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THE CONCEPT OF INFLUENCE AND ITS USE IN STRUCTURING COMPLEX DECISION PROBLEMS

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The generality of the decision analysis methodology permits its application to decision problems regardless of the particular discipline or setting in which the problem occurs. Consequently, the decision analyst may be unfamiliar with the relationships of the variables in the problem. One device for communicating those relationships is a diagram identifying

(continued)

the existence of influences between the variables. This research contributes a general mathematical characterization of the influence between random variables. The influence can be characterized by a matrix that is null if and only if no influence exists and otherwise indicates the degree and type of influence by its nonzero elements.

An electrical engineer uses the schematic diagram of a circuit to conceptualize and communicate the relationship between the voltage at different points of an electronic device. The definition of influence can serve the decision analyst in an analogous manner, helping him to conceptualize and communicate the relationship of the probability distributions on different variables in a probabilistic decision model. The definition of influence supports a calculus of influences that allows one to compute the total influence of one variable on another even when there are several intermediate variables. Using this influence calculus, the importance of a particular variable to the decision model can be determined. An immediate consequence is a recommendation for which variables to include in the model and whether the uncertainty about a variable is important. These recommendations include a new interpretation of deterministic sensitivity.

An important, philosophical result of this research is the demonstration that which variables should be included in the decision model depend on the decision maker's risk attitude. Two decision makers with the same state of information but different risk attitudes should model the same decision problem differently.

Finally, the theoretical basis for the influence definition is different from that of the conventional discretization or decision tree representation for solving decision problems. Since the acceptability of the influence method depends on its accuracy and ease of implementation relative to discretization, the theoretical bases of the influence method and discretization are compared.

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Abstract

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An important, philosophical result of this research is the demonstration that which variables should be included in the decision model depend on the decision maker's risk attitude. Two decision makers with the same state of information but different risk attitudes should model the same decision problem differently.

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"My master is so great because he eats when he is hungry and drinks when he is thirsty."

. . . part of a Buddhist Proverb

"Uncertainty makes me nervous, and certainty makes me unnervous."

. . . Mary Hartman, Mary Hartman

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CHAPTER I

Introduction and Overview

Decision analysis combines the decision maker's uncertainty about problem variables, his structure relating the decisions and outcomes, and his preferences over outcomes to obtain a logically consistent decision. Structuring of the decision problem is part of the foundation of decision analysis.

> "The primary function of the decision analyst is to capture the relationships among the many variables in a decision problem, a process called structuring." [4]

The purpose of this research is to improve the decision analysis structuring methodology. There must be a decision analysis theoretical foundation for the methodology, so that the resulting structure conduces to a solution of the decision problem. At the same time, we require that the methodology be useful for communicating the relationships in a complex decision problem. In the following two sections we discuss these two requirements and review previous research related to each.

1.1 Decision Analysis Theoretical Structuring

To be precise about what is required of the structuring process, let the random variable v represent the profit and d the decision variable. Then, a decision model structure should permit the computation of the conditional profit lottery $\{v|d,\mathcal{S}\}$.* If the decision maker believes the profit lottery depends on a set $\underline{s}_1$ of state variables, it may be convenient to compute the profit lottery through the expansion equation

$$\{v|d,\mathcal{S}\} = \int_{\underline{s}_1} \{v|\ d,\ \underline{s}_1,\mathcal{S}\}\{\underline{s}_1|d,\mathcal{S}\}d\underline{s}_1\ . \tag{1.1.1}$$

The decision maker may believe that some of the variables included in $\underline{s}_1$ depend on other state variables. Let the second set of state variables be denoted as $\underline{s}_2$. Then,

$$\{v|d,\mathcal{S}\} = \int_{\underline{s}_1} \int_{\underline{s}_2} \{v|d,\underline{s}_1,\mathcal{S}\}\{\underline{s}_1|d,\underline{s}_2,\mathcal{S}\}\{\underline{s}_2|d,\mathcal{S}\}d\underline{s}_2 d\underline{s}_1 \tag{1.1.2}$$

The most general probabilistic expansion of this form is

$$\{v|d,\mathcal{S}\} = \int_{\underline{s}_1} \int_{\underline{s}_2} \{v|d,\underline{s}_1,\underline{s}_2,\mathcal{S}\}\{\underline{s}_1|d,\underline{s}_2,\mathcal{S}\}\{\underline{s}_2|d,\mathcal{S}\}ds_1 ds_2\ . \tag{1.1.3}$$

However, since all of the state variables upon which v depends are included in $\underline{s}_1$,

$$\{v|d,\underline{s}_1,\underline{s}_2,\mathcal{S}\} = \{v|d_1,s_1,\mathcal{S}\}\ ,$$

and (1.1.3) reduces to (1.1.2).

---

* The notation $\{v|d,\mathcal{S}\}$ represents the probability density distribution on the variable v conditioned on the variable d . Since we take the subjective view of probability the symbol $\mathcal{S}$ is included to represent conditioning on a particular state of information.

The expansion of (1.1.2) can be repeated to, say, $\underline{s}_n$, until the decision maker is comfortable in assessing $\{\underline{s}_n|d,S\}$ and the other required marginal probability distributions. The resulting expansion is

$$(1.1.4) \quad \{v|d,S\} = \int_{\underline{s}_1} \int_{\underline{s}_2} \cdots \int_{\underline{s}_n} \cdots \int_{\underline{s}_N} \{v|d,\underline{s}_1,S\}\{\underline{s}_1|d,\underline{s}_2,S\} \cdots$$

$$\{\underline{s}_{n-1}|d,\underline{s}_n,S\} \cdots \{\underline{s}_N|d,S\}d\underline{s}_1\, d\underline{s}_2 \cdots d\underline{s}_n \cdots d\underline{s}_N$$

and is represented in Figure 1-1.

Generally, the expansion procedure described by equation (1.1.4) is impractical, because it results in so many dependencies between state variables. If the expansion above includes a total of k state variables, then the number of possible dependencies is $(k-1)!$ For complex decision problems, the cost of including every dependency in the analysis is prohibitive and unwarrantable.

The decision analyst uses his judgment to ignore some dependencies and to include others in striking a balance between excessive detail and unreal simplicity. However, once we admit that the decision model is not going to include every variable relationship that the decision maker identifies, then we must require a theoretical basis on which to select the subset of variable relationships that are to be included. Suppose $\underline{s}_n^*$, $1 \leq n \leq N$, represents a subset of $\underline{s}_n$,

$$(1.1.5a) \qquad \underline{s}_1^* \subseteq \underline{s}_1, \ldots, \underline{s}_n^* \subseteq \underline{s}_n, \ldots, \underline{s}_N^* \subseteq \underline{s}_N .$$

Which subsets of variables $\underline{s}_n^*$ used in equation (1.1.5b),

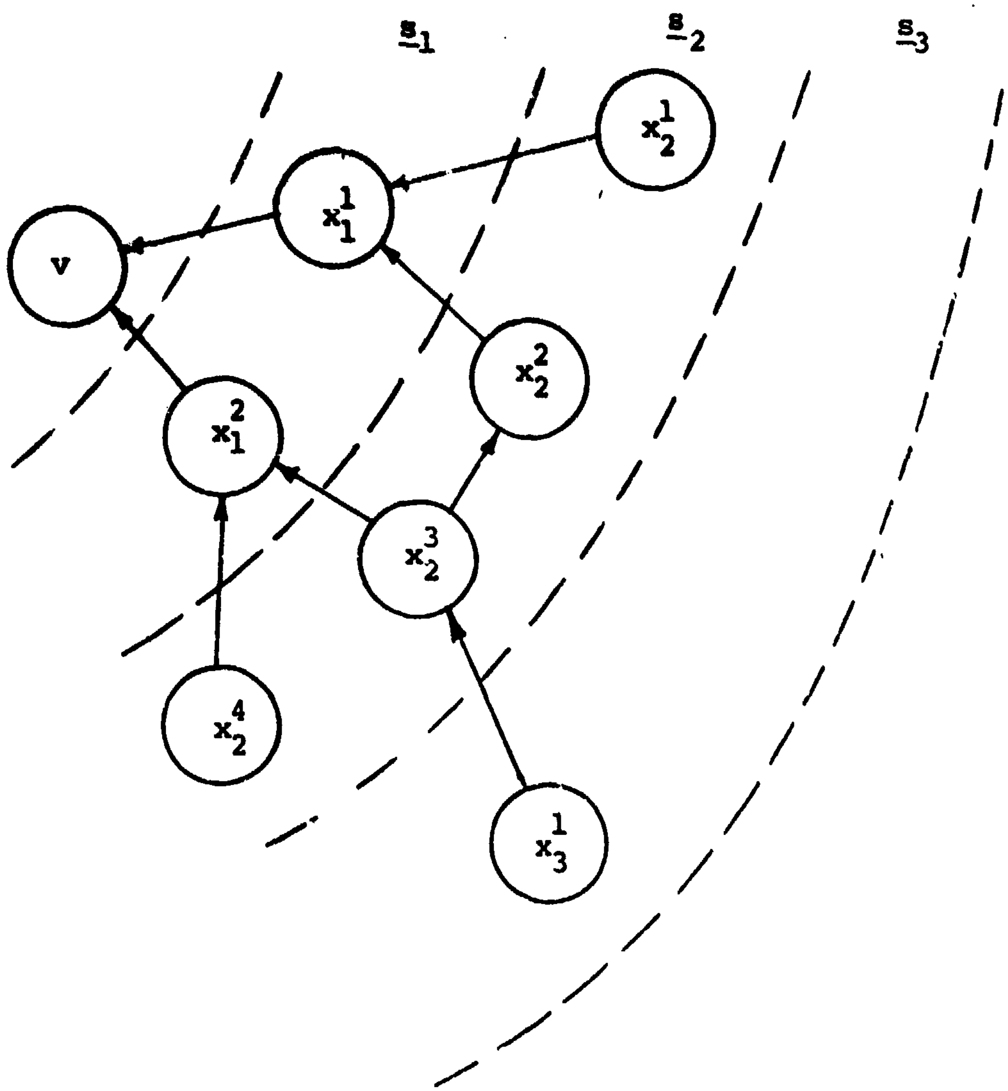


FIGURE 1-1. A Representation Of The Expansion Equation Showing The Sets $\underline{s}_i$ Of State Variables

$$(1.1.5b)\quad \{v|d,S\} \simeq \int_{\underline{s}_1^*}\int_{\underline{s}_2^*}\cdots\int_{\underline{s}_N^*}\{v|d,\underline{s}_1^*,S\}\{\underline{s}_1^*|d,\underline{s}_2^*,S\}\ldots\{\underline{s}_N^*|d,S\}d\underline{s}_1^*\,d\underline{s}_2^*\ldots d\underline{s}_N^*$$

best represents the decision maker's structure, equation (1.1.4), as the total number of state variables decreases?

Review of Related Work in the Theoretical Structuring of Decision Analysis Problems

The purpose of the deterministic phase of the decision analysis cycle is to provide a deterministic structure for a decision problem and, through the deterministic sensitivity, to indicate the important variables for inclusion in the probabilistic model. This procedure is intuitively reasonable and has proven successful in over a decade of application. [1,6,7,9,17] However, so far a theory has not been presented to show that deterministic sensitivity is the best criterion on which to determine the subsets $\underline{s}_n^* \subseteq \underline{s}_n$ of variables to be included in the stochastic decision model. As a result, it is possible to construct some examples of deterministic sensitivity that could make the selection of the stochastic variables ambiguous. For example, consider a decision model with two independent, uncertain state variables x_1 and x_2 . Suppose the variables have identical marginal probability distributions and different deterministic sensitivities shown in Figure 1-2. If the measure of the importance of a variable to the decision model is its affect on the expectation of the profit lottery, then it is

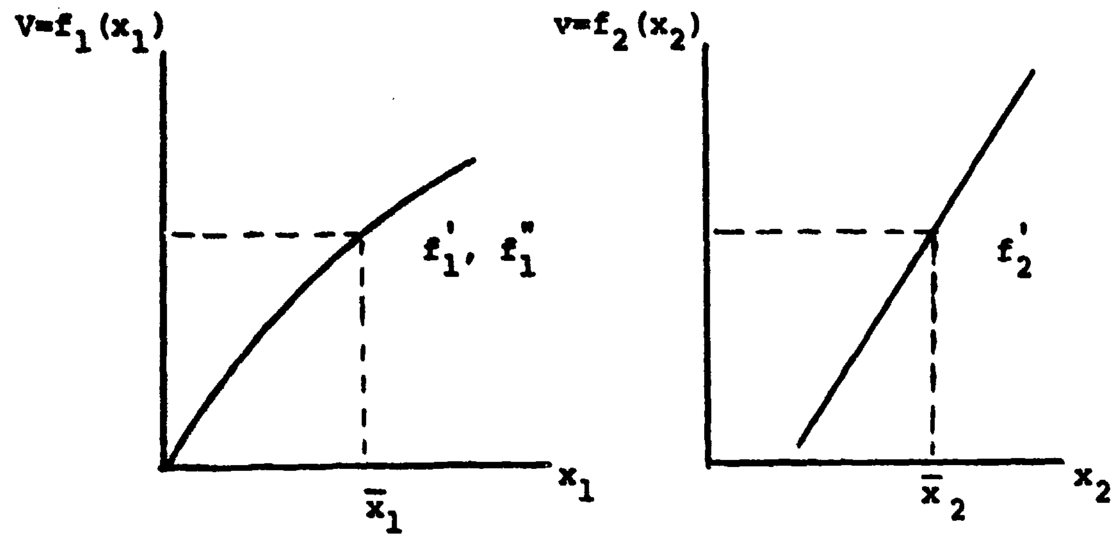


FIGURE 1-2 Deterministic Sensitivity For Two Variables

better to fix x_2 at its mean and allow x_1 to be represented as uncertain than vice versa. Because of the linearity of its deterministic sensitivity, uncertainty about x_2 affects the expectation of the profit lottery in the same way as it affects the expectation of x_2. Hence, only the expectation of x_2 is required to model the expectation of the profit lottery. Analytical support for these contentions is provided in Chapter 3.

Two early articles by Matheson [8] and Smallwood [16] demonstrate that one could use decision analysis to decide among possible decision models. This approach, however, requires the assignment of a prior probability distribution over either the possible profit lotteries resulting from a complete analysis (Matheson) or the space of possible models (Smallwood). While these studies provide an interesting conceptual tool for understanding the structure of a decision problem, they have not been widely used in practice, probably because of the analyst's reluctance to assign the required priors.

More recently Tani [18] suggested a variation of Matheson's approach that can be used to quantify the dissatisfaction with a current model. Rather than encoding a prior over possible profit lotteries, Tani uses the differences between the current lottery and an "authentic" lottery, which is still a difficult assessment. Tani assumes that the marginal probability distribution encoded on state variables are authentic.

With respect to this dissertation, Tani's most important contribution is the establishment of a philosophical criterion by which to judge the "goodness" of a model. He introduces the concept of authenticity as the metric for decision models.

> "Our ideal in decision analysis is not to construct the perfect model, but rather to obtain the authentic profit lottery -- the one that accurately expresses our uncertainty about the future." [18]

An authentic profit lottery is one that accurately and fully expresses the decision maker's beliefs. We have shown elsewhere how an otherwise attractive structuring device is unacceptable, because it is not related to the authenticity of the profit lottery. [13]

1.2 Practical Considerations in Structure Assessment

Three important difficulties oppose our attempts to structure a decision problem: unfamiliarity, complexity, and numerous participants. First, by unfamiliarity we mean the analyst's unfamiliarity with the political, technical, and economic influences in a particular decision problem. The decision analysis profession emerged from the theory with the conviction that the logical methodology is equally applicable to all decision problems from deciding on new business ventures or enacting legislation to buying a house. In fact, it is the decision analyst's initial unfamiliarity with the important relationships of the problem that helps him to maintain the vitally important professional detachment. [3] However, because of this unfamiliarity, the analyst requires tools through which the decision maker can communicate clearly his perception of the problem structure.

A second difficulty in structuring is the complexity of the decision problem. As the number of variables in the decision problem increases, the structuring task becomes more difficult for two reasons. First the number of possible relationships between variables increases as (n-1)!, leading to a corresponding increase in the effort requried for the assessment. Secondly, the decision maker will not apprehend some of the actual variable relationships, because prior to the decision analysis methodology he had no orderly way to address them. By contrast, in a small decision problem, with few variables, the decision maker is aware of all the relationships between variables and has often spent considerable time in analyzing them.

If the decision problem is complex, then there are likely to be many participants in structuring the problem. Even when there is a single decision maker, many experts are likely to be consulted regarding the relationships between variables, as well as, for probability assessment. Moreover, several decision analysts may be involved. Figure 1-3 depicts the various communication paths required for structuring a complex problem and again suggests the need for a powerful communication tool.

<u>Review of Related Work in Structure Assessment</u>

One promising communication tool is the influence diagram. Influence diagrams were developed at SRI International as an automated structuring aid for decision analysis. [10] An influence between two random variables, x and y , is said to exist when the variables are probabilistically dependent, and an arrow is drawn to connect the two variables (Figure 1-4a).

> <u>Definition 1.1.</u> An influence between two random variables x and y exists if for some x with nonzero probability,

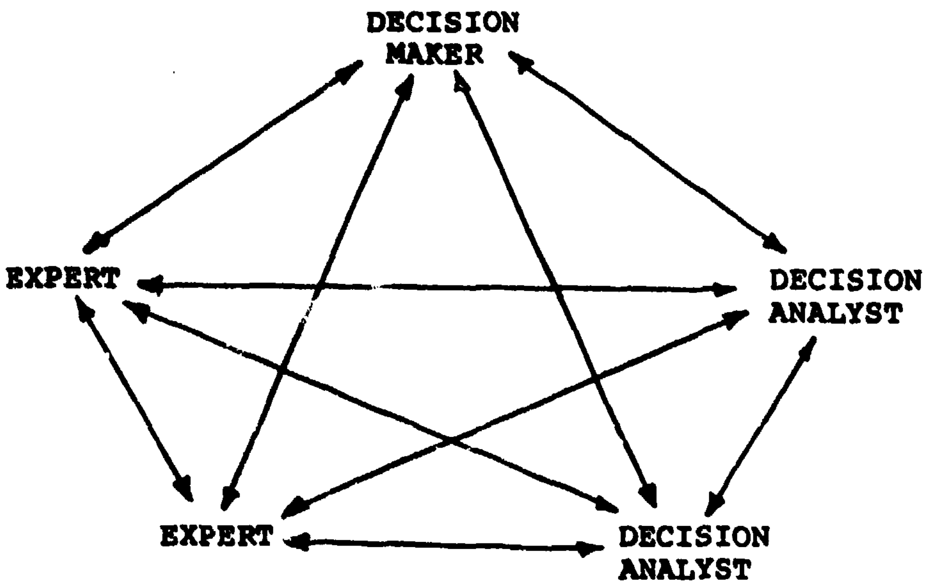
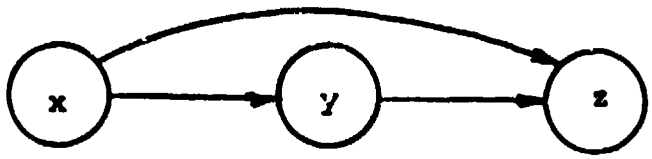


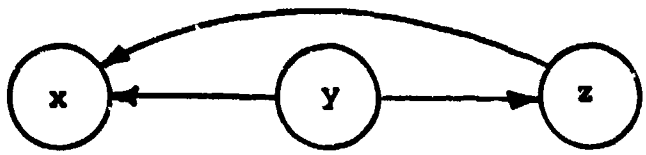
FIGURE 1-3. Communication Paths Required For Structuring A Complex Decision Problem

a. x influences y

b. $\{x,y,z|S\} = \{z|x,y,S\}\{y|x,S\}\{x|S\}$



c. $\{x,y,z|S\} = \{x|y,z,S\}\{z|y,S\}\{y|S\}$



d. $\{x,y,z|S\} = \{x|z,y,S\}\{z|S\}\{y|S\}$

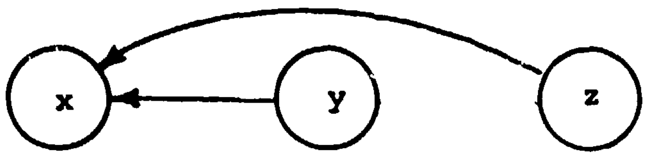


FIGURE 1-4. The Correspondence Between Influence Diagrams And Assertions Of Probablistic Independence

$$\{y|x,S\} \neq \{y|S\}$$

Using this definition, some rules for the manipulation of influence diagrams can be derived and are discussed in reference [10]. Each influence diagram corresponds to a particular expansion of a joint distribution. For example, the influence diagram of 1-4b represents the expansion

$$\{x,y,z|S\} = \{z|x,y,S\}\{y|x,S\}\{x|S\} \ .$$

An alternative expansion, represented by 1-4c, is

$$\{x,y,z|S\} = \{x|y,z,S\}\{z|y,S\}\{y|S\} \ ,$$

and, therefore, 1-4c is an allowable rearrangement of the influences of 1-4b. Comparing 1-4d with 1-4c shows that the influence between y and z has been removed, and

$$\{x,y,z|S\} = \{x|z,y,S\}\{z|S\}\{y|S\} \ .$$

Another important property of definition 1.1 is that it appears to coincide with the decision maker's intuitive use of the word influence. In past applications of influence diagrams for complex decision problems at SRI International, when a decision maker or his expert identified the existence of an influence between variables (even though it was not mathematically defined for them as in definition 1.1), the variables were later determined to be probabilistically dependent. Furthermore, influences that were identified as being strong represented, roughly speaking, more probabilistic dependence than influences that were identified as weak.

Other researchers have demonstrated similar structuring devices, such as the interaction matrix method. [2,15] This method indicates the existence or non-existence of interactions or influences in a matrix form rather than diagrammatically. Rows represent a set of variables x_i ; and columns y_j . An influence between x_i and y_j is indicated by setting the ij element of the matrix to one, otherwise it is set to zero. When there are sequential influences, x_i influencing y_j influencing z_k , the matrices can be multiplied to show the existence of influences between x_i and z_k (Figure 1-5). These interaction matrices may be quite large.

There are several important shortcomings in this approach. First, the term "interaction" is not generally defined. It may mean different things to the decision maker and analyst, and no test is available to compare usage of those words. Secondly, a ranking of interactions between x_i and z_k is determined by the number of influences that exist between the two. This ranking completely ignores the questions of the degree and type of the interaction.

A final criticism of the interaction matrix also applies to the current use of influence diagrams. Both require judgment as to the relative importance of influences or interactions. The analyst laments, "Everything is affected to some degree by everything else," and a limitation exists on the number of influences that can be analyzed. Since interaction is not defined in its use with interaction matrices, we see little chance that anyone's judgment regarding the relative importance of interactions is meaningful. For influence diagrams, a rough notion of more or less dependence exists, but it is not precise. Figure 1-6 shows an example of two possible relationships of x and y . In the first

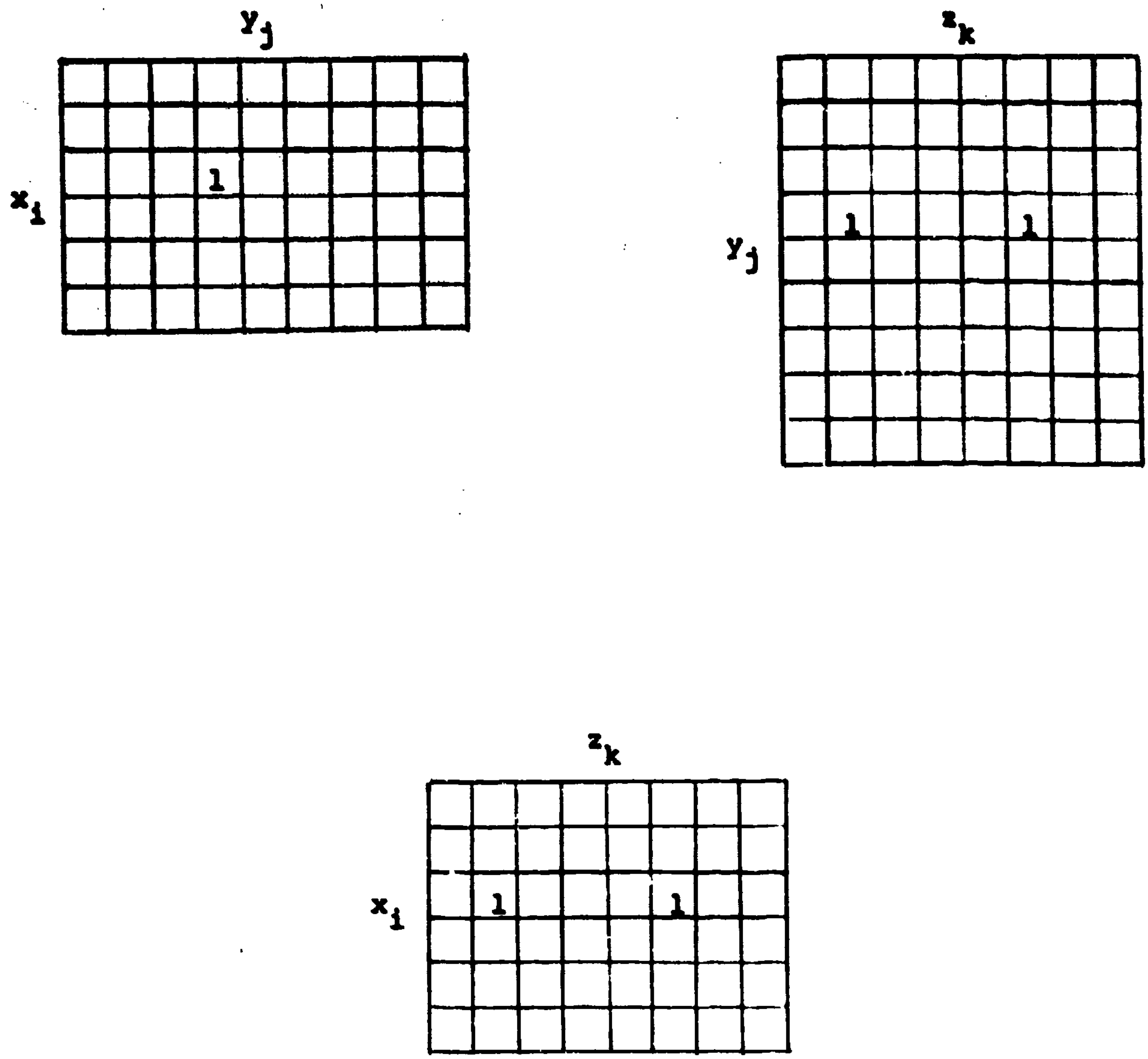


FIGURE 1-5. Multiplication of Interaction Matrices

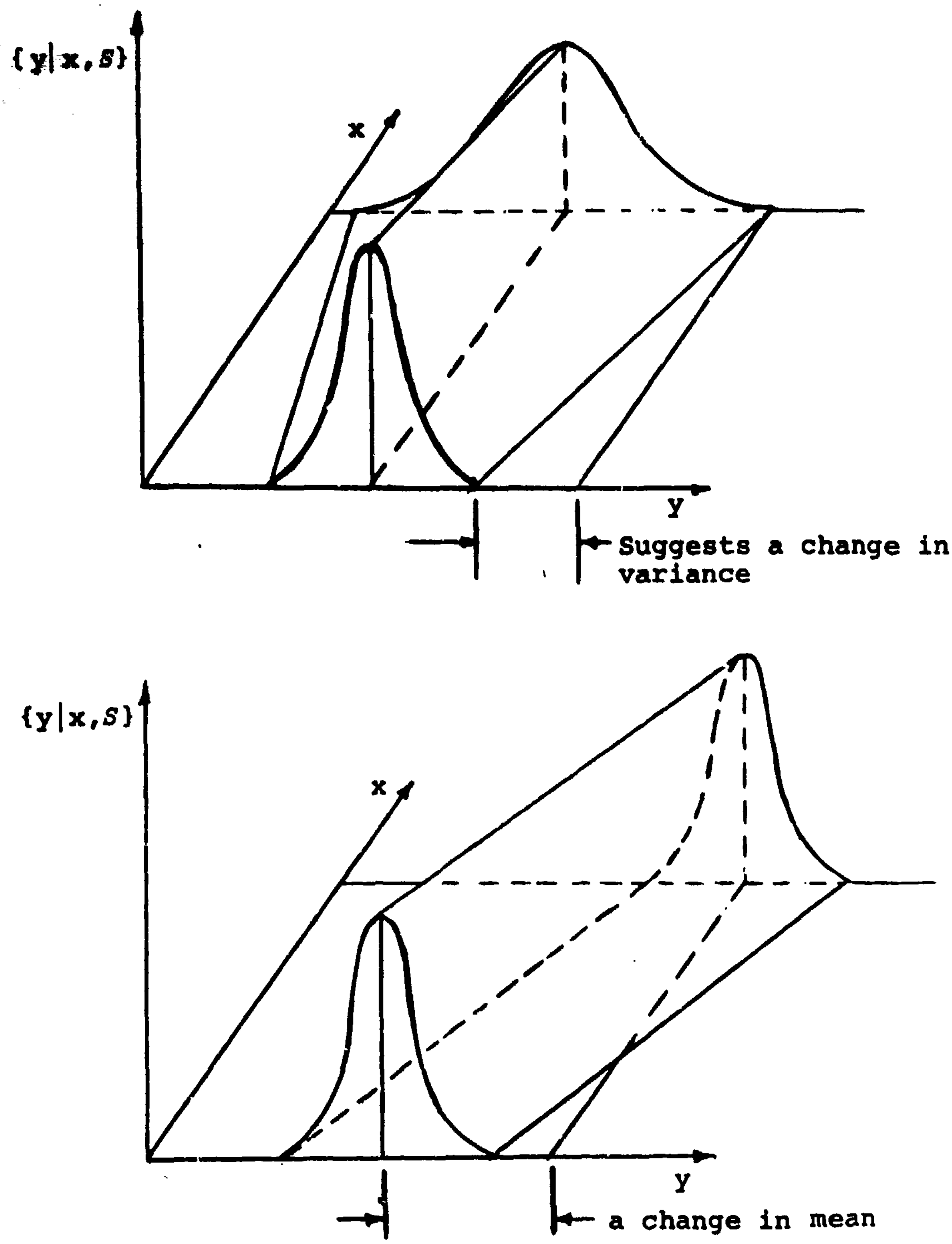


FIGURE 1-6. An Example Demonstrating Ambiguity About The Relative Importance Of Two Influences

example the expectation of the distribution on y is unchanged given the value of x . However, the variance of that distribution changes depending on the value of x . In the second example, the expectation changes depending on the value of x , but the variance does not. Which of the two cases represents the most important influence?

1.3 Summary and Contributions of this Research

The example of Figure 1-6 suggests that definition 1.1 is not a complete description of influence. While it does define the existence of influence, it does not describe the influence itself. Most of Chapter 2 is devoted to the development of a mathematically precise, general description of influence that is consistent with the condition of existence given by definition 1.1. This description is applicable to both continuous and discrete influencing and influenced random variables. In the final section of Chapter 2, we introduce an approximate description of influence to reduce the informational requirement.

Section 2 of Chapter 3 shows that the influence of any variable on the profit lottery can be determined by means of an influence calculus. The notation for the influence of one variable on another is carefully selected so that the equations for the influence calculus can be obtained by inspection of the influence diagram. The implications for the decision model of the various degrees and types of influence are conceptualized in the influence-consequence matrix, which is presented in Section 3.4. We show how the influence matrix can be used to estimate the differences between the profit lottery from the decision model and the decision maker's authentic profit lottery. Probably the most important philosophical result of Chapter 3 is that the selection of variables to compose the decision model should depend on the decision maker's risk attitutde. Several examples are given.

