stationary and Gaussian with mean zero we also have $ED_{2,N} \le ED_{3,N}$, and $ED_{3,N} \le ED_{4,N}$, etc. Therefore

$$ED_{j,N} \le ED_{j+1,N}, \quad \text{all} \quad j = 1,2,\dots .$$

Since the distributions in $\{Z_t\}$ are unaffected by time shifts due to stationarity, the proof is complete. □

We can actually prove a stronger result of which Theorem 1 is only a special case. This pertains to strictly stationary processes in general. In this case no moments assumption is needed at all.

Theorem 2. Every strictly stationary process $\{Z_t\}$ is strictly oscillatory.

Proof. Again, by stationarity it is sufficient to consider the HOC $D_{j,N}$. Define

$$\lambda_1^{(j)} \equiv P(X_t^{(j)} = 1 \mid X_{t-1}^{(j)} = 1), \quad p^{(j)} = P(X_t^{(j)} = 1).$$

Then

$$ED_{j,N} = 2Np^{(j)}(1 - \lambda_1^{(j)}).$$

Now, in general we have (See Lemma 1 below)

$$D_{j,N} \le D_{j+1,N} + 1$$

surely. Therefore $ED_{j,N} \le ED_{j+1,N} + 1$ and so

$$\frac{2Np^{(j)}(1 - \lambda_1^{(j)})}{N} \le \frac{2Np^{(j+1)}(1 - \lambda_1^{(j+1)})}{N} + \frac{1}{N}.$$

As $N \to \infty$ we obtain the inequality

$$2p^{(j)}(1 - \lambda_1^{(j)}) \le 2p^{(j+1)}(1 - \lambda_1^{(j+1)}).$$

Therefore

$$ED_{j,N} = 2Np^{(j)}(1 - \lambda_1^{(j)}) \le 2Np^{(j+1)}(1 - \lambda_1^{(j+1)}) = ED_{j+1,N}.$$

Since by stationarity every stretch of data of fixed length is endowed with the same distribution, it follows that the process is strictly oscillatory. □

Corollary 1. A sequence of independent and identically distributed random variables is strictly oscillatory.

Observe that the original sequence may be completely random but that the higher order differences become highly dependent and by the corollary also more and more oscillatory.

Our next example concerns weakly stationary processes. Here we appeal to the spectral properties of such processes as expressed by their spectral measures. But first we prove a basic lemma which builds on the interesting relation between $\{X_t^{(j)}\}$ and $\{X_t^{(j+1)}\}$.

Lemma 1. For any process $\{Z_t\}$ there exists a t_0, $1 \le t_0 \le N$ and t_0 depends on j, $j = 1,2,\ldots,$ such that for every N

$$\sum_{\substack{t=1 \\ t \ne t_0}}^{N} d_t^{(j)} \le \sum_{\substack{t=1 \\ t \ne t_0}}^{N} d_t^{(j+1)} \tag{5}$$

with probability one.

Proof. Fix j. Suppose $D_{j,N} = 0$; then we are done. So suppose $D_{j,N} > 0$. Then there is at least one symbol change in $\{X_t^{(j)}\}$, $t = 1,\ldots,N$. Note that in general we have the implication

$$\{X_{t-1}^{(j)} \ne X_t^{(j)}\} \subset \{X_t^{(j+1)} = X_t^{(j)}\}. \tag{6}$$

It follows that there exists t_1 such that $X_{t_1}^{(j+1)} = X_{t_1}^{(j)}$. But (6) implies that each symbol change in $\{X_t^{(j)}\}$ produces at least one symbol change in $\{X_t^{(j+1)}\}$ for $t_1 < t \le N$. On the other hand, again by (6) and due to an end effect, the number of symbol changes in $\{X_t^{(j+1)}\}$ always exceeds that in $\{X_t^{(j)}\}$ minus one for $1 \le t \le t_1$. It follows that there exists a t_0 which depends on j such that (5) holds. □

The lemma says that in $\{X_t^{(j+1)}\}$, $1 \le t \le N$, there are $N-1$ locations which give at least as many symbol changes as the corresponding locations in $\{X_t^{(j)}\}$, while no information is available about the remaining location which we denote by t_0. Clearly t_0 may change with j. Note that at t_0 we may lose a symbol change with each difference as is the case with polynomials in t. For by differencing a polynomial of degree n we obtain another polynomial of degree $n-1$.

