



「「「「「」」

sesses (it sesses (it avances () 7777787



AFWAL-TR-87-3006



ON PERIODIC SOLUTIONS OF AN ATWOOD'S PENDULUM

AD-A183 078

DONALD MITTLEMAN

ANAMET LABORATORIES, INC. 3400 INVESTMENT STREET HAYWOOD, CALIFORNIA S4545



MAY 1987

Interim Report for period August 1985 - July 1986.

Approved for public release; distribution is unlimited.

FLIGHT DYNAMICS LABORATORY AIR FORCE WRIGHT AERONAUTICAL LABORATORIES AIR FORCE SYSTEMS COMMAND WRIGHT-PATTERSON AIR FORCE BASE, OHIO 45433-6553

When Government drawings, specifications, or other data are used for any purpose other than in connection with a definitely related Government procurement operation, the United States Government thereby incurs no responsibility nor any obligation whatsoever; and the fact that the government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data, is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture use, or sell any patented invention that may in any way be related thereto.

This report has been reviewed by the Office of Public Affairs (ASD/PA) and is releasable to the National Technical Information Service (NTIS). At NTIS, it will be available to the general public, including foreign nations.

This technical report has been reviewed and is approved for publication.

Robert W. Mordon ROBERT W. GORDON, Aerospace Engineer

Engineering Technology Group

Ott. F. Charry

OTTO F. MAURER, Acting Chief Structural Vibration Branch

FOR THE COMMANDER

Brand (

HENRY A. BONDARUK, JR, Colonel, USAF Chief, Structures Division

"If your address has changed, if you wish to be removed from our mailing list, or if the addressee is no longer employed by your organization please notify AFWAL/FIBG, W-PAFB,OH 45433-6553 to help us maintain a current mailing list".

copies of this report should not be returned unless return is required by security considerations, contractual obligations, or notice on a specific document.