Chapter 4 compares methods for approximating the profit lottery from information about the distribution of the profit lottery conditioned on a state variable and the marginal distribution on the state variable. First, we show that it is the functional form of the conditional surface as a function of the state variable that determines the amount of information required to compute the profit lottery exactly. We also show that the difference between the quadratic approximation, which is the basis for the practical application of the influence concept, and conventional discretization is the assumption about the shape of the conditional surface. The quadratic method assumes this surface is quadratic in the conditioning variable, and discretization assumes it is piecewise linear. Since the quadratic method is shown to be as sound theoretically as discretization and to be comparable in both ease of assessment and accuracy, it should be considered as an alternate method for the solution of decision problems directly from the influence diagram.

After summarizing the results of the previous chapters, Chapter 5 proposes extending the influence calculus to include decision variables in order that the solution to decision problems can be directly obtained from the influence diagram. We show that the influence calculus and influence notation extend in a natural way to accommodate decision variables. Furthermore, the influence vector describing the influence of the decision variable on the profit lottery may be closely related to the solution of the decision problem. These preliminary results lead us to encourage further research in the application of the influence concept to decision problems.

CHAPTER 2

Toward a Theory of Influence

2.1 Introduction

Influence diagrams are an attractive means for assessing the decision maker's structure and communicating it. However, since only the existence of an influence has so far been defined, the diagrams are only useful for specifying the existence of relationships between variables. That is, the decision maker can indicate using influence diagrams which state variables are members of the sets $\underline{s}_1$, $\underline{s}_2$, ... $\underline{s}_N$ of the equation (1.1.4). The influence diagram does not indicate the nature of the conditional relationships among the state variables, $\{v|\underline{s}_1,d,S\}$ or $\{\underline{s}_{n-1}|\underline{s}_n, S\}$, and there is no theoretical basis by which to reduce the potentially large sets $\underline{s}_n^*$ of important state variables (expression 1.1.5).

The purpose of this chapter is to present a definition of influence that will extend the usefulness of influence diagrams. To that end, influence should be defined so that structuring a decision model using the variables with greatest influence will result in a theoretically sound decision model and lead to a solution of the decision problem. We also want a definition that is consistent with the intuition of both the analyst and the decision maker . Finally practical application demands that the degree of influence be either easily assessable or routinely assessed as part of the decision analysis procedure.

Some possible definitions of influence do not result in a useful structuring methodology for decision analysis, because they do not focus on the authenticity of the profit lottery. For example, mutual information is a concept from information theory that is used to measure the dependence of two random variables. The mutual information for variables x and y denoted by $I_{x,y}$ is given by

$$I_{x,y} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{x,y|S\} \log \frac{\{x,y|S\}}{\{x|S\}\{y|S\}} \, dx \, dy .$$

Constructing a decision model on the basis of a variable's mutual information with other model variables does not result in a satisfactory decision model. [13] The reason is that mutual information indicates the importance of a variable to the system model rather than its importance to the profit lottery. This conclusion is congruous with Tani's claim that the purpose of modeling in decision analysis is the attainment of the decision maker's authentic profit lottery.

2.2 Continuous Random Variables Influencing Either Continuous Random Variables or Discrete Random Variables

Since this section is rather lengthy and includes several theorems, lemmas, and their proofs, Table 2-1 outlines the essential argument presented in this section. Our interest is mainly in the matrix $\frac{d\{y|S\}}{d\{x|S\}}$. However, to explain the meaning of the elements of this matrix and to introduce the notation for derivatives of moments of probability distributions, it is necessary to

Table 2-1

Survey of the Argument Presented in Section 2.2

1. Theorem 2.1 (weak): A particular matrix $M = \emptyset$ if and only if the moments of $\{y|S\}$ are independent of $\{x|S\}$.

2. Lemma 2.2: Suppose $\{y|S\}$ has an analytic characteristic function. Then, if its moments are independent of $\{x|S\}$, the distribution $\{y|S\}$ is independent of $\{x|S\}$.

3. Theorem 2.4 (strong): Suppose $\{y|S\}$ has an analytic characteristic function. Then, $M = \emptyset$ if and only if $\{y|S\}$ is independent of $\{x|S\}$.

4. Lemma 2.5: The matrix $M = \emptyset$ if and only if another matrix $\frac{d\{y|S\}}{d\{x|S\}} = \emptyset$.

5. Theorem 2.6: Suppose $\{y|S\}$ has an analytic characteristic function. Then $\frac{d\{y|S\}}{d\{x|S\}} = \emptyset$ if and only if $\{y|S\}$ is independent of $\{x|S\}$.

6. Corollary 2.7: Suppose $\{y|S\}$ has an analytic characteristic function. Then, the condition $\frac{d\{y|S\}}{d\{x|S\}} \neq \emptyset$ is equivalent to probabilistic dependence and the existence of an influence according to Definition 1.1.

first consider the matrix M , which will presently be defined. We also use the matrix M to explain the necessity of probability densities that have analytic characteristic functions.

The main results of this section are theorem 2.6 and corollary 2.7. They show the close relationship between the influence matrix and our previous use of influence and justify our interpretation of the influence matrix as a description of the influence.

To begin our development of a description of an influence, we consider a limiting case of the influence between two variables: no influence. In this case, the authenticity of the profit lottery or any other influenced state variable is unaffected by the authenticity of the influencing variable. What is the mathematical characterization of the nonexistence of an influence?

Let the influenced variable y , have a probability density function or probability mass function $\{y|\mathcal{S}\}$. Our development in the remainder of the chapter will not depend on whether y is continuous or discrete. If y is the value function, then $\{y|\mathcal{S}\}$ is called the profit lottery, and the profit lottery may be conditioned on a setting of the decision variable. However, the development that follows can easily be modified to include the conditioning by a decision. Let x be the influencing state variable with probability density distribution $\{x|\mathcal{S}\}$. Again, this state variable could be conditioned by a setting of the decision variable. We begin with the expansion equation,

(2.2.1). $$\langle y^m|\mathcal{S}\rangle = \int_x \langle y^m|x,\mathcal{S}\rangle \{x|\mathcal{S}\}dx .$$

$$m = 1,2,\ldots$$

Expanding $\langle y^m | x, \mathcal{S} \rangle$ in a Taylor's series about $\langle x | \mathcal{S} \rangle$ and integrating, we obtain

$$\text{(2.2.2)} \qquad \langle y^m | \mathcal{S} \rangle = \langle y^m | x = \bar{x}, \mathcal{S} \rangle + \sum_{n=2} \frac{1}{n!} \left. \frac{d^n \langle y^m | x, \mathcal{S} \rangle}{dx^n} \right|_{\bar{x}} \langle (x - \bar{x})^n | \mathcal{S} \rangle$$

$$m = 1, 2, \ldots$$

Equation (2.2.2) shows the dependence of the m^{th} moment of the profit lottery on the central moments of the marginal distribution of the state variable $\{x | \mathcal{S}\}$, and on the conditional distribution of y given x, $\{y | x, \mathcal{S}\}$. This equation includes ordinary moments of a distribution,

$$\langle x^n | \mathcal{S} \rangle = \int_{-\infty}^{\infty} x^n \{x | \mathcal{S}\} dx$$

and central moments,

$$\langle (x - \bar{x})^n | \mathcal{S} \rangle = \int_{-\infty}^{\infty} (x - x)^n \{x | \mathcal{S}\} dx \ .$$

These two equations are related, since $(x - x)^n$ may be expanded and then the expectation operator applied, e.g.,

$$\langle (x - \bar{x})^2 | \mathcal{S} \rangle = \langle x^2 - 2x\bar{x} + \bar{x}^2 | \mathcal{S} \rangle$$

$$= \langle x^2 | \mathcal{S} \rangle - \langle x | \mathcal{S} \rangle^2$$

Sometimes we denote expectation with a bar over the variable, as for example

$$\langle x | \mathcal{S} \rangle = \bar{x} \ .$$

The m^{th} moment of y is a function of every moment of the state variable distribution according to equation (2.2.2),

$$\text{(2.2.3)} \qquad \langle y^m | \mathcal{S} \rangle = f(\langle x | \mathcal{S} \rangle, \langle (x - \bar{x})^2 | \mathcal{S} \rangle, \langle (x - x)^3 | \mathcal{S} \rangle, \ldots)$$

The functional form of $f(\bullet)$ is determined by the functional form of the conditional distributions $\langle y^m | x, S \rangle$. If the m^{th} moment of the profit lottery is unaffected by a small perturbation in the n^{th} central moment $(n = 2,3,\ldots)$ of the state variable distribution, then it must be true that

$$\left.\frac{\partial \langle y^m | S \rangle}{\partial \langle (x-\bar{x})^n | S \rangle}\right|_{\bar{x}, \langle (x-\bar{x})^2 | S \rangle, \ldots} \equiv \frac{\partial f}{\partial \langle (x-\bar{x})^n | S \rangle} = 0 \qquad n = 2,3,\ldots$$

Differentiating (2.2.2) gives

$$\text{(2.2.4)} \qquad \left.\frac{\partial \langle y^m | S \rangle}{\partial \langle (x-\bar{x})^n | S \rangle}\right|_{\bar{x}} = \frac{1}{n!} \left.\frac{d^n \langle y^m | x, S \rangle}{dx^n}\right|_{\bar{x}} = 0 \,. \qquad n = 2,3,\ldots,$$

Similarly,

$$\text{(2.2.5)} \quad \left.\frac{\partial \langle y^m | S \rangle}{\partial \langle x | S \rangle}\right|_{\bar{x}} = \left.\frac{d \langle y^m | x, S \rangle}{dx}\right|_{\bar{x}} + \sum_{n=2} \frac{1}{n!} \left.\frac{d^{n+1} \langle y^m | x, S \rangle}{dx^{n+1}}\right|_{\bar{x}} \langle (x-\bar{x})^n | S \rangle$$

$$= 0 \,.$$

Both (2.2.4) and (2.2.5) are functions of the expectation of the marginal distribution of x and both are evaluated at the nominal value $\langle x | S \rangle$. Equation (2.2.4) shows that the rate of change of the m^{th} moment of the profit lottery is independent of the magnitude of the n^{th} central moment of the state variable $\langle (x-\bar{x})^n | S \rangle$, $n = 2,3,\ldots$ However, from (2.2.5) the rate of change of the m^{th} moment with respect to $\langle x | S \rangle$ may depend on the magnitude of every moment of the state variable distribution.

Now, suppose that the marginal distribution of the state variable x is changed in such a way as to perturb its mean and central moments slightly. If the profit lottery is unaffected by the change, then equations (2.2.4) and (2.2.5) must hold for

$$m = 1, 2, \ldots$$

In matrix form this condition is $M = 0$ where

$$(2.2.6) \qquad M = \begin{bmatrix} \left.\dfrac{\partial\langle y|\mathcal{S}\rangle}{\partial\langle x|\mathcal{S}\rangle}\right|_{\bar{x}} & \cdots & \left.\dfrac{\partial\langle y^m|\mathcal{S}\rangle}{\partial\langle x|\mathcal{S}\rangle}\right|_{\bar{x}} & \cdots \\ \vdots & & \vdots & \\ \left.\dfrac{\partial\langle y|\mathcal{S}\rangle}{\partial\langle (x-\bar{x})^n|\mathcal{S}\rangle}\right|_{\bar{x}} & \cdots & \left.\dfrac{\partial\langle y^m|\mathcal{S}\rangle}{\partial\langle (x-\bar{x})^n|\mathcal{S}\rangle}\right|_{\bar{x}} & \cdots \\ \vdots & & \vdots & \end{bmatrix}$$

The authenticity of the profit lottery can only be affected by the authenticity of a state variable distribution when the profit lottery is altered as a result of changes in the state variable distribution. Hence, the development of the last few pages suggests the following:

<u>Theorem 2.1 (Weak Form)</u>. The moments (if they exist) of the marginal distribution $\{y|\mathcal{S}\}$ of a variable y are independent of the marginal density distribution $\{x|\mathcal{S}\}$ of a variable x if and only if $M = 0$, where M is defined by equation (2.2.6).

<u>Proof:</u> First, assume $M = 0$. Then, by (2.2.6),

$$(2.2.7) \qquad \frac{1}{n!} \left.\frac{d^n\langle y^m|x,\mathcal{S}\rangle}{dx^n}\right|_{\bar{x}} = 0 \quad \text{for all } m \text{ and } n \geq 1 .$$

Hence, $\langle y^m|x,\mathcal{S}\rangle$ is not a function of x , $\langle y^m|x,\mathcal{S}\rangle = \langle y^m|\mathcal{S}\rangle$, and the moments of $\{y|\mathcal{S}\}$ are independent of $\{x|\mathcal{S}\}$. Next, assume the moments of $\{y|S\}$ are independent of $\{x|S\}$. Let $\{x|S\}$ be an arbitrary probability density function on x . By equation (2.2.2), we have

$$(2.2.8) \qquad \langle y^m|S\rangle = \langle y^m|x = \bar{x},S\rangle + \sum_{n=2} \frac{1}{n!} \left.\frac{d^n\langle y^m|x,S\rangle}{dx^n}\right|_{\bar{x}} \langle(x-\bar{x})^n|S\rangle$$

Suppose $\{x|S\}'$ is another probability density distribution that differs from $\{x|S\}$ only in its n^{th} central moment. Then since the moments of $\{y|S\}$ are unaffected by $\{x|S\}$,

$$(2.2.9) \qquad \langle y^m|S\rangle = \langle y^m|x = \bar{x}',S\rangle + \sum_{n=2} \frac{1}{n!} \left.\frac{d^n\langle y^m|x,S\rangle}{dx^n}\right|_{\bar{x}} \langle(x-x)^n|S\rangle'$$

Equating (2.2.8) and (2.2.9) and noting $\bar{x} = \bar{x}'$ yields,

$$\frac{1}{n!} \left.\frac{d^n\langle y^m|x,S\rangle}{dx^n}\right|_{\bar{x}} \langle(x-\bar{x})^n|S\rangle = \frac{1}{n!} \left.\frac{d^n\langle y^m|x,S\rangle}{dx}\right|_{\bar{x}} \langle(x-x)^n|S\rangle'$$

which is only true if

$$\frac{1}{n!} \left.\frac{d^n\langle y^m|x,S\rangle}{dx^n}\right|_{\bar{x}} = 0 .$$

If the two distributions $\{x|S\}$ and $\{x|S\}'$ differ only in their means, equating (2.1.8) and (2.1.9) gives

$$(2.2.10) \qquad \langle y^m|x = \bar{x},S\rangle = \langle y^m|x = \bar{x}',S\rangle .$$

Since $\{x|S\}$ and therefore $\bar{x}$ are arbitrary, equation (2.1.10) requires that,

$$\left.\frac{d\langle y^m|x,S\rangle}{dx}\right|_{\bar{x}} = 0$$

Theorem 2.1 as presented above falls short of claiming that distribution $\{y|S\}$ is independent of $\{x|S\}$. That claim requires that independence of all of the moments implies independence of the distribution itself. Under certain conditions on $\{y|S\}$, one can show that this independence implication holds, and these conditions lead to a stronger version of Theorem 2.1.

Lemma 2.2: Suppose $\{y|S\}$ has a characteristic function f(is) that is analytic in a neighborhood of $s = 0$, where s is a complex variable. If

$$\langle y^n|S\rangle = \langle y^n|x,S\rangle , \qquad n = 1,2,\ldots$$

then

$$\{y|S\} = \{y|x,S\}$$

Our proof depends on a theorem proven by Neuts [12] and restated here.

Theorem 2.3 [Neuts]. If the distribution $\{y|S\}$ has a characteristic function $f_y(is)$ that is analytic in a neighborhood of $s = 0$, then

$$f_y(is) = \sum_{k=0}^{\infty} \frac{\langle y^k|x,S\rangle}{k!}(is)^k \tag{2.2.11}$$

Proof of Lemma 2.2:

Using the hypothesis of this lemma with equation (2.2.11) we obtain,

$$f_y(is) = \sum_{k=0}^{\infty} \frac{\langle y^k|x,S\rangle}{k!}(is)^k \tag{2.2.12}$$

The right-hand side of (2.2.12) is the Maclauren's series for the characteristic function of $\{y|x,S\}$, $f_{y|x}(\cdot)$. Hence

$$f_y(is) = f_{y|x}(is)$$

Since the characteristic function has a unique inverse,

$$\{y|\mathcal{S}\} = \{y|x,\mathcal{S}\} .$$

While our lemma is concerned only with independence, Theorem 2.3 addresses the broader issue of when a distribution is determined by its moments. A distribution is not generally determined by its moments, and there are several examples that demonstrate the indeterminancy [11]; these same examples prohibit us from claiming in Theorem 2.1 that the <u>distribution</u> $\{y|\mathcal{S}\}$ is independent when $M = 0$. However, with the aid of Lemma 2.2, we introduce another theorem.

> <u>Theorem 2.4. (strong form)</u>. Suppose $\{y|\mathcal{S}\}$ has a characteristic function $f(is)$ that is analytic in a neighborhood of $s = 0$. The marginal distribution $\{y|\mathcal{S}\}$ on a variable y is independent of the marginal density distribution $\{x|\mathcal{S}\}$ on a variable x if and only if $M = 0$.

<u>Proof:</u>

First assume $M = 0$. By Theorem 2.1 the moments of $\{y|\mathcal{S}\}$ are independent of $\{x|\mathcal{S}\}$. By Lemma 2.2 $\{y|\mathcal{S}\} = \{y|x,\mathcal{S}\}$. Next assume $\{y|\mathcal{S}\}$ is independent of $\{x|\mathcal{S}\}$. Then the moments of $\{y|\mathcal{S}\}$ are independent of $\{x|\mathcal{S}\}$. According to Theorem 2.1, $M = 0$.

We defer discussion of the importance of the matrix M and its connection with influence since there is a more useful form of Theorem 2.4. By writing

$$(2.2.13)\quad \langle (y-\bar{y})^m|S\rangle = \langle y^m|S\rangle + \ldots + (-1)^r\binom{m}{(m-r)}\langle y^{m-r}|S\rangle\langle y|S\rangle^r + \ldots + \langle y|S\rangle^m ,$$

$m = 2,3,\ldots$
$o < r < m$

and differentiating we obtain,

$$\left.\frac{\partial\langle (y-\bar{y})^m|S\rangle}{\partial\langle (x-\bar{x})^n|S\rangle}\right|_{\bar{x}} = \left.\frac{\partial\langle y^m|S\rangle}{\partial\langle (x-\bar{x})^n|S\rangle}\right|_{\bar{x}} + \ldots$$

$$(2.2.14)\quad + (-1)^r\binom{m}{(m-r)}\left\{\langle y|S\rangle^r \left.\frac{\partial\langle y^{m-r}|S\rangle}{\partial\langle (x-\bar{x})^r|S\rangle}\right|_{\bar{x}} + r\langle y^{m-r}|S\rangle \langle y|S\rangle^{r-1} \left.\frac{\partial\langle y|S\rangle}{\partial\langle (x-\bar{x})^n|S\rangle}\right|_{\bar{x}}\right\} + \ldots + m\langle y|S\rangle^{m-1}\left.\frac{\partial\langle y|S\rangle}{\partial\langle (x-\bar{x})^n|S\rangle}\right|_{\bar{x}}$$

$m = 2,3,\ldots$
$n = 2,3,\ldots$
$o < r < m$

Each of these terms can be evaluated with the help of (2.2.4) and (2.2.5).

Now consider a matrix denoted by $\frac{d\{y|S\}}{d\{x|S\}}$ and defined by

$$(2.2.15)\quad \frac{d\{y|S\}}{d\{x|S\}} = \begin{bmatrix} \left.\frac{\partial\langle y|S\rangle}{\partial\langle x|S\rangle}\right|_{\bar{x}} & \ldots & \left.\frac{\partial\langle (y-y)^m|S\rangle}{\partial\langle x|S\rangle}\right|_{\bar{x}} & \ldots \\ \vdots & & \vdots & \\ \left.\frac{\partial\langle y|S\rangle}{\partial\langle (x-\bar{x})^n|S\rangle}\right|_{\bar{x}} & \ldots & \left.\frac{\partial\langle (y-\bar{y})^m|S\rangle}{\partial\langle (x-\bar{x})^n|S\rangle}\right|_{\bar{x}} & \ldots \\ \vdots & & \vdots & \end{bmatrix}$$

Lemma 2.5. The matrix $M = 0$ if and only if the matrix $\frac{d\{y|S\}}{d\{x|S\}} = 0$, where M is defined by equation (2.2.6) and $\frac{d\{y|S\}}{d\{x|S\}}$ by (2.2.15).

Proof: For convenience of notation let $N = \frac{d\{y|S\}}{d\{x|S\}}$.

Suppose $M = \emptyset$, then we must show that $N_{nm} = 0$ for all n,m. By equation (2.2.14)

$$\text{(2.2.16)} \quad \frac{d\langle(y-\bar{y})^m|S\rangle}{d\langle\cdot|S\rangle} = \frac{d\langle y^m|S\rangle}{d\langle\cdot|S\rangle} + \ldots + (-1)^r \binom{m}{r}\left\{\bar{y}^r \frac{d\langle y^{m-r}|S\rangle}{d\langle\cdot|S\rangle} + \overline{y^{m-r}}\, r\, \bar{y}^{r-1} \frac{d\langle y|S\rangle}{d\langle\cdot|S\rangle}\right\} + \ldots + m\bar{y}^{m-1} \frac{d\langle y|S\rangle}{d\langle\cdot|S\rangle}$$

where $d\langle\cdot|S\rangle$ represents $d\langle(x-\bar{x})^n|S\rangle$ for $n = 2,3,\ldots$

Since the m^{th} column of N, $\frac{d\langle(y-\bar{y})^m|S\rangle}{d\langle\cdot|S\rangle}$ is a linear combination of the columns of M according to (2.2.16), $M = \emptyset$ implies $N = \emptyset$.

Now, suppose $N = \emptyset$. We show by induction that $M = \emptyset$. Comparing equations (2.2.6) and (2.2.15),

$$M_{n,1} = N_{n,1} \qquad \text{all } n .$$

Let

$$M_{n,j} = N_{n,j} \qquad j < m .$$

By hypothesis,

$$\text{(2.2.17)} \qquad M_{n,j} = 0 \qquad j < m .$$

Since $N_{n,m} = 0$ for all n, equation (2.2.16) becomes

$$0 = \frac{d\langle y^m|\mathcal{S}\rangle}{d\langle\cdot|\mathcal{S}\rangle} + \ldots + (-1)^r \binom{m}{r} \left\{ \bar{y}^r \frac{d\langle y^{m-r}|\mathcal{S}\rangle}{d\langle\cdot|\mathcal{S}\rangle} + \overline{y^{m-r}}\, r\bar{y}^{r-1} \frac{d\langle y|\mathcal{S}\rangle}{d\langle\cdot|\mathcal{S}\rangle} \right\} + \ldots + m\bar{y}^{m-1} \frac{d\langle y|\mathcal{S}\rangle}{d\langle\cdot|\mathcal{S}\rangle} \tag{2.2.18}$$

However, by (2.2.17)

$$\frac{d\langle y^j|\mathcal{S}\rangle}{d\langle\cdot|\mathcal{S}\rangle} = 0 \qquad j < m ,$$

and (2.2.18) reduces to

$$0 = \frac{d\langle y^m|\mathcal{S}\rangle}{d\langle\cdot|\mathcal{S}\rangle} \quad .$$

Consequently

$$M_{n,j} = 0 \qquad j \leq m .$$

By induction, $N = 0$ implies $M = 0$.

Using Lemma 2.5 with Theorem 2.4, we immediately obtain:

> <u>Theorem 2.6.</u> Suppose $\{y|\mathcal{S}\}$ has a characteristic function $f(is)$ that is analytic in a neighborhood of $s = 0$. The marginal distribution $\{y|\mathcal{S}\}$ of a variable y is independent of the marginal density distribution $\{x|\mathcal{S}\}$ of a variable x , if and only if $\frac{d\{y|\mathcal{S}\}}{d\{x|\mathcal{S}\}} = 0$, where $\frac{d\{y|\mathcal{S}\}}{d\{x|\mathcal{S}\}}$ is defined by (2.2.15).

When an influence exists between two variables, a matrix of the form of $\frac{d\{y|\mathcal{S}\}}{d\{x|\mathcal{S}\}}$ characterizes the influence. It is a particular representation of how the authenticity of one distribution depends on the authenticity of another distribution.

The n,m element $(n > 1, m > 1)$ is the effect on the m^{th} central moment of the influenced distribution of a unit change in the n^{th} central moment of the influencing distribution (Equation 2.2.14).

Since the matrix is null when no influence exists and indicates the type of dependency when an influence does exist, we call that matrix the "influence matrix" or simply the "influence" of $\{x|S\}$ on $\{y|S\}$. We denote the influence matrix describing the influence by $\frac{d\{y|S\}}{d\{x|S\}}$.

Many distributions of practical interest, such as the uniform, exponential and normal, do have analytic characteristic functions. However, it is not really necessary that the marginal distribution $\{y|S\}$ have an analytic characteristic function. By using Lemma 2.5 with Theorem 2.1, a weak version of 2.6 can be obtained that gives $\frac{d\{y|S\}}{d\{x|S\}} = 0$ if and only if the moments of $\{y|S\}$ are independent of $\{x|S\}$. In all the work that follows we are only concerned with the moments of the influenced distribution. Furthermore, we use the influence matrix to describe influences when they are known to exist, not to discern their existence.

One implication of the matrix definition of influence is that an influence diagram should be drawn as in 2-1a rather than 2-1b. An influence is the impact of one marginal distribution on another. Looking back to definition 1.1, it is clear that the existence of an influence means that the distribution is influenced rather than the variable. However, definition 1.1 is ambiguous about what the influencing factor is.

a.

b.

FIGURE 2-1. A Comparison Of Our Representation Of Influence With The Conventional Representation

While the difference between 2-1a and 2-1b is not essential, it distinguishes influence diagrams from other structuring devices. By emphasizing that an influence relates two probability distributions, the users of the diagrams are reminded that the relationship may be probabilistic as well as deterministic and noncausal as well as causal.

Influence Matrix Notation is Conducive to An Influence Calculus

The notation for the influence matrix $\frac{d\{y|\mathcal{S}\}}{d\{x|\mathcal{S}\}}$ was selected because it leads to a framework for conceptualizing and analyzing the influences represented by a complex influence diagram. In particular, it supports a calculus of influences.

Suppose three variables x, y, and z are related as shown by the influence diagram below:

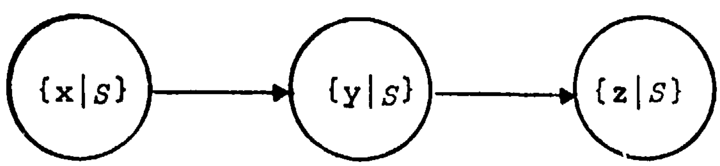


FIGURE 2-2. An Influence Diagram Suggesting An Influence Calculus

Given the influence of $\{x|\mathcal{S}\}$ on $\{y|\mathcal{S}\}$, as described by the matrix $\frac{d\{y|\mathcal{S}\}}{d\{x|\mathcal{S}\}}$, and the influence of $\{y|\mathcal{S}\}$ on $\{z|\mathcal{S}\}$, described by $\frac{d\{z|\mathcal{S}\}}{d\{y|\mathcal{S}\}}$, how can the influence of $\{x|\mathcal{S}\}$ on $\{z|\mathcal{S}\}$ be computed? Recalling Equation (2.2.3)

$$(2.2.3) \qquad \langle y^m|S\rangle \equiv f(\langle x|S\rangle\ ,\ \langle(x-\bar{x})^2|S\rangle\ ,\ \ldots,\ )$$

every moment of $\{y|S\}$ depends on every central moment of $\{x|S\}$ and its expectation. By equation (2.2.13), every central moment of $\{y|S\}$ must also depend on every central moment and the expectation of $\{x|S\}$,

$$(2.2.19) \qquad \langle(y-\bar{y})^m|S\rangle = f_m(\langle x|S\rangle\ ,\langle(x-\bar{x})^2|S\rangle,\ \ldots)$$

Similarly,

$$(2.2.20) \qquad \langle z|S\rangle = g_1(\langle y|S\rangle\ ,\langle(y-\bar{y})^2|S\rangle,\ \ldots)$$

$$\langle(z-\bar{z})^r|S\rangle = g_r(\langle y|S\rangle\ ,\ \langle y-\bar{y})^2|S\rangle\ \ldots)$$

By the chain rule of differential calculus,

$$(2.2.21) \quad \frac{\partial\langle(z-\bar{z})^r|S\rangle}{\partial\langle(x-\bar{x})^n|S\rangle} = \frac{\partial\langle(z-z)^r|S\rangle}{\partial\langle y|S\rangle}\ \frac{\partial\langle y|S\rangle}{\partial\langle(x-\bar{x})^n|S\rangle} + \frac{\partial\langle(z-\bar{z})^r|S\rangle}{\partial\langle(y-\bar{y})^2|S\rangle}\ \frac{\partial\langle(y-\bar{y})^2|S\rangle}{\partial\langle(x-x)^n|S\rangle}$$

$$+ \ldots$$

Since (2.2.21) is the matrix product of the r^{th} column of $\frac{d\{z|S\}}{d\{y|S\}}$, and the n^{th} row of $\frac{d\{y|S\}}{d\{x|S\}}$, we can write

$$(2.2.22) \qquad \frac{d\{z|S\}}{d\{x|S\}} = \frac{d\{y|S\}}{d\{x|S\}}\ \frac{d\{z|S\}}{d\{y|S\}}$$

which is analogous to the chain rule of differential calculus. We pursue the calculus of influences and its form for complex influence diagrams more thoroughly in a later section.

