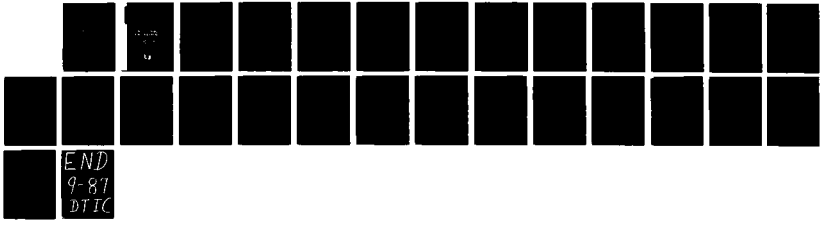
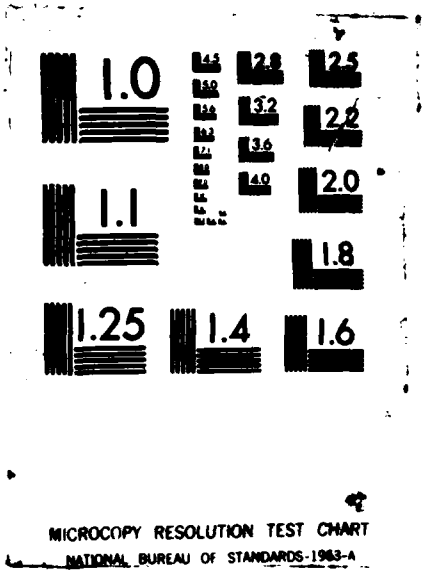
AD-A183 650 OPTIMIZATION OF A CONSTRAINED SPHERICAL LOCATION PROBLEM: THE CASE OF SIN.. (U) NORTH CAROLINA STATE UNIV AT RALEIGH DEPT OF INDUSTRIAL ENGIN.. A YAMANI ET AL. 1/1

UNCLASSIFIED MAR 87 NCSU-IE-TR-87-5 N00014-84-K-0046 F/G 1/2 NL



END
9-87
DTIC


1.0
1.1
1.25
1.4
1.6
2.0
1.8
2.2
2.5
2.8
3.2
3.6
4.0
MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

(12)

Optimization of a Constrained Spherical Locati Problem: The Case of Single Air Craft Mid-Air Refueling

by

Abdulrahman Yamani*
Thom J. Hodgson**
Louis A. Martin-Vega*

Technical Report 87-5

DEPARTMENT OF

INDUSTRIAL ENGINEERING

North Carolina State University
Raleigh, N. C.

Abstract

This paper addresses the problem of a single military transport aircraft that must carry cargo from an origin base to a destination base. Refueling is required and performed by a "tanker" aircraft that originates from and returns to a third base. The objective is to determine the initial fuel required by each aircraft and the location of the refueling point so as to minimize the total fuel consumed subject to restrictions on the range of the transport and the tanker. Based on U.S. Air Force data, analytical relationships are derived which allow the problem to be formulated similar to a constratined spherical Weber problem with two main differences: 1) the objective function is non-linear, and 2) some of the constraints are a function of the decision variables. Spherical convexity of both the objective function and feasible region is shown and used to develop an optimal algorithm. Computational experience is given.

Key words: Spherical, Position fixing, Logistics planning

The problem addressed in this paper is that of a military transport aircraft (A/C) required to transport a cargo of w pounds from an origin base "O" to a destination base "D". The distance between these bases (D_{OD}) is greater than the range of the aircraft thus requiring it to refuel enroute. Refueling is performed in mid-air using aircraft called "tankers". The operation consists of the tanker flying from its base "B" to some point along the route of the transport A/C where it carries out the aerial refueling. Once refueling is completed the tanker returns to "B" while the transport continues on to its destination. The objective is to determine the initial fuel required by each A/C and the location of the refueling point so as to minimize the total fuel consumed by both A/C subject to restrictions on the range of the transport and the tanker. Since the distances over which the aircraft would normally fly in a real application are large, it is appropriate that the geometry of the solution approach be that of a location problem on the sphere (earth).

The closest related problem found in the literature is the spherical Weber problem with maximum distance constraints formulated by Ayken (1983). The problem consists of determining the optimal location of a source so as to minimize the sum of the weighted distance from the source to a finite number of demand points while keeping the source within a distance S_i of a given point U_i. U_i may be any of the demand points and all points are located on the surface of a sphere (earth).

The optimal solution procedure developed by Ayken was based on properties obtained by himself and others for the unconstrained version of the spherical Weber problem (Aly, Kay, and Litwhiler 1979,

Drezner 1981, Drezner and Wesolowsky 1978, Katz and Cooper 1980, Litwhiler and Aly 1979). The problem we are considering differs from the one solved by Ayken in two major considerations. First, the objective function of the problem addressed in this paper is shown to be a non-linear function of the spherical distance whereas Ayken's problem dealt with a linear objective function. Second, some of the constraints of the problem we are considering depend on some of the decision variables. These properties as well as other mathematical lationships between fuel consumption, distance traversed, cargo weight and initial fuel of the transport A/C are derived in the Appendix.

In a recent series of papers, Mehrez, Stern, and Ronen (1983), Mehrez and Stern (1985), and Melkman, Stern, and Mehrez (1986), a related problem is studied. They study the problem of refueling tankers escorting smaller aircraft with the objective of maximizing the the effective range of the smaller aircraft. A central assumption in these papers is that fuel flow is a constant. For larger cargo aircraft, this assumption is difficult to support.

Finally it should be mentioned that the initial motivation for studying the problem addressed herein resulted from various conversations between the authors and personnel at the U. S. Air Force Logistics Management Center at Gunter, AFS, Alabama. Subsequently, the Air Force provided raw data describing A/C performance at different gross weights, altitudes and speeds which was then transformed into the mathematical relationships previously mentioned. Information on these and other details related to the practical origin of this problem is also included in the Appendix.

ASSUMPTIONS AND NOTATION

Prior to formulating the problem, the following assumptions and notation need to be introduced.

