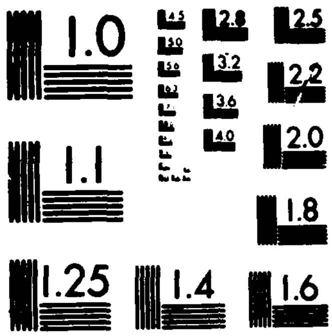

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The theory and practice of the h-p version of finite element method

Ben Qi Guo and Ivo Babuška

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THE THEORY AND PRACTICE OF THE h-p VERSION OF FINITE ELEMENT METHOD

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Abstract: The p and h-p version is a new development in finite element method in recent years. This paper is addressing some theoretical advances and presents numerical illustrations.

1. Introduction

There are three versions of finite element method. The classical h-version achieves the accuracy by refining the mesh while using low degrees p of elements, p = 1,2 usually. The p-version keep the mesh fixed and the accuracy is achieved by increasing the degree p. The h-p version properly combines both approaches.

The h-p version is the new development of finite element method. It was first addressed by Babuška and Dorr [4]. The further analysis and computation for two dimensional problems were made by Guo, Babuška ~~in [20]~~ where the exponential rate of convergence was proved. The one dimensional analysis was given by Gui, Babuška, ~~[18]~~ The improvement of the results for curvilinear boundary and curvilinear elements was made by Babuška, Guo. ~~in [9]~~ The problem with non-homogeneous Dirichlet data was studied by Babuška, Guo. ~~in [10]~~ The h-p version with C^{m-1}-elements for the problem of 2m order was discussed by Guo in [19]. The feedback and adaptive approach was developed by Gui, Babuška ~~[18]~~ and Babuška, Rank, [11].

The p and h-p version for two dimensional problems were implemented in the commercial code PROBE by Noetic Tech., St. Louis. See [27,28]. The commercial program FIESTA (Istituto Sperimentale Modelli e Struttre) has limited p and h-p capabilities in three dimensions. The p and h-p versions of finite element method in three dimensions is being developed by Noetic Tech. and by Aeronautical Research Institute of Sweden.

The practical effectivity of the p and h-p version is closely related to the problems in structural mechanics which are the problems of elliptic partial equations with piecewise analytic data (as domains, boundary conditions). The analysis of the interior regularity was given by Morrey [23], and the behavior of the solution in the neighborhood of corners and edges of the domain was given by Kondrat'ev, Oleynik [21,22] and Grisvard [16,19]. The characterization of the regularity of the solution on nonsmooth domains in the frame of countably normed spaces was ginven in the series of papers by Babuška, Guo in [5,6,7,8] and [9].

The advances in the p-version were discussed in [26]. The computational aspects were addressed in [25].

For a survey of the state of the art of the p and h-p versions we refer also to [2].

2. Finite Element Method and the Approximation

Let $B(u,v)$ be a bilinear form defined on $H_1 \times H_2$, where H_1, H_2 are reflexive Banach spaces equipped with norm $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Let further $F \in H_2^*$, i.e., F be the linear functional on H_2.

We seek the solution $u_0 \in H_1$ such that

$$B(u_0,v) = F(v) \quad \text{for all} \quad v \in H_2. \tag{2.1}$$

If the bilinear form $B(u,v)$ is continuous and satisfies the well-known inf-sup condition (see [3]), then the problem (2.1) has a unique solution.

Let now $S_1 \subset H_1$, $S_2 \subset H_2$. The approximation problem is to find the finite element solution $u_{S_1} \in S_1$ such that

$$B(u_{S_1},v) = F(v) = B(u_0,v) \quad \text{for all} \quad v \in S_2. \tag{2.2}$$

For sake of simplicity we assume that the bilinear form $B(u,v)$ satisfies the inf-sup condition on $S_1 \times S_2$. Then u_0 exists, is unique, and

$$\|u_{S_1} - u_0\|_1 \le C(S_1,S_2) Z(u_0,H_1,S_1) \tag{2.3}$$

where

$$Z(u_0,H_1,S_1) = \inf_{w \in S_1} \|u_0 - w\|. \tag{2.4}$$

For details, see [1,3]. We shall assume that

$$C(S_1,S_2) \le D$$

where D is independent of S_1 and S_2, hence the norm of the error $e = u_{S_1} - u_0$ is completely governed by $Z(u_0,H_1,S_1)$. The accuracy of finite element solution is actually an approximation problem. Obviously the solutions u_0 is not known, but it will be assumed that they belong to a compact solution set $K \subset H_1$. Hence we are interested in $Z(u_0,H_1,S_1)$ for any $u_0 \in K$. The precise characterization of the set K which is made as small as possible is crucial for the most accurate approximation.

