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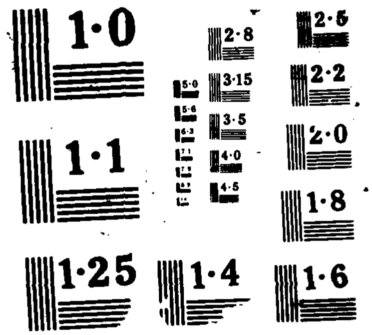
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ON PERIODIC SOLUTIONS OF AN ATWOOD'S PENDULUM

AD-A183 078

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This technical report has been reviewed and is approved for publication.

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An Atwood's pendulum is defined as an Atwood's machine in which one of two masses is allowed to swing as a pendulum while the other remains constrained to move only in the vertical direction. The pendulum motion of the one mass induces a varying tension in the connecting wire; this, in turn, produces motion in the second mass. It is shown that this motion can be made periodic if the ratio of the two masses and the dependency of this ratio on the initial conditions are chosen as prescribed in this report. If this condition is not met, the motion consists of the superposition of two motions. The first is motion in a constant gravitational field where the effective "gravity" is kg; the factor k is determined explicitly. The second is the periodic motion that is the central theme of this report. During the course of the analysis, the fundamental frequency of the periodic motion is determined. It is shown to be slightly higher than the frequency of a pendulum of comparable length swinging in the earth's gravitational field; the factor is given explicitly. This work is restricted to the extent that small angle approximations are introduced initially for trigonometric functions.

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Foreword

This report documents an effort performed by Dr. Don Mittleman, Professor of Mathematics at Oberlin College, Oberlin, Ohio for the Structural Vibration and Acoustics Branch, Structures and Dynamics Division, Flight Dynamics Laboratory, Air Force Wright Aeronautical Laboratories, Wright Patterson Air Force Base, Ohio. The investigation was conducted under Flight Dynamics Laboratory Project 2401, Structural Mechanics; Task 240104, Vibration Prediction and Control, Measurement and Analysis; Work Unit 24010432, Large Space Structures Technology Program.

The work was performed by Dr. Mittleman as a visiting scientist to the Flight Dynamics Laboratory through a contract with Anamet Laboratories, Inc. during the period August 1985 through July 1986. This report is an interim result of Dr. Mittleman's work.

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Section I

Introduction

The Structural Vibration and Acoustics Branch has initiated a program to study the dynamics and control of Large Space Structures (LSS). This Large Space Structures Technology Program (LSSTP) is intended to enable the Flight Dynamics Laboratory to instrument, test, and analyze large space structures on the ground in order to predict their behavior in space. Since the testing is to be done in a ground based laboratory, i.e. under 1-g acceleration, the experiments must be designed so as to counteract this gravitational effect. One proposal is to use soft suspension systems. Long cables can provide pendulum support with low frequency for horizontal motion, but, for a pendulum, the restraint in the vertical direction is rigid. One way to provide soft restraint vertically, while maintaining the pendulum approach, is to counter balance the test model.

The simplest counter-balanced suspension system is the classic Atwood's machine.[1] This consists of two masses m_1 and m_2 connected by an inextensible, flexible wire of negligible mass, draped over a frictionless, massless pulley, which in turn is rigidly suspended from an overhead support. The motion of the two masses is assumed to be constrained to the vertical direction: either there is no motion, the situation that occurs when the two masses are equal and there is no initial velocity, or, as one mass rises the other falls.

In previous reports [2],[3], the Atwood's configuration was studied when the one mass, m_1, remained constrained to move vertically but the other mass, m_2, was swung as a pendulum. The results described in these two reports may be summarized as follows: after deriving the equations of motion, these were programmed and run on a digital computer using a Runga-Kutta-Fehlberg [4] routine. The first numerical experiments assumed the two masses m_1 and m_2 were equal. As one normally does with a pendulum, the mass m_2 was displaced through an initial angle θ_0, released with zero initial velocity, $\dot{\theta}_0(0) = 0$ and the subsequent motion studied. Several runs were made , taking successive values for θ_0 of 0.01, 0.10, 0.20, and 0.50 radians. In all four cases, the mass m_1 initially dropped ever so slightly and then rose monotonically. Obviously, this is equivalent to a lengthening of the pendulum arm. As the pendulum arm increased, the amplitude of swing decreased.