Assumptions:

1. The gross weight of the A/C should not exceed its maximum takeoff weight (MTOW), [i.e., A/C empty weight + cargo weight + fuel weight $\leq$ MTOW].
2. There are no load balance or size restrictions associated with the A/C load.
3. The earth is a perfect sphere.
4. The A/C follows the great circle route (shortest path) in flying from one point to another.
5. Weather at altitude (i.e., the jet stream) is assumed to be negligible.
6. Aerial refueling takes a negligible amount of time. Thus, the region in which fuel transfer takes place is considered to be a single point.
7. Everything takes place at altitude. Thus, take-off and its cost are ignored.

Notation:

EW = Transport A/C Empty Weight

MTOW = Maximum Take-Off Weight of the transport A/C

F_{max} = Maximum fuel capacity of the transport A/C

H_{max} = Maximum fuel capacity of the tanker A/C

g = Initial fuel of the transport A/C (a decision variable)

h = Initial fuel of the tanker A/C (a decision variable)

w = Cargo weight

GW = Gross weight of the transport A/C ($GW = EW+g+w$)

GW_{max} = Maximum allowed gross weight of the transport A/C

$g_{max}(w)$ = The maximum fuel the transport A/C can carry in order to be able to take off with a cargo of weight w

= $Min(F_{max}, MTOW-EW-w)$

$G_{max}(w)$ = The maximum fuel the transport A/C can carry when it is in the air, given that its cargo weight is w

= $Min(F_{max}, GW_{max}-EW-w)$

$MPF(GW)$ = Distance traveled by an A/C in miles per 1,000 lbs of fuel at a given gross weight GW

$d_1(\theta,\phi)$ = The spherical distance between the origin base (θ_o,ϕ_o) and the refueling point (θ,ϕ)

$d_2(\theta,\phi)$ = The spherical distance between the refueling point (θ,ϕ) and the destination (θ_D,ϕ_D)

$d_3(\theta,\phi)$ = The spherical distance between the tanker base (θ_T,ϕ_T) and the refueling point

$R(g,w)$ = Range of the transport A/C when its initial fuel is g and its cargo weight is w

$FC(g,w,d)$ = Fuel consumed by the transport A/C when it flies a distance d and its initial fuel is g and cargo weight is w

$FN(w,d)$ = Exact amount of fuel needed by the transport A/C to fly a distance d when the cargo weighs w

[d must be $\leq R(F_{max},0)$]

$R^*(h,w)$ = Range of the tanker when its initial fuel is h and cargo weight is w (usually w = 0)

$FC^*(h,w,d)$= Fuel consumed by the tanker when it flies a distance d, its initial fuel is h, and cargo weight is w [d must be $\leq R^*(h,w)$, and usually w = 0]

$FN^*(w,d)$ = Exact amount of fuel needed by the tanker to fly a distance d when its cargo weight is w [d must be $\leq R^*(H_{max},0)$, and usually w = 0]

$f_1(\theta,\phi)$ = The fuel consumed by the transport A/C in flying from (θ_o,ϕ_o) to (θ,ϕ) [$f_1(\theta,\phi) = FC(g,w,d_1)$]

$f_2(\theta,\phi)$ = The fuel needed by the transport A/C to go from (θ,ϕ) to (θ_D,ϕ_D) [$f_2(\theta,\phi) = FN(w,d_2)$]

$f_3(\theta,\phi)$ = The fuel consumed by the tanker in going from (θ_D,ϕ_D) to (θ,ϕ) [$f_3(\theta,\phi) = FC^*(h,0,d_3)$]

$f_4(\theta,\phi)$ = The fuel needed by the tanker to go from (θ,ϕ) back to (θ_o,ϕ_o) [$f_4(\theta,\phi) = FN^*(0,d_4)$]

PROBLEM FORMULATION

The problem can now be stated mathematically as follows:

Find (θ,ϕ), g and h that will

$$(P) \quad \text{Minimize } V(\theta,\phi) = \sum_{i=1}^{4} f_i(\theta,\phi)$$

$$\text{S.T.} \quad d_1(\theta,\phi) \leq R[g_{max}(w),w] \tag{1}$$

$$d_2(\theta,\phi) \leq R[G_{max}(w),w] \tag{2}$$

$$d_4(\theta,\phi) \leq R^*(H_{max},0) \tag{3}$$

$$FN(w,d_1) \leq g \leq g_{max}(w) \tag{4}$$

$$FN^*(0,2d_3) \leq h \leq H_{max} \tag{5}$$

$$(h+g) \geq \sum_{i=1}^{4} f_i(\theta,\phi) \tag{6}$$

$$0 \leq \theta \leq 2\pi \quad (7)$$

$$0 \leq \phi \leq \pi \quad (8)$$

The first three constraints represent conditions that must be satisfied by the refueling point. Inequality (1) says that the distance between the origin base and the refueling point should not exceed the maximum range of the transport A/C. Inequality (2) says that the transport A/C must be able to reach its destination from the refueling point. Inequality (3) says that the tanker should be able to make a round trip safely. Constraints (4), (5) and (6) are the conditions that must be satisfied by g and h, that is, g must be at least equal to the amount of fuel needed to fly from the origin to the refueling point. Similarly, h should permit the tanker to make a trip from its base to (θ,ϕ) and back. Also, the total initial fuel for both A/C must be at least equal to the total fuel consumed.

A few more remarks are in order. First, the intersection of the first three inequality constraints (1), (2) and (3) represents a region within which all feasible refueling points must lie. Figure 1 illustrates such a region. Note that even though all feasible points are included in this region, not all the points in this region are necessarily feasible. It can be shown that any feasible point must satisfy (1), (2), (3) and the following constraint:

$$g_{max}(w) - FC[g_{max}(w), w, d_1] + [H_{max} - FC^*(H_{max}, 0, d_2)] \leq f_{max}(\theta,\phi) + f_{max}(\theta,\phi) \quad (9)$$

which is derived from (4), (5), and (6).