UNCLASSIFIED

122

In Propriet Security CLASSIFICATION Inclassified Inclassified Security CLASSIFICATION AUTHORITY Security CLASSIFICATION REPORT NUMBERIES Security CLASSIFICATION Security Security CLASSIFICATION	1	REPORT DOCUME	NTATION PAG	E		
Unclassified for authomity 3. Distribution/Availability of REPORT 2. SECUTY CLASSFIED TON/DOWNGRADING SCHEDULE In Imited. 2. Distribution/Availability of REPORT NUMBER(S) 3. Distribution/Availability of REPORT NUMBER(S) 2. Distribution/Downgrading Schedule In Imited. 2. PERFORMING ORGANIZATION REPORT NUMBER(S) 5. MONITORING ORGANIZATION REPORT NUMBER(S) Anamet Laboratories, Inc. If applicable 2. Aname of PERFORMING ORGANIZATION B. OFFICE SVMBOL 3. Aname of PERFORMING ORGANIZATION B. OFFICE SVMBOL 3. Adde of PERFORMING ORGANIZATION B. OFFICE SVMBOL 3. Adde of PERFORMING ORGANIZATION B. OFFICE SVMBOL 3. Adde of PERFORMING ORGANIZATION B. OFFICE SVMBOL 3. Distribution/State and ZIP Codel The NAME OF PUNCINGREPONSORING 3. AFWAL/FIBG F33615-86-C-216 AFWAL/FIBG F33615-86-C-216 AFWAL/FIBG F33615-86-C-216 AFWAL/FIBG F33615-86-C-216 AFWAL/FIBG FAGE COMP Marging Perform The Codel AFWAL/FIBG FAGE COMP Marging Perform The Codel AFWAL/FIBG FAGE COMP Marging Perform The Codel AFWAL/FIBG FAGE COMP Marging Perform The Nume Code Perform Marging P	1. REPORT SECURITY CLASSIFICATION		16. RESTRICTIVE M	ARKINGS		
Approved for public release; distribution in the construction of the one mass induces a varying testion of the one mass. It is shown the of the one mass induces a varying testion of the one mass induces a varying testion of the one mass induces a varying testion of the one mass. It is shown the second testion of the one mass induces a varying testion of the one mass. It is shown the second testion and the second testion and the of the second testion and testing testion and testingenergies. The testion and testion and testi	Unclassified		3. DISTRIBUTION/A	VAILABILITY O	FREPORT	
b DECLASSFICATION/DOWNGRADING SCHEDULE unlimited. * PERFORMING ORGANIZATION REPORT NUMBERISI A MONITORING ORGANIZATION REPORT NUMBERISI * PERFORMING ORGANIZATION REPORT NUMBERISI A MONITORING ORGANIZATION REPORT NUMBERISI Anamet Laboratories, Inc. Monitories, Anamet Caboratory (AFWAL/FIE * ADDRESS (CID, Sake and ZIP Cake) Name of MONITORING ORGANIZATION * ADDRESS (CID, Sake and ZIP Cake) Name of MONITORING ORGANIZATION * ADDRESS (CID, Sake and ZIP Cake) Name of MONITORING ORGANIZATION * ADDRESS (CID, Sake and ZIP Cake) AFWAL/FIBG * TO TE OF REPORT ISS. THACE COVERED FROMT INTERCENT FILL * TO TE OF REPORT ISS. THACE COVERED FROMT INTERCENT FILL * TO TE OF REPORT ISS. THACE COVERED FROMT INTERCENT FILL (SAKE AND	2. SECURITY CLASSIFICATION AUTHORITY		Approved for public release; distribution i unlimited.			
# PERFORMING ORGANIZATION REPORT NUMBERISI 5. MONITORING ORGANIZATION REPORT NUMBERISI # PERFORMING ORGANIZATION REPORT NUMBERISI 5. MONITORING ORGANIZATION REPORT NUMBERISI # AMME OF PERFORMING ORGANIZATION AL PERFORMING ORGANIZATION REPORT NUMBERISI 7. NAME OF MUTTORING ORGANIZATION REPORT NUMBERISI # Anamet Laboratories, Inc. If sphicebili 7. NAME OF MUTTORING ORGANIZATION REPORT NUMBERISI # ADDRESS (Cir), State and ZIP Codel ADDRESS (Cir), State and ZIP Codel AFWAL/FIBG # ADDRESS (Cir), State and ZIP Codel AFWAL/FIBG PROCEMENT INSTRUMENT IOENTIFICATION NUMBER # ADDRESS (Cir), State and ZIP Codel AFWAL/FIBG PROCEMENT INSTRUMENT IOENTIFICATION NUMBER March Dynamics Laboratory AFWAL/FIBG PROCEMENT INSTRUMENT IOENTIFICATION NUMBER On Periodic Solutions of an Atwood's Pendulum 62201F 2401 04 12: PERGONAL AUTHORIS PROCEMENT NO. NO. NO. NO. 11: Title (Horder Sweth) Interim 135. TIME COVERED MAY 1987 15 PAGE COUNT 13: Stree or REPORT ISSUERCET TERMS (Common on WARD of MAY 1987 16 PAGE COUNT NO. NO. 11: Title (Horder Sweth) MAXD 1985_OJUL 1986 14 DATE OF REPORT (Y. MO. Dav) 15 PAGE COUNT 11: Titl	2b. DECLASSIFICATION/DOWNGRADING SCHEDULE					
ARMAE OF PERFORMING ORGANIZATION Anamet Laboratories, Inc. b. OFFICE SYMBOL If applicable/ If a	4. PERFORMING ORGANIZATION REPORT NUMB	ER(S)	5. MONITORING OF	GANIZATION RI	PORT NUMBER)