While not concerned per se with counter-balanced suspension systems, material of collateral interest may be found in [5],[6]. These two references treat problems whereby a swinging mass is being hoisted by a winch.

It is intuitively obvious that if the mass m_1 is made slightly greater than m_2 that this monotonic lengthening of the pendulum arm with time would be slowed and that if m_1 were made much greater than m_2, the length of the pendulum arm would actually decrease. Thus, there must be some critical relation between m_1 and m_2 for which the length of the pendulum arm is either constant or a periodic function of time. Further numerical experiments [2] indicated that this relationship is:

$$m_1 = m_2[1.000 + 7.451\times10^{-8}\,\theta_0 + 0.250\,\theta_0^2]$$

It was found using a least square fit to data points corresponding to θ_0=0.1, 0.2, 0.3, 0.4, and 0.5 radians.

In this report, a theoretical derivation of the relationship between m_1 and m_2 is obtained under the added restriction, however, that the initial angular displacement θ_0 is small. The technique used is classical perturbation theory.[7]

Section II

The Mathematical Statement

The geometry of the configuration studied is given in Figure 1. From this geometry, assuming that the masses of the pulley and the wire are negligible and that the radius of the pulley may be neglected also, the differential equations describing the motion of the mass m_2 are:

$$\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta} + g\sin\theta = 0 \qquad (1)$$

$$(m_1 + m_2)\ddot{\rho} - m_2\rho\dot{\theta}^2 + m_1 g - m_2 g\cos\theta = 0 \qquad (2)$$

where dots indicate differentiation with respect to the time ***t***. Equations 1 and 2 may be simplified slightly if we observe that by letting $\rho = \rho g$, both equations contain g as a factor and we may divide through by it. This is equivalent to choosing units so that g = 1. Furthermore, by using the small angle approximation, these equations then become:

$$\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta} + \theta = 0 \qquad (3)$$

$$(m_1 + m_2)\ddot{\rho} - m_2\rho\dot{\theta}^2 + (m_1 - m_2) + \tfrac{1}{2}m_2\theta^2 = 0 \qquad (4)$$

Case i) Constant ρ.

We prove now that there is no solution of Equations 3 and 4 for which $\rho = \rho_0 =$constant. Actually, we shall show that assuming that ρ = constant leads to a contradiction. Using this assumption, Equations 3 and 4 become

$$\rho_0\ddot{\theta} + \theta = 0 \qquad (3.1)$$

$$m_2\rho_0\dot{\theta}^2 - (m_1 - m_2) - \tfrac{1}{2}m_2\theta^2 = 0 \qquad (4.1)$$

Differentiate Equation 4.1 with respect to the time ***t*** and divide through by m_2 to get:

$$2\rho_0 \ddot{\theta}\dot{\theta} - \theta\dot{\theta} = 0 \tag{5}$$

If $\dot{\theta} \equiv 0$, then Equation 3.1 implies that $\theta \equiv 0$ and Equation 4.1 then implies that $m_1 = m_2$, the classic Atwood's machine case with equal masses. For this classic example there is indeed a solution $\rho = \rho_0$, a constant. If $\dot{\theta} \not\equiv 0$, then Equation 5 implies that

$$2\rho_0 \ddot{\theta} = \theta \tag{6}$$

Equations 3.1 and 6 are incompatible. There is, therefore, no solution of Equations 3 and 4 for which ρ is constant.

We proceed now to discover the relationship between m_1 and m_2 for which $\rho(\boldsymbol{t})$ and $\theta(\boldsymbol{t})$ are periodic solutions of Equations 3 and 4.

Case ii) The General Case.