3. Characterization of the Solution for Elliptic Problem with Piecewise Analytic Data

As indicated in the previous section, selection of the h,p,h-p versions and the performance of three versions strongly depend on the solution set K. Let us characterize this set for elliptic problems with piecewise analytic (input) data.

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A-1

Let $\Omega \in \mathbb{R}^2$ be a bounded domain (curvilinear polygon) shown in Figure 3.1 with vertices A_i, $1 \le i \le M$, the boundary $\partial\Omega$ be a piecewise analytic curve $\Gamma = \bigcup_{i=1}^{M} \Gamma_i$ where Γ_i's are open arc with endpoint A_i and A_{i+1}. We denote the internal angle by ω_i, $0 < \omega_i \le 2\pi$, $1 \le i \le M$.

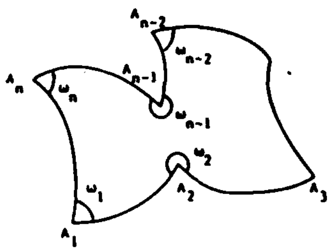

Fig. 3.1. The domain with piecewise analytic boundary.

Let $\Gamma^0 = \bigcup_{i \in D} \bar{\Gamma}_i$ and $\Gamma^1 = \Gamma - \Gamma^0$ be the Dirichlet and Neumann boundary respectively. For simplicity we consider only the problem

$$-\Delta u + u = f \quad \text{in} \tag{3.1a}$$

$$u = g^0 \quad \text{on} \quad \Gamma^0 \tag{3.1b}$$

$$\frac{\partial u}{\partial n} = g^1 \quad \text{on} \quad \Gamma^1 \tag{3.1c}$$

If $f \in L_2(\Omega)$, $g^1 \in L_2(\Gamma^1)$, $g^0|_{\Gamma_i} \in H^1(\Gamma_i)$, $i \in D$, g^0 is continuous on Γ^0, the problem (3.1) has unique solution (weak sense) $u_0 \in H^1(\Omega)$. No matter how smooth f, g^0, g^1 are, the singularity appears at the corners of the domain. Hence the standard Sobolev spaces are not a powerful tool for this type of problem, and various weighted Sobolev norms were introduced.

Let $\beta = (\beta_1, \ldots, \beta_M)$ be an M-tuple of real numbers $0 < \beta_i < 1$, $1 \le i \le M$. For any integer $k \ge 0$ we shall write $\beta + k = (\beta_i + k, \ldots, \beta_M + k)$. By $r_i(x)$ we denote the Euclidean distance between $x \in \Omega$ and the vertex A_i, $1 \le i \le M$. We denote then $\Phi_{\beta+k}(x) = \prod_{i=1}^{M} r_i^{\beta_i + k}(x)$.

Define for $k \ge \ell \ge 0$, $H_\beta^{k,\ell}(\Omega) = \{u \in H^{\ell-1}(\Omega)$, $\Phi_{\beta+k-\ell} D^\alpha u \in L_2(\Omega)$, $\ell \le |\alpha| \le k\}$ (if $\ell = 0$, the condition that $u \in H^{\ell-1}(\Omega)$ is absent) and $B_\beta^\ell(\Omega) = \{u \in H_\beta^{k,\ell}(\Omega)$, $\|\Phi_{\beta+k-\ell} D^\alpha u\|_{L_2(\Omega)} \le C d^{k-\ell}(k-\ell)!$, $k = \ell, \ell+1, \ell+2, \ldots, |\alpha| = k$, C and d independent of $k\}$. As usual, we denote $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2$, $\alpha_i \ge 0$, $i = 1,2$, integers and $D^\alpha u = \dfrac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} = u_{x_1^{\alpha_1} x_2^{\alpha_2}}$.