2.3 Discrete Random Variables Influencing Either Continuous or Discrete Random Variables

There are many discrete events, such as whether or not it rains tomorrow and the number of senators voting for a particular treaty. Sometimes the probability mass function for these discrete events influences a probability density function of a continuous random variable or a probability mass function. To extend the influence matrix concept to include this situation, let $\{x|S\}$ be the influencing probability mass function, and let $\{y|S\}$ be either a probability mass function or a probability density function. The quantity x takes on the values x_k for $1 \leq k \leq K$ with probability $P_x(x_k)$. Our approach is to fit a continuous curve through the points $\langle y^m|x_k,S\rangle$ at each point x_k. Then, these curves can be used to derive equations for $\langle y^m|S\rangle$ in the same way as in the previous section. Let the function $f^m(x)$ satisfy the following conditions:

(2.3.1) $$f^m(x_k) = \langle y^m|x_k,S\rangle \quad k = 1,2,\ldots,K$$

The function $f^m(x)$ is not unique, and such a function always exists. For example, polynomials of degree greater than $K-1$ can be found which satisfy (2.3.1). Writing the Taylor's series for $f^m(x)$ about $x = \bar{x}$ gives

(2.3.2) $$f^m(x) = f^m(\bar{x}) + \sum_{n=1} \frac{1}{n!} f^m(\bar{x})(x-\bar{x})^n .$$

where $$f_n^m = \frac{d^n f^m(x)}{dx^n} .$$

This expression leads to,

$$\langle y^m|S\rangle = \sum_K \langle y^m|x_k,S\rangle P_x(x_k)$$

(2.3.3) $$= \sum_k [f^m(\bar{x}) + \sum_{n=1} \frac{1}{n!} f_n^m(\bar{x})(x_k-\bar{x})^n] P_x(x_k)$$

$$= f^m(\bar{x}) + \sum_{n=2} \frac{1}{n!} f_n^m(\bar{x}) \langle (x-\bar{x})^n|S\rangle$$

Equation (2.3.3) allows us to define the elements of the influence matrix for a discrete influencing variable with equations similar to those we use when the influencing variable is continuous. In particular, we can write

$$\text{(2.3.4)} \qquad \frac{\partial\langle y^m|S\rangle}{\partial\langle (x-\bar{x})^n|S\rangle} = \frac{1}{n!}\, f_n^m(\bar{x})$$

$$\text{(2.3.5)} \qquad \frac{\partial\langle y^m|S\rangle}{\partial\langle x\,|S\rangle} = f_1^m(x) + \sum_{n=2} \frac{1}{n!}\, f_{n+1}^m(x)\ \langle (x-\bar{x})^n|S\rangle$$

and

$$\left.\frac{\partial\langle (y-\bar{y})^m|S\rangle}{\partial\langle (x-\bar{x})^n|S\rangle}\right|_{\bar{x}} = \left.\frac{\partial\langle y^m|S\rangle}{\partial\langle (x-\bar{x})^n|S\rangle}\right|_{\bar{x}} + \ldots$$

$$\text{(2.3.6)} \qquad + (-1)^r \binom{m}{(m-r)} \left\{ \langle y|S\rangle^r \left.\frac{\partial\langle y^{m-r}|S\rangle}{\partial\langle (x-\bar{x})^r|S\rangle}\right|_{\bar{x}} + r\langle y^{m-r}|S\rangle \right.$$

$$\left. \langle y|S\rangle^{r-1} \left.\frac{\partial\langle y|S\rangle}{\partial\langle (x-\bar{x})^n|S\rangle}\right|_{\bar{x}} \right\} + \ldots + m\langle y|S\rangle^{m-1} \left.\frac{\partial\langle y|S\rangle}{\partial\langle (x-\bar{x})^n|S\rangle}\right|_{\bar{x}}$$

$$m = 2,3,\ldots$$

$$n = 2,3,\ldots$$

Equations (2.3.4) and (2.3.5) differ from (2.2.4) and (2.2.5) only in the replacement of the conditional moment $\langle y^m|x,S\rangle$ by the function $f^m(x)$. Equation (2.3.6) is identical to (2.2.14).

The similarity of these sets of equations means that the theorems and corollaries presented earlier for the case of a continuous influencing variable are applicable to the case of the discrete influencing variable. No essential changes are required in the proofs. Hence, the influence matrix $\frac{d\{y|S\}}{d\{x|S\}}$, with its elements appropriately defined, can describe the influence of one random variable on another, whether the variables are continuous or discrete.

2.4 Approximation to Obtain Finite Influence Matrices

One difficulty with the influence matrices presented so far is that they require specifying an infinite number of elements. To obtain the information necessary to compute the elements, the decision analyst would need to assess the marginal distribution on the influencing state variable and the entire conditional surface. Table 2-2 shows that all of the moments of the state variable distribution are required. Obtaining all of the derivatives of the conditional surface practically necessitates assessing the entire conditional surface. The moments or probability densities of the influenced variable are also required, but in Chapter 4 we show how to obtain these moments from other information.

Since our intention is to use the influence matrix to structure the decision problem, an approximation to the influence matrix may be sufficient and desirable. The analysis of the influence matrices is likely to show that some state variables are unimportant to the problem. If a variable is determined to be unimportant, then most or all of the information obtained about the unimportant variable will be irrelevant to the remainder of the decision analysis. Hence, it is important to determine the structure of the decision problem with as little information as possible.

The numbers of nonzero elements of the influence matrix can be reduced to four by making the following approximations:

Table 2-2

Information required to specify the Complete Influence Matrix.

- All derivatives of every conditional moment,

$$\left.\frac{d^n \langle y^m | x, S\rangle}{dx^n}\right|_{\bar{x}} \qquad \begin{array}{l} m = 1,2,\ldots \\ n = 1,2,\ldots \end{array}$$

- All moments of the state variable distribution.

$$\langle x|S\rangle \ , \ \langle(x-x)^n|S\rangle \qquad n = 2,3,\ldots$$

1). **The profit lottery can be adequately described by its mean and variance.**

2). **Every conditional expectation and second moment is quadratic in the conditioning variable.**

The severity of the first approximation depends on the intended use of the profit lottery. With only the mean and variance, one can adequately represent the mean and the dispersion about the mean. In many instances the certain equivalent can be determined to within a few percent using the approximation, [14]

(2.4.1) $$\tilde{}\langle y|S\rangle = \langle y|S\rangle - \frac{1}{2}\,\gamma\,{}^{v}\langle y|S\rangle \quad .$$

This equation states that the amount the decision maker would accept for certain, rather than face the uncertainty of the profit lottery, is approximately equal to the expectation of the profit lottery less an amount proportional to the variance of the lottery. The constant of proportionality γ is a measure of the decision maker's attitude toward risk, and it is called his risk attitude. Questions about the cumulative probability, which are sometimes important, however, can only be crudely answered using these approximations.

By the first approximation, the influence matrix of Figure 2-3a simplifies to a matrix with only two nonzero columns (Figure 2-3b). Approximation 2 is not an approximation of a probability distribution by a quadratic function. Rather, it is the approximation of two functions of the conditioning variable by two different quadratic functions,

a.

$$\frac{d(y|S)}{d(x|S)} = \begin{bmatrix} \left.\frac{\partial\langle y|S\rangle}{\partial\langle x|S\rangle}\right|_{\bar{x}} & \left.\frac{\partial\langle (y-\bar{y})^2|S\rangle}{\partial\langle x|S\rangle}\right|_{\bar{x}} & \left.\frac{\partial\langle (y-\bar{y})^3|S\rangle}{\partial\langle x|S\rangle}\right|_{\bar{x}} & \cdots \\ \left.\frac{\partial\langle y|S\rangle}{\partial\langle (x-\bar{x})^2|S\rangle}\right|_{\bar{x}} & \left.\frac{\partial\langle (y-\bar{y})^2|S\rangle}{\partial\langle (x-\bar{x})^2|S\rangle}\right|_{\bar{x}} & \left.\frac{\partial\langle (y-\bar{y})^3|S\rangle}{\partial\langle (x-\bar{x})^2|S\rangle}\right|_{\bar{x}} & \cdots \\ \left.\frac{\partial\langle y|S\rangle}{\partial\langle (x-\bar{x})^3|S\rangle}\right|_{\bar{x}} & \left.\frac{\partial\langle (y-\bar{y})^2|S\rangle}{\partial\langle (x-\bar{x})^3|S\rangle}\right|_{\bar{x}} & \left.\frac{\partial\langle (y-\bar{y})^3|S\rangle}{\partial\langle (x-\bar{x})^3|S\rangle}\right|_{\bar{x}} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

b.

$$\frac{d(y|S)}{d(x|S)} = \begin{bmatrix} \left.\frac{\partial\langle y|S\rangle}{\partial\langle x|S\rangle}\right|_{\bar{x}} & \left.\frac{\partial\langle (y-\bar{y})^2|S\rangle}{\partial\langle x|S\rangle}\right|_{\bar{x}} & 0 & \cdots \\ \left.\frac{\partial\langle y|S\rangle}{\partial\langle (x-\bar{x})^2|S\rangle}\right|_{\bar{x}} & \left.\frac{\partial\langle (y-\bar{y})^2|S\rangle}{\partial\langle (x-\bar{x})^2|S\rangle}\right|_{\bar{x}} & 0 & \cdots \\ \left.\frac{\partial\langle y|S\rangle}{\partial\langle (x-\bar{x})^3|S\rangle}\right|_{\bar{x}} & \left.\frac{\partial\langle (y-\bar{y})^2|S\rangle}{\partial\langle (x-\bar{x})^3|S\rangle}\right|_{\bar{x}} & 0 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

c.

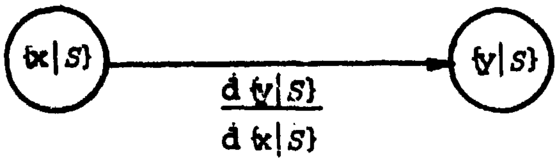
$$\frac{d(y|S)}{d(x|S)} = \begin{bmatrix} \left.\frac{\partial\langle y|S\rangle}{\partial\langle x|S\rangle}\right|_{\bar{x}} & \left.\frac{\partial\langle (y-\bar{y})^2|S\rangle}{\partial\langle x|S\rangle}\right|_{\bar{x}} & \left.\frac{\partial\langle (y-\bar{y})^3|S\rangle}{\partial\langle x|S\rangle}\right|_{\bar{x}} & \cdots \\ \left.\frac{\partial\langle y|S\rangle}{\partial\langle (x-\bar{x})^2|S\rangle}\right|_{\bar{x}} & \left.\frac{\partial\langle (y-\bar{y})^2|S\rangle}{\partial\langle (x-\bar{x})^2|S\rangle}\right|_{\bar{x}} & \left.\frac{\partial\langle (y-\bar{y})^3|S\rangle}{\partial\langle (x-\bar{x})^2|S\rangle}\right|_{\bar{x}} & \cdots \\ 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

FIGURE 2-3. The Effects Of Approximations On The Influence Matrix

$$(2.4.2) \qquad \langle y^m | x, S \rangle = k_0^m + k_1^m(x-\bar{x}) + k_2^m(x-\bar{x})^2 \quad m = 1,2$$

Using only this approximation, the influence matrix of 2-3a simplifies to the matrix with zero elements as shown in Figure 2-3c. Applying both approximations gives the four element matrix of Figure 2-4.

Approximating the influence matrix with a 2×2 matrix considerably reduces the informational requirement. The approximate matrix only requires four derivatives that describe the conditional surface, and the mean of the state variable distribution.



$$\frac{d\{y|\mathcal{S}\}}{d\{x|\mathcal{S}\}} = \begin{bmatrix} \left.\dfrac{d<y|\mathcal{S}>}{d<x|\mathcal{S}>}\right|_{\bar{x}} & \left.\dfrac{d^V<y|\mathcal{S}>}{d<x|\mathcal{S}>}\right|_{\bar{x}} \\ \left.\dfrac{d<y|\mathcal{S}>}{d^V<x|\mathcal{S}>}\right|_{\bar{x}} & \left.\dfrac{d^V<y|\mathcal{S}>}{d^V<x|\mathcal{S}>}\right|_{\bar{x}} \end{bmatrix}$$

$$= \begin{bmatrix} h_x & h_x^* - 2<y|\mathcal{S}>h_x \\ .5h_{xx} & .5h_{xx}^* - <y|\mathcal{S}>h_{xx} \end{bmatrix}$$

where

$$h_x = \left.\frac{d<y|x,\mathcal{S}>}{dx}\right|_{\bar{x}} \qquad h_{xx} = \left.\frac{d^2<y|x,\mathcal{S}>}{dx^2}\right|_{\bar{x}}$$

$$h_x^* = \left.\frac{d<y^2|x,\mathcal{S}>}{dx}\right|_{\bar{x}} \qquad h_{xx}^* = \left.\frac{d^2<y^2|x,\mathcal{S}>}{dx^2}\right|_{\bar{x}}$$

FIGURE 2-4. The Elements Of The Approximate Influence Matrix For The Influence Of $\{x|\mathcal{S}\}$ On $\{y|\mathcal{S}\}$

CHAPTER 3

Developing Decision Models from Influence Matrices

3.1 Introduction

The influence matrix, as developed in Chapter 2, provides a new way to conceptualize the influence of a variable's probability distribution on the profit lottery. In this chapter, we show how this conceptualization can be used in the development of a decision model. Though judgment still remains an essential part of the decision model, intuition about the importance of influences in the model is not required.

3.2 Influence Calculus

Chapter 2 demonstrates that an influence calculus exists for general influence matrices. The simple example of $\{x|\mathcal{S}\}$ influencing $\{y|\mathcal{S}\}$ influencing $\{z|\mathcal{S}\}$ suggests that the influence calculus could be used to reduce complicated influence diagrams to simple diagrams involving only a few variables. This simplification would allow one to describe the total influence of any variable in the diagram on the profit lottery.

These possibilities motivate the more thorough development of the influence calculus presented in this section. We begin by demonstrating a general procedure for computing the total influence of a variable on any other variable appearing in the influence diagram. The total influence is the sum of its direct influence on the variable and its indirect influences, through intermediate variables. Next, we derive the mathematical forms for the elements of the influence matrices for several basic influence diagrams.

In the computation of the elements of the influence matrices, we sacrifice the generality of the infinite, exact influence matrix for the reduced complexity of the 2×2 approximate matrix, though a more general and more complicated development is possible. The influence calculus is derived for influencing variables that are continuous. However, it is also applicable for discrete random variables and mixtures of discrete and continuous random variables.

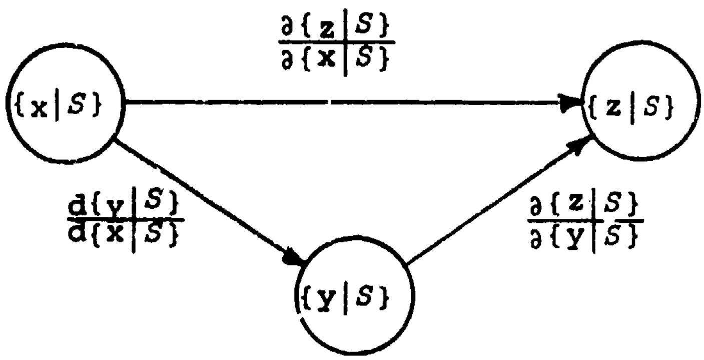
<u>An Influence Evaluation Procedure</u>

If each of the arrows of an influence diagram is labeled with the proper influence matrix, then the equation for the total influence of one variable on another can be easily determined directly from the diagram. The rule for labeling influences is very simple. An influence is labeled as a partial influence, for example $\frac{\partial\{z|S\}}{\partial\{x|S\}}$ if the influenced state variable is influenced by more than one state variable. If it is influenced by only one other state variable, then it is labeled as a total influence, for example $\frac{d\{y|S\}}{d\{x|S\}}$. Table 3-1 presents and illustrates a procedure for determining the total influence equation that uses the influence diagram and is applicable to complicated diagrams. The results of the procedure for several complex diagrams are shown in Figure 3-1.

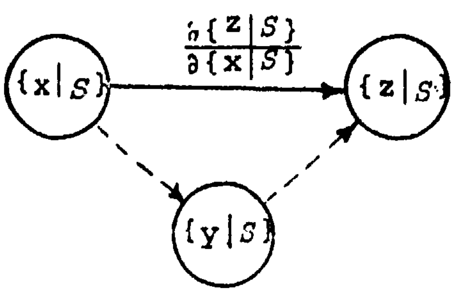
TABLE 3-1

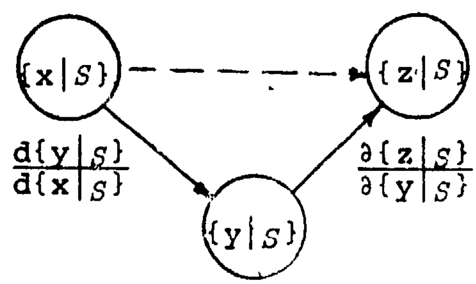
Procedure for Determining the Equation for the Total Influence of One Variable Upon Another

1. Label all arrows with the proper influence matrix. An arrow is labeled as a partial influence if the influenced state variable is influenced by more than one state variable.



2. Determine the direct and indirect influence paths between the two nodes





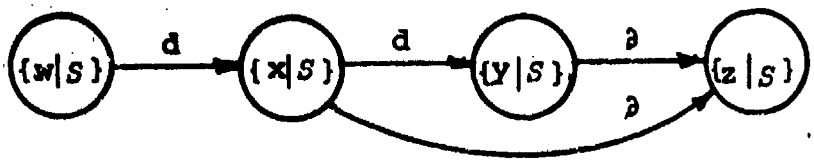
3. Compute the path influences by taking the product (in the direction of the arrows) of all the influences on the path

$$\frac{\partial\{z|S\}}{\partial\{x|S\}} \qquad\qquad \frac{d\{y|S\}}{d\{x|S\}}\,\frac{\partial\{z|S\}}{\partial\{y|S\}}$$

4. The total influence is the sum of the path influences

$$\frac{d\{z|S\}}{d\{x|S\}} = \frac{\partial\{z|S\}}{\partial\{x|S\}} + \frac{d\{y|S\}}{d\{x|S\}}\,\frac{\partial\{z|S\}}{\partial\{y|S\}}$$

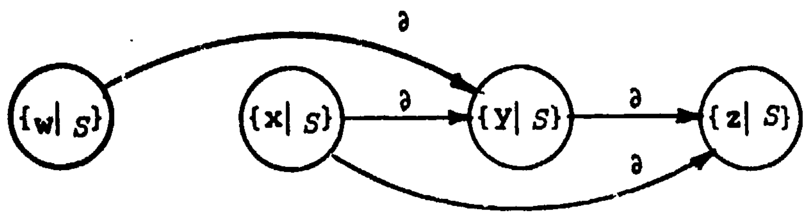
a.



$$\frac{d\{z|S\}}{d\{w|S\}} = \frac{d\{x|S\}}{d\{w|S\}} \cdot \frac{\partial\{z|S\}}{\partial\{x|S\}} + \frac{d\{x|S\}}{d\{w|S\}} \frac{d\{y|S\}}{d\{x|S\}} \frac{\partial\{z|S\}}{\partial\{y|S\}}$$

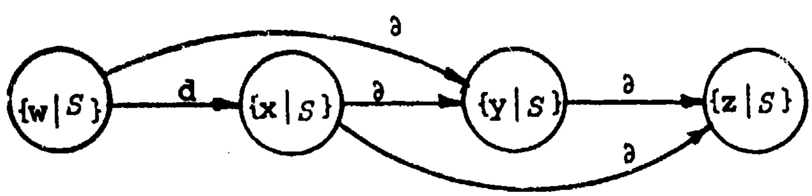
$$= \frac{d\{x|S\}}{d\{w|S\}} \frac{d\{z|S\}}{d\{x|S\}}$$

b.



$$\frac{d\{z|S\}}{d\{w|S\}} = \frac{\partial\{y|S\}}{\partial\{w|S\}} \frac{\partial\{z|S\}}{\partial\{y|S\}}$$

c.



$$\frac{d\{z|S\}}{d\{w|S\}} = \frac{d\{x|S\}}{d\{w|S\}} \frac{\partial\{z|S\}}{\partial\{x|S\}} + \frac{d\{x|S\}}{d\{w|S\}} \frac{\partial\{y|S\}}{\partial\{x|S\}} \frac{\partial\{z|S\}}{\partial\{y|S\}}$$

$$+ \frac{\partial\{y|S\}}{\partial\{w|S\}} \frac{\partial\{z|S\}}{\partial\{y|S\}}$$

$$= \frac{d\{x|S\}}{d\{w|S\}} \frac{d\{z|S\}}{d\{x|S\}} + \frac{\partial\{y|S\}}{\partial\{w|S\}} \frac{\partial\{z|S\}}{\partial\{y|S\}}$$

FIGURE 3-1. Examples Of The Influence Calculus Using The Procedure Of Table 3-1

The only difficulty in evaluating the influence of one variable on another is that all of the elements of every intermediate influence matrix must be assessed.

Expressions for those elements for several common influence configurations are derived next.

A Direct and An Indirect Influence

Consider the problem of determining the total effect of a change in the marginal distribution of a state variable on the distribution of another state variable. Suppose that the influencing state variable has both a direct influence and an indirect influence, through another state variable. An example of this problem is calculating the total influence of $\{x|S\}$ on $\{z|S\}$ in Figure 3-2.

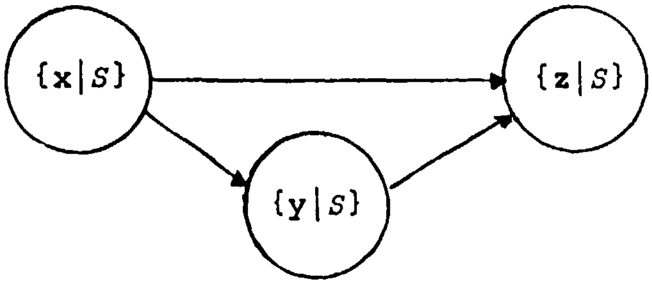


FIGURE 3-2. An Influence Diagram with a Direct and An Indirect Influence

To compute the total influence, we begin by expanding $\langle z|x,y,S\rangle$ in a Taylor's series, multiplying by $\{x,y|S\}$, and integrating. The result is

$$(3.2.1)\quad \langle z|S\rangle = \langle z|\bar{x},\bar{y},S\rangle + \frac{1}{2} f_{xx}\, {}^{v}\langle x|S\rangle + \frac{1}{2} f_{yy}\, {}^{v}\langle y|S\rangle + f_{xy}\mathrm{Cov}\,(x,y)\ ,$$

where

$$f_x = \left.\frac{\partial\langle z|x,y,S\rangle}{\partial x}\right|_{\bar{x},\bar{y}}\ ,\quad f_{xx} = \left.\frac{\partial^2\langle z|x,y,S\rangle}{\partial x^2}\right|_{\bar{x},\bar{y}}\ ,\ \text{etc.}$$

Taking the partial derivative with respect to $\langle x|S\rangle$ gives

$$\left.\frac{\partial\langle z|S\rangle}{\partial\langle x|S\rangle}\right|_{\bar{x},\bar{y}} = f_x + f_{xy}\left.\frac{\partial\,\mathrm{Cov}(x,y)}{\partial\langle x|S\rangle}\right|_{\bar{x},\bar{y}} . \tag{3.2.2}$$

Notice that f_{xx} , f_{yy} , and f_{xy} are constants by the assumptions made in Chapter 2. Equation (3.2.2) is the direct influence of $\langle x|S\rangle$ on $\langle z|S\rangle$. To evaluate it, the change in the covariance due to a change in the mean of $\{x|S\}$ must be determined.

First, we can express $\mathrm{Cov}(x,y)$ in terms of the moments of $\{x|S\}$, according to Appendix A,

$$\mathrm{Cov}(x,y) = \sum_{n=1}^{\infty}\frac{1}{n!}\left.\frac{d^n\langle y|x,S\rangle}{dx^n}\right|_{\bar{x}}\langle(x-\bar{x})^{n+1}|S\rangle . \tag{3.2.3}$$

A similar expression could be derived for terms of the form $\langle(x-\bar{x})^n(y-\bar{y})^m|S\rangle$, which would arise if terms of greater than order two were included in equation (3.2.1). Hence, a more general development of this type is possible.

If the conditional surfaces are approximated with a quadratic function in the conditioning variable (approximation 2 of Chapter 2), then

$$\mathrm{Cov}(x,y) = h_x\,{}^{V}\!\langle x|S\rangle + \tfrac{1}{2}h_{xx}\langle(x-\bar{x})^3|S\rangle \tag{3.2.4}$$

where

$$h_x = \left.\frac{\partial\langle y|x,S\rangle}{\partial x}\right|_{\bar{x}} \quad\text{and}\quad h_{xx} = \left.\frac{\partial^2\langle y|x,S\rangle}{\partial x^2}\right|_{\bar{x}}$$

Equation (3.2.4) expresses $\mathrm{Cov}(x,y)$ in terms of $\langle x|S\rangle$, and ${}^{V}\!\langle x|S\rangle$. However, $\mathrm{Cov}(x,y)$ can be written as a function of

$\langle x|S\rangle$ and $\langle y|S\rangle$ using

$$\langle y|S\rangle = \langle y|\bar{x},S\rangle + \frac{1}{2} h_{xx} \, {}^{V}\langle x|S\rangle \; . \tag{3.2.5}$$

Solving (3.2.5) for ${}^{V}\langle x|S\rangle$ and substituting into (3.2.4) produces,

$$\mathrm{Cov}(x,y) = \frac{2h_x}{h_{xx}} (\langle y|S\rangle - \langle y|\bar{x},S\rangle) + \frac{1}{2} h_{xx} \langle (x-\bar{x})^3|S\rangle \tag{3.2.6}$$

In this equation $\langle x|S\rangle$ and $\langle y|S\rangle$ are independent variables.

As a result, the required derivatives of $\mathrm{Cov}(x,y)$ can be evaluated,

$$\begin{aligned} \frac{\partial \mathrm{Cov}(x,y)}{\partial \langle x|S\rangle} &= 2(\langle y|S\rangle - \langle y|\bar{x},S\rangle) - 2\frac{h_x^2}{h_{xx}} \\ &= h_{xx} \, {}^{V}\langle x|S\rangle - \frac{2h_x^2}{h_{xx}} \end{aligned} \tag{3.2.7}$$

and

$$\frac{\partial \mathrm{Cov}(x,y)}{\partial \langle y|S\rangle} = 2\frac{h_x}{h_{xx}} \; . \tag{3.2.8}$$

Again, h_{xx} is constant by our approximations in Chapter 2. Substituting (3.2.7) into (3.2.2) yields,

$$\left.\frac{\partial \langle z|S\rangle}{\partial \langle x|S\rangle}\right|_{\bar{x}} = f_x + f_{xy}\left(h_{xx} \, {}^{V}\langle x|S\rangle - \frac{2h_x^2}{h_{xx}}\right) \tag{3.2.9}$$

A similar expression for the partial derivative with respect to $\langle y|S\rangle$ can be obtained with the help of (3.2.8),

$$\left.\frac{\partial \langle z|S\rangle}{\partial \langle y|S\rangle}\right|_{\bar{x}} = f_y + 2f_{xy} \frac{h_x}{h_{xx}} \; . \tag{3.2.10}$$

This equation gives the direct influence of $\langle y|S\rangle$ on $\langle z|S\rangle$.

The total influence of $\langle x|S\rangle$ on $\langle z|S\rangle$ can be obtained by using (3.2.9) and (3.2.10) in

$$\text{(3.2.11)}\quad \left.\frac{d\langle z|S\rangle}{d\langle x|S\rangle}\right|_{\bar{x},\bar{y}} = \left.\frac{\partial\langle z|S\rangle}{\partial\langle x|S\rangle}\right|_{\bar{x},\bar{y}} + \left.\frac{d\langle y|S\rangle}{d\langle x|S\rangle}\right|_{\bar{x},\bar{y}} \left.\frac{\partial\langle z|S\rangle}{\partial\langle y|S\rangle}\right|_{\bar{x},\bar{y}}$$

$$+ \left.\frac{d^V\langle y|S\rangle}{d\langle x|S\rangle}\right|_{\bar{x},\bar{y}} \left.\frac{\partial\langle z|S\rangle}{\partial^V\langle y|S\rangle}\right|_{\bar{x},\bar{y}}$$

Equation (3.2.11) says that the total influence of $\langle x|S\rangle$ on $\langle z|S\rangle$ is the sum of its direct influence and its indirect influence. The factor $\left.\frac{\partial\langle z|S\rangle}{\partial^V\langle y|S\rangle}\right|_{\bar{x},\bar{y}}$ is found from (3.2.1) to be $\frac{1}{2} f_{yy}$.

A similar derivation leads to an expression for the influence of $^V\langle x|S\rangle$ on $\langle z|S\rangle$:

$$\left.\frac{d\langle z|S\rangle}{d^V\langle x|S\rangle}\right|_{\bar{x},\bar{y}} = \left.\frac{\partial\langle z|S\rangle}{\partial^V\langle x|S\rangle}\right|_{\bar{x},\bar{y}} + \left.\frac{d\langle y|S\rangle}{d^V\langle x|S\rangle}\right|_{\bar{x},\bar{y}} \left.\frac{\partial\langle z|S\rangle}{\partial\langle y|S\rangle}\right|_{\bar{x},\bar{y}}$$

(3.2.12)

$$+ \frac{d^V\langle y|S\rangle}{d^V\langle x|S\rangle} \left.\frac{\partial\langle z|S\rangle}{\partial^V\langle y|S\rangle}\right|_{\bar{x},\bar{y}}$$

where $\left.\frac{\partial\langle z|S\rangle}{\partial^V\langle x|S\rangle}\right|_{\bar{x},\bar{y}} = \frac{1}{2} f_{xx}$, $\frac{\partial\langle z|S\rangle}{\partial^V\langle y|S\rangle} = \frac{1}{2} f_{yy}$, and

$$\left.\frac{\partial\langle z|S\rangle}{\partial\langle y|S\rangle}\right|_{\bar{x},\bar{y}} = f_y + f_{xy}\frac{h_x}{h_{xx}}.$$

Equations (3.2.11) and (3.2.12), along with expressions for

$$\frac{d^V\langle z|S\rangle}{d\langle x|S\rangle} = \frac{d\langle z^2|S\rangle}{d\langle x|S\rangle} - 2\langle z|S\rangle\frac{d\langle z|S\rangle}{d\langle x|S\rangle}$$

and

$$\frac{d^V\langle z|S\rangle}{d^V\langle x|S\rangle} = \frac{d\langle z^2|S\rangle}{d^V\langle x|S\rangle} - 2\langle z|S\rangle \frac{d\langle z|S\rangle}{d^V\langle x|S\rangle}$$

can be combined in the convenient influence matrix form for the total influence of $\{x|S\}$ on $\{z|S\}$, $\frac{d\{z|S\}}{d\{x|S\}}$.

$$\frac{d\{z|S\}}{d\{x|S\}} = \begin{bmatrix} \left.\frac{d\langle z|S\rangle}{d\langle x|S\rangle}\right|_{\bar{x},\bar{y}} & \left.\frac{d^V\langle z|S\rangle}{d\langle x|S\rangle}\right|_{\bar{x},\bar{y}} \\ \left.\frac{d\langle z|S\rangle}{d^V\langle x|S\rangle}\right|_{\bar{x},\bar{y}} & \left.\frac{d^V\langle z|S\rangle}{d^V\langle x|S\rangle}\right|_{\bar{x},\bar{y}} \end{bmatrix}$$

$$= \frac{\partial\{z|S\}}{\partial\{x|S\}} + \frac{d\{y|S\}}{d\{x|S\}}\frac{\partial\{z|S\}}{\partial\{y|S\}} \qquad (3.2.13)$$

The total influence is the sum of the direct influence, $\frac{\partial\{z|S\}}{\partial\{x|S\}}$, and the indirect influence $\frac{d\{y|S\}}{d\{x|S\}}\frac{\partial\{z|S\}}{\partial\{y|S\}}$. Figures 3-3a and 3.3b detail the elements of selected matrices of this equation.