The perturbation technique requires that Equations 3 and 4 be modified and rewritten as follows:

$$\ddot{\rho} = \epsilon[\mathbf{m}_2\rho\dot{\theta}^2 - (\mathbf{m}_1 - \mathbf{m}_2) - \tfrac{1}{2}\mathbf{m}_2\theta^2] \tag{7}$$

$$\rho\ddot{\theta} + \theta = -2\dot{\rho}\dot{\theta} \tag{8}$$

where we have introduced a parameter ϵ and written $\mathbf{m}_1 = m_1/(m_1+m_2)$ and $\mathbf{m}_2 = m_2/(m_1+m_2)$. In Equations 7 and 8, we change the time variable ***t*** to a pseudo-time **t** by means of the formula $\boldsymbol{t} = \sqrt{\omega}\,\mathbf{t}$ and get

$$\rho'' = \epsilon[\mathbf{m}_2\rho\theta'^2 + \omega(\mathbf{m}_2 - \mathbf{m}_1) - \tfrac{1}{2}\mathbf{m}_2\omega\theta^2] \tag{9}$$

$$\rho\theta'' + \omega\theta = -2\rho'\theta' \tag{10}$$

where the $'$ indicates differentiation with respect to **t**. Assuming that ρ, θ, ω and $\mathbf{m}_1$ are analytic functions of ϵ:

$$\rho = \mathbf{r}_0 + \epsilon \mathbf{r}_1 + \epsilon^2 \mathbf{r}_2 + \ldots$$
$$\theta = \theta_0 + \epsilon \theta_1 + \epsilon^2 \theta_2 + \ldots$$
$$\omega = \omega_0 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \ldots$$
$$\mathbf{m}_1 = \mu_0 + \epsilon \mu_1 + \epsilon^2 \mu_2 + \ldots$$

We introduce these expansions into Equations 9 and 10 and collect terms in like powers of ϵ. Since ϵ is an arbitrary parameter, each coefficient of the powers of ϵ must be zero. This results in setting up an infinite system of pairs of second order differential equations which can be solved successively and recursively. The first set of pairs is obtained by setting $\epsilon=0$. The two equations obtained are:

$$\mathbf{r}_0'' = 0 \qquad (11)$$

$$\mathbf{r}_0 \theta_0'' + \omega_0 \theta_0 = -2\mathbf{r}_0' \theta_0' \qquad (12)$$

The solution of Equation 11 is $\mathbf{r}_0 = r_0$, a constant, since we assume as an initial condition that $\mathbf{r}_0'(0) = 0$. The solution of Equation 12 then becomes

$$\theta_0 = \theta_0 \cos(\mathbf{kt}) \qquad (13)$$

where θ_0 is the initial angular displacement, $k^2 = \omega_0 / r_0$ and we have assumed that $\theta_0'(0) = 0$. It will turn out that ω_0 plays no role in the final form of the formulas for $\rho(\boldsymbol{t})$ and $\theta(\boldsymbol{t})$. In terms of the physical variables, $k^2 = g/\rho(0)$.

If Equation 9 is differentiated with respect to the parameter ϵ and then ϵ is set equal to zero, we obtain:

$$\mathbf{r}_1'' = \mathbf{m}_2\mathbf{r}_0\theta_0'^2 - \mu_0\omega_0 + \mathbf{m}_2\omega_0(1 - \tfrac{1}{2}\theta_0^2) \tag{14}$$

When the values of θ_0 and θ_0', as given from Equation 13 and its derivative, are substituted into Equation 14, and remembering that $\mathbf{r}_0 = r_0$, a constant, then, after some algebraic manipulation, Equation 14 may be written as:

$$\mathbf{r}_1'' = \omega_0[-\mu_0 + \mathbf{m}_2 + (1/4)\mathbf{m}_2\theta_0^2 - (3/4)\mathbf{m}_2\theta_0^2\cos(2\mathbf{k}\mathbf{t})] \tag{15}$$