Let γ be the union of some edges of Ω. The space $H_\beta^{k-1/2,\ell-1/2}(\Omega)$ (resp. $B_\beta^{\ell-1/2}(\gamma)$) is defined as the trace of $H_\beta^{k,\ell}(\Omega)$ (resp. $B_\beta^\ell(\Omega)$) on γ. If $g \in H_\beta^{k-1/2,\ell-1/2}(\gamma)$ (resp. $B_\beta^{\ell-1/2}(\gamma)$), then there is $G \in H_\beta^{k,\ell}(\Omega)$ (resp. $B_\beta^\ell(\Omega)$) such that $G|_\gamma = g$. We define

$$\|g\|_{H_\beta^{k-1/2,\ell-1/2}(\gamma)} = \inf_{G|_\gamma = g} \|G\|_{H_\beta^{k,\ell}(\Omega)}.$$

Theorem 3.1. Let Ω be a polygon, $f \in B_\beta^0(\Omega)$, $g^1 \in B_\beta^{1/2}(\Gamma^1)$, $g^0 \in B_\beta^{3/2}(\Gamma^0)$, $0 < \beta_i < 1$, $\beta_i > 1 - \frac{\pi}{\omega_i}$ (resp. $1 - \frac{\pi}{2\omega_i}$ if different types of boundary condition are imposed on Γ_i and Γ_{i+1}), $1 \le i \le M$, then the problem (3.1) has a unique solution $u \in H^1(\Omega)$ and $u \in B_\beta^2(\Omega)$. For proof, see [5].

If Ω is a curvilinear polygon we introduce $C_\beta^2(\Omega) = \{u \in H_\beta^{2,2}(\Omega)$, $|D^\alpha u(x)| \le C d^{k-1}(k-1)!\,[\Phi_k(x)]^{-1}$, $\forall\, \alpha$, $|\alpha| = k \ge 1$, C and d independent of $k\}$. Then we have

Theorem 3.2. Let Ω be a curvilinear polygon and Γ_i's are analytic arcs, $1 \le i \le M$, $f \in B_\beta^0(\Omega)$, $g^{3/2-\ell} \in B_\beta^{3/2-\ell}(\Gamma^\ell)$, $\ell = 0,1$, $0 < \beta_i < 1$, $\beta_i > 1 - \frac{\pi}{\omega_i}$ (resp. $1 - \frac{\pi}{2\omega_i}$), $1 \le i \le M$. Then the problem (3.1) has a unique solution $u \in H_\beta^1(\Omega)$ and $u \in C_\beta^2(\Omega)$. For proof, see [9].

Remark 3.1. Since $B_\beta^2(\Omega) \subset C_\beta^2(\Omega) \subset B_{\beta+\epsilon}^2(\Omega)$ for arbitrary $\epsilon > 0$ the result of Theorem 3.2 is weaker than that of Theorem 3.1. Nevertheless, it will not affect the asymptotic rate of convergence for the h-p version.

Remark 3.2. Theorems 3.1 and 3.2 are also valid for generaly strongly elliptic equation and system with analytic coefficients satisfying inf-sup condition [5,6]. The interface problems with piecewise analytic interfaces and the eigenvalue problems have the same properties too, see [8]. The solution of elliptic problem of $2m$ order on polygonal domain belongs to $B_\beta^{m+1}(\Omega)$ [7]. Hence for this class of problems including many structural problems the solution set $K = B_\beta^\ell(\Omega)$ or $C_\beta^\ell(\Omega)$, $\ell \ge 2$.

The definition of $B_\beta^{\ell-1/2}(\gamma)$ does not give the structure of the space, and it is often difficult to verify in general whether g belongs to this space. Hence further characterization of the structure of $B_\beta^{\ell-1/2}(\gamma)$ is important for application.

Let $I = (a,b) \subset \mathbb{R}^1$. Analogously as before we shall define the spaces $H_\gamma^{k,\ell}(I)$ and $B_\gamma^\ell(I)$. Let $\rho_1 = |x-a|$, $\rho_2 = |x-b|$ and $\gamma = (\gamma_1, \gamma_2)$ be the 2-tuple of real numbers, $0 < \gamma_i < 1$, $i = i,2$. For any integer $k \ge 0$ we shall write $\gamma + k = (\gamma_1 + k, \gamma_2 + k)$, and denote $\Psi_{\gamma+k} = \prod_{i=1}^{M} \rho_i^{\gamma_i+k}$. For $k \geq \ell \geq 0$, we define

$H_\gamma^{k,\ell}(I) = \{u \in H^{\ell-1}(I),\ \Psi_{\gamma+m-\ell}u^{(m)} \in L_2(I),\ \ell \leq m \leq k\}$

(if $\ell = 0$, the condition that $u \in H^{\ell-1}(I)$ is absent) and

$$B_\gamma^\ell(I) = \{u \in H_\gamma^{k,\ell}(I),\ \|\Psi_{\gamma+k-\ell}u^{(k)}\|_{L_2(I)} \leq Cd^{k-\ell}(k-\ell)!$$

$$k = \ell,\ell+1,\ldots,\ C \text{ and } d \text{ independent of } k\}.$$

Analogously we define spaces $B_{\gamma_i}^\ell(\Gamma_i)$ on Γ_i with γ_i instead of γ. We have the following theorem.