Theorem 3. Let $\{X_t\}$ be a weakly stationary process with spectral distribution function F. Assume that π is in the spectral support. Then $\{\nabla^j Z_t\}$ becomes strictly oscillatory as $j \to \infty$.

Proof. Again by stationarity, it is sufficient to consider the higher order crossings. The normalized spectral measure of $\{\nabla^j Z_t\}$ is given by

$$\nu_j(d\omega) = \frac{(\sin\frac{1}{2}\omega)^{2j} dF(\omega)}{\int_{-\pi}^{\pi} (\sin\frac{1}{2}\lambda)^{2j} dF(\lambda)}$$

and from Kedem and Slud (1982) we know that ν_j converges weakly to a measure supported at $\pm\pi$. That is,

$$\nu_j \Rightarrow \frac{1}{2}\delta_{-\pi} + \frac{1}{2}\delta_{\pi}$$

where δ_u denotes the point mass at u. Therefore as $j \to \infty$

$$\mathrm{Corr}(\nabla^j Z_{t-1}, \nabla^j Z_t) = \int_{-\pi}^{\pi} \cos(\omega)\nu_j(d\omega) \to -1.$$

But this means that for every t_0

$$E d_{t_0}^{(j)} = P(d_{t_0}^{(j)} = 1) \to 1, \quad j \to \infty,$$

or

$$E(d_{t_0}^{(j+1)} - d_{t_0}^{(j)}) \to 0, \quad j \to \infty,$$

and apply the lemma. $\square$

To prove that a process is not strictly oscillatory it is sufficient to show that $Ed_t^{(j)}$ does not increase with j for some t.

Example 1. Let ε be a zero mean continuous random variable. Define $\{Z_t\}$ by

$$Z_1 = \varepsilon$$
$$Z_2 = \varepsilon$$
$$Z_t = 0, \quad t \neq 1,2.$$

Then the differences yield

	...	-1	0	1	2	3	4	5	6	7	8	...
Z	...	0	0	ε	ε	0	0	0	0	0	0	...
∇Z	...	0	0	ε	0	$-\varepsilon$	0	0	0	0	0	...
$\nabla^2 Z$	...	0	0	ε	$-\varepsilon$	$-\varepsilon$	ε	0	0	0	0	...
$\nabla^3 Z$	...	0	0	ε	-2ε	0	2ε	$-\varepsilon$	0	0	0	...
$\nabla^4 Z$	...	0	0	ε	-3ε	2ε	2ε	-3ε	ε	0	0	...
$\nabla^5 Z$	...	0	0	ε	-4ε	5ε	0	-5ε	4ε	$-\varepsilon$	0	...
		⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

We see that

$$Ed_4^{(3)} = 1 \quad \text{but} \quad Ed_4^{(4)} = P(\varepsilon < 0) < 1$$

and

$$Ed_5^{(5)} = 1 \quad \text{but} \quad Ed_5^{(6)} = P(\varepsilon > 0) < 1.$$

Therefore $\{Z_t\}$ is not strictly oscillatory.

Example 2. The above example can be generalized in various ways. Let $\{\varepsilon_t\}$ be a sequence of continuous random variables each with mean zero. Define

$$Z_t = \begin{cases} \varepsilon_t, & t \le 0 \\ \varepsilon_1, & t = 1,2 \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{array}{lcccccccc}
Z & \cdots & \varepsilon_0 & \varepsilon_1 & \varepsilon_1 & 0 & 0 & 0 & \cdots \\
\nabla Z & & & \varepsilon_1-\varepsilon_0 & 0 & -\varepsilon_1 & 0 & 0 & \cdots \\
\nabla^2 Z & & & & \varepsilon_0-\varepsilon_1 & \boxed{-\varepsilon_1} & \boxed{\varepsilon_1} & 0 & \cdots \\
\nabla^3 Z & & & & & \boxed{-\varepsilon_0} & \boxed{2\varepsilon_1} & -\varepsilon_1 & \cdots
\end{array}$$

and so

$$1 = Ed_4^{(3)} \ge Ed_4^{(4)} = P(\varepsilon_0 \text{ and } \varepsilon_1 \text{ have the same sign})$$

and the process is not strictly oscillatory.