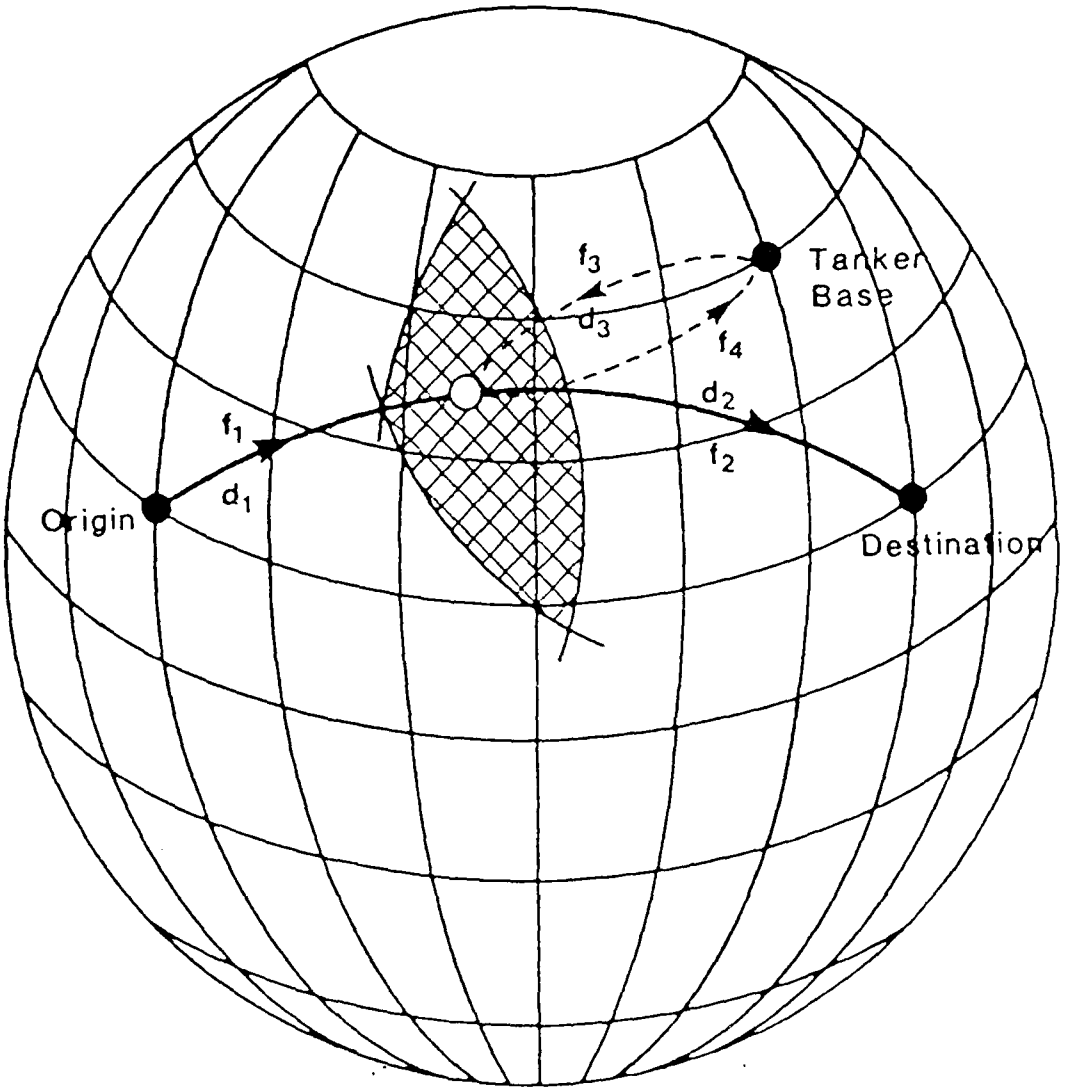


Figure 1: The region inside which all feasible refueling points lie.

Second, since values of g and h greater than needed will increase fuel consumption unnecessarily, we can restrict attention to the case where the equality holds for constraint (6). This means that there is only one value of $h(\theta,\phi)$ for every choice of $g(\theta,\phi)$. Consequently, the best value of $g(\theta,\phi)$, (denoted by $g^*(\theta,\phi)$) that will result in the lowest total fuel consumption for that point is the solution to the following non-linear subproblem in one dimension (g):

$$\text{(SP1)} \qquad \text{Minimize } g + FN^*[h_r(g), d_m] + h_r(g) \tag{10}$$

$$\text{S.T.} \qquad g_{min}(\theta,\phi) \leq g(\theta,\phi) \leq g_{max}(w) \tag{11}$$

where

$$g_{min}(\theta,\phi) = \begin{cases} FN(w, d_1) & \text{if } g_r(\theta,\phi) \leq 0, \\ FN(g_r+w, d_1) + g_r & \text{if } g_r(\theta,\phi) \geq 0, \end{cases} \tag{12}$$

and

$$g_r(\theta,\phi) = f_m(\theta,\phi) + f_1(\theta,\phi) - [H_{max} - FC^*(H_{max}, 0, d_m)] \tag{13}$$

$$h_r(g) = f_m(\theta,\phi) + f_1(\theta,\phi) - [g(\theta,\phi) - f_1(\theta,\phi)] \tag{14}$$

$$h(\theta,\phi) = FN^*[h_r(g), d] + h_r(g) \tag{15}$$

An important result due to Drezner and Wesolowsky (1978) says that the distance from a given point r within a circle of radius $\pi/2$ and center r_1 is a spherically convex function. Since the right hand side of constraints (1), (2) and (3) are usually much less than $\pi/2$, each of these constraints represent a spherically convex set. The region resulting from their intersection will also be a spherically convex set. Therefore, Problem (P), without constraint (9), will have a spherically convex feasible region.

Our fourth remark is related to Aly et.al. (1979) who showed that the search for an optimal solution to the spherical Weber problem can be restricted to the spherically convex hull of the demand points. This result can be extended to the case at hand.

Property 1: The search for the optimal refueling point can be restricted to the spherically convex hull of the three base points (the origin, the destination, and the tanker base).