Theorem 3.3. Let Ω be a polygon, further

(1) If g^0 is continuous on Γ^0, $g_i^0 = g^0|_{\Gamma_i} \in B_{\gamma_i}^1(\Gamma_i)$, $i \in D$, with $\gamma_i = (\gamma_{i,1},\gamma_{i,2})$, $0 < \gamma_{i,\ell} < 1/2$, $\ell = 1,2$, then there is a function $G^0 \in B_\beta^2(\Omega)$ such that $G^0|_{\Gamma^0} = g^0$, and $\beta_i = \max\{\gamma_{i,1},\gamma_{i-1,2}\} + 1/2$ for $i,i-1 \in D$, $\beta_i = 1/2 + \gamma_{i-1,2}$, for $i-1 \in D$ and $i \notin D$, $0 < \beta_i < 1$ are arbitrary for $i,i-1 \notin D$.

(2) If $g_i^1 = g^1|_{\Gamma_i} \notin B_{\gamma_i}^0(\Gamma_i)$, with $\gamma_i = (\gamma_{i,1},\gamma_{i,2})$, $0 < \gamma_{i,\ell} < 1/2$, $i \in D$, $\ell = 1,2$, then there is a function $G^1 \in B_\beta^1(\Omega)$ such that $G^1|_{\Gamma^1} = g^1$, and $\beta_i = \max\{\gamma_{i,1},\gamma_{i-1,2}\} + 1/2$ for $i,i-1 \notin D$, $\beta_i = \gamma_{i,1}+1/2$, for $i \notin D$, $i-1 \in D$, $0 < \beta_i < 1$ are arbitrary for $i,i-1 \in D$.

Remark 3.3. Theorem 3.3 also holds for curvilinear polygon with piecewise analytic boundary (see [10]).

Remark 3.4. Theorem 3.3 allows us to verify the boundary conditions mentioned in Theorems 3.1 and 3.2.

4. The Mesh and Finite Element Spaces

Mesh design is very crucial to the accuracy of method and depends very much on the solution set $K = B_\beta^2(\Omega)$ and $\mathcal{C}_\beta^2(\Omega)$. We assume for simplicity that Ω is a polygon contained in a unit disc centered at origin which coincides with the vertex A_1 of Ω, and $K = B_\beta^2(\Omega)$ with $\Phi_\beta = r^\beta$, i.e., assume that the singularity appears only at one vertex of Ω.

Mesh typically used in the h-p version is such that domain is divided into several layers by geometric progression. the jth layer, $1 \leq j \leq n+1$ consists of elements $\Omega_{i,j}$, $1 \leq i \leq I(j)$. In addition to the usual conditions in the theory of finite element method, the main characterization of the (geometric) mesh $\Omega_\sigma^n = \{\Omega_{i,j},\ 1 \leq i \leq I(j),\ 1 \leq j \leq n+1\}$ is following:

(C1) Let mesh factor σ be an arbitrary number, $0 < \sigma < 1$, and let $d_{i,j}$ be the distance between origin and $\Omega_{i,j}$, $h_{i,j}$ and $\underline{h}_{i,j}$ be the maximum and minimum of length of edges of $\Omega_{i,j}$, then $d_{i,j}$, $\underline{h}_{i,j}$, $h_{i,j}$ satisfy

$$\sigma^{n+2-j} \leq d_{i,j} < \sigma^{n+1-j},$$

$$\kappa_1 d_{i,j} \leq \underline{h}_{i,j} \leq h_{i,j} \leq \kappa_2 d_{i,j},$$

for $1 \leq i \leq i(j)$, $1 < j \leq n+1$ and

$$d_{i,1} = 0,$$

$$\kappa_3\sigma^{n+1} \leq \underline{h}_{i,1} \leq h_{i,j} \leq \kappa_4\sigma^n$$

for $1 \leq i \leq i(1)$. $\kappa_m, 1 \leq m \leq 4$ are independent of i and j.