Ga NAME OF PERFORMING ORGANIZATION Anamet Laboratories, Inc. Bis. OFFICE SYMBOL ("Papeleoble") Name OF MONITORING ORGANIZATION AIT FORCe Wright Aperoautical Laborat Flight Dynamics Laboratory (AFWAL/FIB Wright-Patterson AFB OH 45433-6553 8. ADDRESS (Cir), State and ZIP Coder 3400 Investment Street Haywood CA 94545 Defice SYMBOL ("Papeleoble") No. DEFICE SYMBOL ("Papeleoble") 8. ADDRESS (Cir), State and ZIP Coder Advanization Flight Dynamics Laboratory Bis. OFFICE SYMBOL ("Papeleoble") No. DEFICE SYMBOL ("Papeleoble") 8. ADDRESS (Cir), State and ZIP Coder AFWAL/FIBG Bis. OFFICE SYMBOL ("Papeleoble") No. DEFICE SYMBOL ("Papeleoble") No. DEFICE SYMBOL ("Papeleoble") 8. ADDRESS (Cir), State and ZIP Coder AFWAL/FIBG Bis. OFFICE SYMBOL ("Papeleoble") No. DEFICE SYMBOL ("Papeleoble") No. DEFICE SYMBOL ("Papeleoble") 10. SOURCE OF FUNCTION NUMBE ("Papeleoble") Interime Topentification? (Do Pariodic Solutions of an Atwood's Pendulum Donald Mittleman No. Yask ("Papeleoble") 12. PERSONAL AUTHORIS Donald Mittleman Inter imme ACC 1985ro JUL 1986 ("Adwood's pendulum Deriodic motion definition are chosen as an Adwood's machine in which one of two masses allowed to swing as a pendulum while the other remains constrained to move only in vertical direction. The pendulum motion of the one mass induces a varying tension connecting wire; this, in turn, produces motion in the second mass. It is shown th motion can be made periodic if the ratio of the two masses and the dependency of the other initial conditions are chosen as prescribed in this report. If this conditi			AFWAL-TR-8	37-3006		
Anamet Laboratories, Inc. If applicable/ Air Porce Wright Acronautical Laboratory (AFWAL/FIE 62. ADDRESS (Cir, State and ZiP Code) The ADDRESS (Cir, State and ZiP Code) 3400 Investment Street AFWAL/FIES Haywood CA 94545 AFWAL/FIES 62. ADDRESS (Cir, State and ZiP Code) AFWAL/FIES 64. ADDRESS (Cir, State and ZiP Code) AFWAL/FIES AFWAL/FIEG F33615-86-C-3216 64. ADDRESS (Cir, State and ZiP Code) Is Sounce of FUNDING MOS. AFWAL/FIEG F33615-86-C-3216 71. Tit E: direch Security Commentation Is Sounce of FUNDING MOS. 10. Doration Solid (ir, State and ZiP Code) Is Sounce of FUNDING MOS. 11. Title: Greate Security Commentation G2201F 2401 12. FERSONAL AUTHORIS Is SUBJECT TERMS (Continue on nuere if necessory and identify by Mark number) AEC 1985/rg.JUL 1986 13. VPG or REFORT Iss. TIME COVERED Is SUBJECT TERMS (Continue on nuere if necessory and identify by Mark number) 14. SUBJECT TERMS (Continue on nuere if necessory and identify by Mark number) AEC 1985/	6. NAME OF PERFORMING ORGANIZATION	66. OFFICE SYMBOL	78. NAME OF MONI	TORING ORGAN		
Flight Dynamics Laboratory (FPAL/FIE Flight Dynamics Laboratory AFWAL/FIES Widet Dynamics Laboratory Flight Dynamics Laboratory AFWOL/FIE Flight Dynamics Laboratory	Anamet Laboratories. Inc	(If applicable)	Air Force Wright Aeronautical Laboratorie			
3400 Investment Street AFWAL/FIBG Haywood CA 94545 Wright-Patterson AFB OH 45433-6553 Stands of FUNDINGSPONSORING ORGANIZATION Bb. OFFICE SYMBOL (If applicable) PROCUMENTINSTRUMENT IDENTIFICATION NUMBER (If applicable) Flight Dynamics Laboratory AFWAL/FIBG F33615-86-C-3216 BC ADDRESS (City, Nice and ZIP Code) AFWAL/FIBG F33615-86-C-3216 AFWAL/FIBG Is source of FUNDING NOS. AFWAL/FIBG PROGRAM Wright-Patterson AFB OH 45433-6553 It FITCE Include Security Claustering) On Periodic Solutions of an Atwood's Pendulum 62201F 2401 Danald Mittleman 13. TYPE of Reform 13. THE COVERED FROM_AUTHORS) 14 DATE OF REPORT (Yr. No., Day) 15. PAGE COUNT 21 COSATI CODES 18 SUBJECT TERMS (Continue on NUMERE (Interest) MAY 1987 24 19. SUPPLEMENTARY NOTATION Is Subject TERMS (Continue on NUMERE (Interest) and identify by block number! 24 19. SUPPLEMENTARY NOTATION Is Subject TERMS (Continue on NUMERE (Interest) and identify by block number! 24 19. SUPPLEMENTARY NOTATION Is subject TERMS (Continue on NUMERE (Interest) and identify by block number! 15. PAGE COUNT 19. SUPPLEMENTARY NOTATION Is Subject TERMS (Continue on NUMERE (I	6c. ADDRESS (City, State and ZIP Code)		Flight Dynar 75. ADDRESS (City,	NICS Labora State and ZIP Cod	tory (Arwal)	/F1BG)