Proof: Let V be the spherically convex hull of the three points and X be a point on the sphere such that $X \notin V$. Let d_1, d_2, d_3 be the spherical distance between X and the origin, the destination, and the tanker base, respectively. Aly et al. (1979) proved that there exists a point $X' \in V$ that dominates X in the spherical distance with respect to the three base points. That is, if d'_1, d'_2, and d'_3 are the distance between X' and the three points, then $d'_i \leq d_i$ for i = 1,2,3. Since fuel consumption increases with distance, [i.e., $FN(.,d_2)$, $FC(.,.,d_1)$, $FN^*(.,d_3)$ and $FC^*(.,.,d_3)$ are all increasing functions of the distance], then the objective function value at X' is less than or equal to the objective function value at X. Q.E.D.

Taking the first and fourth remarks together, it can be seen that the search for the optimal refueling point can be restricted to the intersection of the region described by (1), (2) and (3) with the spherically convex hull of the three bases.

Finally, the fifth remark concerns the objective function. It is a nonlinear function of the spherical distance (the great circle distance). Using the fuel function (see Appendix), it is seen that

$$f_1 = g + a/a_1 - [(a+a_1 g)^2 - 2a_1 d_1]^{1/2} / a_1$$

$$f_2 = -a/a_1 + (a^2 + 2a_1 d_2)^{1/2} / a_1$$

$$f_3 = h + a'/a_1 - [(a+a_1 h)^2 - 2a_1 d_3]^{1/2} / a_1$$

$$f_4 = -a'/a_1 + (a'^2 + 2a_1 d_4)^{1/2} / a_1$$

$$f_5 = h+g-1/a_1 \left\{ \begin{array}{c} [(a+a_1 g)^2-2a_1 d_1]^{1/2}-[a^2+2a_1 d_2]^{1/2} \\ +[(a+a_1 h)^2-2a_1 d_3]^{1/2}-[a'^2+2a_1 d_4]^{1/2} \end{array} \right\}$$

The way the objective function is written, it is difficult to verify its spherical convexity either by using the second derivative test or by applying the definition of a spherically convex function directly. Nevertheless, spherical convexity of the objective function can be shown by rewriting the objective function using some spherically convex function like FN(.,.). To do this, define for any point (θ,ϕ) the following:

WFL[$g(\theta,\phi)$,d_1] = weight of the fuel left in the transport A/C when it flies a distance d_1 from the origin to (θ,ϕ) given that its initial fuel is $g(\theta,\phi)$.

Thus,

$$WFL[g(\theta,\phi),d_1] = g(\theta,\phi) - FC[g(\theta,\phi),w,d_1] \qquad (16)$$

WFL(.,.) is spherically convex because FC(.,.,.) is spherically concave.

Let

$$WFL_{max}(d_1) = \text{Max } WFL[g(\theta,\phi),d_1] = WFL[g_{max}(w),d_1] \quad (17)$$

Note that

$$0 \leq WFL[g_m(\theta,\phi),d_1] \leq WFL_{max}(d_1), \quad (18)$$

and that $WFL_{max}(.)$ is spherically convex also.

Since $g^*(\theta,\phi)$ is the best value of $g(\theta,\phi)$, then using the spherically convex functions FN AND WFL, the elements of the objective function can be rewritten as

$$f_1(\theta,\phi) = FN(w+WFL[g^*(\theta,\phi),d_1],d_1) \quad (19)$$

$$f_m(\theta,\phi) = FN(w,d_m) \quad (20)$$

$$f_4(\theta,\phi) = FN^*(0,d_3) \quad (21)$$

$$f_3(\theta,\phi) = FN^*(f_m+f_4-WFL[g^*(\theta,\phi),d_1],d_3) \quad (22)$$

and the total fuel consumption = $V(\theta,\phi) = \sum_{i=1}^{4} f_i(\theta,\phi)$.

Both FN(.,.) and FN*(.,.) are convex and increasing functions of their arguments, which, in turn, are spherically convex functions. Therefore, the resulting objective function is spherically convex (Greenberg and Pierskalla 1971, Rockafellar 1970).

MOTIVATION FOR THE SOLUTION PROCEDURE OF (P0)

For this problem, the optimal point (θ^*,ϕ^*) and its optimal $g^*(\theta^*,\phi^*)$ (and consequently, its optimal $h^*(\theta^*,\phi^*)$) are to be found. Since $g^*(\theta^*,\phi^*)$ depends upon (θ,ϕ) and we do not know this in advance (note that (Sp1) needs to be solved to find this out!), it is difficult to use (19), (20), (21) and (22) directly. That is, we can not search for both (θ^*,ϕ^*) and $g^*(\theta^*,\phi^*)$ simultaneously. However, this difficulty can be overcome by creating an upper bound function that can be improved from one iteration to another until its minimum value coincides with the minimum of the original function. To illustrate this, let

$$a(\theta,\phi) = WFL[g(\theta,\phi),d_1] \,/\, WFL_{max}(d_1) \quad (23)$$

then,

$$0 \leq a(\theta,\phi) \leq 1 \text{ for all } (\theta,\phi) \quad (24)$$

Thus, $g^*(\theta,\phi)$ would result in $a^*(\theta,\phi)$, where $a^*(\theta,\phi)$ is the best value of $a(\theta,\phi)$, i.e.,

$$a^*(\theta,\phi) = WFL[g^*(\theta,\phi),d_1] \,/\, WFL_{max}(d_1) \quad (25)$$

Also, from (25), we have

$$WFL[g(\theta,\phi),d_1] = a(\theta,\phi)WFL_{max}(d_1) \quad (26)$$

The elements of the objective function can be rewritten using $a^*(\theta,\phi)WFL_{max}(d_1)$ instead of $WFL[g^*(\theta,\phi),d_1]$ as

$$f_1(\theta,\phi) = FN[w+a^*(\theta,\phi)WFL_{max}(d_1),d_1]$$

$$f_2(\theta,\phi) = FN(w,d_2)$$

$$f_4(\theta,\phi) = FN^*(0,d_3)$$

$$f_3(\theta,\phi) = FN^*[f_2+f_4-a^*(\theta,\phi)WFL_{max}(d_1),d_3]$$

Now setting $a(\theta,\phi) = a$ for all points, and denoting the resulting function by $UB(\theta,\phi,a)$, one gets