(C2) Let $M = (M_{i,j},\ 1 \leq i \leq I(j),\ 1 \leq j \leq n+1)$, $M_{i,j}$ is a one-to-one mapping of standard square S (resp. standard triangle T) onto $\Omega_{i,j}$. Let P_ℓ and γ_ℓ denote the vertex and side of $\Omega_{i,j}$, then $M_{i,j}^{-1}(P_\ell)$ and $M_{i,j}^{-1}(\gamma_\ell)$ are the vertex and side of S (resp. T), $1 \leq \ell \leq 4$ (resp. $1 \leq \ell \leq 3$). Moreover, if $M_{i,j}$ and $M_{m,k}$ map S (resp. T) onto element $\Omega_{i,j}$ and $\Omega_{m,k}$ with common side $\gamma_1 = A_1A_2$, then

$$\operatorname{dist}(M_{i,j}^{-1}(A), M_{i,j}^{-1}(A_\ell)) = \operatorname{dist}(M_{m,k}^{-1}(A), M_{m,k}^{-1}(A_\ell)),$$

for any $A \in \gamma_1$, $\ell = 1,2$. We assume each side γ_ℓ of $\Omega_{i,j}$ is analytic curve, $1 \leq \ell \leq 4$ (resp. $1 \leq \ell \leq 3$),

$$\gamma_\ell : \begin{array}{l} x = h_{i,j}\varphi_{i,j,\ell}(\chi) \\ y = h_{i,j}\psi_{i,j,\ell}(\chi) \end{array} \quad \chi \in I = (0,1)$$

and

$$|\varphi_{i,j,\ell}^{(k)}|,\ |\psi_{i,j,\ell}^{(k)}| \leq CL^k k!$$

where C and L are independent of ℓ,i,j. Accordingly, the mapping $M_{i,j}$ of S (resp. T) onto $\Omega_{i,j}$ is analytic on $\bar{S}$ (resp. $\bar{T}$) and can be extended to $S^* \supset \bar{S}$. Let $J_{i,j}$ be the Jacobian of $M_{i,j}$. We shall assume that

$$C_1\underline{h}_{i,j} \leq J_{i,j} \leq C_2 h_{i,j}$$

with constants C_1,C_2 independent of i,j.

Remark 4.1. Figure 6.3 is an example of the geometric mesh for the problem with singularity at one corner, but the mesh can be analogously generalized for problems with singularity at every corner.

Remark 4.2. If mesh Ω_σ^n contains triangular elements some additional assumptions have to be imposed. In the practice these assumptions can easily be satisfied, see [9].

Let $P = (p_{i,j},\ 1 \leq i \leq I(j),\ 1 \leq j \leq n+1)$ and $Q = (q_{i,j},\ 1 \leq i \leq I(j),\ 1 \leq j \leq n+1)$ be the degree vector with integer $p_{i,j}$ and $q_{i,j} \geq 0$.

We define the finite element spaces

$S^{P,Q}(\Omega_\sigma^n) = \{\phi \mid \phi(x,y) = \phi_{i,j}(M_{i,j}^{-1}(x,y))$ for $(x,y) \in \Omega_{i,j}$, $\phi(\xi,\eta)$ is the polynomial of degree $\le p_{i,j}$ in ξ and of degree $\le q_{i,j}$ in $\eta\}$

and

$$S^{P,Q,1}(\Omega_\sigma^n) = S^{P,Q}(\Omega_\sigma^n) \cap H^1(\Omega),$$

$$\overset{\circ}{S}{}^{P,Q,1}(\Omega_\sigma^n) = S^{P,Q}(\Omega_\sigma^n) \cap \overset{\circ}{H}{}^1(\Omega)$$

where $\overset{\circ}{H}{}^1(\Omega) = \{u \in H^1(\Omega),\ u|_{\Gamma^0} = 0\}$.

By N we denote $\dim(\overset{\circ}{S}{}^{P,Q,1}(\Omega_\sigma^n))$, the number of degree of freedom.

5. Basic Approximation Theorems of the h-p Version

We will list some basic approximation results in the case that $H_1 = H_2 = \overset{\circ}{H}{}^1(\Omega)$, $K = B_\beta^2(\Omega)$ or $C_\beta^2(\Omega)$ and $S_1 = S_2 = \overset{\circ}{S}{}^{P,Q,1}(\Omega_\sigma^n)$, i.e. we seek the estimates of $Z(u, \overset{\circ}{H}{}^1(\Omega), \overset{\circ}{S}{}^{P,Q,1}(\Omega_\sigma^n))$ for $u \in K$.