A Special Case

A special case of two influencing variables occurs when either of the following conditions holds:

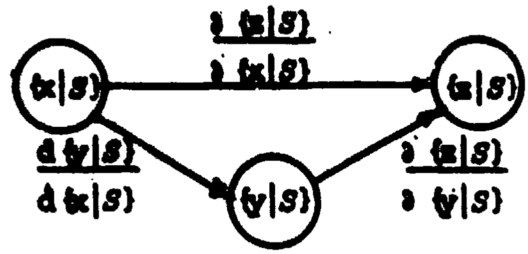
a) $Cov(x,y) = 0$

b) $\left.\frac{\partial^2\langle z|x,y\rangle}{\partial x\,\partial y}\right|_{\bar{x},\bar{y}} = \left.\frac{\partial^2\langle z^2|x,y\rangle}{\partial x\,\partial y}\right|_{\bar{x},\bar{y}} = 0$

In that case, Equation (3.2.1) reduces to

$$\langle z|S\rangle = \langle z|\bar{x},\bar{y},S\rangle + f_{xx}\,{}^V\langle x|S\rangle + f_{yy}\,{}^V\langle y|S\rangle ,$$

from which we find



$$\frac{\partial\{z|S\}}{\partial\{x|S\}} = \begin{bmatrix} \left.\dfrac{\partial <z|S>}{\partial <x|S>}\right|_{\bar{x},\bar{y}} & \left.\dfrac{\partial\,{}^{V}<z|S>}{\partial <x|S>}\right|_{\bar{x},\bar{y}} \\ \\ \left.\dfrac{\partial <z|S>}{\partial\,{}^{V}<x|S>}\right|_{\bar{x},\bar{y}} & \left.\dfrac{\partial\,{}^{V}<z|S>}{\partial\,{}^{V}<x|S>}\right|_{\bar{x},\bar{y}} \end{bmatrix}$$

$$= \begin{bmatrix} f_x + f_{xy}(h_{xx}\,{}^{V}<x|S> - 2h_x^2/h_{xx}) & f_x^* - 2<z|S>f_x + (h_{xx}\,{}^{V}<x|S> - 2h_x^2/h_{xx})(f_{xy}^* - 2<z|S>f_{xy}) \\ \\ .5f_{xx} & .5f_{xx}^* - <z|S>f_{xx} \end{bmatrix}$$

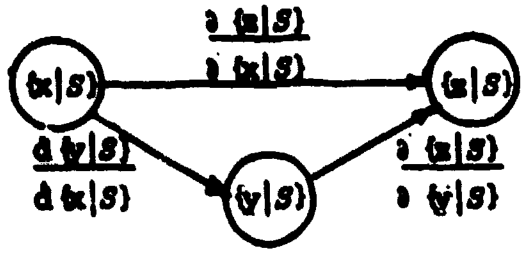
where

$$h_x = \left.\frac{d<y|x,S>}{dx}\right|_{\bar{x}} \qquad h_{xx} = \left.\frac{d^2<y|x,S>}{dx^2}\right|_{\bar{x}}$$

$$f_x = \left.\frac{\partial<z|x,y,S>}{\partial x}\right|_{\bar{x},\bar{y}} \qquad f_{xx} = \left.\frac{\partial^2<z|x,y,S>}{\partial x^2}\right|_{\bar{x},\bar{y}} \qquad f_{xy} = \left.\frac{\partial^2<z|x,y,S>}{\partial x\,\partial y}\right|_{\bar{x},\bar{y}}$$

$$f_x^* = \left.\frac{\partial<z^2|x,y,S>}{\partial x}\right|_{\bar{x},\bar{y}} \qquad f_{xx}^* = \left.\frac{\partial^2<z^2|x,y,S>}{\partial x^2}\right|_{\bar{x},\bar{y}} \qquad f_{xy}^* = \left.\frac{\partial^2<z^2|x,y,S>}{\partial x\,\partial y}\right|_{\bar{x},\bar{y}}$$

FIGURE 3-3a. The Elements Of The Influence Matrices Of Equation (3.2.13)



$$\frac{\partial\{z|S\}}{\partial\{y|S\}} = \begin{bmatrix} \left.\dfrac{\partial <z|S>}{\partial <y|S>}\right|_{\bar{x},\bar{y}} & \left.\dfrac{\partial^{v} <z|S>}{\partial <y|S>}\right|_{\bar{x},\bar{y}} \\ \left.\dfrac{\partial <z|S>}{\partial^{v} <y|S>}\right|_{\bar{x},\bar{y}} & \left.\dfrac{\partial^{v} <z|S>}{\partial^{v} <y|S>}\right|_{\bar{x},\bar{y}} \end{bmatrix}$$

$$= \begin{bmatrix} f_y + 2f_{xy}h_x/h_{xx} & f_y^{*} - 2<z|S>f_y + 2g_x/g_{xx}(h_{yx} - 2<z|S>f_{xy}) \\ .5f_{yy} & .5f_{yy}^{*} - <z|S>fyy \end{bmatrix}$$

where

$$f_y = \left.\frac{\partial <z|x,y,S>}{\partial y}\right|_{\bar{x},\bar{y}} \qquad f_{yy} = \left.\frac{\partial^2 <z|x,y,S>}{\partial y^2}\right|_{\bar{x},\bar{y}}$$

$$f_y^{*} = \left.\frac{\partial <z^2|x,y,S>}{\partial y}\right|_{\bar{x},\bar{y}} \qquad f_{yy}^{*} = \left.\frac{\partial^2 <z^2|x,y,S>}{\partial y^2}\right|_{\bar{x},\bar{y}}$$

FIGURE 3-3b. The Elements Of The Influence Matrices Of Equation (3.2.13)

$$\frac{\partial\langle z|S\rangle}{\partial\langle x|S\rangle} = \left.\frac{\partial\langle z|x,y,S\rangle}{\partial x}\right|_{\bar{x},\bar{y}} \quad .$$

These conditions produce a simplification in the influence matrix, which is evident comparing $\frac{\partial\{z|S\}}{\partial\{x|S\}}$ as shown in Figure 3-3a with

$$\frac{\partial\{z|S\}}{\partial\{x|S\}} = \begin{bmatrix} f_x & f_x^* - 2\langle z|S\rangle f_x \\ 1/2\, f_{xx} & 1/2\, f_{xx}^* - \langle z|S\rangle f_{xx} \end{bmatrix} \tag{3.2.14}$$

The direct influence matrix of (3.2.14) can be determined without information about $\langle y|x,S\rangle$. Furthermore, under these conditions we have

$$f_x = \left.\frac{\partial\langle z|x,y,S\rangle}{\partial x}\right|_{\bar{x},\bar{y}}$$

$$= \left.\frac{\partial\langle z|x,S\rangle}{\partial x}\right|_{\bar{x}} \quad ,$$

and similar expressions for f_{xx} , f_x^* , and f_{xx}^* . Hence, the elements of (3.2.14) can be determined independently of the influence of $\{y|S\}$

Of course, condition a) holds for the situation depicted in Figure 3-4. In that case there are no indirect influences, and equation (3.2.13) simplifies to

$$\frac{d\{z|S\}}{d\{x|S\}} = \frac{\partial\{z|S\}}{\partial\{x|S\}} \tag{3.2.15}$$

where $\frac{\partial\{z|S\}}{\partial\{x|S\}}$ is given by (3.2.14).

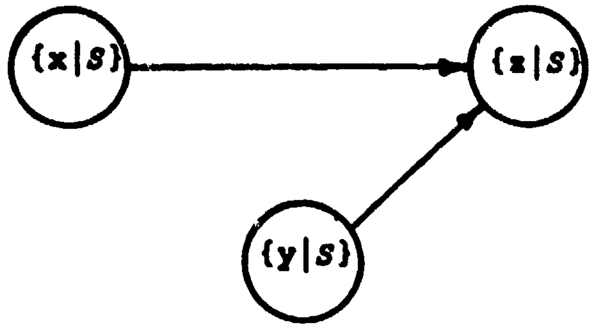


FIGURE 3-4. Two Independent Influences on a Variable

A Direct Influence and Several Indirect Influences

Equation (3.2.13) and Figure 3-3 show the proper definition of the influence matrices when there is one direct and one indirect influence on a variable. The development of the equation can readily be extended to a higher number of indirect influences.

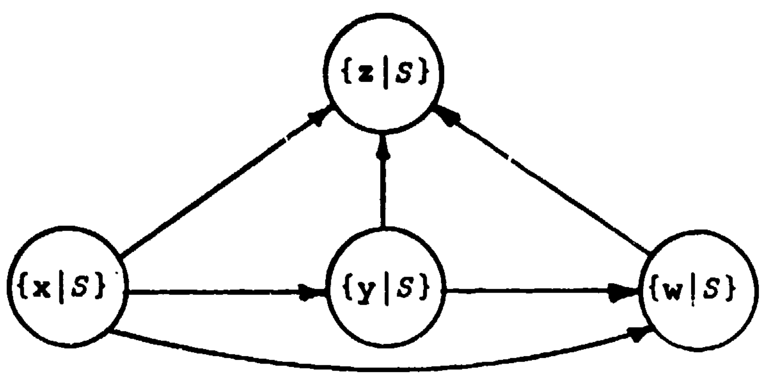


FIGURE 3-5. An Influence Diagram with a Direct and Two Indirect Influences

For example, for the influence configuration of Figure 3-5, the proper influence equation is,

$$\frac{d\{z|S\}}{d\{x|S\}} = \frac{\partial\{z|S\}}{\partial\{x|S\}} + \frac{d\{y|S\}}{d\{x|S\}}\frac{\partial\{z|S\}}{\partial\{y|S\}} + \frac{d\{w|S\}}{d\{x|S\}}\frac{\partial\{z|S\}}{\partial\{w|S\}} \tag{3.2.16}$$

The elements of selected matrices are displayed in Figure 3-6a and 3-6b.

The covariances between each variable that influences $\{z|S\}$ and all of the other influencing state variables must be considered in computing the direct influence of that variable.

Evaluation of the elements of (3.2.16) requires that assessments be made of the mean and variance of z conditioned on the three variables $x, y,$ and w. It can be easily extended to the case of more than three influences on the variable z. In fact, the appropriate equations can be written down by inspection of (3.2.13) and (3.2.16). The practical limitation on the extensions of (3.2.16) is the difficulty of assessing moments conditioned on more than three variables.

3.3 Influence Vectors

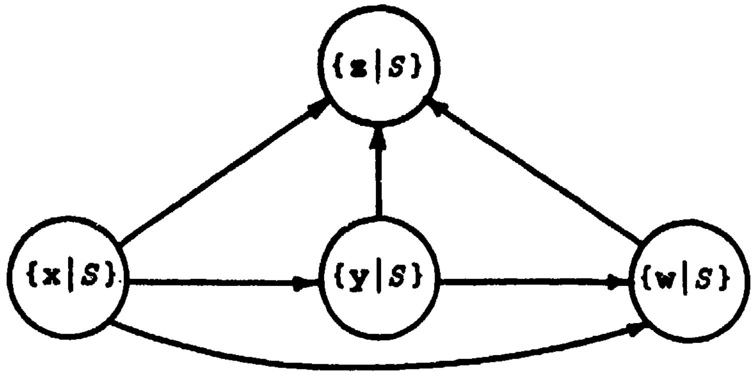
When the influenced marginal distribution is the profit lottery, it is sometimes unnecessary to describe the influence on the entire distribution. Often the certain equivalent adequately characterizes the profit lottery. In those instances a description of the influence on the certain equivalent may be sufficient.

Howard [5] showed that the certain equivalent could be represented as a power series:

$$^{v}\langle z|S\rangle = \sum_{k=1}^{\infty} \frac{^{k}z}{k!} (-\gamma)^{k-1}$$

where

γ = the decision maker's risk aversion coefficient



$$\frac{\partial\{z|\mathcal{S}\}}{\partial\{x|\mathcal{S}\}} = \begin{bmatrix} f_x + & f_x^{*} - 2\langle z|\mathcal{S}\rangle f_x + \\ f_{xy}\left(h_{xx}^{v}\langle x|\mathcal{S}\rangle - \frac{2h_x^{2}}{h_{xx}}\right) + & \left(h_{xx}^{v}\langle x|\mathcal{S}\rangle - \frac{2h_x^{2}}{h_{xx}}\right)\left(f_{xy}^{*} - 2\langle z|\mathcal{S}\rangle f_{xy}\right) + \\ f_{xw}\left(g_{xx}^{v}\langle x|\mathcal{S}\rangle - \frac{2g_x^{2}}{g_{xx}}\right) & \left(g_{xx}^{v}\langle x|\mathcal{S}\rangle - \frac{2g_x^{2}}{g_{xx}}\right)\left(f_{xw}^{*} - 2\langle z|\mathcal{S}\rangle f_{xw}\right) \\ .5f_{xx} & .5f_{xx}^{*} - \langle z|\mathcal{S}\rangle f_{xx} \end{bmatrix}$$

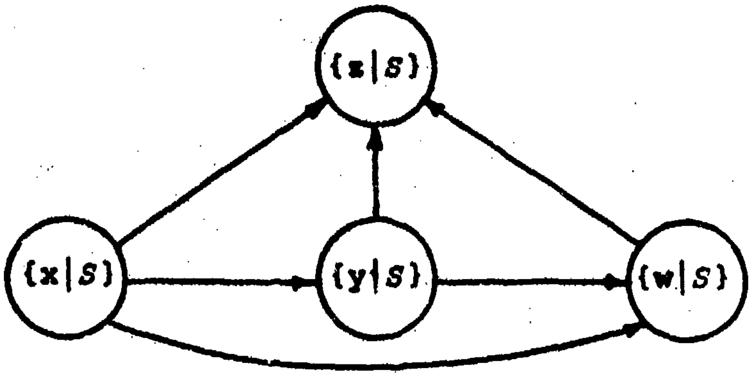
where

$$f_x = \left.\frac{\partial\langle z|w,x,y,\mathcal{S}\rangle}{\partial\ x}\right|_{\bar{w},\bar{x},\bar{y}} \qquad f_x^{*} = \left.\frac{\partial\langle z^2|w,x,y,\mathcal{S}\rangle}{\partial\ x}\right|_{\bar{w},\bar{x},\bar{y}}$$

$$h_x = \left.\frac{\partial\langle y|x,\mathcal{S}\rangle}{\partial\ x}\right|_{\bar{x}} \qquad g_x = \left.\frac{\partial\langle w|x,\mathcal{S}\rangle}{\partial\ x}\right|_{\bar{x}}$$

etc.

FIGURE 3-6a. The Elements Of Selected Influence Matrices Of Equation (3.2.16)



$$\frac{\partial\{z|\mathcal{S}\}}{\partial\{y|\mathcal{S}\}} = \begin{bmatrix} f_y + 2f_{xy}\dfrac{h_x}{h_{xx}} + & f_x^* - 2\langle z|\mathcal{S}\rangle f_y + \\ f_{yw}\left(k_{yy}^{\,v}\langle y|\mathcal{S}\rangle - \dfrac{2k_y^{\,2}}{k_{yy}}\right) & 2\dfrac{h_x}{h_{xx}}\left(f_{xy}^* - 2\langle z|y,\mathcal{S}\rangle f_{xy}\right) + \\ & \left(k_{yy}^{\,v}\langle y|\mathcal{S}\rangle - \dfrac{2k_y^{\,2}}{k_{yy}}\right)\left(f_{yw}^* - 2\langle z|\mathcal{S}\rangle f_{yw}\right) \\ & \\ .5f_{yy} & .5f_{yy}^* - \langle z|\mathcal{S}\rangle f_{yy} \end{bmatrix}$$

where

$$k_y = \left.\frac{\partial\langle w|y,\mathcal{S}\rangle}{\partial\, y}\right|_{\bar{y}} \qquad \text{etc.}$$

FIGURE 3-6b. The Elements Of Selected Influence Matrices Of Equation (3.2.16)

k_z = the k^{th} cumulant of the profit lottery.

By truncating the series and evaluating the cumulants, one can obtain an expression of the form

$$\text{(3.3.1)} \qquad {}^{\sim}\langle z|S\rangle = C_1\langle z|S\rangle + \sum_{k=2}^{K} C_k \langle (z-\bar{z})^k|S\rangle \, \gamma^{k-1}$$

For example, with $K=2$, $C_1=1$ and $C_2 = -\frac{1}{2}$. If we let

$$\text{(3.3.2)} \qquad C = [C_1 \quad C_2\gamma \quad C_3\gamma^2 \ldots C_k\gamma^k \quad 0 \quad 0 \quad 0]^T ,$$

then we can define an influence vector

$$\text{(3.3.3)} \qquad \frac{d\,{}^{\sim}\langle z|S\rangle}{d\{x|S\}} = \frac{d\{z|S\}}{d\{x|S\}} \; C = \left[\frac{\partial\,{}^{\sim}\langle z|S\rangle}{\partial\langle x|S\rangle}\Bigg|_{\bar{x}} \quad \frac{\partial\,{}^{\sim}\langle z|S\rangle}{\partial\langle (x-\bar{x})^2|S\rangle}\Bigg|_{\bar{x}} \quad \cdots \right]^T$$

According to this equation, the influence of $\{x|S\}$ on the certain equivalent of the profit lottery is described by a vector of changes in ${}^{\sim}\langle z|S\rangle$ resulting from changes in the expectation and central moments of $\{x|S\}$.

The influence calculus applies to influence vectors as well as influence matrices. For example, if

$$\frac{d\{z|S\}}{d\{x|S\}} = \frac{d\{y|S\}}{d\{x|S\}} \cdot \frac{d\{z|S\}}{d\{y|S\}} ,$$

then

$$\frac{d\,{}^{\sim}\langle z|S\rangle}{d\{x|S\}} = \frac{d\{z|S\}}{d\{x|S\}} \; C$$

$$\text{(3.3.4)} \qquad = \frac{d\{y|S\}}{d\{x|S\}} \cdot \frac{d\,{}^{\sim}\langle z|S\rangle}{d\{y|S\}} .$$

A special case of Equation (3.3.3) occurs when $k=1$ in equation (3.3.1), then (3.3.3) becomes

$$(3.3.5) \qquad \frac{d\langle z|S\rangle}{d\{x|S\}} = \frac{d\{z|S\}}{d\{x|S\}} \begin{bmatrix} 1 & 0 & 0 & \cdots \end{bmatrix}^T$$

$$= \left[\left. \frac{d\langle z|S\rangle}{d\langle x|S\rangle} \right|_{\bar{x}} \quad \left. \frac{d\langle z|S\rangle}{d\langle (x-\bar{x})^2|S\rangle} \right|_{\bar{x}} \quad \cdots \right]^T .$$

3.4 An Example Introducing the Influence Consequence Matrix

We have shown in preceding sections that the influence matrices when combined using the influence calculus establish how changes in the marginal probability distribution of a state variable affect the profit lottery. We have also suggested that the relative importance of these changes is measurable by the change in the certain equivalent, e.g. $\frac{d\tilde{~}\langle z|S\rangle}{d\{x|S\}}$. In this section we propose a method for combining the theory of influence with the judgment of the analyst as an aid in developing the decision model.

To illustrate the method, we make use of a modified version of the entrepreneur's decision problem, posed by Howard [5]. The entreprenuer must decide upon the price of his product P in the face of uncertainty about demand, $q(P)$, and the cost of his product $C(P)$. His profit function is

$$\pi(C,q,P) = Pq(P) - Cq(P)$$

The variables C and q are assumed to be uncertain. Letting

$$\Delta C = C - \bar{C}$$

$$\Delta q = q - \bar{q}$$

allows definition of $\Delta\pi(\Delta C, \Delta q, P)$ as

$$(3.4.1) \quad \Delta\pi(\Delta C,\Delta q,P) = P[q(P) + \Delta q] - C[q(P) + \Delta q] - \Delta C - \pi(C,q,P)$$

which is the change in profit due to fluctuations of C and q about their means. Both $\{\Delta q|S\}$ and $\{\Delta C|S\}$ have means of zero and

$$^{v}\langle\Delta C|S\rangle = 10000$$
$$^{v}\langle\Delta q|S\rangle = 100 \quad .$$

An influence diagram for the original problem with modifications is shown in Figure 3-7. The random variables γ , ε , and δ have been added and the dependencies of the moments of $\{\Delta C|\gamma,\varepsilon,S\}$ and $\{\Delta q|\delta,\varepsilon,S\}$ are presented in Figure 3-8. For example, ε can be thought of as a general economic indicator. As ε increases both $\langle\Delta C|\gamma,\varepsilon,S\rangle$ and $\langle\Delta q|\delta,\varepsilon,S\rangle$ are assumed to increase linearly, but the corresponding variances do not increase. A similar remark holds for the relationship between ΔC and ε . The following marginal probability densities have been assumed:

$$(3.4.2) \qquad \begin{aligned} \{\gamma|S\} &= \frac{1}{50} & 0 \le \delta \le 10 \\ \{\varepsilon|S\} &= \frac{1}{2} & -1 \le \varepsilon \le 1 \\ \{\delta|S\} &= 2\delta & 0 \le \delta \le 1 \quad . \end{aligned}$$

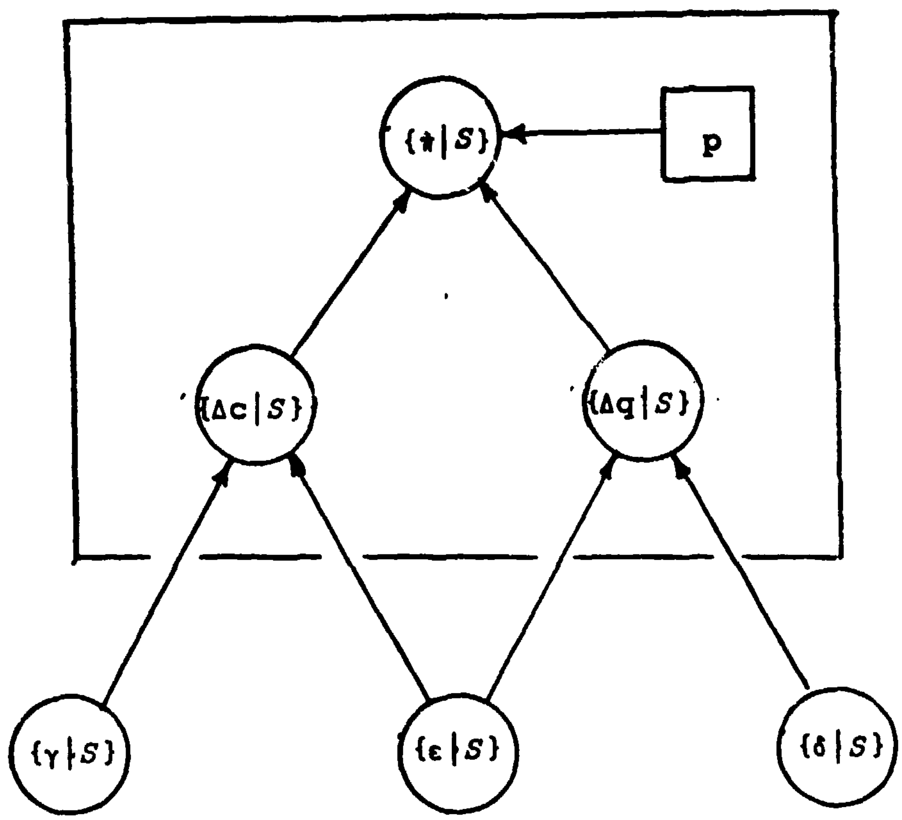
From the original entreprenuer's problem, we find

$$(3.4.3) \qquad \Delta\pi(0,\Delta q,P^*) = 17.58\Delta q + .0249(\Delta q)^2$$

and

$$(3.4.4) \qquad \Delta\pi(\Delta C,0,P^*) = -\Delta C$$

where P^* is the optimal setting of P when $\Delta C = \Delta q = 0$. All



Boundary of original entreprenuer's problem ————

FIGURE 3-7. Influence For The Modified Entreprenuer's Problem

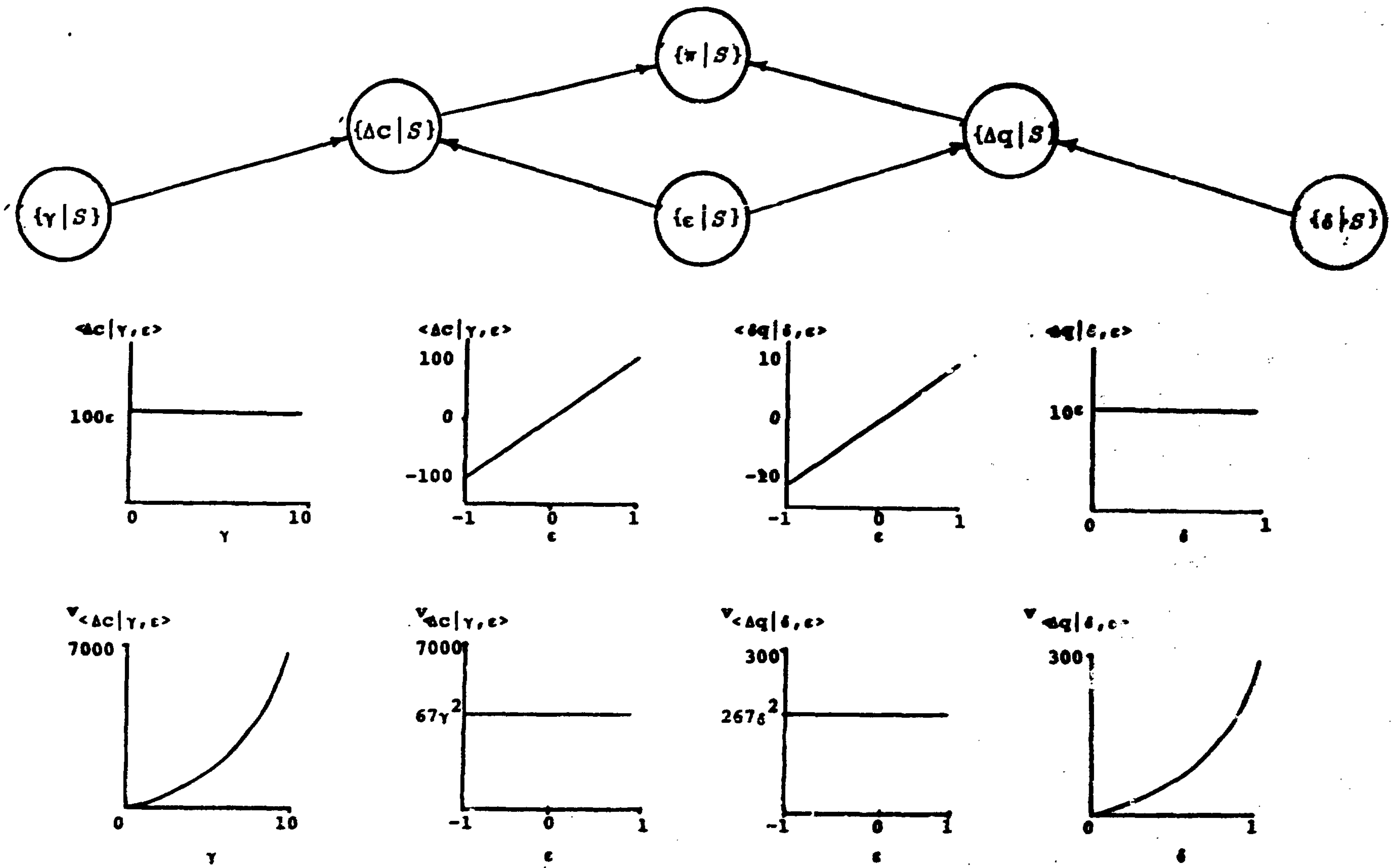


FIGURE 3-8. Dependence Of The Conditional Means And Variance, For The Modified Entreprenuer's Problem

of the following is conditioned on $P = P^*$.

If we assume the decision maker is risk neutral, then by (3.3.4) the influence vectors are

$$\frac{\partial\langle\Delta\pi|P^*,S\rangle}{\partial\{\Delta q|S\}} = \left[\frac{\partial\langle\pi|P^*,S\rangle}{\partial\langle\Delta q|S\rangle}\Bigg|_{\overline{\Delta q}} \quad \frac{\partial\langle\pi|P^*,S\rangle}{\partial^{v}\langle\Delta q|S\rangle}\Bigg|_{\overline{\Delta q}}\right]^T$$

(3.4.5)

$$\frac{\partial\langle\Delta\pi|P^*,S\rangle}{\partial\{\Delta C|S\}} = \left[\frac{\partial\langle\pi|P^*,S\rangle}{\partial\langle\Delta C|S\rangle}\Bigg|_{\overline{\Delta C}} \quad \frac{\partial\langle\pi|P^*,S\rangle}{\partial\langle\Delta q|S\rangle}\Bigg|_{\overline{\Delta C}}\right]^T$$

The cross-partial $\frac{\partial^2\langle\Delta\pi|P^*,S\rangle}{\partial\Delta C\partial\Delta q}$ is zero according to (3.4.1). Therefore, (3.4.5) can readily be evaluated, using (3.4.3) and (3.4.4), as

$$\text{(3.4.6)} \quad \frac{\partial\langle\Delta\pi|P^*,S\rangle}{\partial\{\Delta q|S\}} = \left[f_{\Delta q} \quad \frac{1}{2}f_{\Delta q\Delta q}\right]^T$$

$$= \left[17.6 \quad .025\right]^T$$

and

$$\frac{\partial\langle\Delta|P^*,S\rangle}{\partial\{\Delta C|S\}} = \left[f_{\Delta C} \quad \frac{1}{2}f_{\Delta C\Delta C}\right]^T$$

$$= \left[-1 \quad 0\right]^T .$$

The elements of the influence matrices for state variables γ,δ, and ε can be determined from Figure 3-8. The matrices are shown in Figure 3-9, along with the computations for the total influence on the profit lottery.

The influence matrices can be useful in explaining the type and strength of relationships between state variables and the profit lottery. Examining the influence matrices shows that $\{\gamma|S\}$ affects only $^{v}\langle\Delta C|S\rangle$, and $^{v}\langle\Delta C|S\rangle$ does not affect

$$\frac{d\langle \Pi | S\rangle}{d\{\gamma | S\}} = \frac{\partial\{\Delta c | S\}}{\partial'\{\gamma | S\}} \; \frac{d\langle \pi | S\rangle}{d\{\Delta c | S\}}$$

$$= \begin{bmatrix} 0 & 3556 \\ 0 & 67 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{d\langle \pi | S\rangle}{d\{\delta | S\}} = \frac{\partial'\{\Delta q | S\}}{\partial'\{\delta | S\}} \; \frac{d\langle \pi | S\rangle}{d'\{\Delta q | S\}}$$

$$= \begin{bmatrix} 0 & 356 \\ 0 & 267 \end{bmatrix} \begin{bmatrix} 17.6 \\ .025 \end{bmatrix} = \begin{bmatrix} 8.7 \\ 6.6 \end{bmatrix}$$

$$\frac{d\langle \pi | S\rangle}{d\{\varepsilon | S\}} = \frac{\partial'\{\Delta c | S\}}{\partial'\{\varepsilon | S\}} \; \frac{d\langle \pi | S\rangle}{d\{\Delta c | S\}} + \frac{\partial'\{\Delta q | S\}}{\partial'\{\varepsilon | S\}} \; \frac{d\langle \pi | S\rangle}{d\{\Delta q | S\}}$$

$$= \begin{bmatrix} 100 & 1334 \\ 0 & 1335 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} 17.6 \\ .025 \end{bmatrix}$$

$$= \begin{bmatrix} 76 \\ .25 \end{bmatrix}$$

FIGURE 3-9. The Influence Matrices For The Influences Of γ, δ, and ε.

$\langle\pi|P^*,S\rangle$. Consequently, neither $\langle\gamma|S\rangle$ nor $^v\langle\gamma|S\rangle$ affect the expectation of the profit lottery. On the other hand, $\langle\pi|P^*,S\rangle$ is affected by $^v\langle\Delta q|S\rangle$. Therefore, $\{\delta|S\}$ affects the profit lottery even though it affects only $^v\langle\Delta q|S\rangle$. The effect of $\langle\delta|S\rangle$ on $\langle\pi|P^*,S\rangle$ is small, because it acts though its influence on $^v\langle\Delta q|S\rangle$. $\langle\varepsilon|S\rangle$ has a much larger effect on $\langle\pi|P^*,S\rangle$, because it acts though its influence on $\langle\Delta q|S\rangle$. The effect of $\langle\varepsilon|S\rangle$ through $\langle\Delta q|S\rangle$ is partially offset by an opposing effect through $\langle\Delta C|S\rangle$.