The last two examples show that we can alter a sequence of random variables and even a strictly oscillatory sequence and render it non-oscillatory in our sense. This leads to the observation that an oscillatory process in the sense of Priestley (1981) need not be strictly oscillatory.

Theorem 4. Suppose a process $\{Z_t\}$ is an oscillatory process with the representation (2). Then it need not be strictly oscillatory.

Proof. It is sufficient to construct an example. Let $\{\varepsilon_t\}$ be a second order Gaussian stationary process with mean zero and correlation function ρ_k. Let $\{c_t\}$ be a sequence of nonnegative constants. Define

$$Z_t = c_t \varepsilon_t, \quad t = 0, \pm 1, \ldots .$$

Then Z_t is oscillatory with respect to $\{c_t e^{i\omega t}\}$. Assume $\rho_2 = 0$ and consider the time points 7, 8, say. Then

$$\mathrm{Corr}(Z_7, Z_8) = \rho_1$$

while

$$\mathrm{Corr}(\nabla Z_7, \nabla Z_8) \to 0, \quad \text{as} \quad c_7 \to 0.$$

Thus, when $\rho_1 \to -1$ and $c_7 \to 0$ we have

$$Ed_8^{(1)} \geq Ed_8^{(2)}. \qquad \square$$

It should be noted that by a proper choice of c_t, Z_t may still be strictly oscillatory. For example, we can choose c_t to be a slowly time varying sequence.

5. Convergence of Rate Processes

Consider the rate processes in the above figures. It is seen empirically that from a certain point, say N_0, we have

$$\bar{D}_1(N) \leq \bar{D}_2(N) \leq \bar{D}_3(N) \leq \ldots \leq 1, \quad \forall\, N > N_0,$$

where N_0 is quite small, e.g. $N_0 = 100$. This is rather expected since by Lemma 1, for all N

$$\bar{D}_j(N) \leq \bar{D}_{j+1}(N) + \frac{1}{N} \quad \text{surely.} \tag{7}$$

Now fix N_0 and consider $\bar{D}_j(N)$, $N > N_0$. Then by (7), as $N_0 \to \infty$, $\bar{D}_j(N)$ for $N > N_0$ will tend to increase with j. More precisely, (7) entails that

$$c_j \equiv \limsup_{N\to\infty} \bar{D}_j(N)$$

converges monotonically with probability one as $j \to \infty$ to a limit, c, say, such that $0 \leq c \leq 1$. This convergence, though useful in spectrum analysis as is shown in the next section, ignores the oscillatory behavior of the process for low and moderate N. We therefore take a different route.

Inspired by the figures, we shall assume that a limiting rate process exists and let $D(\cdot)$ be the corresponding point process.

A convenient way to describe the monotone convergence of the rate processes, is to prove the convergence of the corresponding point processes $D_1(\cdot)$, $D_2(\cdot)$, ..., obtained from strictly oscillatory processes, to $D(\cdot)$. To do so, we follow the work of Kallenberg (1976).

For $\mu \in N$ we write μf for the integral

$$\int_S f(s)\mu(ds).$$

For each fixed f, this integral is a mapping from N into R. Let f be an arbitrary continuous function $f: S \to [0,\infty)$ with compact support. The class of all finite intersections of N-sets of the form

$$\{\mu: s < \mu f < t\}$$

where s and t are real numbers, form a base for a topology on N to which we refer as the vague topology. Then μ_n tends to μ in this topology iff $\mu_n f \to \mu f$ for all continuous functions f from S to $[0,\infty)$ with compact support. Then N becomes a metric space and we can think of the $D_j(\cdot)$ as random elements in (N,F). By definition $D_j(\cdot) \overset{d}{\to} D(\cdot)$ if $Eg(D_j) \to Eg(D)$ for all bounded continuous functions $g: N \to R$.