Theorem 5.1. Let Ω be a polygon and $u \in B_\beta^2(\Omega) \cap \overset{\circ}{H}{}^1(\Omega)$, then for any $\sigma \in (0,1)$, $P = Q$, $\nu j \le p_{i,j} \le \mu n$, $0 \le \nu \le \mu < \infty$ and $p_{i,j} \ge 1$. We have

$$Z(u, \overset{\circ}{H}{}^1(\Omega), S^{P,Q,1}(\Omega_\sigma^n)) \le Ce^{-bN^{1/3}} \quad (5.1)$$

where b and C are independent of $N = \dim(\overset{\circ}{S}{}^{P,Q,1}(\Omega_\sigma^n))$, the number of degree of freedom. For proof, see [9].

Theorem 5.2. If $u \in C_\beta^2(\Omega) \cap \overset{\circ}{H}{}_\beta^1(\Omega)$, Ω is a curvilinear polygon, the boundary of domain is piecewise analytic, then the result of the previous theorem holds. For proof, see [9].

Remark 5.1. Mesh factor σ can be any number $\in (0,1)$ the computation shows that $\sigma = 0.15$ is the optimal value. In [18] it has been proved that $\sigma = (\sqrt{2}-1)^2 = 0.17$ is the optimal mesh factor in one dimensional setting. The value $\sigma = 0.15$ in two dimensional problems reflects the fact the solutions in the neighborhood have essentially one dimensional character.

Remark 5.2. If $g_i^0 = g^0|_{\Gamma_i} \in B_{\gamma_i}^1(\Gamma_i)$, $i \in D$ with $\gamma_i = (\beta_i - 1/2, \beta_{i+1} - 1/2)$ are non-homogeneous Dirichlet boundary condition, the theorems above hold provided g^0 is properly projected on the trace of finite element space $S^{P,Q,1}(\Omega_\sigma^n)$.

Remark 5.3. For problems of order $2m$, the theorems hold when geometric mesh contains only parallelogram and triangular elements. For details, see [22].

6. Numerical Results

We will present some numerical results for the solution of a plane strain elasticity problem. We selected the model of crack panel loaded by traction that the exact solution is the first (symmetric) and second (antisymmetric) mode of stress intensity factor solution. This problem was selected because it characterizes the usual difficulties of engineering computation. Due to the symmetry and antisymmetry we need only to solve the problem in the upper half of the panel shown in Figure 6.1. The solution has singular behavior at the tip of the crack, i.e., the displacement $U = (u,v)$ has the expression $(r^{1/2}\phi_1(\theta), r^{1/2}\phi_2(\theta))$ near the origin. Obviously $u,v \notin H^2(\Omega)$ and $u,v \in B_\beta^2(\Omega)$ for $\beta > 1/2$.

The energy of U is defined as

$$G(U) = \frac{E}{2(1-2\nu)(1+\nu)}\int_\Omega ((1-\nu)((\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial y})^2) + 2\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + (1/2-\nu)((\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})^2)dxdy = \|U\|_E^2$$

where E and ν are the Young's modulus of elasticity and Poisson ratio. The error $e = U - U_{FE}$ is measured in energy norm, and by (5.1)

$$\|e\|_E \le Ce^{-bN^{1/3}}. \quad (6.1)$$

The relative error is defined as

$$\|e\|_{E,R} = \|e\|_E / \|U\|_E \times 100\%.$$

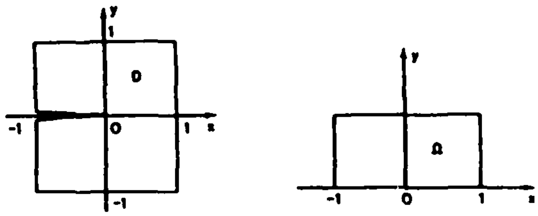

Figure 6.1. The crack panel.

The computation of the p and h-p version were made by program PROBE [27]. The computation of the h-version with $p = 1$ was made by the adaptive program FEARS developed at University of Maryland [24]. We will compare the performance of the three versions of finite element method.

Meshes A_i, $1 \le i \le 6$ which are refined near the tip by geometric progression with factor $\sigma = 0.15$ are shown in Figure 6.2.