$$f_1'(\theta,\phi) = FN[w+aWFL_{max}(d_1),d_1] \quad (27)$$

$$f_2'(\theta,\phi) = f_2(\theta,\phi) = FN(w,d_2) \quad (28)$$

$$f_4'(\theta,\phi) = f_4(\theta,\phi) = FN^*(0,d_3) \quad (29)$$

$$f_3'(\theta,\phi) = FN^*[f_2+f_4-aWFL_{max}(d_1),d_3] \quad (30)$$

where $0 \leq a \leq 1$

and $$UB(\theta,\phi,a) = \sum_{i=1}^{4} f_i'(\theta,\phi) \quad (31)$$

$UB(\theta,\phi,a)$ has the following characteristics:

1) It is spherically convex because it is the sum of spherically convex functions;

2) $UB[\theta,\phi,a] \geq V(\theta,\phi)$ (32)

In other words, since any g other than $g^*(\theta,\phi)$ would result in a

higher cost for $V(\theta,\phi)$, and also produce an "a" that is different from $a^*(\theta,\phi)$, then any "a" used in $UB(\theta,\phi,a)$ would make $UB(\theta,\phi,a) \leq V(\theta,\phi)$. Therefore UB is an upper-bound for the objective function;

3) $UB[\theta,\phi,a^*(\theta,\phi)] = V(\theta,\phi)$ for any (θ,ϕ). That is, the equal sign of eq. (32) holds true when $a = a^*(\theta,\phi)$

Therefore

$$UB[\theta^*,\phi^*,a^*(\theta^*,\phi^*)] = V(\theta^*,\phi^*) \qquad (33)$$

4) $UB(\theta,\phi,a) = V(\theta,\phi)$ at the boundary points where $d_1(\theta,\phi) = R[g_{max}(w),w]$ because, at those points, $WFL_{max}(d_1) = 0$. Thus the value of a does not matter.

How can we use this UB function to solve (P)? Given a particular value of a, we can optimize the UB function to find the best (θ,ϕ) that corresponds to it. Now, if at some iteration, we have the value of $a^*(\theta^*,\phi^*)$ [without knowing (θ^*,ϕ^*)], and we optimize the UB function, then we will end up with (θ^*,ϕ^*). This is true since if (θ^*,ϕ^*) minimizes the true objective function, it will also minimize $UB[\theta,\phi,a^*(\theta^*,\phi^*)]$.

SOLUTION PROCEDURE OF (P)

1. Initialization:

   Let k=0, and a^{0}=1. Start with a point (θ^{0},ϕ^{0}) that satisfies (1), (2) and (3) [(See Yamani 1986 for the seletion of a good starting point).]

2. Let k=K+1. Find the point that will minimize $UB(\theta,\phi,a^{k-1})$ s.t. (1), (2), and (3). Let the solution be (θ^{k},ϕ^{k}).

3. For (θ^{k},ϕ^{k}), solve (SP1) to find the best $g^*(\theta^{k},\phi^{k})$. Use

equation (25) to find $a^*(\theta^k,\phi^k)$.

4. If $a^*(\theta^k,\phi^k) \neq a^{k-1}$, we have an improvement in the objective function. Therefore, let $a^k = a^*(\theta^k,\phi^k)$ and go to step 2. Otherwise, the solution is (θ^k,ϕ^k), $g^*(\theta^k,\phi^k)$, and $h^*(\theta^k,\phi^k)$ is calculated from eqs. (14) and (15).

Convergence Proof

We need to show that

$$\sum_{i=1}^{4} f_i(\theta^{k-1},\phi^{k-1}) \geq \sum_{i=1}^{4} f_i(\theta^k,\phi^k) \tag{34}$$

Assume, at iteration k-1, that

$$a^*(\theta^{k-1},\phi^{k-1}) \neq a^{k-2}.$$

(Otherwise, we would have stopped there.)

By setting $a^{k-1} = a^*(\theta^{k-1},\phi^{k-1})$, we have

$$\sum_{i=1}^{4} f_i(\theta^{k-1},\phi^{k-1}) = UB(\theta^{k-1},\phi^{k-1},a^{k-1}) \tag{35}$$

At iteration k, since (θ^k,ϕ^k) minimizes $UB(\theta,\phi,a^{k-1})$,

$$UB(\theta^{k-1},\phi^{k-1},a^{k-1}) \geq UB(\theta^k,\phi^k,a^{k-1}) \tag{36}$$

But,

$$UB(\theta^k,\phi^k,a^{k-1}) \geq \sum_{i=1}^{4} f_i(\theta^k,\phi^k) \tag{37}$$

and inequality (34) follows.

Moreover,

$$\sum_{i=1}^{4} f_i(\theta^{k-1},\phi^{k-1}) \geq \sum_{i=1}^{4} f_i(\theta^k,\phi^k), \tag{38}$$

if $$UB(\theta^k,\phi^k,a^{k-1}) \geq \sum_{i=1}^{4} f_i(\theta^k,\phi^k). \tag{39}$$

The last inequality holds true if $a^*(\theta^k,\phi^k) \neq a^{k-1}$. Therefore, the true objective function improves from one iteration to another. Since the problem is spherically convex, the solution procedure will converge to the global optimum solution (Drezner and Wesolowsky

1978). Q.E.D.

Numerical Results

The above procedure was programmed on a Vax 11/750 in BASIC. There was no attempt at program efficiency. Several test problems were solved to represent a variety of possible geographic configurations and complexities. The minimum CPU time was approximately 0.5 second (2 iterations) and the maximum was almost 1.0 second (4 iterations) for convergence within one degree longitude and latitude. Convergence within one minute longitude and latitude required only one more iteration in each case with a maximum CPU time of 1.3 seconds. The results of these test problems are summarized in Table 2, with Table 1 providing a key for understanding the entries in Table 2.