Observe that for any $B \in B$ we can have at most a finite number of integer points and thus also a finite number of zero-crossings. Therefore

$$\lim_{r\to\infty} \limsup_{j\to\infty} P(D_j(B) > r) = 0.$$

Next, by definition of strictly oscillatory processes, for any $I = (a,b]$, a,b finite, we have by monotonicity

$$ED_j(I) \to ED(I).$$

Third consider, without loss of generality, the higher order crossings $D_{j,N}$ and $D_{j+1,N}$. Then by Lemma 1, there exists t_0, D_j' and D_{j+1}', such that

$$D_{j,N} = D'_j + d^{(j)}_{t_0},$$

$$D_{j+1,N} = D'_{j+1} + d^{(j+1)}_{t_0}$$

and $$D'_j \leq D'_{j+1} \quad \text{surely.}$$

By strict oscillation

$$P(d^{(j)}_{t_0} \geq 1) \leq P(d^{(j+1)}_{t_0} \geq 1).$$

Therefore, by Lemma 1,

$$P(D'_j + d^{(j)}_{t_0} \geq 1) \leq P(D'_{j+1} + d^{(j+1)}_{t_0} \geq 1)$$

or

$$P(D_{j,N} = 0) \geq P(D_{j+1,N} = 0)$$

and so the sequence $P(D_{j,N} = 0)$ converges monotonically as $j \to \infty$ for any N. By appealing to Theorem 4.7 in Kallenberg (1976) it follows that $D_j(\cdot)$ converges in distribution to the point process $D(\cdot)$ corresponding to the limiting rate process.

<u>Theorem 5</u>. Let $\{Z_t\}$ be strictly oscillatory and let $D(\cdot)$ be the point process corresponding to the limiting rate procsss. Then $D_j(\cdot) \xrightarrow{d} D(\cdot)$ with respect to the vague topology on N.

6. The Stationary Gaussian Case.

Theorem 6. Let $\{Z_t\}_{t=1}^{\infty}$ be a zero mean stationary Gaussian process, and suppose $\omega^* \leq \pi$ is the highest frequency in the spectral support. If $\lim_{N\to\infty} \bar{D}_j(N) \leq \omega^*/\pi$ with probability one, $j = 1,2,\ldots,$ then

$$\lim_{j\to\infty} \lim_{N\to\infty} \bar{D}_j(N) = \frac{\omega^*}{\pi} \quad \text{a.s.}$$

Proof. Let ν_j be as in Theorem 3. Then from Kedem and Slud (1982) $\nu_j \Rightarrow \frac{1}{2}\delta_{-\omega^*} + \frac{1}{2}\delta_{\omega^*}$, $j \to \infty$. But the Gaussian assumption implies that

$$\cos(\pi E\bar{D}_j(N)) = \int_{-\pi}^{\pi} \cos(\omega)\nu_j(d\omega) \to \cos(\omega^*), \quad j \to \infty.$$

Therefore $E\bar{D}_j(N) \to \omega^*/\pi$, $j \to \infty$, uniformly in N.

By strict stationarity we can define the a.s. limit

$$c_j \equiv \lim_{N\to\infty} \bar{D}_j(N).$$

Then by Lemma 1 $c_j \leq c_{j+1}$ a.s., and so $\lim_{j\to\infty} c_j \equiv c \leq \omega^*/\pi$ a.s. Thus by bounded convergence $E(c_j) \to E(c)$, $j \to \infty$. Strict stationarity and Fatou's Lemma yield

$$E(c_j) \geq \limsup_{N\to\infty} E\bar{D}_j(N).$$

Since $Ec_j \to Ec$ and $E\bar{D}_j(N) \to \omega^*/\pi$,

$$E(c) \geq \omega^*/\pi .$$

But $c \leq \omega^*/\pi$. Therefore $c = \omega^*/\pi$ a.s. □

<u>Corollary 2</u>. If $\omega^* = \pi$, then

$$\lim_{j\to\infty} \lim_{N\to\infty} \bar{D}_j(N) = 1$$

<u>Proof</u>. In this case $0 \le \bar{D}_j(N) \le 1$ surely. □

Let $\rho^{(j)}(1)$ be the first correlation in $\{\nabla^j Z_t\}$.

<u>Theorem 7</u>. Let $\{Z_t\}_1^\infty$ be an ergodic zero mean stationary Gaussian process. Then

$$\lim_{N\to\infty} \bar{D}_j(N) = \frac{1}{\pi} \cos^{-1}(\rho^{(j+1)}(1)) \quad \text{a.s.}$$

<u>Proof</u>. The proof follows from the fact that

$$\rho^{(0)}(1) = \cos(\pi E \bar{D}_1(N)).$$ □

Theorem 7 shows that in some cases we can expect the rate processes to converge into straight lines, a fact that can be used in classification and discrimination of stationary processes.

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