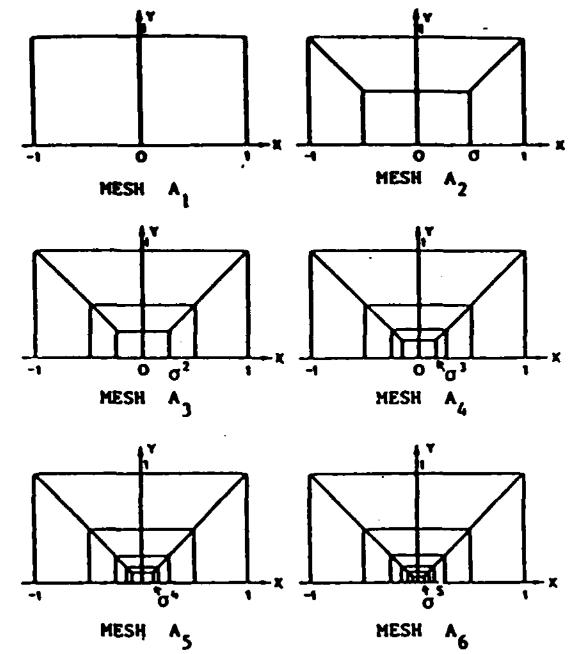

Figure 6.2. Mesh

The table 6.1 shows the relationship between N,n,p and $\|e\|_{E,R}$ where n is the number of layers, p is the element degree and N is the number of degree of freedom. The relationship is plotted in Ln $\|e\|_{E,R} \times N^{1/3}$ scale and shown in Figure 6.3. The curve characterizing the convergence of the h-p version is the envelope of the six curves of the p-version on the Mesh A_i, $1 \le i \le 6$ and is nearly a straight line. The slope of the line is b in (6.1) and is numerically 0.67.

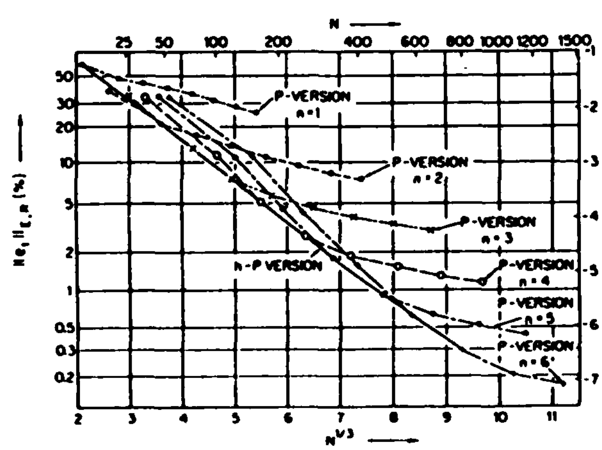

Figure 6.3. Relative error in energy norm vs. number of degree of freedom. The symmetric problem (E = 1, ν = 0.3) on Mesh A_i, $1 \le i \le 6$, σ = 0.15, p = n.

Table 6.1. Relationship between $\|e\|_{E,R}$, N,n,p,b and C for the h-p version. The symmetric problem (E=1,v=0.3) on Mesh A_i, $1 \le i \le 6$, σ = 0.15, p = n.

Mesh	p	N	$N^{1/3}$	$\|e\|_{E,R}$%	b	$C/\|U_0\|_E$
A_1	1	9	2.08	60.92	0.741	1.455
A_2	2	48	3.63	20.23	0.740	2.303
A_3	3	121	4.95	7.61	0.776	2.098
A_4	4	256	6.35	2.57	0.720	1.810
A_5	5	477	7.81	0.90	0.670	1.683
A_6	6	808	9.31	0.33	0.670	1.688

In Figure 6.4 we show the dependence of the error on σ. We see the best value of σ is close to $(\sqrt{2}-1)^2 = 0.17$ which is the theoretical optimal value in one dimension.

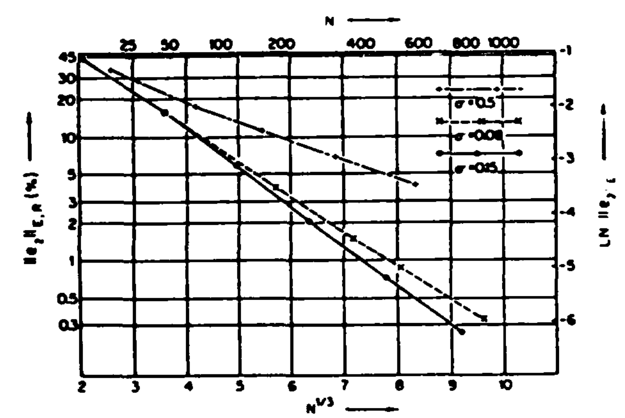

Figure 6.4. Dependence of Relative error of the h-p version in energy on the mesh factor σ. The anti-symmetric problem (E=1,v=0.3) on Mesh A_i, $1 \le i \le 6$.