Summary

In this paper we have shown that the single aircraft refueling problem is a non-linear convex location problem on the sphere. The algorithm that has been presented is an optimal solution procedure for this problem. In the sequel to this paper, these results are used as the basis for solving the more complex multi-aircraft refueling problem.

TABLE 1: Key for trial runs chart of Table 2

Location	Longitude	Latitude	Code
New Jersey	75W	40N	A
Delaware	75W	38N	B
North Carolina	78W	35N	C
Puerto Rico	66W	18N	D
Azores Islands	25W	37N	E
Iceland	20W	65N	F
Germany	10E	50N	G
Turkey	30E	40N	H
Saudi-Arabia	47E	25N	I
Egypt	28E	30N	J
England	0E/W	52N	K

W = Cargo Weight	Code
100,000 lbs	1
200,000 lbs	2

Fuel ACCR = Fuel Accuracy

10 means to within 10 lbs

POS ACCR = Position Accuracy

1 means to within 1 degree

Table 2: Trial runs for Test Problem

Run No.	"O"	"D"	"B"	Initial Refueling Point (Guess)	W	Fuel ACCR	Pos ACCR	g_o (lbs)	Fuel Transferred to Transport (lbs)	Fuel Consumed By Tanker (lbs)	Total Fuel (lbs)	Optimal Location	Time (Sec)	Iterations
1	A	H	D	40W 35N	2	10	1	113,521	136,704	149,055	399,280	49W 40N	0.6	2
2	G	C	F	35W 40N	2	10	1	79,431	130,541	1,058	211,030	20W 65N	1.0	4
3	B	I	E	30W 30N	1	10	1	137,142	144,464	5,866	287,472	25W 37N	0.8	2
4	B	J	D	30W 25N	2	10	1	118,712	152,613	156,783	428,008	42W 36N	0.5	2
5	C	K	F	30W 50N	2	10	1	129,882	54,425	19,269	203,576	29W 63N	0.6	2

"O" = Origin

"D" = Destination

"B" = Tanker Base

g_o = Initial fuel for transport A/C

h_o = Initial fuel for tanker A/C

= Fuel consumed by tanker + fuel transferred to transport

Appendix A

Derivation and Characterization of the Fuel Consumption Function and Other Related Functions

Here, mathematical representation of the various relationships between fuel consumption[1], distance traveled, cargo weight, and initial fuel of a typical jet A/C are derived.

The United States Air Force provided raw data describing A/C performance at different gross weights, different altitudes, and different speeds. These raw data have to be transformed before they are useful. A sample of these data is shown in Figure 2 for a C-5A transport A/C flying at an altitude of 31,000 feet. Different curves are presented for different A/C weights, and they show the distance traveled per 1,000 pounds of fuel burned at a given speed and at that given gross weight (GW). Consider the points on the curve indicating 99 percent maximum specific range. It is assumed, without loss of generalilty, that the transport A/C will operate at those points since speed is maximized, while fuel consumption is only 1 percent greater than the minimum possible.

The distance traveled in miles per 1,000 pounds of fuel burned when the A/C gross weight is GW, denoted here as MPF(GW), is plotted against gross weight GW in Figure 3 using the 99 percent line in Figure 2. Least square linear and quadratic fits to the data were tried, and both fit well as seen in Figure 3.

For the linear fit,

$$MPF(GW) = a_0 + a_1 GW,$$

where

$$a_0 = 36.2829,\ a_1 = -0.027$$

Model C-5A

SPECIFIC RANGE

4 Engines 31,000 Feet

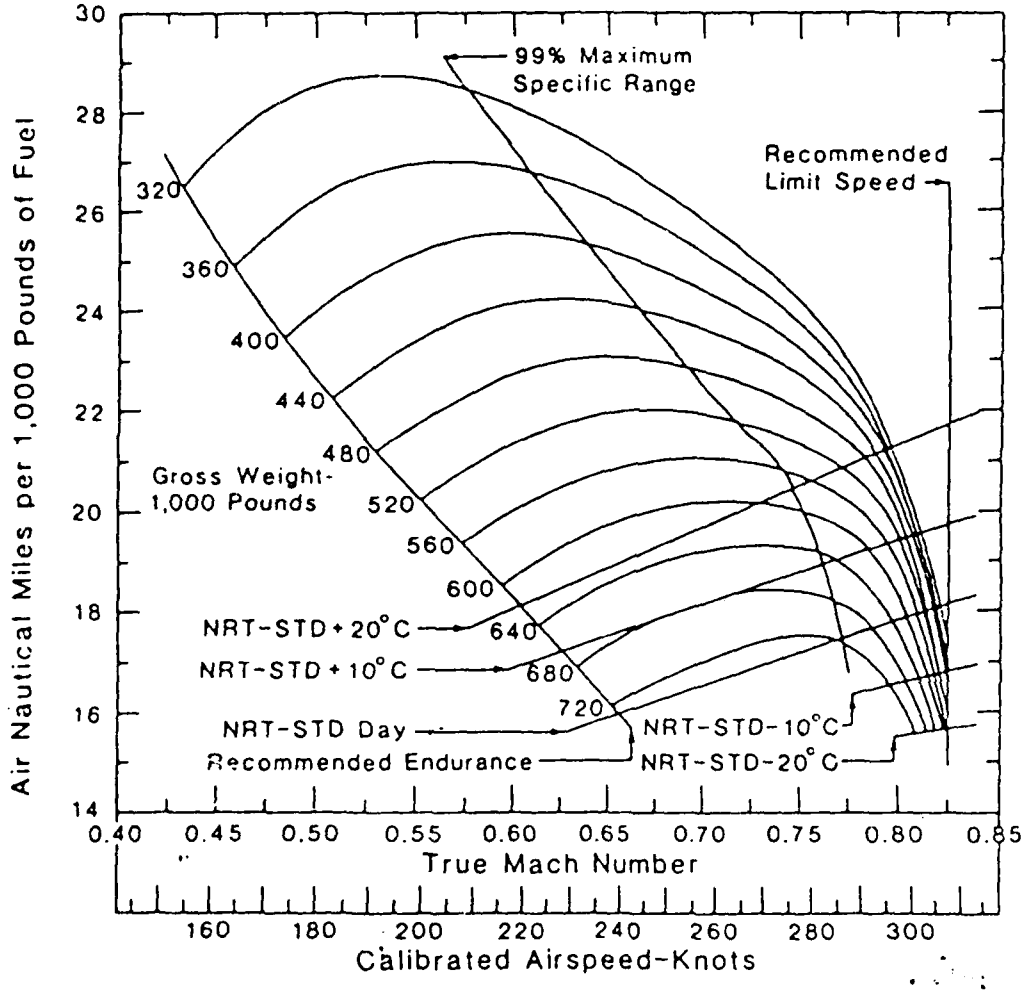


Figure 2: Data for C-5A aircraft.