Since we are seeking a *periodic* solution for $\mathbf{r}$, the secular term in the general solution of Equation 15, the term that would give rise to a quadratic increase or decrease with time in $\mathbf{r}_1$, will be eliminated if we set:

$$\mu_0 = \mathbf{m}_2 + (1/4)\mathbf{m}_2\theta_0^2 \tag{16}$$

Otherwise, $\mathbf{r}_1$ will contain a term of the form $(-\mu_0 + \mathbf{m}_2 + (1/4)\mathbf{m}_2\theta_0^2)\mathbf{t}^2/2$ and, depending on the relative magnitude chosen for μ_0 and $(\mathbf{m}_2 + (1/4)\mathbf{m}_2\theta_0^2)$, $\mathbf{r}_1$ increases or decreses quadratically with time. The solution of Equation 15, for the initial conditions $\mathbf{r}_1(0) = \mathbf{r}_1'(0) = 0$, is:

$$\mathbf{r}_1 = (3/16)r_0\mathbf{m}_2\theta_0^2(\cos(2\mathbf{k}\mathbf{t}) - 1) \tag{17}$$

When Equation 10 is differentiated with respect to ϵ and then ϵ is set equal to zero, we obtain:

$$\mathbf{r}_0\theta_1'' + \omega_0\theta_1 = -\mathbf{r}_1\theta_0'' - \omega_1\theta_0 - 2\mathbf{r}_1'\theta_0' \tag{18}$$

Since $\mathbf{r}_0$, θ_0 and $\mathbf{r}_1$ are known, we substitute their respective values into the right-hand side of Equation 18 to obtain:

$$r_0\theta_1'' + \omega_0\theta_1 = \theta_0\{-[\omega_1 + (15/32)\mathbf{m}_2\theta_0^2\omega_0]\cos(\mathbf{k}\mathbf{t}) + [(15/32)\mathbf{m}_2\theta_0^2\omega_0]\cos(3\mathbf{k}\mathbf{t})\} \tag{19}$$

We are interested in obtaining the particular solution of Equation 19 that excludes the secular term. This is done by choosing

$$\omega_1 = -(15/32)\mathbf{m}_2\theta_0^2\omega_0 \tag{20}$$

With this done, the particular solution of Equation 20, with initial conditions $\theta_1(0) = \theta_1{}'(0) = 0$, is:

$$\theta_1 = (15/256)\theta_0(\mathbf{m}_2\theta_0^2)[\cos(k\mathbf{t}) - \cos(3k\mathbf{t})] \tag{21}$$

Note that if ω_1 is not chosen as in Equation 20, then the solution for θ_1 will go to infinity with time. We interpret this by saying that the mathematical procedure fails. Thus, in order to retain mathematical viability, we must pick ω_1 as given in Equation 20.

If we were to proceed no further, we would have the following approximate solutions to Equations 3 and 4 (after setting $\epsilon = 1$):

$$\rho = r_0[1 + (3/16)\mathbf{m}_2\theta_0^2(\cos(2k\mathbf{t}) - 1)] \tag{22}$$

$$\theta = \theta_0\left\{\cos(k\mathbf{t}) + (15/256)(\mathbf{m}_2\theta_0^2)[\cos(k\mathbf{t}) - \cos(3k\mathbf{t})]\right\} \tag{23}$$

where

$$\mathbf{m}_1 = \mathbf{m}_2 + (1/4)\mathbf{m}_2\theta_0^2 \tag{24}$$

$$\omega = \omega_0[1 - (15/32)\mathbf{m}_2\theta_0^2] \tag{25}$$

$$k^2 = \omega_0/r_0 \tag{26}$$

We proceed to get the next higher order terms in the expansions for $\rho, \theta, \mathbf{m}_1$, and ω.