Figure 6.5 shows that the h-p version in insensitive to change of Poisson ratio. The slope of the curves of the h-p version for v = 0.3 and 0.49 are almost the same. The locking problem never occured.

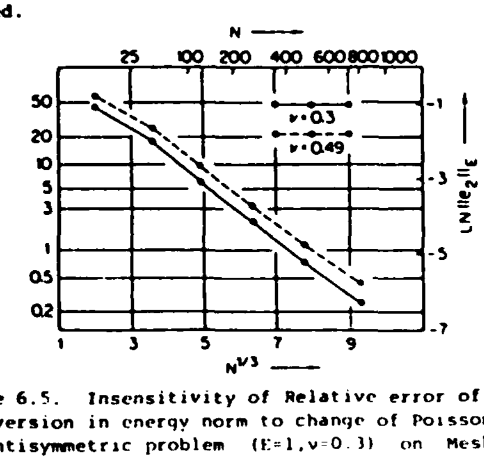

Figure 6.5. Insensitivity of Relative error of the h-p version in energy norm to change of Poisson ratio. The antisymmetric problem (E=1,v=0.3) on Mesh A_i, $1 \le i \le 6$.

Table 6.2. Estimated error of the h-p version. The symmetric problem ($E = 1$, $\nu = 0.3$) on Mesh A_i, $1 \le i \le 6$, $\sigma = 0.15$, $p = n$.

n	$\Vert\tilde{e}\Vert_E$	$\Vert e\Vert_E$	$\Vert\tilde{e}\Vert_{E,R}$%	$\Vert e\Vert_{E,R}$%	$(\Vert e\Vert_E - \Vert\tilde{e}\Vert_E)/\Vert e\Vert_E$%
1	2.9596E-1	2.9662E-1	60.83	60.92	0.2189
2	9.8774E-2	9.8511E-2	20.28	20.23	−0.2669
3	3.7033E-2	3.7055E-2	7.606	7.611	0.0606
4	1.2489E-2	1.2500E-2	2.565	2.567	0.0926
5	4.3359E-3[a]	4.3691E-3	0.891[a]	0.897	0.6689

In Figure 6.6 we compare the performance of the h,p and h-p versions in ln $\Vert e\Vert_{E,R}$ × ln N scale. We see that the accuracy 0.5~1.0% is very expensive and probably is not achievable at all for the p-version and h version with $p = 1$. The h-p version allows us to use a relatively very small number of elements to obtain high accuracy.

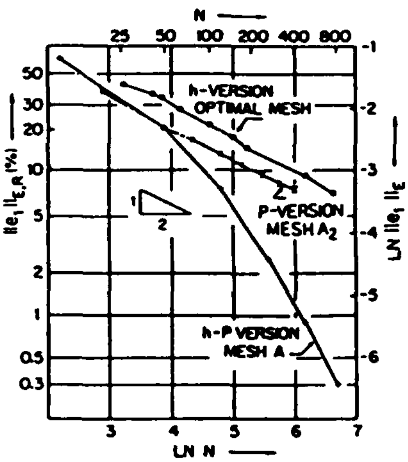

Figure 6.6. Relative error in energy norm vs. number of degree of freedom for the h,p,h-p versions. The symmetric problem on which mesh factor $\sigma = 0.15$.

Since the exponential rate of convergence can be achieved for low p and n as well as high p and n, we can use the computational results from three successive computations to obtain estimation of error in energy norm. Table 6.2 shows that the estimated error is very reliable with relative error less than 1%. For the formulation of estimated error, see [20].

7. Conclusions

From theoretic analysis and computation we can conclude the following:

(1) The asymptotic theory reflects accurately the computational practice in the entire range of engineering accuracy. The exponential rate of convergence is achieved in computation in industrial practice.

(2) The optimal geometric factor of mesh refinement is close to 0.15.

(3) The performance of the p and h-p versions is not influenced when the Poisson ratio $\nu \approx 0.5$ (i.e., when the material is nearly incomprehensible).

(4) Industrial experience with the method (by program PROBE) indicates high effectivity and advantages of the h-p version, see [15].

(5) Preliminary computation and theoretical analysis show that in the three dimensions the p and h-p have superior qualities in practical computation of problems in structural mechanics.

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