Selecting Variables for the Model

Sometimes the influence matrix itself will indicate that a particular variable is unimportant to the problem and need not be included in the model. For example, since $\langle\pi|P^*,S\rangle$ is unaffected by the distribution on $\{\delta|S\}$ in the example, the decision maker will not suffer if $\{\delta|S\}$ is omitted from the model.

In general, the analyst must use his judgment to interpret the influence matrix. Figure 3-9 shows that $\langle\pi|P^*,S\rangle$ is more sensitive to the mean of $\{\varepsilon|S\}$ than to the mean of $\{\delta|S\}$, but is less sensitive to the variance of $\{\varepsilon|S\}$ than to the variance of $\{\delta|S\}$. Should the uncertainty of both of these variables be included in the model, or only one? If one is to be uncertain, which one should it be?

To clarify the roles of the influence matrices and the analyst's judgment, we introduce the expression

$$\left| \frac{d\langle z|S\rangle}{d\langle x|S\rangle}\,\Delta\langle x|S\rangle \right| \qquad (3.4.6)$$

The factor $\frac{d^{\sim}\langle z|S\rangle}{d\langle x|S\rangle}$ is the first term of the influence vector $\frac{d^{\sim}\langle z|S\rangle}{d\{x|S\}}$ of equation (3.3.3).

In the context of the influence matrix, we think of it as the change in the certain equivalent due to a change in the mean of $\{x|S\}$. However, it may also be interpreted as the deterministic sensitivity of the certain equivalent to a unit change in the variable x about its mean. The factor $\Delta\langle x|S\rangle$ expresses the decision analyst's judgment about the difference between the mean of the distribution assigned to x and the mean of the decision maker's authentic distribution on x . This difference may be due to difficulties in the assessment procedure, a hasty assessment, or the fact that the state variable is influenced by many other variables. Therefore, the quantity given by (3.4.6) is the change in the certain equivalent that results from obtaining an authentic mean of the state variable distribution.

If the value of expression (3.4.6) for a particular state variable is a large fraction of the certain equivalent or the difference between certain equivalents for different decision settings, additional modeling or more careful assessment of that state variable is warranted. If the value is small, then other state variables should be considered for additional work.

The number of variables that are treated as uncertain in a decision analysis problem is usually constrained to six or seven due to time and budget limitations. Therefore, identifying which variables should be represented as uncertain is an important concern. While (3.4.6) can be used to indicate the importance of further modeling, a similar expression indicates whether a particular state variable should be treated as a random variable or whether the uncertainty can be ignored and the variable fixed at its mean value. The factor $\left.\frac{d^{\sim}\langle z|S\rangle}{d^{v}\langle x|S\rangle}\right|_{\bar{x}}$ in expression (3.4.7),

$$(3.4.7)\qquad \left|\left.\frac{d^{\sim}\langle z|S\rangle}{d^{v}\langle x|S\rangle}\right|_{\bar{x}} \; {}^{v}\langle x|S\rangle\right| ,$$

is the second element of the influence vector $\frac{d^{\sim}\langle z|S\rangle}{d\{x|S\}}$

We could have multiplied $\frac{d^{\sim}\langle z|S\rangle}{d^{v}\langle x|S\rangle}$ by a factor $\Delta^{v}\langle x|S\rangle$, the decision analyst's judgment about the difference between the variance of the assigned distribtuion and the authentic distribution on x . However, by multiplying by the variance ${}^{v}\langle x|S\rangle$ rather than the change in variance, Equation (3.4.7) approximates the change in certain equivalent that results from fixing x at its mean and not treating it as a random variable. Appendix B gives a proof that under the approximations made earlier

$$(3.4.8)\qquad \frac{\partial^{\sim}\langle z|S\rangle}{\partial^{v}\langle x|S\rangle}\, {}^{v}\langle x|S\rangle \approx {}^{\sim}\langle z|S\rangle - {}^{\sim}\langle z|x=\bar{x}, S\rangle .$$

The influence-consequence matrix of Figure 3-10 conceptualizes the implications of the various combinations of magnitude for the modeling indicators of (3.4.6) and (3.4.7). The top portion of each matrix entry recommends an action regarding the improvement of the authenticity of the distribution assigned to the state variable. In the lower portion of each entry is a

$$\left| \frac{d\langle z|S\rangle}{d\langle x|S\rangle} \; \Delta\langle x|S\rangle \right|$$

$$\left| \frac{d^{-}\langle z|S\rangle}{d^{V}\langle x|S\rangle} \; V\langle x|S\rangle \right|$$

	LARGE	SMALL
LARGE	Model Further	No Further Modeling
	Uncertain	Uncertain
SMALL	Model Further	No Further Modeling
	Fixed At Mean	Fixed At Mean

FIGURE 3-10. The Influence-Consequence Matrix

recommendation about whether that state variable should be considered as uncertain or should be fixed at its mean in the decision model.

Of particular interest is the lower left element, which corresponds to a large change in the certain equivalent due to anticipated changes in the expectation of the state variable, but little change when its uncertainty is ignored. In this case, further modeling is recommended to more accurately determine the mean of the state variable. A model of the influences on the mean of that state variable might include uncertainty, even though the influenced variable itself is not treated as uncertain in the decision model. To our knowledge, this option has not been exercised in decision models, and it is a new result of our analysis of influences.

Returning to the illustrative entreprenuer's problem, Table 3-2 shows the values of (3.4.6) and (3.4.7) for the state variables γ, δ, and ε. Since a numerical estimate of the difference between the assessed mean and authentic mean is required for each state variable, we have assumed

$$\Delta \langle x|S\rangle = .1\ {}^{v}\langle x|S\rangle \ .$$

This expression reflects the feeling that the mean of a broad distribution is less authentic than the mean of a narrow distribution. Of course, other functional forms might also be used.

As we mentioned above, γ is irrelevant and may be ignored. Treating ε as uncertain is not necessary, since its uncertainty only contributes .08 to the expected value of 200.5

Table 3-2

The Influence of the Variables γ, ϵ, and δ and the Modeling Consequence

	γ	δ	ϵ
$\frac{\partial\langle\pi \mid S\rangle}{\partial\langle\cdot \mid S\rangle}$	0	8.7	76
$\frac{\partial\langle\pi \mid S\rangle}{\partial^{V}\langle\cdot \mid S\rangle}$	0	6.6	.25
$\frac{\partial\langle\pi \mid S\rangle}{\partial\langle\cdot \mid S\rangle} \times .1 \times {}^{\sigma}\langle\cdot \mid S\rangle$	0	.16	4.4
$\frac{\partial\langle\pi \mid S\rangle}{\partial^{V}\langle\cdot \mid S\rangle} \times {}^{V}\langle\cdot \mid S\rangle$	0	.37	.08
CONSEQUENCE	No Further Modeling Fixed At Mean	No Further Modeling Uncertain	Further Modeling Fixed At Mean

(from the reference [5]). ε may be fixed at its mean. However, our estimate of the difference between the mean of the authentic distribution and the mean of the assigned distribution on ε leads to a change of 4.4 in the expected value. Though this is a small percentage of the expected value, further modeling of the mean of $\{\varepsilon|S\}$ is more important than further modeling of either the mean or variance of the other variables. Of the three variables, δ should be treated as uncertain in the decision model. Again, its effect on the expectation of the profit lottery is small.

Estimation of the Error in the Certain Equivalent Using Influence Matrices

In addition to their application in the influence-consequence matrix, the total influence vector can be used to give an estimate for the difference between the certain equivalent of the model and the authentic certain equivalent. If $\Delta^{\sim}\langle x|S\rangle$ is that difference, then

$$(3.4.9) \qquad \Delta^{\sim}\langle\pi|S\rangle = \frac{\partial^{\sim}\langle\pi|S\rangle}{\partial\langle x|S\rangle}\,\Delta\langle x|S\rangle + \frac{\partial^{\sim}\langle\pi|S\rangle}{\partial^{v}\langle x|S\rangle}\,\Delta^{v}\langle x|S\rangle$$

where $\Delta\langle x|S\rangle$ is, as defined above, the analyst's estimate of the difference in the means of the assigned distribution on x and the authentic distribution. The factor $\Delta^{v}\langle x|S\rangle$ is defined similarly to $\Delta\langle x|S\rangle$ as the difference in the variances of the assigned distribution and the authentic distribution. A more general form for (3.4.9) is

$$\Delta^{\sim}\langle\pi|S\rangle = \sum_{x_i} \frac{\partial^{\sim}\langle\pi|S\rangle}{\partial\langle x_i|S\rangle} \Delta\langle x_i|S\rangle + \frac{\partial^{\sim}\langle\pi|S\rangle}{\partial^{v}\langle x_i|S\rangle} \Delta^{v}\langle x_i|S\rangle ,$$

where x_i are the state variables on which marginal distributions are encoded. In influence matrix form,

$$(3.4.10) \qquad \Delta^{\sim}\langle\pi|S\rangle = \sum_{x_i} \frac{d^{\sim}\langle\pi|S\rangle}{d\{x_i|S\}} \Delta\{x_i|S\}$$

where $\Delta\{x_i|S\}$ is the vector of differences between central moments of the assessed distribution and the authentic distribution,

$$\Delta\{x_i|S\} = [\Delta\langle x_i|S\rangle \;\; \Delta^{v}\langle x_i|S\rangle \;\; \Delta\langle(x_i-x_i)^3|S\rangle \;\; . \; . \; . \;] .$$

Equation (3.4.10) relates dissatisfaction with the profit lottery to estimates of the error in the assessment of state variable distributions. Other researchers have related that dissatisfaction to estimates of the error in the profit lottery itself, which is difficult to estimate. We believe the first element $\Delta\langle x_i|S\rangle$ of the vector $\Delta\{x_i|S\}$ can be estimated by the analyst, though estimating $\Delta^{v}\langle x_i|S\rangle$ is probably too difficult. As a practical matter, the difference between the certain equivalent of the modeled profit lottery and the authentic profit lottery could be approximated as

$$(3.4.11) \qquad \Delta^{\sim}\langle v|S\rangle \simeq \sum_{x_i} \frac{\partial^{\sim}\langle v|S\rangle}{\partial\langle x_i|S\rangle} \Delta\langle x_i|S\rangle \quad .$$

Using the information in Figure 3-9 for the entrepreneur's problem we estimate the error in the certain equivalent to be

$$\tilde{\Delta}\langle v|S\rangle \simeq \frac{\partial^{\sim}\langle v|S\rangle}{\partial\langle\gamma|S\rangle}\Delta\langle\gamma|S\rangle + \frac{\partial^{\sim}\langle v|S\rangle}{\partial\langle\delta|S\rangle}\Delta\langle\delta|S\rangle + \frac{\partial^{\sim}\langle v|S\rangle}{\partial\langle\varepsilon|S\rangle}\Delta\langle\varepsilon|S\rangle$$

$$\simeq 0. + .16 + 4.4$$

$$\simeq 4.56$$

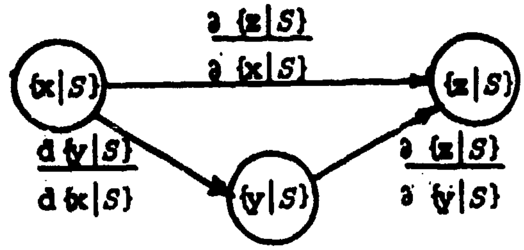
3.5 The Influence of Deterministically Related Variables

Most decision problems involve deterministic relationships between some of the problem variables. Usually the value function is a deterministic function of some of the state variables. In the entrepreneur's problem, for example, the profit is determined by ΔC and Δq . Because of the frequency of such relationships, this special case of influence is treated in this section.

The approximate influence matrices for $\{x|S\}$ deterministically influencing $\{z|S\}$ and for $\{x|S\}$ and $\{y|S\}$ deterministically influencing $\{z|S\}$ are shown in Figures 3-11a and 3-11b. These elements can be obtained directly by differentiating the expressions for $\langle v(x)|S\rangle$ and ${}^{v}\langle v(x)|S\rangle$ given as Equations (4.2) and (4.5) of reference [5]. Using the deterministic influence matrix we can write the deterministic influence vector,

$$\frac{d^{\sim}\langle z|S\rangle}{d\{x|S\}} = \left[f_x - \gamma f_x f_{xx}\ {}^{v}\langle x|S\rangle \quad \frac{1}{2} f_{xx} - \frac{\gamma}{2} f_x^2\right]^T \tag{3.5.1}$$

Let x_1 and x_2 be two independent state variables that have identical marginal probability distributions, but have different deterministic sensitivities (Figure 3-12). v varies quadratically with x_1 according to $v(x_1) = f_1(x_1)$ and linearly



$$\frac{\partial\{z|\mathcal{S}\}}{\partial\{x|\mathcal{S}\}} = \begin{bmatrix} \left.\dfrac{\partial\langle z|\mathcal{S}\rangle}{\partial\langle x|\mathcal{S}\rangle}\right|_{\bar{x},\bar{y}} & \left.\dfrac{\partial\,{}^{V}\langle z|\mathcal{S}\rangle}{\partial\langle x|\mathcal{S}\rangle}\right|_{\bar{x},\bar{y}} \\[2em] \left.\dfrac{\partial\langle z|\mathcal{S}\rangle}{\partial\,{}^{V}\langle x|\mathcal{S}\rangle}\right|_{\bar{x},\bar{y}} & \left.\dfrac{\partial\,{}^{V}\langle z|\mathcal{S}\rangle}{\partial\,{}^{V}\langle x|\mathcal{S}\rangle}\right|_{\bar{x},\bar{y}} \end{bmatrix}$$

$$= \begin{bmatrix} g_x & \begin{array}{l} 2g_x\,{}^{V}\langle x|\mathcal{S}\rangle + 2g_y g_{xy}\,{}^{V}\langle y|\mathcal{S}\rangle + \\ 2(g_{xx}g_y + g_x g_{xy})h_x\,{}^{V}\langle x|\mathcal{S}\rangle + \\ 2g_x g_y(h_{xx}\,{}^{V}\langle x|\mathcal{S}\rangle - 2h_x^2/h_{xx}) \end{array} \\[3em] .5g_{xx} & f_x^{\,2} \end{bmatrix}$$

where

$$g_x = \left.\frac{\partial z(x,y)}{\partial x}\right|_{\bar{x},\bar{y}} \qquad g_{xx} = \left.\frac{\partial^2 z(x,y)}{\partial x^2}\right|_{\bar{x},\bar{y}} \qquad g_y = \left.\frac{\partial z(x,y)}{\partial y}\right|_{\bar{x},\bar{y}} \quad \text{etc.}$$

$$h_x = \left.\frac{d\langle y|x,\mathcal{S}\rangle}{dx}\right|_{\bar{x},\bar{y}} \quad \text{etc.}$$

FIGURE 3-11b. The Approximate Influence Matrix For Deterministic Influences

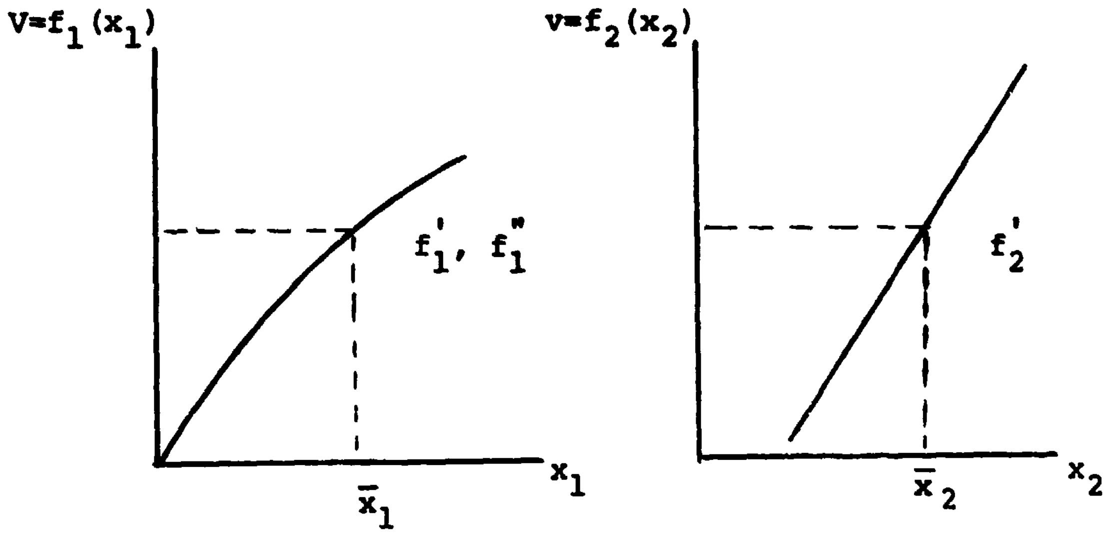


FIGURE 3-12. Deterministic Sensitivity For Two Variables

with x_2 , $v(x_2) = f_2(x_2)$. By expression (3.4.7), the change in the certain equivalent due to fixing x_1 and x_2 at their means are, respectively,

$$\frac{d^{\sim}\langle v|S\rangle}{d^{v}\langle x_1|S\rangle} \; {}^{v}\langle x_1|S\rangle = -\frac{f_1''}{2} + \frac{\gamma}{2} f_1'^{2} \; {}^{v}\langle x_1|S\rangle \qquad (3.5.2)$$

$$\frac{d^{\sim}\langle v|S\rangle}{d^{v}\langle x_2|S\rangle} \; {}^{v}\langle x_2|S\rangle = +\frac{\gamma}{2} f_2'^{2} \; {}^{v}\langle x_2|S\rangle \qquad (3.5.3)$$

Now, suppose that the budget constraint on the analysis is such that only one of these two variables can be included in the model as uncertain, and the other must be fixed as its mean. Policy regions for this decision as a function of the decision maker's risk tolerance $\rho = \frac{1}{\gamma}$ and f_2' can be obtained by equating (3.5.2) and (3.5.3). Figure 3-13 displays the resulting diagram.

<u>Risk Neutral Decision Maker Is Concerned About Nonlinearity</u>

The most surprising feature of the figure is that for risk neutral decision makers, x_1 is the variable that should be treated as uncertain in order to minimize the error in the certain equivalent of the profit lottery. This result is easily seen by setting $\gamma = 0$ in Equations (3.5.2) and (3.5.3). It is surprising because the choice between x_1 and x_2 does not depend on the slope of the deterministic sensitivity plot. Even though the change in the certain equivalent between extreme values of x_2 could be greater than the change between extreme values of x_1, it is x_1 that should be included as uncertain.

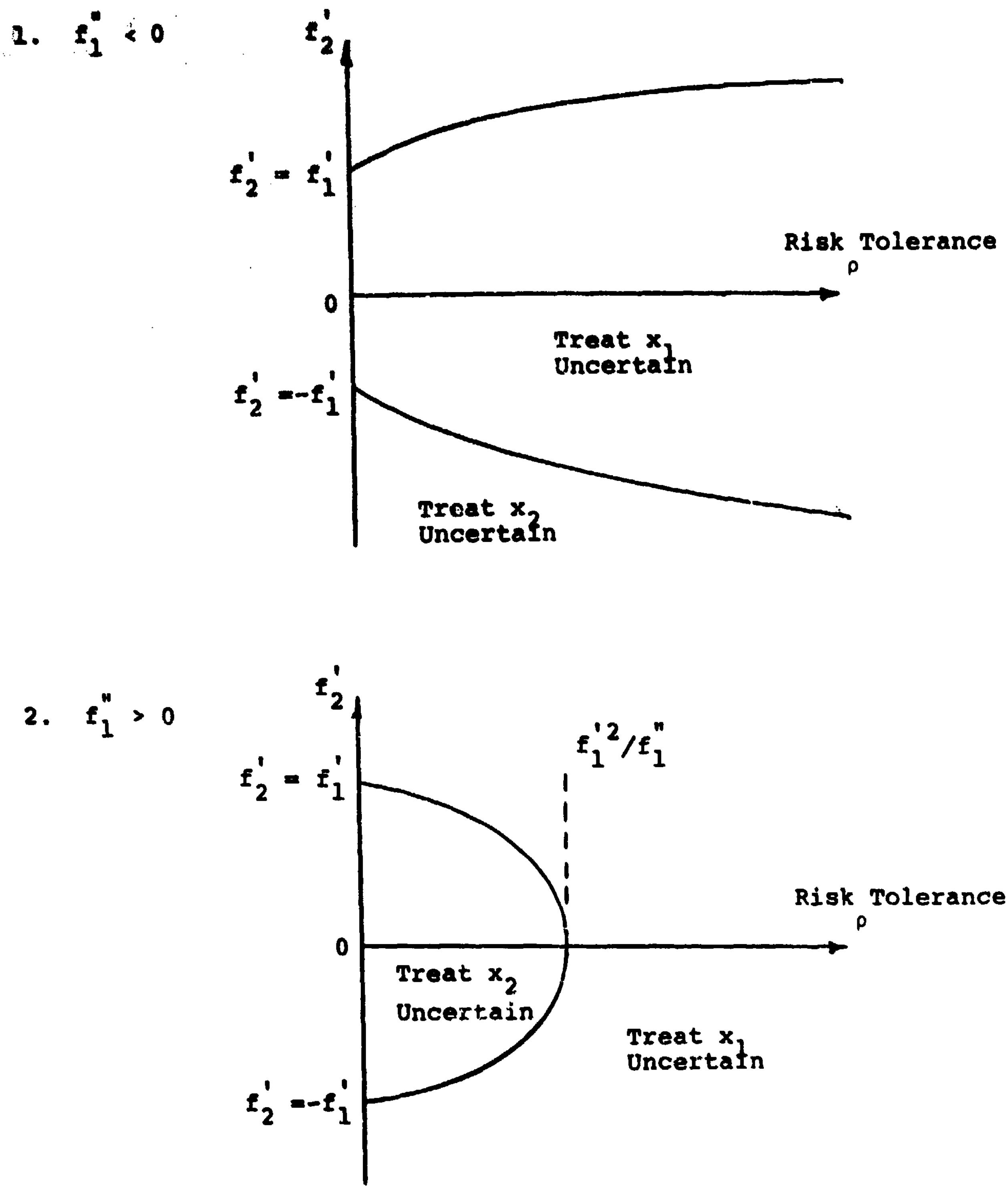


FIGURE 3-13. Modeling Policy Diagram For Treating The Variable x_1 or x_2 As Uncertain On The Basis Of Their Deterministic Sensitivities

The linear variations of the functions $f_1(x)$ and $f_2(x)$ do not appear in the expression for $\langle v(x)|S\rangle$. Consequently, the risk neutral decision maker needs to concentrate only on modeling the nonlinear dependency of the value function on the state variable x_1 , because that dependency alters the expectation of the profit lottery. However, the linear variations of $f_1(x)$ and $f_2(x)$ do affect the variance and risk premium. As the decision maker's risk tolerance decreases, he is increasingly concerned with the risk premium and the linear terms. The risk averse decision maker must balance the error that is made by ignoring the effect of $f_2(x)$ on the risk premium against the errors made by ignoring the effects of $f_1'(x)$ and $f_1''(x)$ on the risk premium and the effect of $f_1''(x)$ on the expectation of the profit lottery.

Dependency of the Decision Model on the Decision Maker's Risk Attitude

Perhaps a more important philosophical issue suggested by Figure 3-13 is the implication that the choice of variables for the decision model should depend on the decision maker's risk attitude. Two decision makers with the same information about a problem but with different risk attitudes may model a decision problem differently.

Decision analysts have always distinguished themselves from other modelers by insisting that the specifications of a model be determined by a particular decision. Our work suggests that the specifications of the model must not only be determined by a particular decision, but also by a particular decision maker. For decision models, as for probability, there is no objectively correct model. The appropriate model depends on the preferences and beliefs of the decision maker.

An Example Showing How Risk Attitude Affects the Decision Model

In order to determine the magnitude of the impact of risk attitude on the decision model, in this section we apply our results to a decision analysis of an actual investment, which has not been published because of its proprietary nature.

The decision problem was to select strategies to maximize the net present value of a new service venture. Figure 3-14 shows the deterministic sensitivity results for the seven most important variables: service price, equilibrium plant capacity, start-up date, governmental restriction, transfer payment, future competition, and initial plant capacity. A brief explanation of each variable is given in Table 3-3. The analysis is not sensitive to the cost of the project since the service facility has been constructed.

Suppose that a total of five variables can be treated as uncertain and four of the five, x_1, x_2, x_3, and x_4 , have been selected. What recommendation can be made regarding the variables x_5, x_6, and x_7? The functional forms of the deterministic sensitivities to these variables are presented in Figure 3-15. Table 3-4 lists the first and second derivatives of these sensitivities, the means and variances of the state variable probability distributions, and the values of the expressions

TABLE 3-3. Brief Explanation of the State Variables in the Example Problem

x_1	Service Price	-	Since this service is unique in the U.S., a large uncertainty exists about the demand curve.
x_2	Equilibrium Plant Capacity	-	Though the plant that provides the service is constructed, it has not operated, and new technology is involved.
x_3	Start-up Date	-	Even if the owners decide to begin plant operation as soon as possible, the actual start-up date is uncertain. Delayed start-up reduces the present value of the profit stream.
x_4	Governmental Restriction	-	Because the client is currently the only company able to provide this service, the possibility of government restriction of profit exists.
x_5	Transfer Payment	-	Part of the service is performed by another company. The cost of that service is uncertain.
x_6	Future Competition	-	Competitors may enter the market in the future.
x_7	Initial Plant Capacity	-	The plant is expected to start production at less than equilibrium capacity. Early capacity is uncertain, however.

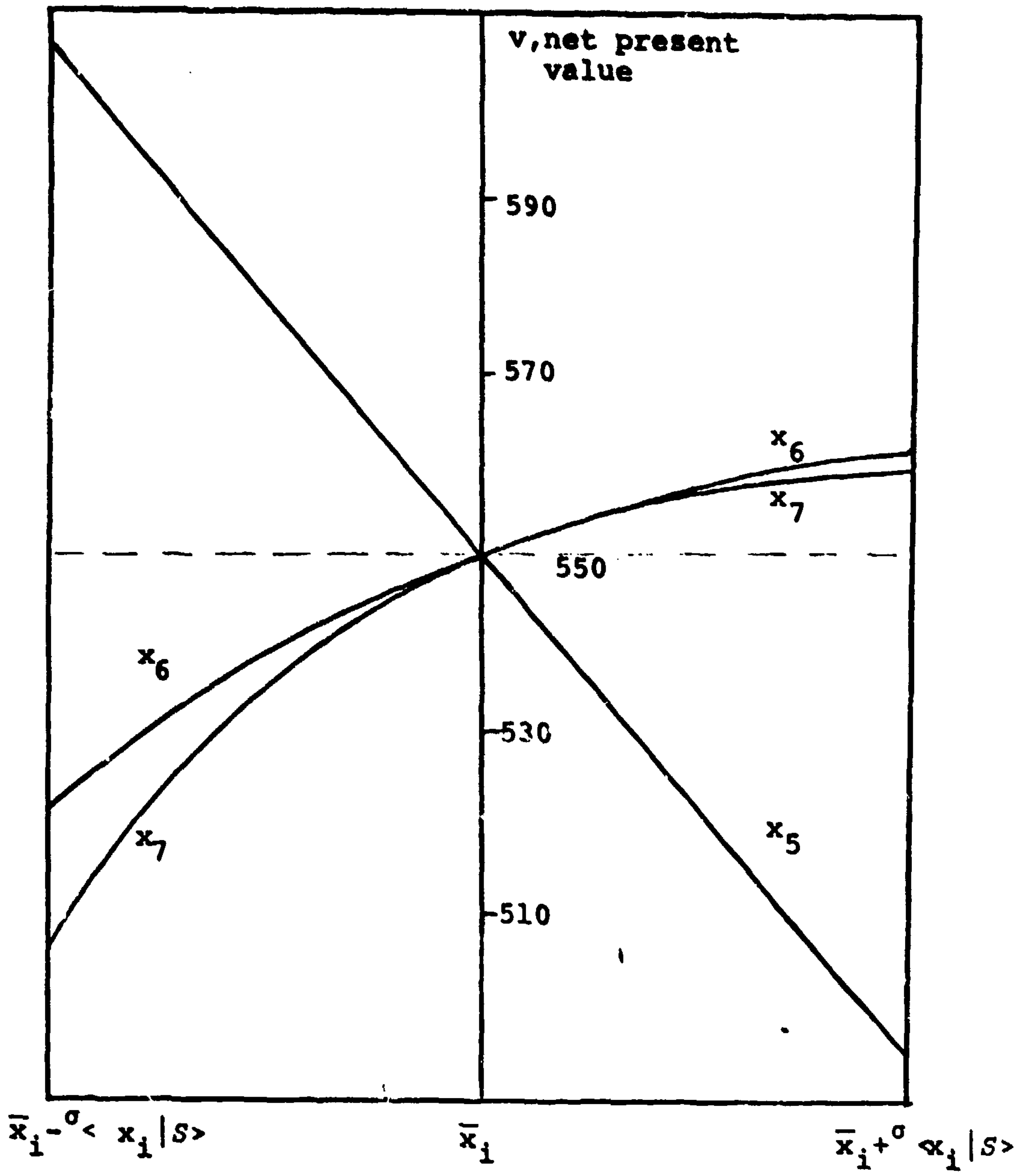


FIGURE 3-15. Functional Forms For The Deterministic Sensitivities of x_5, x_6, and x_7.

TABLE 3-4. Numerical Data For An Example Showing The Impact Of Risk Attitude On The Decision Model

x_i	$\ll x_i\|S\rangle$	$v_{\langle x_i\|S\rangle}$	f_i'	f_i''	$\frac{d\langle v\|S\rangle}{d\langle x_i\|S\rangle}\ .1^{\sigma}\langle x_i\|S\rangle$	$\frac{d\langle v'\|S\rangle}{d\langle x_i\|S\rangle}\ v \ll x_i\|S\rangle$
x_5 Transfer Payment	54	117	-5.18	0	5.99	-6.27
x_6 Reneg. Price	42.25	45.3	2.94	-.159	1.98	2.87
x_7 Initial Capacity	599	35233	.13	-.00049	2.44	-9.78

$$\left| \frac{d^{\sim}\langle v|S\rangle}{d\langle x_i|S\rangle} \right|_{\bar{x}_i} (.1)^{\sigma}\langle x|S\rangle \Bigg| \quad \text{and} \quad \left| \frac{d^{\sim}\langle v|S\rangle}{d^{v}\langle x_i|S\rangle} \right|_{\bar{x}} {}^{v}\langle x_i|S\rangle \Bigg| .$$

Recall these last two expressions approximate, respectively, the change in the certain equivalent due to a change in the mean of a state variable and the change in the certain equivalent due to ignoring uncertainty in the state variable.

This table reveals that the variable with the largest deterministic sensitivity x_5 is not the first one that should be treated probabilistically. To the contrary, x_7 , with the smallest sensitivity would produce the greatest change in certain equivalent if its uncertainty was ignored. As we discussed, in the previous section this effect is the result of the non-linearity of the determinis sensitivity of x_7 , as seen in Figure 3-15.

As we might expect, obtaining an authentic mean for variable x_5 is more important than for variables x_6 and x_7 . Upon inspection of the influence consequence m : (Figure 3-10), of the three variables x_5, x_6, and x_7, x_5 and x_6 should be fixed at their means, but x_5 should be carefully assessed or futher modeled.

Finally, the modeling policy diagram for the decision to treat x_5 or x_7 as uncertain is shown in Figure 3-16.