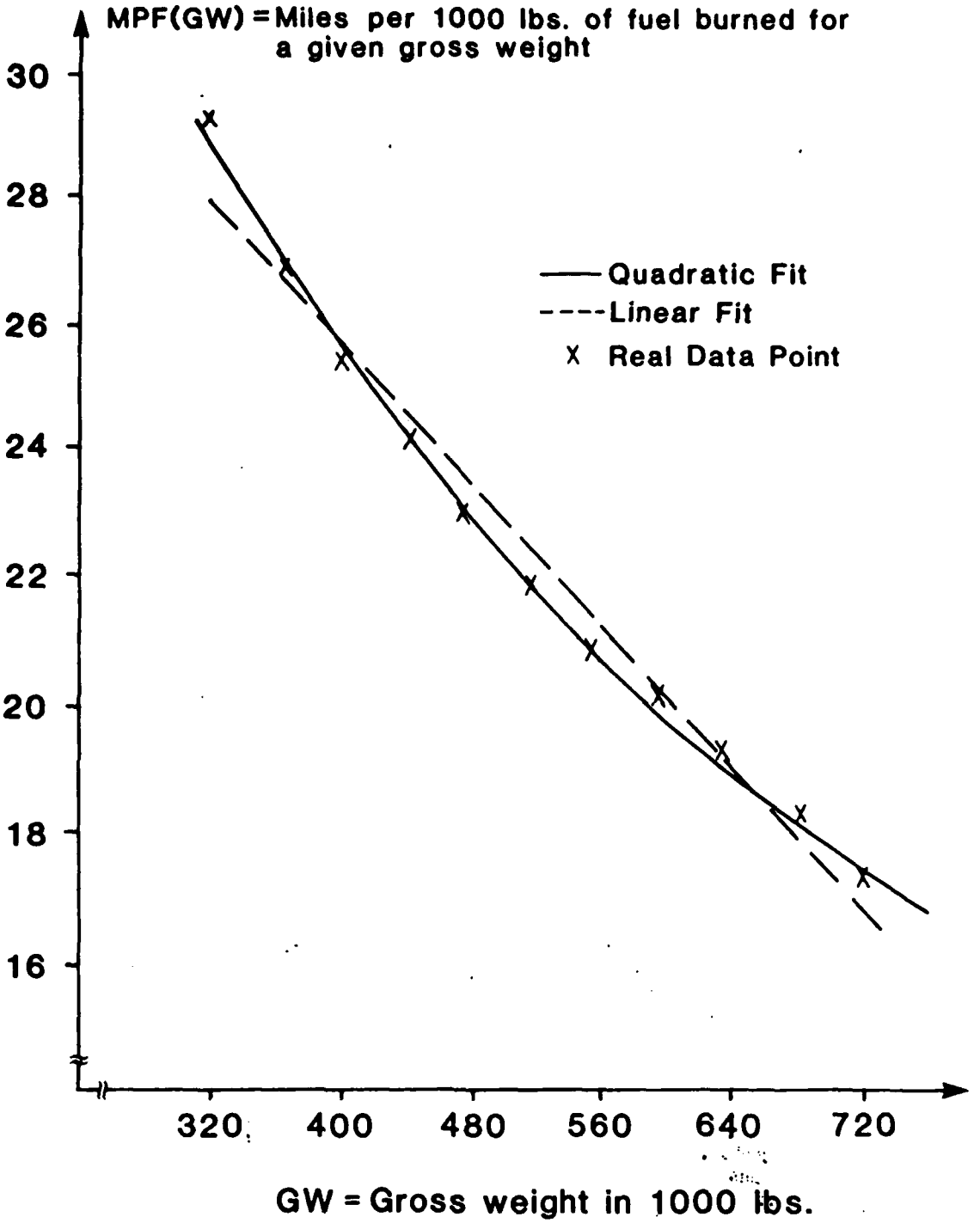


Figure The MPF function versus GW.

and correlation coefficient = -0.9919.

For the quadratic fit,

$$MPF(GW) = b_0 + b_1 GW + b_2 GW^2$$

where

$$b_0 = 43.7616,\ b_1 = -0.0576,\ b_2 = 2.94 \times 10^{-5}$$

and correlation coefficient = -0.9983.

Only the linear form of the function will be used here. The use of the quadratic form is similar (Yamani 1986).

Note that at any moment during the flight,

$$GW = EW + w + f$$

where f = the present (instantaneous) amount of fuel and EW = A/C empty weight. During flight, f changes due to fuel burning; thus, GW changes while EW and w stay the same. So,

$$dGW = df$$

The Range Function $R(g,w)$

Initial Gross Weight = EW + w + g

Final Gross Weight = EW + w

Then,

$$R(g,w) = \int_{EW+w}^{EW+w+g} MPF(GW)\ dGW$$

$$= \int_{0}^{g} MPF(GW)\ df$$

$$= \int_{0}^{g} MPF(EW+w+f)\ df$$

The Fuel Consumption Function $FC(g,w,d)$

Let g_r equal the final amount of fuel left.