By equating the coefficients of ϵ^2 from both sides of Equation 9, we get

$$\mathbf{r}_2{}'' = -[\omega_0\mu_1 + \omega_1(\mu_0 - \mathbf{m}_2)] + \mathbf{m}_2[\mathbf{r}_1\theta_0'^2 + 2\mathbf{r}_0\theta_0'\theta_1'] - \tfrac{1}{2}\mathbf{m}_2[\omega_1\theta_0^2 + 2\omega_0\theta_0\theta_1] \tag{27}$$

After substituiting the several quantities already found for their respective values as given in the right-hand side of Equation 27, it may be rewritten as:

$$\mathbf{r}_2'' = (9/512)\mathbf{m}_2^2\theta_0^4\omega_0\,[4\cos(2k\mathbf{t}) + 9\cos(4k\mathbf{t})] \tag{28}$$

where, in order to preclude the introduction of a secular term in $\mathbf{r}_2$, we had set:

$$\mu_1 = (63/512)\mathbf{m}_2^2\theta_0^4 \tag{29}$$

The solution of Equation 28, subject to the initial conditions $\mathbf{r}_2(0) = \mathbf{r}_2'(0) = 0$, is:

$$\mathbf{r}_2 = (9/2^{13})(\mathbf{m}_2\theta_0^2)^2 r_0[25 - 16\cos(2k\mathbf{t}) - 9\cos(4k\mathbf{t})] \tag{30}$$

By equating the coefficients of ϵ^2 from both sides of Equation 10, we get:

$$r_0\theta_2'' + \omega_0\theta_2 + [\mathbf{r}_1\theta_1'' + \omega_1\theta_1 + 2\mathbf{r}_1'\theta_1'] + [\mathbf{r}_2\theta_0'' + \omega_2\theta_0 + 2\mathbf{r}_2'\theta_0'] = 0 \tag{31}$$

After substituting the values already found for $\mathbf{r}_0, \theta_0, \mathbf{r}_1, \theta_1, \omega_0, \omega_1$ and their derivatives where appropriate, we get:

$$r_0\theta_2'' + \omega_0\theta_2 = (9/2^{14})m_2^2\theta_0^5\omega_0[163\cos(3k\mathbf{t}) - 291\cos(5k\mathbf{t})] \tag{32}$$

and where, in order to preclude the introduction of a secular term in θ_2, we have set:

$$\omega_2 = (9/128)\mathbf{m}_2^2\theta_0^4\omega_0 \tag{33}$$

The value of θ_2, subject to the initial conditions $\theta_2(0) = \theta_2'(0) = 0$, found by integrating Equation (32) is:

$$\Theta_2 = (9/2^{17})\mathbf{m}_2^2\Theta_2^5\,[\,66\cos(\mathbf{kt}) - 163\cos(3\mathbf{kt}) + 97\cos(5\mathbf{kt})] \tag{34}$$

Summarizing the results obtained thus far, we see that after setting ϵ = 1, we get the following approximate solutions for $\rho(\mathbf{t})$ and $\Theta(\mathbf{t})$:

$$\rho(\mathbf{t}) = r_0\Big[1 + (3/16)\mathbf{m}_2\Theta_2^2(\cos(2\mathbf{kt}) - 1) + (9/2^{13})\mathbf{m}_2^2\Theta_0^4(25 - 16\cos(2\mathbf{kt}) - 9\cos(4\mathbf{kt})\Big] \tag{35}$$

$$\Theta(\mathbf{t}) = \Theta_0\Big\{\cos(\mathbf{kt}) + (15/256)\mathbf{m}_2\Theta_0^2[\cos(\mathbf{kt}) - \cos(3\mathbf{kt})] + (9/2^{17})\mathbf{m}_2^2\Theta_2^4\,[\,66\cos(\mathbf{kt}) - 163\cos(3\mathbf{kt}) + 97\cos(5\mathbf{kt})]\Big\}, \tag{36}$$

where

$$\mathbf{m}_1 = \mathbf{m}_2 + (1/4)\mathbf{m}_2\Theta_0^2 + (63/512)\mathbf{m}_2^2\Theta_0^4 \tag{37}$$

$$\omega = \omega_0[1 - (15/32)\mathbf{m}_2\Theta_0^2 + (9/128)\mathbf{m}_2^2\Theta_0^4\,] \tag{38}$$