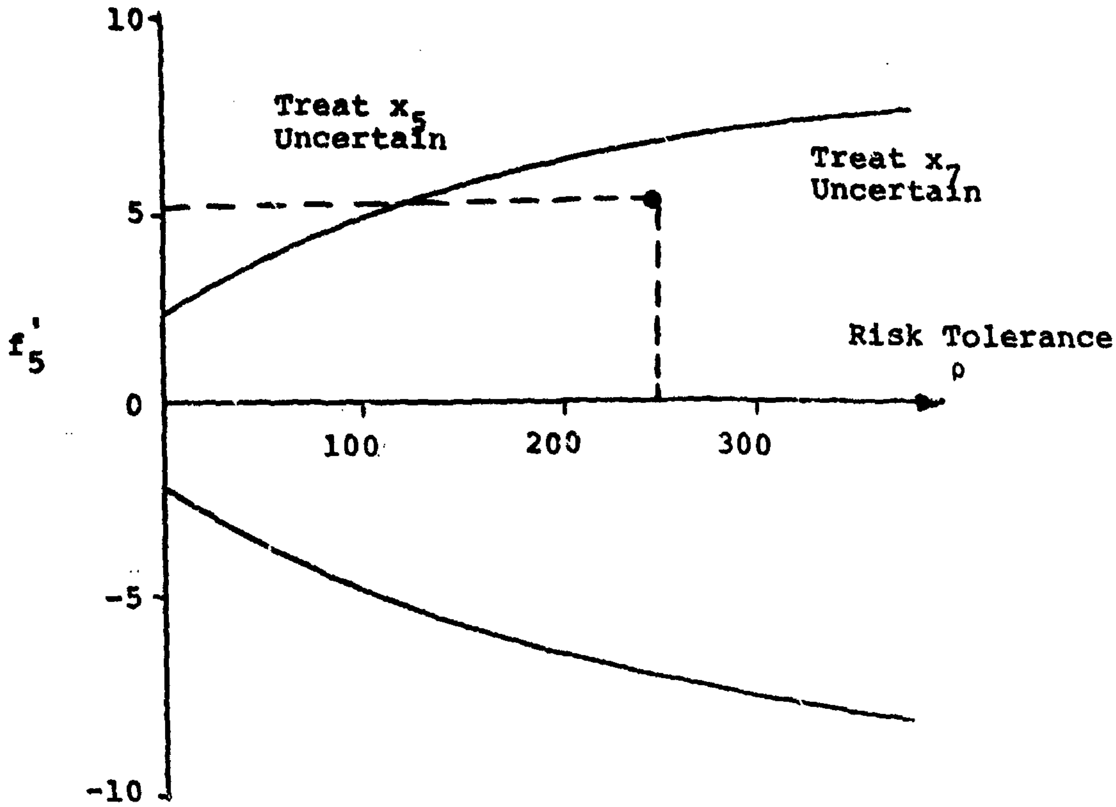


FIGURE 3-16. Modeling Policy Diagram For Treating The Variable x_5 Or x_7 As Uncertain Showing The Location Of The Transfer Price Deterministic Sensitivity

CHAPTER 4

A Comparison of the Quadratic Approximation with Discretization

4.1 Introduction

We begin this chapter by showing what factor determines the information required to compute exactly the moments of the profit lottery. If the conditional moments of the profit lottery are polynomials of degree k in the conditioning variable, then those moments must be assessed at $k+1$ values of the state variable. Furthermore, only k moments of the marginal distribution are required. Hence, it is the functional form of the conditional surface that determines the required information.

Our use of influence matrices depends on approximating the conditional moments by quadratic functions. In Section 4.3 of this chapter, we present procedures for assessing the information required by the quadratic approximation.

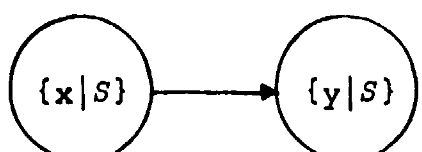
Since current practice for computing the profit lottery is to discretize the conditional distribution, in Section 4.4 we compare discretization with the quadratic approximation. We find that neither the quadratic approximation nor discretization holds a clear advantage. Our sample problem does demonstrate a deficiency in discretization that is not generally appreciated. While discretization of the marginal distribution preserves the mean of the marginal distribution, it does not generally yield the correct mean of the profit lottery.

A related result of this chapter is a derivation of discretization showing under what conditions discretization of a state variable will lead to the correct moments of the profit lottery. This derivation is identical to the derivation of the quadratic procedure. The only difference between discretization and the quadratic procedure is in the assumption about the conditional moments of the profit lottery given the value of the state variable.

Since the theoretical derivation of the quadratic method differs from discretization in the assumption about the conditional moments and since the quadratic method is comparable to discretization in both ease of assessment and accuracy, it should be considered as an alternate method for the solution of decision problems directly from their influence diagrams.

4.2 An Overview of Two Proposed Assessment Procedures

Consider the problems of estimating the marginal probability distribution on a variable y that is influenced by another variable x.



$\{y|\mathcal{S}\}$ might be the profit lottery and x a state variable that influences y through several intermediate state variables, or y might be another state variable directly influenced by $\{x|\mathcal{S}\}$.

In either case, our earlier result, equation (2.2.2), indicates that

$$\langle y^m|\mathcal{S}\rangle = \langle y^m|x = \bar{x},\mathcal{S}\rangle + \sum_{n=2} \frac{1}{n!} \left.\frac{d^n\langle y^m|x,\mathcal{S}\rangle}{dx^n}\right|_{\bar{x}} \langle (x - \bar{x})^n|\mathcal{S}\rangle$$

Now, suppose that $\langle y^m|x,\mathcal{S}\rangle$ is approximated by a polynomial of degree k . Then this equation becomes a finite sum of terms,

$$\text{(4.2.1)} \quad \langle y^m|S\rangle \simeq \langle y^m|x = \bar{x},S\rangle + \sum_{n=2}^{k} \frac{1}{n!} \left.\frac{d^n\langle y^m|x,S\rangle}{dx^n}\right|_{\bar{x}} \langle (x-\bar{x})^n|S\rangle$$

The error in the estimation of $\langle y^m|S\rangle$ due to the approximation is

$$-\sum_{n=k+1}^{\infty} \frac{1}{n!} \left.\frac{d^n\langle y^m|x,S\rangle}{dx^n}\right|_{\bar{x}} \langle (x-\bar{x})^n|S\rangle$$

If $\langle y^m|x,S\rangle$ and $\{x|S\}$ are known, this expression can be used to obtain the value of $\langle y^m|S\rangle$ within any desired degree of accuracy. It should be clear from equation (4.2.1) that as one increases k and the accuracy of the approximation, more and more information is required about the conditional surface. Using equation (4.2.1) required knowledge of $\langle y^m|x,S\rangle$ at $k+1$ values of x in order to determine the k^{th} derivative of $\langle y^m|x,S\rangle$.

In the remainder of the chapter we discuss assessment procedures for the case of $k = 2$, in which the moments of y are approximated as quadratic functions of x . These procedures can be extended for higher values of k , though additional assessments are required.

If we approximate the conditional moment by a quadratic function of x and restrict our attention to means and variances, (4.2.1) becomes

$$\text{(4.2.2a)} \quad \langle y|S\rangle \simeq \langle y|x = \bar{x},S\rangle + \frac{1}{2} \left.\frac{d^2\langle y|x,S\rangle}{dx^2}\right|_{\bar{x}} v\langle x|S\rangle$$

$$\text{(4.2.2b)} \quad \langle y^2|S\rangle \simeq \langle y^2|x = \bar{x},S\rangle + \frac{1}{2} \left.\frac{d^2\langle y^2|x,S\rangle}{dx^2}\right|_{\bar{x}} v\langle x|S\rangle$$

We propose two procedures for obtaining the information required by (4.2.2). First, by assessing $\{y|x = \bar{x}, S\}$, the analyst can compute $\langle y|x = \bar{x}, S\rangle$ and ${}^{\sigma}\langle y|x = \bar{x}, S\rangle$. If the decision maker is familiar with the concepts of conditional mean and conditional standard deviation, the analyst can then ask for the shape of $\langle y|x, S\rangle$ and ${}^{\sigma}\langle y|x, S\rangle$ as functions of x. The two points $\langle y|x = \bar{x}, S\rangle$ and ${}^{\sigma}\langle y|x = \bar{x}, S\rangle$ are combined with the respective shapes to obtain $\langle y|x, S\rangle$ and ${}^{\sigma}\langle y|x, S\rangle$. From these two curves the required second derivatives can be approximated. This entire procedure is discussed more thoroughly and demonstrated in the next section.

A second procedure for less technical decision makers is to assess $\{y|x, S\}$ at two values of x in addition to $\bar{x}$. For example, if $\{y|\bar{x} + {}^{\sigma}\langle x|S\rangle, S\}$, $\{y|\bar{x}, S\}$, and $\{y|\bar{x} - {}^{\sigma}\langle x|S\rangle, S\}$ are assessed, corresponding values of the conditional mean and second moments can be computed. The second derivatives required for (4.2.2) could be obtained using a three-point approximation.

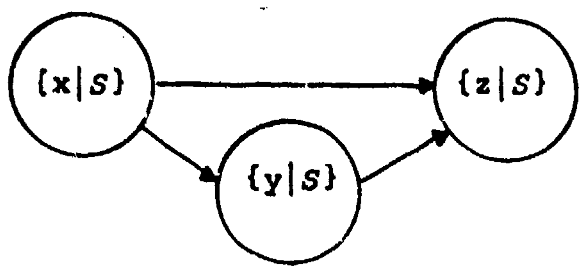
However, when the entire distribution $\{y|x, S\}$ is assessed at three points, it is possible to estimate the marginal distribution $\{y|S\}$, rather than just its mean and variance. Instead of (4.2.2), a similar equation with a similar derivation is used,

$$(4.2.3) \qquad \{y|S\} \simeq \{y|x = \bar{x}, S\} + \frac{1}{2} \left.\frac{\partial^2\{y|x, S\}}{\partial x^2}\right|_{\bar{x}} {}^{v}\langle x|S\rangle$$

This approximation is exact when $\{y|x, S\}$ is quadratic in x. The second procedure, which estimates $\{y|S\}$, is also demonstrated in the next section.

Notice that both quadratic procedures can be applied repetitively. That is, for the situation of $\{x|S\}$ influencing $\{y|S\}$ influencing $\{z|S\}$, equations (4.2.2) or (4.2.3) can be used to obtain $\langle y|S\rangle$ and $^{v}\langle y|S\rangle$. These two moments and the appropriate assessments of $\{z|y,S\}$ can be used in equations like (4.2.2) or (4.2.3) to obtain either $\langle z|S\rangle$ and $\langle z^2|S\rangle$ or the entire distribution $\{z|S\}$.

Using the approximation for the covariance introduced in Chapter 3, equation (3.2.4), both equations can be extended to include situations where the profit lottery is influenced by more than one variable. For example, corresponding to the influence diagram,



the equation for $\{z|S\}$ is

$$\{z|S\} \simeq \{z|\bar{x},\bar{y},S\} + \frac{1}{2}\left.\frac{\partial^2\{z|x,y,S\}}{\partial x^2}\right|_{\bar{x},\bar{y}} {}^{v}\langle x|S\rangle + \frac{1}{2}\left.\frac{\partial^2\{z|x,y,S\}}{\partial y^2}\right|_{\bar{x},\bar{y}} {}^{v}\langle y|S\rangle$$

(4.2.4)

$$+ \left.\frac{\partial^2\{z|x,y,S\}}{\partial x\partial y}\right|_{\bar{x},\bar{y}} \left.\frac{\partial\langle y|x,S\rangle}{\partial x}\right|_{\bar{x}} {}^{v}\langle x|S\rangle + \left.\frac{\partial^2\langle y|x,S\rangle}{\partial x^2}\right|_{\bar{x}} \langle (x-\bar{x})^3|S\rangle$$

Again, this equation can be applied repetitively.

The information that must be assessed to obtain the marginal distribution by (4.2.4) or (4.2.3) or the moments by (4.2.2) is the same information required to obtain the influence matrices in those cases. Hence, the assessments required to determine all of the influence matrices for a given influence diagram can also be used to estimate the profit lottery.

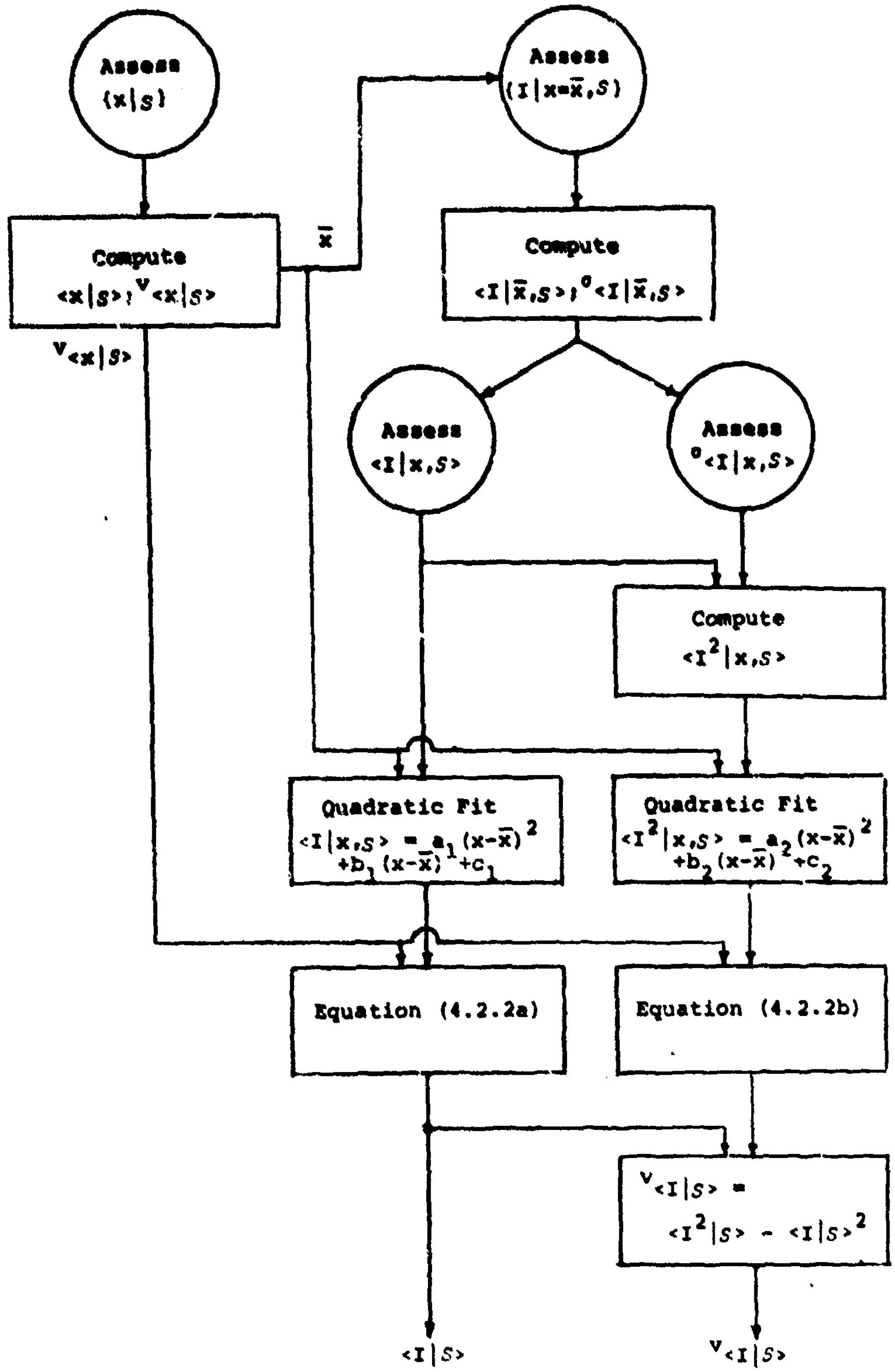
4.3. An Example Demonstrating the Two Quadratic Procedures

In a demonstration of the two quadratic procedures for computing the properties of the profit lottery, consider the age and income distributions for males ages 25 through 64 according to the U. S. Census [19]. Suppose that the decision maker is familiar with the age distribution of that population and the conditional income distribution given age. Then, two procedures, based on equations (4.2.2) and (4.2.3) are available to approximate the marginal income distributions.

Procedure I: Assessment of Conditional Mean and Conditional Variance

Equation (4.2.2) can be used to approximate the mean and variance of the marginal distribution on income with the assessment of only two probability distributions. Figure 4-1 is a flow chart for this procedure. The marginal distribution on age for males ages 25 through 64, shown in Figure 4-2, is assessed in order to compute its mean and variance. According to the figure, these values are 43.7 and 126, respectively. After the mean of the age distribution is determined, the distribution of income conditional on the mean age is assessed. Then, the mean and variance of income conditioned on the mean age can be computed.

The mean income at the mean age is used as a benchmark in obtaining the curve of mean income as a function of age. The decision maker is asked how his estimate of the mean income would


Assess
{x|S}
Compute
<x|S>; v<x|S>
x̄
v<x|S>
Assess
{I|x=x̄,S}
Compute
<I|x̄,S>; σ<I|x̄,S>
Assess
<I|x,S>
Assess
σ<I|x,S>
Compute
<I²|x,S>
Quadratic Fit
<I|x,S> = a1(x-x̄)² +b1(x-x̄)+c1
Quadratic Fit
<I²|x,S> = a2(x-x̄)² +b2(x-x̄)+c2
Equation (4.2.2a)
Equation (4.2.2b)
v<I|S> = <I²|S> - <I|S>²
<I|S>
v<I|S>

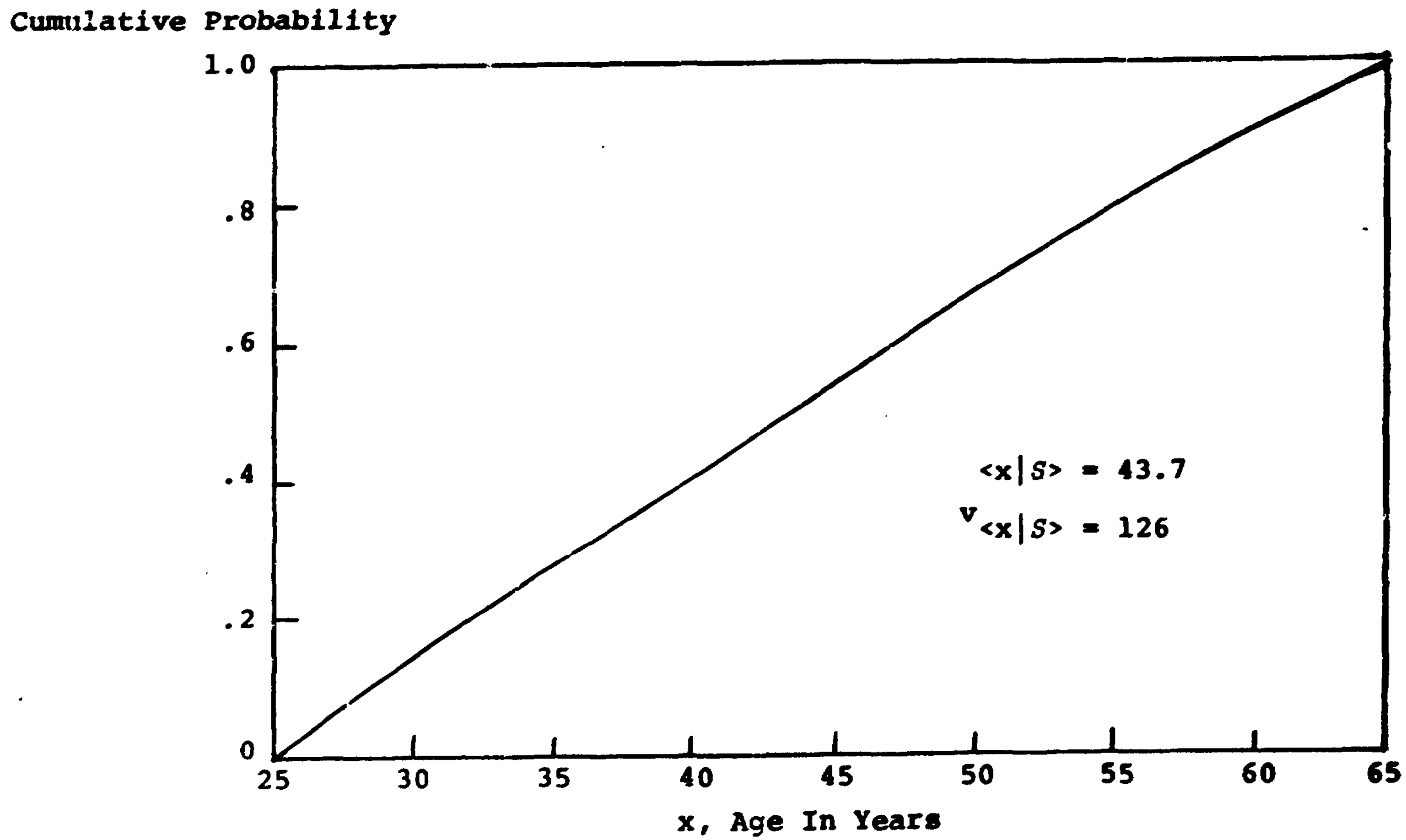


FIGURE 4-2. The Marginal Population Distribution For Males Ages 25 Through 65

change as the revealed value of age increases and decreases from the mean of the age distribution. Similarly, the standard deviation of income at the mean age provides a benchmark for assessing the conditional standard deviation as a function of age. The curves of conditional mean income and income standard deviation computed for the census data are presented in Figure 4-3.

Of course, the conditional standard deviation is a difficult concept, and not all decision makers will be able to draw the required graph, even with the benchmark. It may be necessary to do additional assessment in order to help the decision maker estimate the conditional standard deviation. However, if it is necessary to assess carefully the entire conditional distribution at several values of the conditioning variable, then the second procedure, discussed below, should be used.

Once the curves of conditional mean and standard deviation have been assessed, the conditional second moment can be computed as a function of the conditioning variable. The conditional second moment of income given age $\langle I^2|x,S\rangle$ is also shown in Figure 4-3. Each curve, $\langle I|x,S\rangle$ and $\langle I^2|x,S\rangle$, is fit with a quadratic function and, the coefficients are then used in equation (4.2.2) to compute the moments of the influenced marginal distribution. For our example, we have

$$\langle I|S\rangle = \langle I|x,S\rangle + \frac{1}{2}\frac{d^2\langle I|x,S\rangle}{dx^2}\bigg|_{\bar{x}} v\langle x|S\rangle$$

$$= 10350 + \frac{1}{2}(-17.3)(126)$$

$$= 9260$$

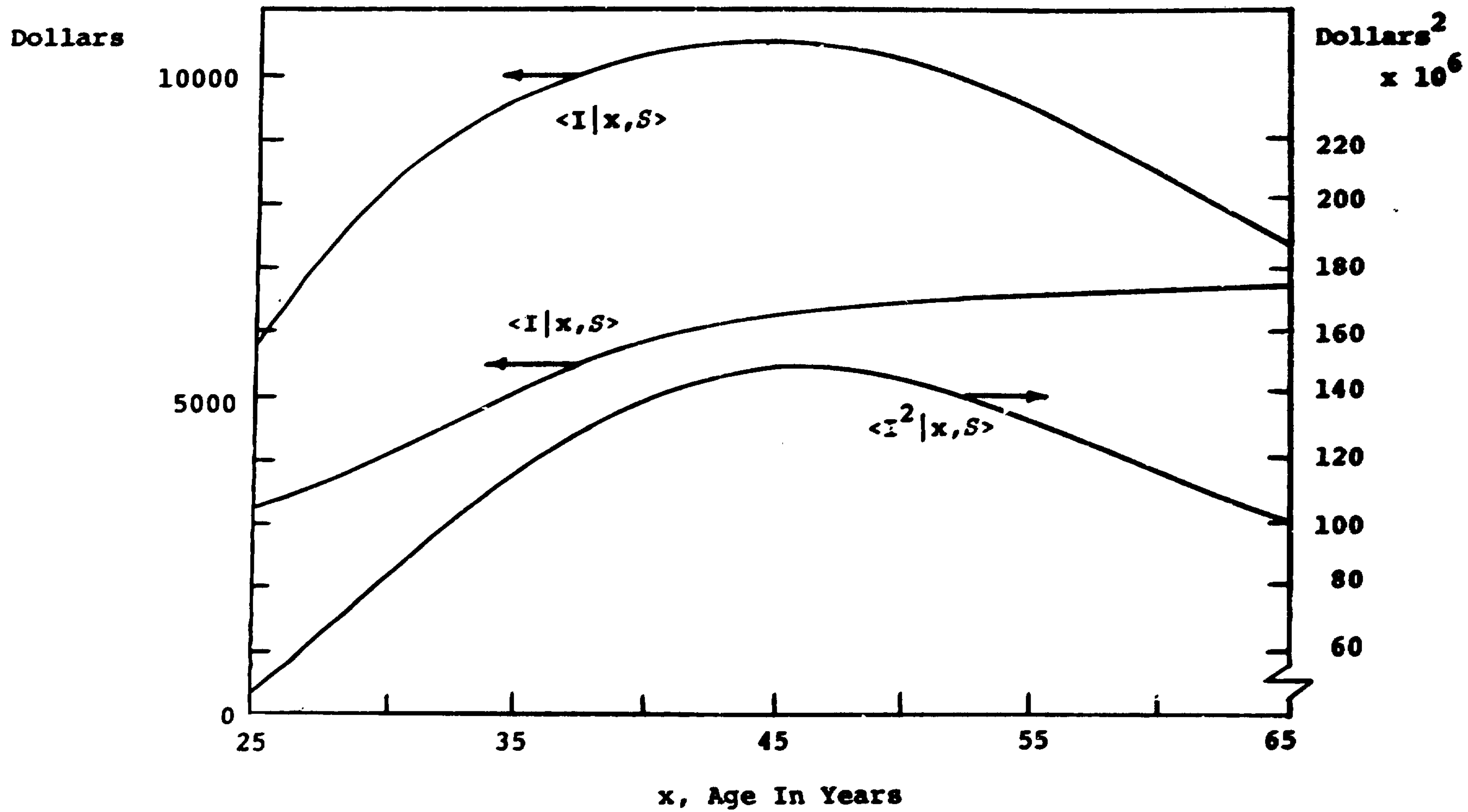


FIGURE 4-3. Conditional Mean, Standard Deviation, And Second Moment Of Income Given Age

$$\langle I^2|S\rangle = \langle I^2|x,S\rangle + \frac{1}{2} \left.\frac{d^2\langle I^2|x,S\rangle}{dx^2}\right|_{\bar{x}} v_{\langle x|S\rangle}$$

$$= 147 \times 10^6 \cdot \frac{1}{2} (-\ .494 \times 10^6)\ (126)$$

$$= 116 \times 10^6$$

and

$$v_{\langle I^2|S\rangle} = \langle I^2|S\rangle - \langle I|S\rangle$$

$$= 30.0 \times 10^6$$

<u>Procedure II: Assessment of Conditional Distributions</u>

Approximating the age distribution using equation (4.2.3) requires the assessment of four complete distributions. Figure 4-4 displays a flow chart for this procedure. The necessary assessments are the marginal age distribution $\{x|S\}$ and three distributions for income conditioned on particular values of age. The conditioning values of age are selected to give a good quadratic approximation to the age-income surface. For example, the following equation works well

$$\left.\frac{\partial^2\{I|x,S\}}{\partial x^2}\right|_{\bar{x}} \simeq \frac{\dfrac{\{I|x=\bar{x}+\sigma_{\langle x|S\rangle},S\}-\{I|x=\bar{x},S\}}{\sigma_{\langle x|S\rangle}} - \dfrac{\{I|x=x,S\}-\{I|x=\bar{x}-\sigma_{\langle x|S\rangle},S\}}{\sigma_{\langle x|S\rangle}}}{\sigma_{\langle x|S\rangle}}$$

(4.3.1)

Therefore, the conditioning values of age for the three assessments of income distribution depend on the mean and standard deviation of the age distribution.