Then,

$$FC(g,w,d) = g - g_r$$

Thus, finding FC reduces to finding g_r, which can be accomplished

by solving for g_r in the following equation:

$$d = \int_{g_r}^{g} MPF(EW+w+f)\, df$$

$$d = \int_{g_r}^{g} [a_0 + a_1(EW+f+w)]\, df$$

which results in

$$g_r = -a/a_1 \pm [(a+a_1 g)^2 - 2a_1 d]^{1/2} / a1$$

Thus,

$$FC(g,w,d) = g + a/a_1 - [(a+a_1 g)^2 - 2a_1 d]^{1/2} / a1$$

where

$$a = a_0 + a_1(EW+w)$$

The Fuel Requirement Function FN(w,d)

If the distance d to be flown and the cargo weight w are known in advance, then the amount of initial fuel g must be at least as great as the amount of fuel needed. If g is set equal to FN(w,d), then we can use the range function R(g,w) to solve for g. That is, set

$$R(g,w) = d = [a' + a_1 w + (a_1/2)g]g,$$

where

$$a' = a_0 + a_1 EW$$

Solving for g results in the following equation:

$$FN(w,d) = g_r = -w-a'/a_1 + [(a'+a_1 w)^2 + 2a_1 d]^{1/2} / a_1$$

Table 3 summarizes the various fuel functions.

TABLE 3: Summary of the fuel-related functions

Function Name	Function Value Using a Linear Fit for MPF	Function Value Using a Quadratic Fit for MPF
MPF(GW)	$a_o + a_1 GW$	$b_o + b_1 GW + b_2 GW^2$
$R(g_o, w_o)$ = Range	$a'_o + a_1 w_o + \frac{a_1}{2} g_o) g_o$	$b'_o + b'_1 w_o + b_2 w_o + (b_2 w_o + \frac{b_1}{2}) g_o + b_2 g_o^2] g_o$
$FC(g_o, w_o, d)$ = Fuel Consumed	$g_o + \frac{a}{a_1} - \frac{\sqrt{(a + a_1 g_o)^2 - 2a_1 d}}{a_1}$	(Roots of a cubic equation) See Yamani (1986)
$FN(w_o, d)$ = Fuel needed	$- w_o - \frac{a'_o}{a_1} + \frac{\sqrt{(a'_o + a_1 w_o)^2 + 2a_1 d}}{a_1}$	(Roots of a cubic equation) See Yamani (1986)

$a'_o = a_o + a_1 EW$ $\qquad b'_o = b_o + b_1 EW + b_2 EW^2$

$a = a_o + a_1(EW + w_o) = a'_o + a_1 w_o$ $\qquad b'_1 = b_1 + 2b_2 EW$

Characteristics of the fuel functions

(1) The function FN(w,d) is a strictly convex and increasing function of the distance d for any given value of the cargo weight w. This can be shown by taking the first and second derivatives of FN with respect to d.

$$\frac{dFN}{d\,d} = 1/[(a'+a_1w)^2 + 2a_1d]^{1/2} > 0$$

$$\frac{d^2FN}{d\,d^2} = -a_1/[(a'+a_1w)^2 + 2a_1d]^{3/2} > 0$$

because $a_1<0$. Therefore, FN is strictly convex in d. Moreover, since the first derivative is always positive, FN is an increasing function of d.

(2) The function FN is a strictly convex and increasing function of the cargo weight w for any given value of d. This can be shown in a straightforward fashion as in the previous case. Yamani (1986) proved that if the distance d is measured along the surface of a sphere (earth), then the function FN(w,d) is spherically convex (s-convex) over a spherical disc of radius $\pi/4$.

(3) The function FC(g,w,d) is a concave and increasing function of the distance d for any given values of the initial fuel g and the cargo weight w.

(4) The function FC(g,w,d) is a convex and increasing function of w for any given values of g and d.

Again, both cases (3) and (4) can be shown to be true in a fashion similar to that of FN(w,d).

References

Aly, A.A., D.C. Kay and D.W. Litwhiler, 1979, "Location Dominance on Spherical Surface," Operations Research, Vol. 27, No. 5, 972-981.

Ayken, Turgut, 1983, "Some Aspects in the Large Region Location Problems on the Surface of the Earth," Ph.D. Dissertation, State University of New York at Buffalo.

Bazaraa, M.S., and C.M. Shetty, 1979, Nonlinear Programming: Theory and Algorithms, John Wiliey and Cons, Inc., New York.

Drezner, A., and G.O. Wesolowsky, 1978, "Facility Location on a Sphere," Journal of Operational Research Society, Vol. 29, No. 10, 997-1004.

Drezner, Z., 1981, "On Location Dominance on Spherical Surface," Operations Research, Vol. 29, No. 6, 1218-1219.

Greenberg, H.J., and W.P. Pierskalla, 1971, "A Review of Quasi-Convex Functions," Operations Research, Vol. 19, No. 7.

Katz, I.N. and L. Cooper, 1980, "Optimum Location on a Sphere," Computers and Operations Research, Vol. 6, No. 2, 175-196.

Litwhiler, D.W., and A.A. Ally, 1979, "Large Region Location Problems," Computers and Operations Research, Vol. 6, No. 1, 1-12.

Mehrez, A., H.I. Stern and D. Ronen, (1983), "Vehicle Refueling Strategies to Maximize Operational Range," Naval Research Logistics Quarterly, Vol. 30, pp. 319-342.

Mehrez, A. and H.I. Stern, (1985), "Optimal Refueling Strategies for a Mixed Vehicle Fleet," Naval Research Logistics Quarterly, Vol. 32, pp. 315-328.

Melkman, A.A., H.I. Stern and A. Mehrez, (1986), "Optimal Refueling Sequence for a Mixed Fleet with Limited Refuelings," Naval Research Logistics Review, Vol. 33, pp. 759-762.

Rockafellar, T.R., 1970, Convex Analysis, Princeton University Press, Princeton, New Jersey.

Selby, S.W. (ed)., 1975, Standard Mathematical Tables 23rd edition., CRC Press, Cleveland, Ohio.

Yamani, A., 1986, Analysis of An Air Transportation System, Ph.D. Dissertation, Department of Industrial and Systems Engineering, University of Florida, Gainesville, Florida.

END

9-87

DTIC