Since $\mathbf{t} = (\omega)^{-\frac{1}{2}}\boldsymbol{t}$, $\rho(\mathbf{t}) \Rightarrow \rho(\omega^{-\frac{1}{2}}\boldsymbol{t})$, $\Theta(\mathbf{t}) \Rightarrow \Theta(\omega^{-\frac{1}{2}}\boldsymbol{t})$, i.e. in formulas (35) and (36) replace $\mathbf{t}$ by $\omega^{-\frac{1}{2}}\boldsymbol{t}$. Finally,

$$\rho(\boldsymbol{t}) = g\rho(\omega^{-\frac{1}{2}}\boldsymbol{t})$$

and

$$\theta(\boldsymbol{t}) = \Theta(\omega^{-\frac{1}{2}}\boldsymbol{t})$$

Section III

Discussion

An understanding of the results presented in the previous section may be enhanced by a numerical example. Before doing this, however, we want to recall the relationships:

$$\rho(\boldsymbol{t}) = g\rho(\boldsymbol{t}) \qquad \text{and} \qquad \boldsymbol{t} = \sqrt{\omega}\,t$$

These imply that $\ddot{\rho} = g\,\ddot{\rho} = g\,\rho''/\omega$. Using these we can calculate the acceleration acting on the mass m_1. We do this first using the approximate solutions given by Equations 22 and 23. In particular, we examine Equation 22 and take the second derivative with respect to **t** to get:

$$\rho''(\mathbf{t}) = -(3/4)\mathbf{m}_2\theta_0^2\omega_0\cos(2k\mathbf{t}) \tag{39}$$

and where we have used $k^2 = \omega_0/r_0$. Then, using the value of ω as given in Equation 25 :

$$\ddot{\rho} = -\{(3/4)[\mathbf{m}_2\theta_0^2]/[1-(15/32)\mathbf{m}_2\theta_0^2]\}\cos(2k\mathbf{t}) \tag{40}$$

Remembering that $\mathbf{m}_1 = 1 - \mathbf{m}_2$, we solve Equation 24 for $\mathbf{m}_2$ and substitute that value into Equation (40) to get:

$$\ddot{\rho} = -[24\theta_0^2/(64 - 7\theta_0^2)]\cos(2k\mathbf{t}) \tag{41}$$

For values of θ_0^2 of interest, say not to exceed 0.1 radians, the expression within the [...] in Equation 41 is a monotonically increasing function of θ_0 and will therefore attain its maximum value at the end-point of the interval of interest. If we take that to be $0 < \theta_0 \leq 0.1$ radians, then for $\theta_0 = 0.1$,

$$\ddot{\rho}(t) = -.0037541\ g \cos(2kt) \tag{42}$$

Thus, the acceleration of the mass m_1 is only, at most, 0.00375 times the acceleration of gravity, a very small quantity. While the mass m_1 does not move as if it were in free space, it is subject to a gravitational influence that is only .00375 times the effect it would normally sense on earth.

The description given above is based on using only two terms in the expansion for $\rho(t)$. Some change in the calculated value of the acceleration of the mass m_1 should be expected if three terms are used in the expansion for $\rho(t)$. While it is possible to carry out these computations algebraically, no new insight into the motion is gained. Accordingly, so as not to get lost in a maze of algebraic manipulation and symbolism, we limit our discussion to a numerical example, indicating along the way how other numerical examples may be handled.

We start with Equation 35 and calculate:

$$\rho''(\mathbf{t}) = \omega_0 (\mathbf{m}_2\theta_0^2) [A_1 \cos(2k\mathbf{t}) + A_2 \cos(4k\mathbf{t})] \tag{43}$$

where:

$$A_1 = [-(3/4) + (9/128)\ \mathbf{m}_2\theta_0^2]$$

$$A_2 = (81/512)\ \mathbf{m}_2\theta_0^2$$

and again we have used $k^2 = \omega_0 / r_0$. Then:

$$\ddot{\rho}(\mathbf{t}) = A_0[A_1 \cos(2k\mathbf{t}) + A_2 \cos(4k\mathbf{t})] \tag{44}$$

where:

$$A_0 = (\mathbf{m}_2\theta_0^2\ \omega_0)/\ \omega$$

or:

$$A_0 = \mathbf{m}_2\Theta_0^2/[1-(15/32)\mathbf{m}_2\Theta_0^2 + (9/128)\mathbf{m}_2^2\Theta_0^4]$$

where the value of ω is taken from Equation 38 .