From the marginal distribution on age (Figure 4-2), the mean age of that population is 43.7, and its variance is 126. The three required distributions of income conditioned on age are shown in Figure 4-5. These distributions are used in equation (4.3.1) to

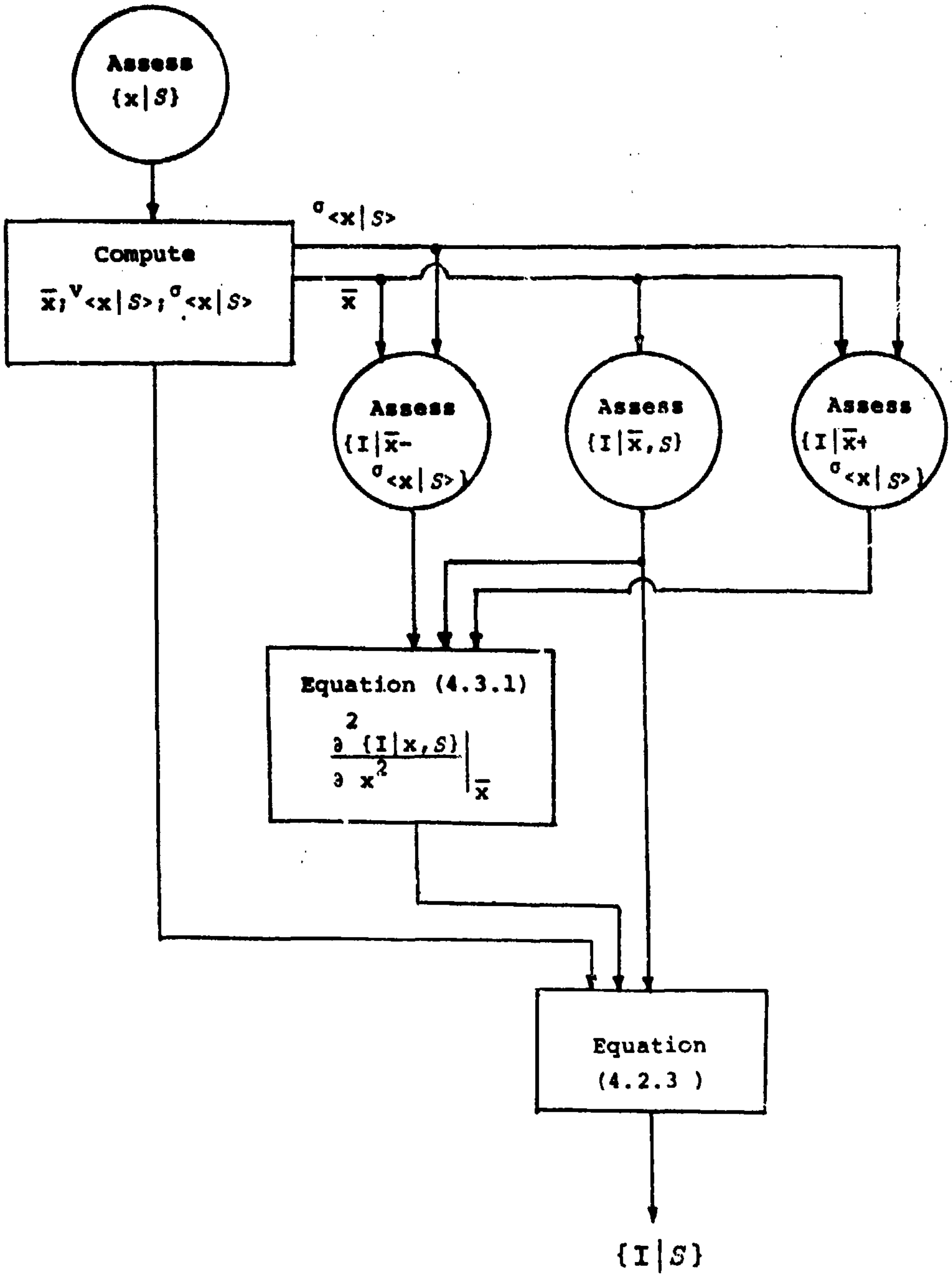


FIGURE 4-4. Flow Chart For The Quadratic Approximation Procedure Using Three Conditional Distributions

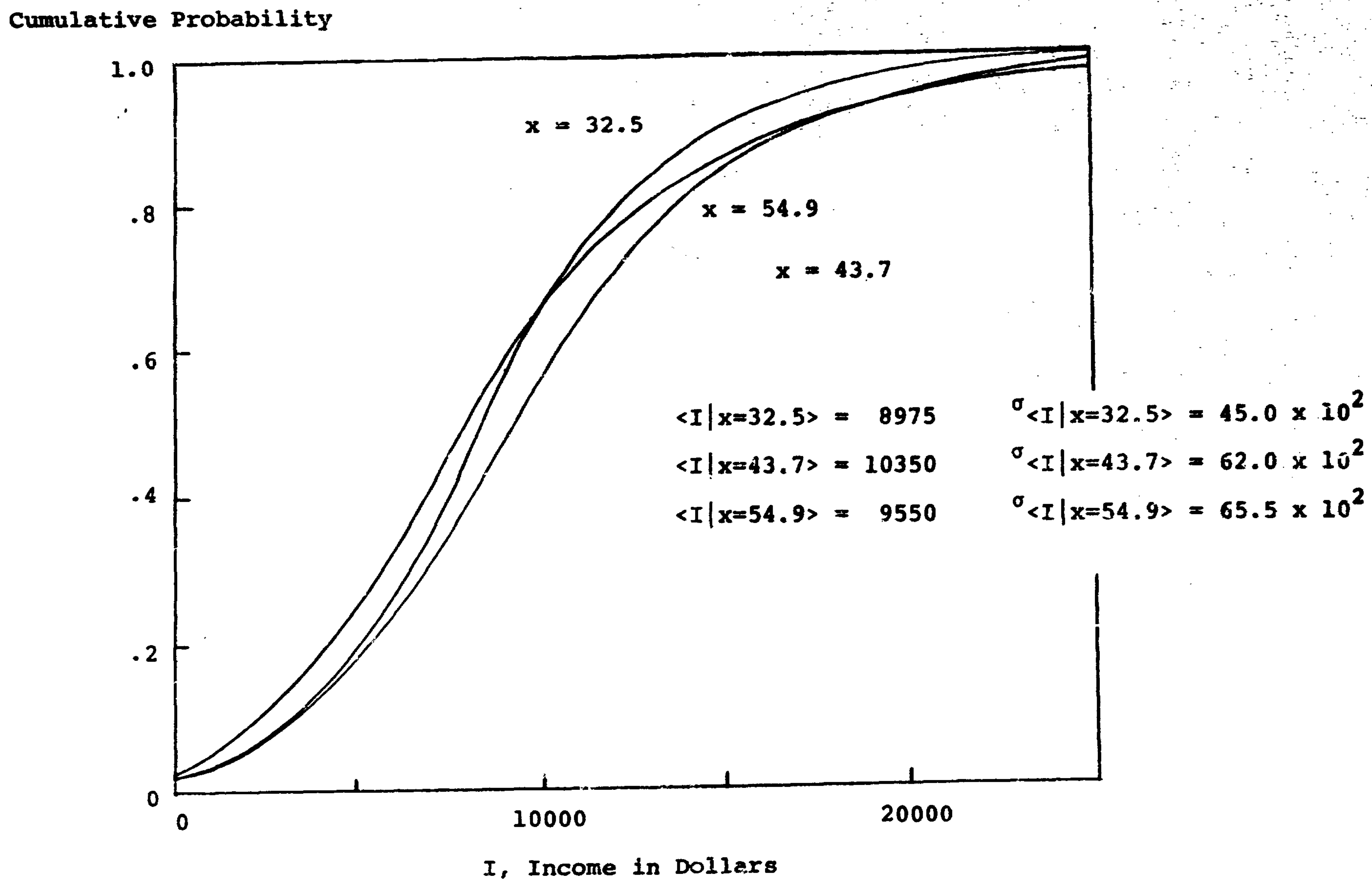


FIGURE 4-5. Three Conditional Distributions On Income Given Age

find $\left.\frac{\partial^2\{I|x,S\}}{\partial x^2}\right|_{\bar{x}}$ which is used along with $\{I|x=\bar{x},S\}$ in equation (4.2.3) in obtaining $\{I|S\}$. Figure 4-6 displays the uneveness of the surface $\{I|x,S\}$ and also shows the quadratic approximation to the surface. According to Figure 4-7, the quadratic approximation for the marginal income distribution closely matches the exact distribution.

<u>The Conventional Procedure: Discretization</u>

In contrast to the quadratic methods, Figure 4-8 diagrams the steps in approximating $\{I|S\}$ using the conventional discrete method. Again, four complete distributions must be assessed: the marginal distribution on age and three conditional income distributions. Following the assessments, each distribution is discretized to find the expectation in each interval.

The marginal distribution on age used in the discretization (Figure 4-2) is the same one used for the quadratic approximation. It is discretized at the points

$$\langle x|x \leq {}_{.275}x,S\rangle = 30$$

$$\langle x|\,{}_{.275}x < x \leq {}_{.80}x,S\rangle = 43.7$$

$$\langle x|\,{}_{.80}x < x,S\rangle = 60$$

in order to reduce the amount of data interpolation required. The notation ${}_px$ refers to the p-fractile of $\{x|S\}$. Conditional distributions $\{I|x,S\}$ are assessed at each of the three ages given in (4.3.2) and discretized.

In Figure 4-9, the discretization of the age and conditional income distributions is displayed as an event tree. By taking the

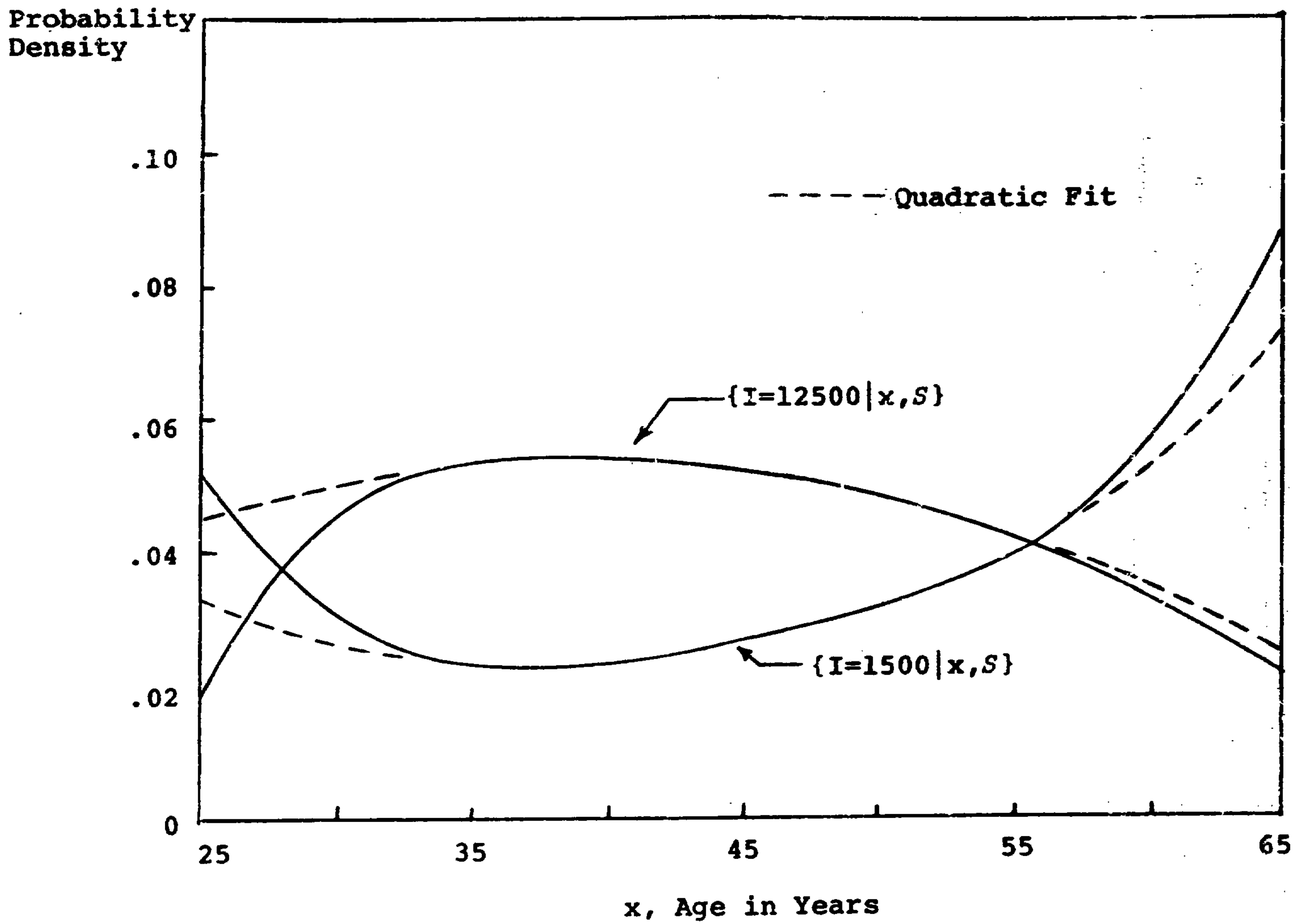


FIGURE 4-6. Two Slices Through The Conditional Probability Surface And The Quadratic Approximation

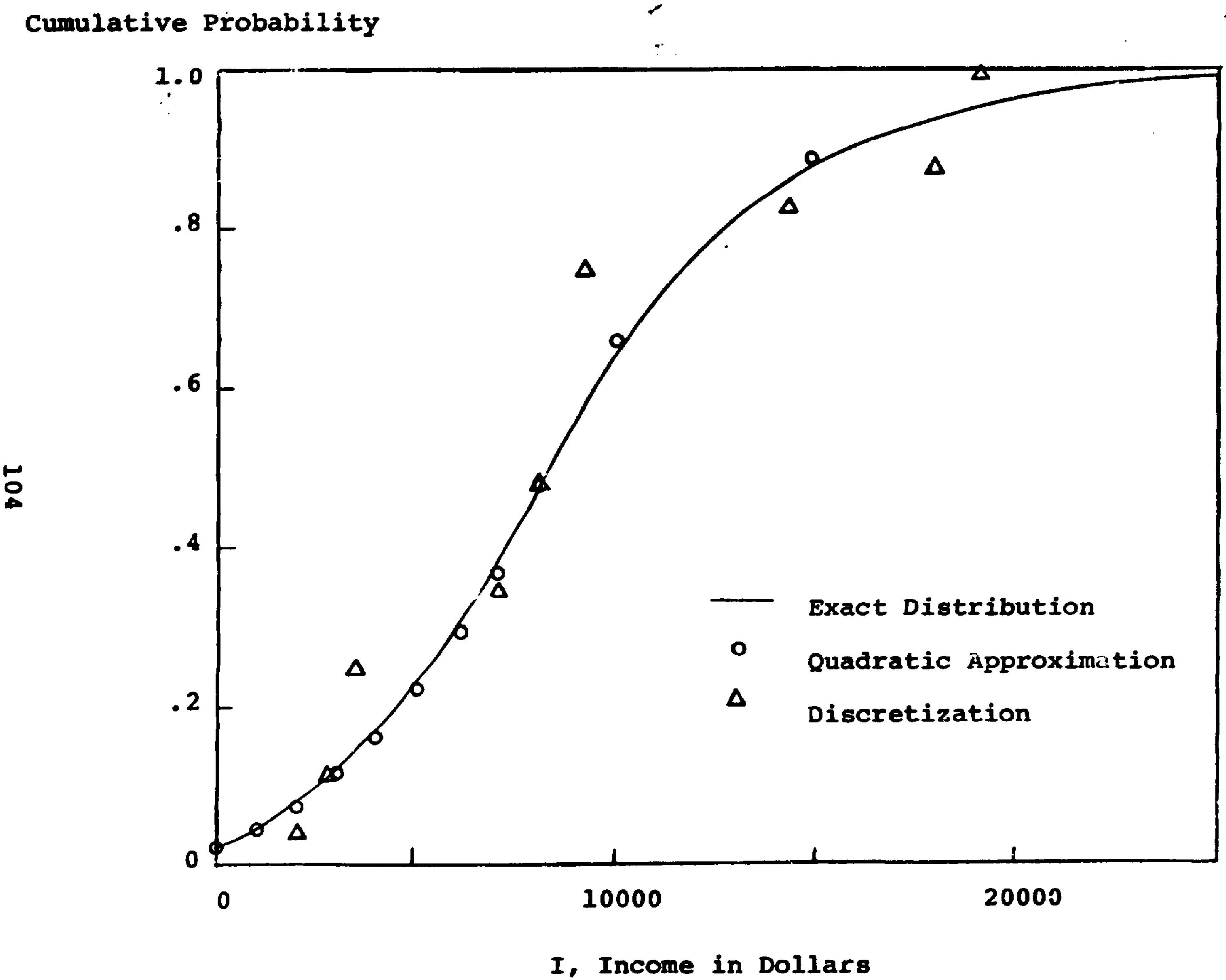


FIGURE 4-7. Comparison Of The Exact Marginal Distribution On Income With The Quadratic And Discrete Approximations

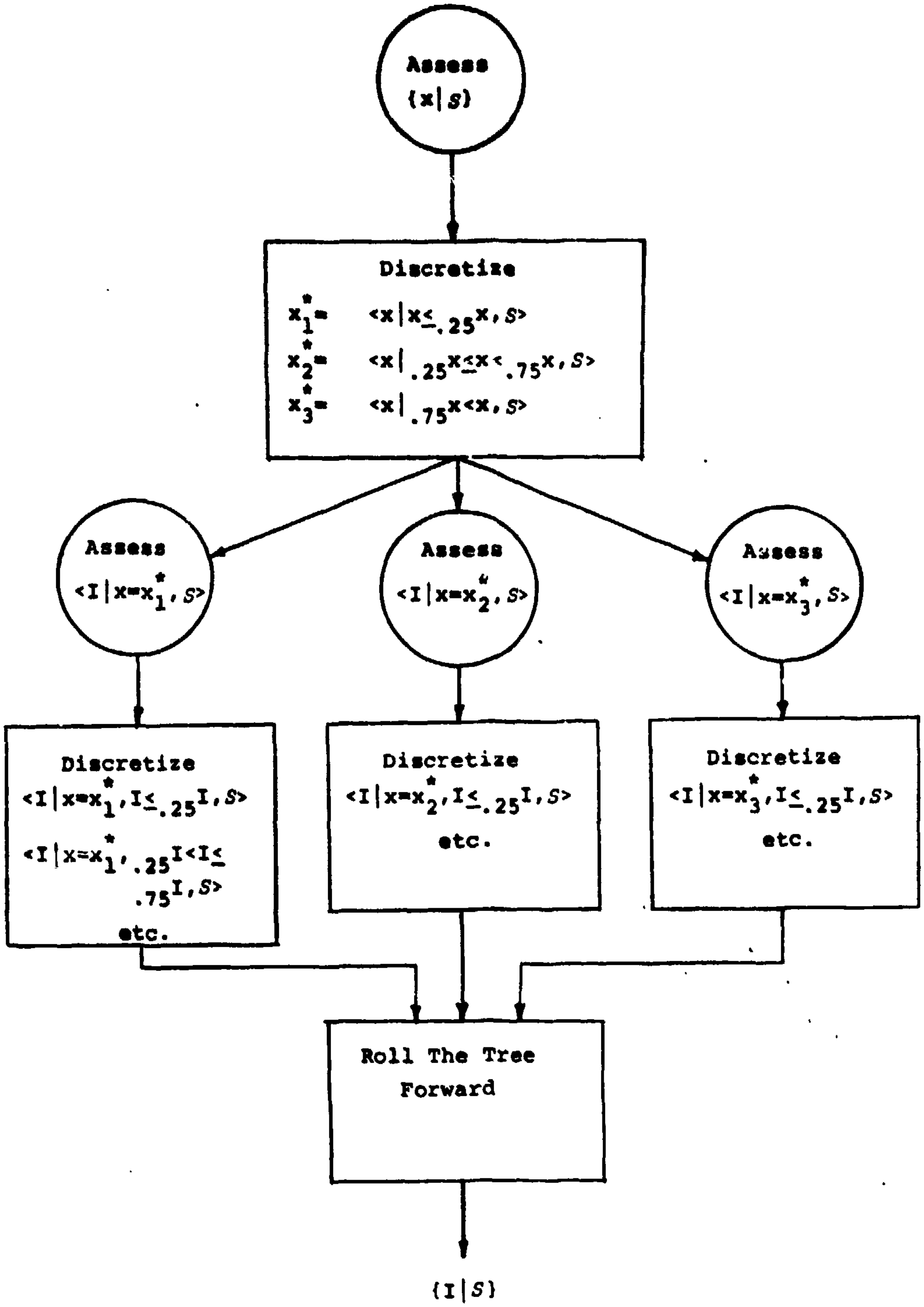


FIGURE 4-8. Flow Diagram For The Conventional Method Of Discretization

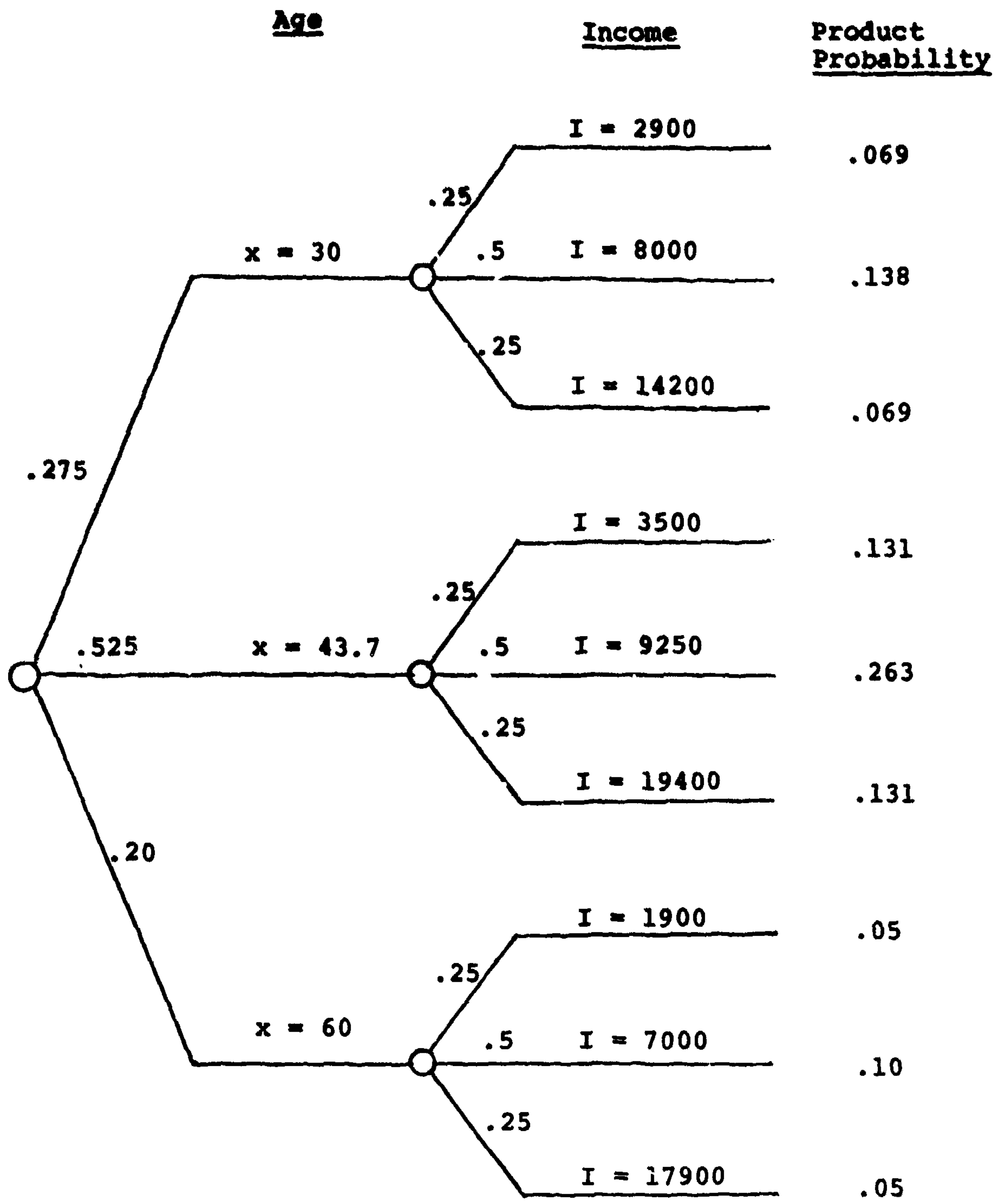


FIGURE 4-9. Event Tree Resulting From Discretization

product of the event tree probabilities, "rolling the tree forward", the marginal distribution on income can be approximated. The discrete approximation is plotted along with the quadratic approximation and the exact distribution in Figure 4-7.

It is evident from Figure 4-7 that the quadratic approximation is quite accurate, and the discrete approximation is less accurate. However, the tabulation of Table 4-1, comparing the moments of the approximation with the exact values, shows the discrete and continuous approximations to be comparable and acceptable. In the next section the theoretical basis for discretization is investigated, and the reason for its overestimation of the expected income is explained.

4.4 Comparison of Discretization With the Quadratic Approximation

The degree of confidence that we have in a model structure resulting from influence matrices and our use of the associated estimate of the profit lottery depends on the accuracy of the quadratic approximation. If the approximation is crude, then the influence matrices can only provide the analyst with a means to conceptualize the modeling process. The associated estimate of the profit lottery would need to be verified by more exact methods. On the other hand, if the quadratic approximation is sufficiently accurate, then the influence method would yield a model structure in which the analyst is confident. Furthermore, additional research to determine how to use the estimates of the profit lottery to solve the decision problem directly from the influence diagram would be warranted.

TABLE 4-1

Comparison of the Discrete and Quadratic Methods with the Exact Distribution

Quantity	Exact	Discrete	Quadratic
Expectation $\langle I\|S\rangle$	9218	9399	9256
	-	2%	0%
Second Moment $\langle I2\|S\rangle$	119×10^6	124×10^6	116×10^6
	-	4%	- 3%
$^{V}\langle I\|S\rangle$	34	36	30
	-	+ 6%	- 12%

The most frequently used method for computing the moments of a distribution that is dependent on another variable is discretization of the influencing variable. Discretization means dividing the range of x into intervals and associating the probability weight from an interval with a point within that interval. For example, if $\{x|S\}$ is a normal distribution, the analyst can divide x into three intervals at the 25% and 75% fractiles of $\{x|S\}$, denoted ${}_{.25}x$ and ${}_{.75}x$, respectively. Then, he can assign probability weights of .25, .50, and .25 to the points

$$\langle x| -\infty \leq x < {}_{.25}x\rangle \simeq {}_{.10}x$$

$$\langle x| {}_{.25}x \leq x < {}_{.75}x\rangle \simeq {}_{.50}x$$

$$\langle x| {}_{.75}x \leq x < \infty \rangle \simeq {}_{.90}x$$

This procedure preserves the expectation of $\{x|S\}$, since

$$\langle x|S\rangle = \langle x|x < {}_{.25}x,S\rangle \,{}^{\leq}\{{}_{.25}x|S\}$$
$$+ \langle x|{}_{.25}x \leq x < {}_{.75}x,S\rangle \,[{}^{\leq}\{{}_{.75}x|S\} - {}^{\leq}\{{}_{.25}x|S\}]$$
$$+ \langle x|{}_{.75}x \leq x,S\rangle \,[1 - {}^{\leq}\{{}_{.75}x|S\}]$$

When $\{x|S\}$ influences $\{y|S\}$, the moments of $\{y|S\}$ can be approximated by repeating the procedure. At each of the points given in (4.4.1) the analyst discretizes the conditional distribution $\{y|x,S\}$. The discretization preserves the moments of the <u>conditional</u> distribution, since,

$$\langle y^m|x,S\rangle = \langle y^m|y < {}_{.25}y,x,S\rangle \,{}^{\leq}\{{}_{.25}y|x,S\}$$
$$+ \langle y^m|{}_{.25}y \leq y < {}_{.75}y,x,S\rangle \,[{}^{\leq}\{{}_{.75}y|x,S\} - {}^{\leq}\{{}_{.25}y|x,S\}]$$
$$+ \langle y^m|{}_{.75}y \leq y,x,S\rangle \,[1 - {}^{\leq}\{{}_{.75}y|x,S\}]$$

However, the moments of the marginal distribution, $\{y|S\}$, are only approximated as,

$$\langle y^m|S\rangle \simeq \langle y^m|x = \langle x|x < {}_{.25}x\rangle ,S\rangle \; {}^{\leq}\{{}_{.25}x|S\}$$

$$(4.4.3) \qquad + \langle y^m|x = \langle x|{}_{.25}x \leq x < {}_{.75}x\rangle ,S\rangle [{}^{\leq}\{{}_{.75}x|S\} - {}^{\leq}\{{}_{.25}x|S\}]$$

$$+ \langle y^m|x = \langle x|{}_{.75}x \leq x\rangle ,S\rangle [1 - {}^{\leq}\{{}_{.75}x|S\}]$$

or more generally

$$(4.4.4) \quad \langle y^m|S\rangle \simeq \sum_{n=1}^{N+1} \langle y^m|x = \langle x|x_{n-1} \leq x < x_n\rangle ,S\rangle \; [{}^{\leq}\{x_n|S\} - {}^{\leq}\{x_{n-1}|S\}]$$

where $x_0 = -\infty$ and $x_{N+1} = +\infty$.

While discretization of $\{x|S\}$ preserves the moments of the state variable distribution $\{x|S\}$, it only approximates the momonts of the profit lottery $\{y|S\}$.

An Example Showing that the Moments of the Profit Lottery May Be Overestimated or Underestimated by Discretization

Suppose that $\{x|S\}$ is a normal distribution with standard deviation σ_x and that

$$\langle y^m|x,S\rangle = a_m x^2 + b_m x + c_m \; .$$

Using equation (4.4.3), the approximate value obtained by discretization at the points of (4.4.1) is

$$\langle y^m|D,S\rangle = a_m \bar{x}^2 + .82 \; a_m \; {}^{v}\langle x|S\rangle + b_m \bar{x} + c_m$$

Since the exact value is $a_m \bar{x}^2 + b_m \bar{x} + c_m$, the error due to discretization is

$$(4.4.5) \qquad \langle y^m|S\rangle - \langle y^m|D,S\rangle = + \; .18 \; a_m \; {}^{v}\langle x|S\rangle$$

where D denotes the value computed using the discretization procedure. From (4.4.5) the errors in the mean and variance of $\{y|S\}$ are found to be

$$(4.4.6)\quad \langle y|S\rangle - \langle y|D,S\rangle = .18\, a_1\, {}^v\langle x|S\rangle$$

$${}^v\langle y|S\rangle - {}^v\langle y|D,S\rangle = +\, .18\, a_2\, {}^v\langle x|S\rangle - .36\, a_1\, {}^v\langle x|S\rangle\, \langle y|S\rangle - .097\, a_1^2\, {}^v\langle x|S\rangle^2 \, .$$

The implication of these equations is displayed in Figure 4-10. There will always be an error in the mean of the profit lottery unless $a_1 = 0$, and whether it is overestimated or underestimated depends on the sign of a_1. In either case, the variance of the profit lottery may be overestimated or underestimated.

<u>Derivation of the Discrete Approximation</u>

These considerations motivate an investigation of the conditions under which discretization, as in equation (4.4.4), is exact. We begin with the expansion equation

$$\langle y^m|S\rangle = \int_x \langle y^m|x,S\rangle\, \{x|S\}\, dx \; ,$$

and divide the integration into regions

$$(4.4.7)\quad \langle y^m|S\rangle = \int_{x_0}^{x_1} \langle y^m|x,S\rangle\{x|S\}dx + \dots \int_{x_{k-1}}^{x_k} \langle y^m|x,S\rangle\{x|S\}dx + \dots + \int_{x_{K-1}}^{x_K} \langle y^m|x,S\rangle\{x|S\}dx$$

Within each region the conditional moment can be expanded in a Taylor's series about the expectation of x in that interval,

$$(4.4.8)\quad \langle y^m|x,S\rangle = \langle y^m|x = \langle x|x\varepsilon k\rangle,S\rangle + \sum_{n=1}^{\infty} \frac{1}{n!} \left.\frac{d^n\langle y^m|x,S\rangle}{d^n x}\right|_{x = \langle x|x\varepsilon k\rangle} (x - \langle x|x\varepsilon k\rangle)^n$$

where $k = \{x\colon x_{k-1} < x \le x_k\}$. Substituting (4.4.8) into (4.4.7)

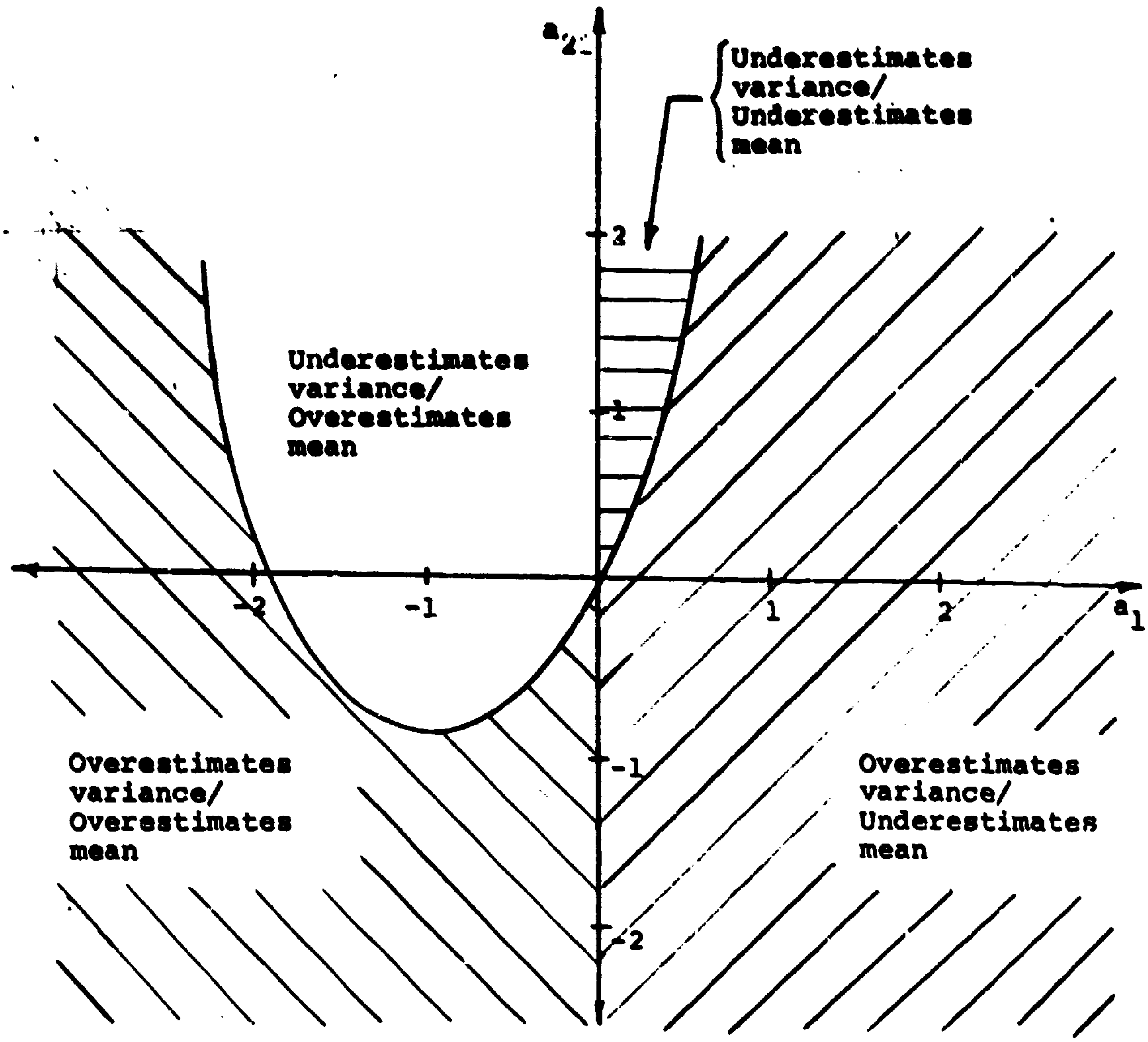


FIGURE 4-10. An Example Showing The Dependence Of The Error Due To Discretization On The Curvature Of The Conditional Mean a_1 And The Conditional Variance a_2

gives

$$(4.4.9)\quad \langle y^m|S\rangle = \sum_{k=1}^{K}\Big[\langle y^m|x = \langle x|x\epsilon k\rangle, S\rangle \{x\epsilon k|S\}$$

$$+ \sum_{n=2}^{\infty} \frac{1}{n!} \left.\frac{d^n\langle y^m|x,S\rangle}{dx^n}\right|_{x = \langle x|x\epsilon k\rangle} \langle (x-\bar{x})^n|x\epsilon k,S\rangle\{x\epsilon k|S\}\Big]$$

If all terms of $n \geq 2$ are ignored, then equation (4.4.9) is equivalent to equation (4.4.4), the usual discrete method for finding the moments of $\{y|S\}$. A similar derivation leads to an expression for the discrete approximation to $\{y|S\}$ itself.

For quadratic conditional moments, as in the two earlier examples, equation (4.4.9) reduces to

$$(4.4.10)\quad \langle y^m|S\rangle = \sum_{k=1}^{K} \langle y^m|x = \langle x|x\epsilon k\rangle, S\rangle\{x\epsilon k|S\}$$

$$+ \frac{1}{2} \left.\frac{d^2\langle y^m|x,S\rangle}{dx^2}\right|_{\langle x|x\epsilon k\rangle} {}^{v}\langle x|x\epsilon k,S\rangle\{x\epsilon k|S\}$$

Consistent with our earlier observations, the sign and magnitude of the error due to discretization depend on the curvature of $\langle y^m|x,S\rangle$. Furthermore, the error will have the same sign in every interval regardless of the marginal distribution $\{x|S\}$.

The usual method of discretization, represented by equation (4.4.4) approximates the conditional moments $\langle y^m|x,S\rangle$ by a piece-wise linear function of x (Figure 4-11). Equation (4.2.2) proposed in this research approximates the conditional moments by a quadratic function of x . Which of these approaches is more accurate depends on the number of intervals in the discrete approximation and the actual shape of the function being approximated.

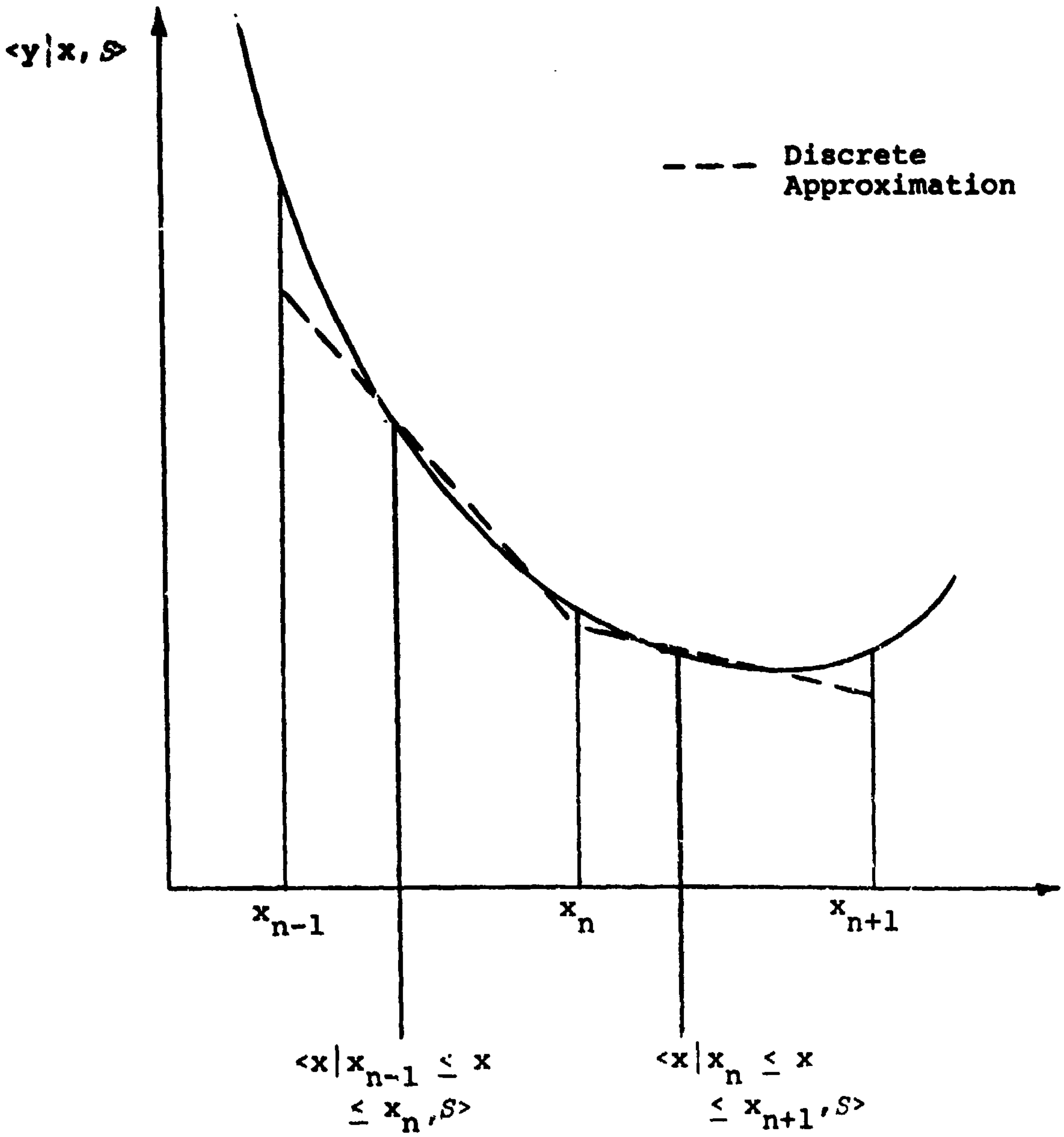


FIGURE 4-11. Discretization Approximates Conditional Moments By A Piecewise Linear Function

We conclude that discretization is not inherently more accurate than the quadratic method. The better method should be determined by the particular application and the ease of assessment.

CHAPTER 5

Summary and Suggestions for Further Research

Our characterization of influence allows the decision analyst to conceptualize the relationship between the probability distributions on different variables in a decision model. One can think of an influence between two probability distributions $\{x|S\}$ and $\{y|S\}$ as transforming the moments of one distribution into the moments of the other. The strength and type of the influence can be measured by a set of derivatives denoted as

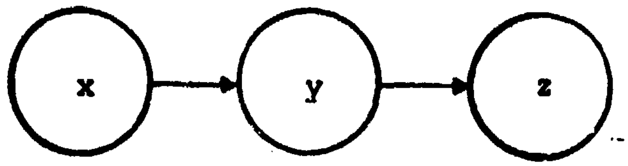
$$(5.1) \qquad \left.\frac{d\langle (y-\bar{y})^m|S\rangle}{d\langle (x-\bar{x})^n|S\rangle}\right|_{\bar{x}} \qquad n > 1,\ m > 1$$

which extends in an obvious way for $n = 1$ and $m = 1$.

Therefore, the influence of x on y is characterized by an infinite matrix that has its n, m element given by (5.1), and is denoted by

$$\frac{d\{y|S\}}{d\{x|S\}} .$$

This influence matrix has several useful properties. First, the matrix is null if and only if the moments of $\{y|S\}$ are unaffected by the moments of $\{x|S\}$ and y is probabilistically independent of x. Hence, the matrix is null if and only if no influence exists. Second, when the matrix has nonzero elements, those elements specify the type and strength of the influence. Finally, because the elements of the matrix follow the rules of differential calculus, a calculus of influences appears possible. For example, in the diagram:



the influence matrix of x on z may be computed as

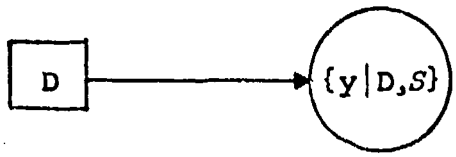
$$\frac{d\{z|S\}}{d\{x|S\}} = \frac{d\{y|S\}}{d\{x|S\}} \; \frac{d\{z|S\}}{d\{y|S\}}$$

Using several examples, we have demonstrated that influence matrices can be used to structure decision problems. The influence-consequence matrix introduced in Chapter 3 shows that the elements of the influence matrix can be used to determine which areas of the decision model require further modeling and which variables should be treated as uncertain in the decision model. These recommendations amount to a new interpretation of deterministic sensitivity.

We have also used the influence matrix to demonstrate that there is no single correct decision model. The appropriate decision model depends on the risk attitude of the decision maker.

Finally, we investigated the theoretical basis for discretizating marginal probability distributions to compute the profit lottery. Discretization and the quadratic method, on which influence matrices are based, differ only in the assumption about the functional form of the profit lottery conditioned on the state variable. Since the two methods are comparable in case of assessment and accuracy, we conclude that the quadratic method should be considered as an alternate method for the solution of decision problems directly from their influence diagrams without the intermediate step of constructing a decision tree.

To develop this possibility would require extending the influence theory and influence calculus presented in Chapters 2 and 3 to include decision variables. We illustrate our approach to this problem with the following example. Consider a decision variable D influencing the profit lottery $\{y|D,\mathcal{S}\}$:



Both the mean and variance of the profit lottery will generally depend on D. Limiting our attention to those two moments, we write

$$\langle y|D,\mathcal{S}\rangle \simeq \langle y|D_0,\mathcal{S}\rangle + \left.\frac{d\langle y|D,\mathcal{S}\rangle}{dD}\right|_{D_0}(D - D_0) + \frac{1}{2}\left.\frac{d^2\langle y|D,\mathcal{S}\rangle}{dD^2}\right|_{D_0}(D - D_0)^2 \tag{5.2}$$

$${}^{v}\langle y|D,\mathcal{S}\rangle \simeq {}^{v}\langle y|D_0,\mathcal{S}\rangle + \left.\frac{d\,{}^{v}\langle y|D,\mathcal{S}\rangle}{dD}\right|_{D_0}(D - D_0) + \frac{1}{2}\left.\frac{d^2\,{}^{v}\langle y|D,\mathcal{S}\rangle}{dD^2}\right|_{D_0}(D - D_0)^2 \tag{5.3}$$

where D_0 is an arbitrary nominal setting of the decision variable. Let us define the influence of D on $\{y|D,\mathcal{S}\}$, analogous to our earlier definition for the influence between state variables, as

$$\text{(5.4)} \qquad \frac{d\{y|\mathcal{S}\}}{dD} = \begin{bmatrix} \left.\frac{d\langle y|D,\mathcal{S}\rangle}{dD}\right|_{D_0} & \left.\frac{d^{v}\langle y|D,\mathcal{S}\rangle}{dD}\right|_{D_0} \\ \left.\frac{d^{2}\langle y|D,\mathcal{S}\rangle}{dD^{2}}\right|_{D_0} & \left.\frac{d^{2}v\langle y|D,\mathcal{S}\rangle}{dD^{2}}\right|_{D_0} \end{bmatrix}$$

The elements of this matrix can be obtained by an assessment procedure similar to that discussed in Chapter 4.

Now, the first order condition for the optimum decision D^* is

$$\left.\frac{d^{\sim}\langle y|\mathcal{S}\rangle}{dD}\right|_{D^*} = 0$$

or approximately

$$\text{(5.5)} \qquad \left.\frac{d\langle y|D,\mathcal{S}\rangle}{dD}\right|_{D^*} - \frac{1}{2}\gamma \left.\frac{d^{v}\langle y|D,\mathcal{S}\rangle}{dD}\right|_{D^*} = 0 \;.$$

Expressions for the two derivatives are obtained by differentiating (5.2) and (5.3). Upon substitution into (5.5), we obtain an estimate for the difference between D_0 and the optimum decision D^*,

$$\text{(5.6)} \qquad D^* - D_0 = - \frac{\left.\frac{d\langle y|D,\mathcal{S}\rangle}{dD}\right|_{D_0} - \frac{1}{2}\gamma \left.\frac{d^{v}\langle y|D,\mathcal{S}\rangle}{dD}\right|_{D_0}}{\left.\frac{d^{2}\langle y|D,\mathcal{S}\rangle}{dD^{2}}\right|_{D_0} - \frac{1}{2}\gamma \left.\frac{d^{2}v\langle y|D,\mathcal{S}\rangle}{dD^{2}}\right|_{D_0}}$$

Equation (5.6) suggests that the optimal decision can be determined from the influence of the decision on the profit lottery, equation (5.4), and the decision maker's risk attitude. In fact, the right side of equation (5.6) is the ratio of the elements of the influence vector

$$\frac{d^{\sim}\langle y|S\rangle}{dD} = \frac{d\{y|S\}}{dD}\,[1 \quad - \tfrac{1}{2}\gamma]^T \,. \tag{5.7}$$

Of course, considerable theoretical development is required to show how to solve general decision problems from the influence diagram. Our simple example assumes a continuous decision variable, which has derivatives, but in practice decision variables are often discrete. Furthermore, our simple influence diagram consists of a single decision variable and a single state variable. Research is needed to show how to solve sequential decision problems with several state variables by the influence method. Another important related question is how to determine the value of information directly from the influence diagram.

Our work with the influence method for state variables, presented in Chapters 2, 3, and 4, and our simple example in this chapter, lead us to believe that additional research and experience with the influence method will lead to more developments of practical value and theoretical interest. We encourage further research in the application of the influence concept to decision problems.

References

[1]. Cook, W. Howard, "Decision Analysis for Product Development," IEEE Transactions on Systems Science and Cybernetics, Vol. SSC-4, No. 3, September 1968, pp. 324-354.

[2]. Hill, J. Doublas and Richard G. Ollila, "Analysis of Complex Decision Making Processes," IEEE Transactions on Systems, Man, and Cybernetics, Vol. SMC-8, No. 3, March 1978, pp. 193-204.

[3]. Howard, Ronald A., "Decision Analysis: Applied Decision Theory," Proceedings of the Fourth International Conference on Operational Research, Wiley-Interscience, 1966, pp. 55-71.

[4]. Howard, Ronald A., "The Foundations of Decision Analysis," IEEE Transactions on Systems Science and Cybernetics, Vol. SSC-4, No. 3, September 1968, pp. 211-219.

[5]. Howard, Ronald A., "Proximal Decision Analysis," Management Science, Theory, Vol. 17, No. 9, May 1971, pp. 507-541.

[6]. Howard, Ronald A., J. E. Matheson, and D. W. North, "The Decision to Seed Hurricanes," Science, Vol. 176, 16 June 1972, pp. 1191-1202.

[7]. Howard, Ronald A., "Social Decision Analysis," Proceedings of the IEEE, Vol. 63, No. 3, March 1975, pp. 359-371.

[8]. Matheson, James E., "The Economic Value of Analysis and Computation," IEEE Transactions on System Science and Cybernetics, Vol. SSC-4, No. 3, September 1968, pp. 325-332.

[9]. Matheson, James E., "Decision Analysis Practice: Examples and Insights," Proceedings of the Fifth International Conference on Operational Research, Venice, 1969, Travistock Publications, 1970, pp. 677-691.

[10]. Miller, Allen C., et al, Development of Automated Aids for Decision Analysis, Research Report to the Defense Advanced Research Projects Agency by SRI International, Menlo Park, Ca., May 1976, pp. 115-174.

[11]. Moran, P. A. P., An Introduction to Probability Theory, Clarendon Press, Oxford, 1968, p. 319.

[12]. Neuts, Marcel F., Probability, Allyn and Bacon, Inc., Boston, 1973, p. 277.

[13]. Owen, Daniel L., "Mutual Information as a Potential Definition of Influence," unpublished paper.

[14]. Pratt, John W., "Risk Aversion in the Small and in the Large," Econometrica, Vol. 32, No. 1-2, January-April, 1964.

[15]. Sage, Andrew P., Methodology for Large Scale Systems, McGraw-Hill Book Company, New York, 1977, pp. 13-164.

[16]. Smallwood, R. D., "A Decision Analysis of Model Selection," IEEE Transactions on Systems Science and Cybernetics, Vol. SSC-4, No. 3, September 1968, pp. 333-342.

[17]. Spetzler, Carl S., and Ramon M. Zamora, "Decision Analysis of a Facilities Investment and Expansion Problem," in Readings in Decision Analysis ed. by Ronald A. Howard, James E. Matheson, and Katherine L. Miller, SRI International, Menlo Park, Ca., 1976, pp. 205-233.

[18]. Tani, Stephen N., Modeling and Decision Analysis, Ph.D. dissertation, Department of Engineering-Economic Systems, Stanford University, June 1975.

[19] U. S. Bureau of the Census, Census of Population: 1970 Vol. 1, Characteristics of the Population, Part I, United States Summary - Section 2, U. S. Government Printing Office, Washington, D.C., 1973.

Appendix A

A Series Representation and Approximation for the Covariance

An expression for $Cov(x,y)$ in terms of the moments of $\{x|\mathcal{S}\}$ can be obtained by writing,

$$\text{(A.1)} \qquad Cov(x,y) = \int_x\int_y (x-\bar{x})(y-\bar{y})\{x,y|\mathcal{S}\}\,dydx$$

$$\text{(A.2)} \qquad = \int_x (x-\bar{x})\langle y|x,\mathcal{S}\rangle\{x|\mathcal{S}\}dx$$

$$\text{(A.3)} \qquad = \int_x x\langle y|x,\mathcal{S}\rangle\{x|\mathcal{S}\}dx - \bar{x}\bar{y}$$

Expanding $\langle y|x,\mathcal{S}\rangle$ in a Taylor's series gives

$$\text{(A.4)} \qquad \langle y|x,\mathcal{S}\rangle = \langle y|x,\mathcal{S}\rangle + \sum_{n=1}^{\infty} \frac{1}{n!} \left.\frac{d^n\langle y|x,\mathcal{S}\rangle}{dx^n}\right|_{\bar{x}} (x-\bar{x})^n$$

Upon substitution of (A.4) into (A.2) and completion of the integration, one obtains

$$\text{(A.5)} \qquad Cov(x,y) = \sum_{n=1}^{\infty} \frac{1}{n!} \left.\frac{d^n\langle y|x,\mathcal{S}\rangle}{dx^n}\right|_{\bar{x}} \langle (x-\bar{x})^{n+1}|\mathcal{S}\rangle \; .$$

When $\langle y|x,\mathcal{S}\rangle$ is linear in x, the covariance is equal to the first term of (A.5),

$$\text{(A.6)} \qquad Cov(x,y) = \left.\frac{d\langle y|x,\mathcal{S}\rangle}{dx}\right|_{\bar{x}} V_{\langle x|\mathcal{S}\rangle}$$

This expression can be interpreted graphically. Equation

(A.3) shows that the covariance is a comparison of the expected area $\langle y|x,\mathcal{S}\rangle\, x$ with the area $\bar{x}\bar{y}$. Because of the linearity of $\langle y|x,\ \rangle$, we have

$$\langle y|\mathcal{S}\rangle = \langle y|\bar{x},\mathcal{S}\rangle \ .$$

The variance of $\{x|\mathcal{S}\}$ can be interpreted roughly as an oscillation of x about $\langle x|\mathcal{S}\rangle$, and it induces a similar oscillation of $\langle y|x,\mathcal{S}\rangle$ about $\langle y|\bar{x},\mathcal{S}\rangle$. Since the increase in the area $\langle y|x,\mathcal{S}\rangle\, x$ is larger when x is greater than $\bar{x}$ than the corresponding reduction when x is less than $\bar{x}$, uncertainty in x increases the expectation of $\langle y|x,\mathcal{S}\rangle\, x$. The amount of the increase of the expectation of $\langle y|x,\mathcal{S}\rangle x$ over $\bar{y}\bar{x}$, which by Equation (A.2) is the covariance, depends on the slope $\left.\frac{d\langle y|x,\mathcal{S}\rangle}{dx}\right|_{\bar{x}}$.

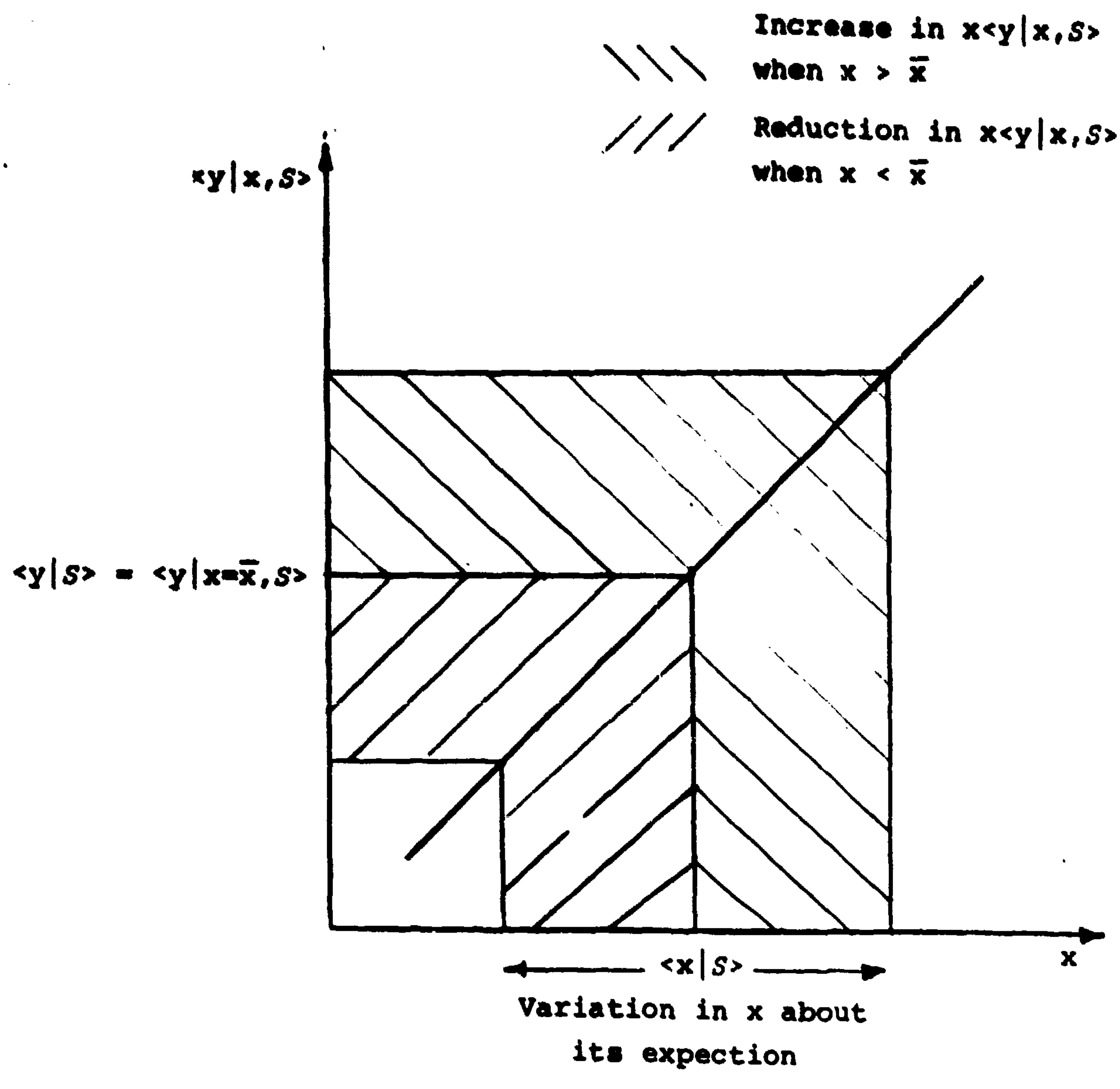


FIGURE A-1. An Interpretation Of The Covariance Using The Approximation Developed In Appendix A

Appendix B

A proof that the change in the certain equivalent of $\{\pi|S\}$ due to fixing x at its mean is approximated by

$$\left.\frac{\partial^{\sim}\langle\pi|S\rangle}{\partial^{v}\langle x|S\rangle}\right|_{\bar{x}} {}^{v}\langle x|S\rangle \; .$$

Taking the first two twerms of equation (2.2.2) for $m = 1,2$ gives

$$\text{(B.1)} \qquad \langle\pi|S\rangle \approx \langle\pi|x = \bar{x},S\rangle + \frac{1}{2} h_{xx}\ {}^{v}\langle x|S\rangle$$

$$\text{(B.2)} \qquad \langle\pi^2|S\rangle \approx \langle\pi^2|x = \bar{x},S\rangle + \frac{1}{2} h^{*}_{xx}\ {}^{v}\langle x|S\rangle \; ,$$

where

$$h_{xx} = \left.\frac{d^2\langle\pi|x,S\rangle}{dx^2}\right|_{\bar{x}} \; , \quad h^{*}_{xx} = \left.\frac{d^2\langle\pi^2|x,S\rangle}{dx^2}\right|_{\bar{x}}$$

Using (B.1) and (B.2), we obtain

$$\text{(B.3)} \qquad {}^{v}\langle\pi|S\rangle = \langle\pi^2|S\rangle - \langle\pi|S\rangle^2$$

$$\approx \langle\pi^2|x = \bar{x},S\rangle - \langle\pi|x = \bar{x},S\rangle^2$$

$$+ \left(\frac{1}{2} h^{*}_{xx} - \langle\pi|x = \bar{x},S\rangle h_{xx} - \frac{1}{4} h^2_{xx}\ {}^{v}\langle x|S\rangle\right){}^{v}\langle x|S\rangle$$

The approximation for the certain equivalent is

$$\text{(B.4)} \qquad {}^{\sim}\langle\pi|S\rangle \approx \langle\pi|S\rangle - \frac{\gamma}{2}\ {}^{v}\langle\pi|S\rangle$$

Using (B.1) and (B.3) in (B.4) yields

$$\text{(B.5)} \qquad {}^{v}\langle\pi|S\rangle \approx \langle\pi|x = \bar{x},S\rangle - \frac{\gamma}{2}\left(\langle\pi^2|x = \bar{x},S\rangle - \langle\pi|x = \bar{x},S\rangle^2\right)$$

$$+\left[\frac{1}{2} h_{xx} - \frac{\gamma}{2}\left(\frac{1}{2}h^{*}_{xx} - \langle\pi|x = \bar{x},S\rangle h_{xx} - \frac{1}{4} h^2_{xx}\ {}^{v}\langle x|S\rangle\right)\right]{}^{v}\langle x|S\rangle$$

The first two terms on the right of (B.5) approximate the certain equivalent of π when $x = \bar{x}$. Hence,

$$\text{(B.6)} \quad {}^{\sim}\langle\pi|S\rangle \quad {}^{\sim}\langle\pi|x = \bar{x},S\rangle \approx [\tfrac{1}{2} h_{xx} - \tfrac{\gamma}{2}(\tfrac{1}{2} h_{xx}^2 - \langle\pi|x = \bar{x},S\rangle h_{xx} - \tfrac{1}{4} h_{xx}^2 \, {}^{v}\langle x|S\rangle)] \; {}^{v}\langle x|S\rangle.$$

According to equation (3.3.3), the influence vector is

$$\frac{d^{\sim}\langle\pi|S\rangle}{d\{x|S\}} = \begin{bmatrix} \dfrac{d^{\sim}\langle\pi|S\rangle}{d\langle x|S\rangle} & \dfrac{d^{\sim}\langle\pi|S\rangle}{d^{v}\langle x|S\rangle} \end{bmatrix}^T$$

$$= \frac{d\{\pi|S\}}{d\{x|S\}} \begin{bmatrix} 1 & -\tfrac{1}{2}\gamma \end{bmatrix}^T$$

The elements of the influence matrix $\frac{d\{\pi|S\}}{d\{x|S\}}$ can be found in Figure 2-4 and used in this equation:

$$\frac{d^{\sim}\langle\pi|S\rangle}{d\{x|S\}} = \begin{bmatrix} h_x & h_x^* - 2\langle\pi|S\rangle h_x \\ \tfrac{1}{2} h_{xx} & \tfrac{1}{2} h_{xx}^* - \langle\pi|S\rangle h_{xx} \end{bmatrix} [1 \quad -\tfrac{1}{2}\gamma]^T.$$

The second element of this product is

$$\text{(B.7)} \quad \frac{d^{\sim}\langle\pi|S\rangle}{d^{v}\langle x|S\rangle} = \tfrac{1}{2} h_{xx} - \tfrac{\gamma}{2} (\tfrac{1}{2} h_{xx}^* - \langle\pi|S\rangle h_{xx})$$

Substituting for $\langle\pi|S\rangle$ according to B.1,

$$\text{(B.8)} \quad \frac{d^{\sim}\langle\pi|S\rangle}{d^{v}\langle x|S\rangle} = \tfrac{1}{2} h_{xx} - \tfrac{\gamma}{2} (\tfrac{1}{2} h_{xx}^* - \langle\pi|x = \bar{x},S\rangle h_{xx} - \tfrac{1}{2} h_{xx}^2 \, {}^{v}\langle x|S\rangle$$

Multiplying (B.8) by ${}^{v}\langle x|S\rangle$ and comparing with (B.6), we find,

$$\frac{d^{\sim}\langle\pi|S\rangle}{d^{v}\langle x|S\rangle}\; {}^{v}\langle x|S\rangle - \frac{\gamma}{8}\, h_{xx}^2\; {}^{v}\langle x|S\rangle^2 \simeq {}^{\sim}\langle\pi|S\rangle - {}^{\sim}\langle\pi|\bar{x},S\rangle$$

Consequently, the expression

(B.9) $$\frac{d^{\sim}\langle\pi|S\rangle}{d\,\langle x|S\rangle}\; {}^{v}\langle x|S\rangle$$

approximates the change in the certain equivalent of $\{\pi|S\}$ due to fixing x at its mean value provided the term

$$\frac{\gamma}{8}\, h_{xx}^2\; {}^{v}\langle x|S\rangle^2$$

is small. If it is not small it can be added to the expression (B.9) to improve the approximation.