The acceleration is stationary when $2k\mathbf{t}_1 = 0 + n\pi$ and when $\cos(2k\mathbf{t}_2) = -A_1/(4A_2)$. For these values of $\mathbf{t}_1$, the maximum value of the acceleration cannot exceed $|A_0|g[\,|A_1| + |A_2|]$; for values of Θ_0 of interest, the magnitude of $A_1/(4A_2)$ is greater than 1 and there is no real value for $\mathbf{t}_2$.

In order to calculate values for A_0, A_1, and A_2 , we need to know $\mathbf{m}_2\Theta_0^2$. For a given value of Θ_0 , say $\Theta_0 = 0.1$ radians, and remembering that $\mathbf{m}_1 + \mathbf{m}_2 = 1$, we find, using Equation 37, that $\mathbf{m}_2 = .4993742479515557$. We then find that $A_0 = 5.0054505 \times 10^{-3}$, $A_1 = -.74964888$, $A_2 = 7.9002567 \times 10^{-4}$ and the maximum value of the acceleration is at most 3.7483759×10^{-3} g.

A comparison of the results when 2 or 3 terms are used in the determination of the acceleration of m_1 indicates that the difference is so small, of the order of 5.7×10^{-6} g, as to be negligible. For all practical purposes, the solution to the problem is given by the Equations 22 - 26 and Equation 41.

When the fundamental frequency of the system is compared to the frequency of a pendulum of length $\rho(0)$, there is a change in frequency given by the multiplicative factor $[(64 + 8\Theta_0^2)/(64 - 7\Theta_0^2)]^{1/2}$. For $\Theta_0 = 0.1$ radians, this factor is 1.0011725.

The solution to Equations 1 and 2 as given by Equations 22-26 or, if you prefer, by Equations 35-38, are *periodic* solutions. As already noted, to obtain a periodic solution for ρ it was necessary to carefully adjust the ratio of the two masses m_1 and m_2. If values of m_1 and m_2 different from those determined by means of Equation 22 (or Equation 37) are used, then the solution for the pendulun arm $\rho(\boldsymbol{t})$ will contain an additional term involving

t^2. The case for $m_1 = m_2$ was studied in reference [8].

If a value of ω different from the ones given in Equations 25 or 38 is used, then the mathematical process fails because secular terms, which physically and experimentally we know do not have meaning, are introduced into the equation for θ, Equations 23 or 36.

Section IV

Conclusions

A theoretical description of an Atwood's pendulum machine is given. Implicit in these calculations were the assumptions that the pulley is frictionless and that its radius could be neglected. Also, the wire connecting the two masses is assumed to be perfectly flexible, inextensible and of negligible mass. The following results were deduced:

(1) If the ratio of the two masses are chosen as prescribed, then the vertical motion of the mass m_1, which is the only motion possible for this mass, is periodic;

(2) Equivalently, the change in length of the pendulum arm associated with the swinging of the mass m_2 is exactly the same as the change in length of the wire supporting m_1 except that as the one increases the other decreases;

(3) If the two masses are not picked as prescribed so as to insure periodic motion, then the motion of m_1 consists of the superimposition of two motions:

(a) an accelerated motion as if the mass m_1 were in a gravitational field of magnitude $< g$, the magnitude of this deviation from g depending on just how much the two masses differ from the ratio required for the periodic motion, and

(b) the periodic motion superimposed (just added onto) the motion described in (a);

(4) While the conclusions (1),(2), and (3) given above were obtained by considering only two terms in the expansion of the relevant functions, the effect of using three terms in these expansions was shown to modify the conclusions only quantitatively, not qualitatively, and the change was of an

order less than 1%.

In mathematical terms, this periodic solution is referred to as a limit cycle; it is, however, unstable. Any set of conditions that differ ever so slightly from the ones needed to establish the limit cycle will produce a trajectory that increasingly diverges from the limit cycle and this translates into the statement that the pendulum arm goes either to zero or to infinity[9],[10].

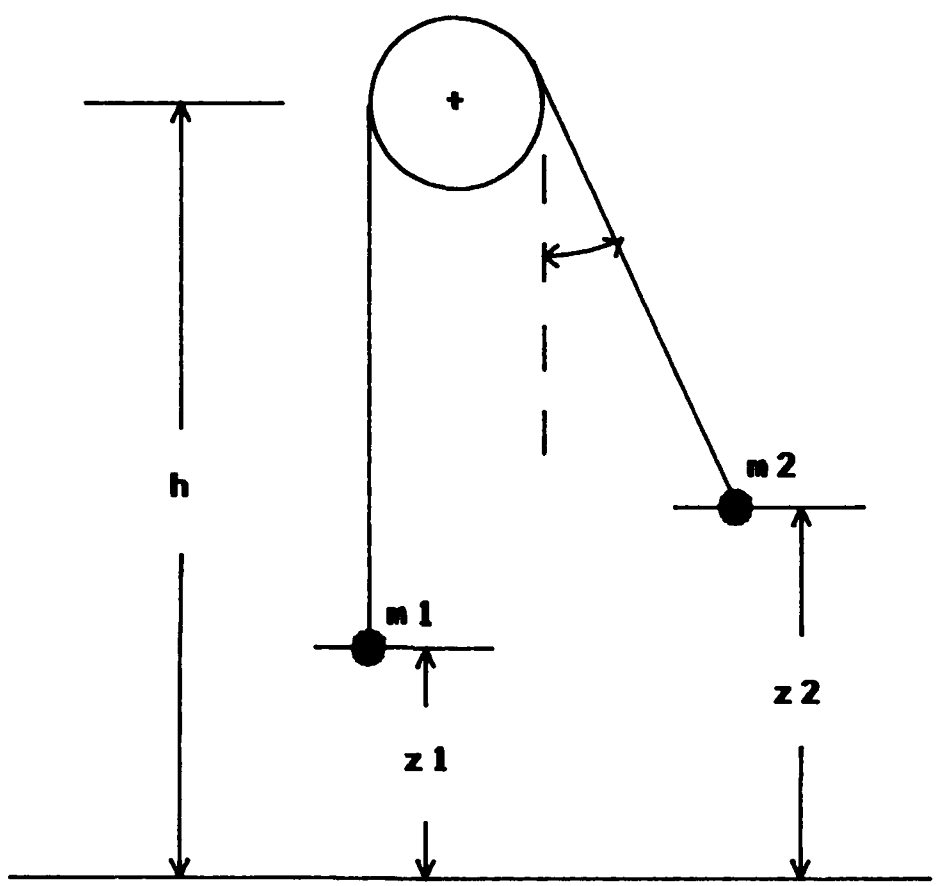

Figure 1
A Schematic for an Atwood's Pendulum

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List of Symbols

m_1 = mass 1

m_2 = mass 2

$\mathbf{m}_1$ = reduced mass1 = $m_1/(m_1+m_2)$

$\mathbf{m}_2$ = reduced mass 2 = $m_2/(m_1+m_2)$

t = real time, seconds

$\mathbf{t}$ = pseudo-time

θ_0 = initial angular displacement

$\dot{\theta}_0$ = initial angular velocity

ρ = length of pendulum arm, meters

θ = $\mathit{\theta}$ = angular displacement of pendulum arm; two symbols are used to make the notation uniform

ρ = ρ/g = length of pendulum arm in a system of units in which g = 1

ω = a parameter relating t and $\mathbf{t}$; eventually shown to be a factor specifying frequency

ϵ = a parameter used to establish asymptotic expansion

$\mathbf{r}_i$ = a term in the expansion of ρ

θ_i = a term in the expansion of θ

ω_i = a term in the expansion of ω

μ_i = a term in the expansion of $\mathbf{m}_1$

r_0 = initial length of pendulum arm in system of units for which g = 1

k = a parameter relating to frequency in the system in which g = 1; $k^2 = \omega_0 / r_0$

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