Haywood CA 94545 Wright-Patterson AFB OH 45433-6553 Wright-Patterson AFB OH 45433-6553 Wright-Patterson AFB OH 45433-6553 Wright-Patterson AFB OH 45433-6553 FROGUREMENT INSTRUMENT IDENTIFICATION NUMBER Program AFWAL/FIBG B. OFFICE SYMBOL ONGANZATION Wright-Patterson AFB OH 45433-6553 II SUDJECT FENDING NOS. II TITE (Include Security Chamification) On Periodic Solutions of an Atwood's Pendulum 62201F 2401 II TITE (Include Security Chamification) On Periodic Solutions of an Atwood's Pendulum 62201F 2401 II TYPE OF REPORT Interim Tab. TIME COVERED FROM AUG 1985 To JUL 1986 Tab. TIME COVERED MAY 1987 Tab. TIME COVERED MAY 1987 II SUBJECT TERMS (Continue on newer (Incremery and identify by block number) Tab. TIME COVERED Atwood's pendulum Tab. TIME COVERED MAY 1987 II SUBJECT TERMS (Continue on newer (Incremery and identify by block number) Tab. TIME COVERED Atwood's pendulum Tab. TIME COVERED MAY 1987 II SUBJECT TERMS (Continue on newer (Incremery and identify by block number) Tab. TIME COVERED TINET (Incremery and identify by block number) II SUBJECT TERMS (Continue on newer (Incremery and identify by block number) Tab. TIME COVERED TINET (Incremery (Incremery and identify by block number) II SUBJECT TERMS (Continue on newer (Incremery and identify by block number) Tab. TIME COVERED TINET (Incremery (Inc	3400 Investment Street		AFWAL/FTBG			
4. NAME OF FUNCING/SPONSORING ORGANIZATION Flight Dynamics Laboratory 4. DORESS (Cir., State and ZIP Code) AFWAL/FIBG 9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER F33615-86-C-3216 4. ADDRESS (Cir., State and ZIP Code) AFWAL/FIBG F33615-86-C-3216 AFWAL/FIBG MAWAL/FIBG INTIGE (Finded Starty Completion (Complete) Donald Mittleman PROGRAM ELEMENTNO PROJECT NO. 11 TTLE (Finded Starty Completion (Complete) Donald Mittleman 10. SOURCE OF FUNDING NOS. FELEMENTNO PROGRAM ELEMENTNO PROJECT NO. 12. PERSONAL AUTHORIS Donald Mittleman 13. TIME COVERED FIND AUG 1985TO JUL 1986 14. DATE OF REPORT (Vr. Mo., Day) 15. PAGE COUNT SCIENCE (Complete) Interim 15. PAGE COUNT FIND AUG 1985TO JUL 1986 12. COSATI CODES 18. SUBJECT TERMS (Continue on NUMER (Interement and Identify by block number) 15. PAGE COUNT Atwood's pendulum periodic motion dynamics 14. DATE OF REPORT (Interement and identify by block number) Atwood's pendulum periodic motion dynamics 14. DATE OF REPORT (Vr. Mo., Day) 15. PAGE COUNT SCIENCE (Continue on NUMER (Interement and Identify by block number) 14. SUPPLEMENTARY NOTATION 18. SUBJECT TERMS (Continue on NUMER (Interement and Identify by block number) 15. PAGE COUNT Atwood's pendulum periodic motion 14. DATE OF REPORT (Interement and Identify by Diock number) 18. SUBJECT TERMS (Continue on NUMER (Interement and Identify by block number) 14. DATE OF REPORT (Interement and Identify by Di	Haywood CA 94545		Wright-Patterson AFB OH 45433-6553			
OPGANIZATION If applicable F33615-86-C-3216 Flight Dynamics Laboratory AFWAL/FIBG F33615-86-C-3216 ACDRESS (City, State and ZIP Code) Is SUDJECT TERMS (Continue no control to conto to conto to conto to control to control to conto to conto to co	B. NAME OF FUNDING/SPONSORING	86. OFFICE SYMBOL	9. PROCUREMENT	NSTRUMENT ID	ENTIFICATION N	JMBER
Be: ADDRESS (City, State and ZIP Code) 10: SOURCE OF FUNDING NOS. AFWAL/FIBG PROGRAM PROJECT Wright-Patterson AFB 0H 45433-6553 FMOGRAM PROJECT 11: MITLE (Include Security Claumifection) 00 04 00 Periodic Solutions of an Atwood's Pendulum 62201F 2401 04 12: PERSONAL AUTHORIS 00ald Mittleman 13b. TIME COVERED 14: DATE OF REPORT (Yr. Mo., Day) 15: PAGE COUNT 13: TYPE OF REPORT 13b. TIME COVERED Hawood's Pendulum 62201F 2401 04 14: DATE OF REPORT 13b. TIME COVERED MAY 1987 15: PAGE COUNT 14: DATE OF REPORT 13b. TIME COVERED MAY 1987 15: PAGE COUNT 15: SUBJECT TERMS (Continue on reverse if necessary and identify by block number) 24 16: SUPLEMENTARY NOTATION 15: PAGE COUNT 24 17: COSATI CODES 18: SUBJECT TERMS (Continue on reverse if necessary and identify by block number) 15: PAGE COUNT 10: ABSTRACT (Continue on reverse if necessary and identify by block number) 14: Atwood's pendulum 15: PAGE COUNT 14: ABSTRACT (Continue on reverse if necessary and identify by block number) 15: Shown the insteam of the the other remains constrained to move only in vertical direction	Flight Dynamics Laboratory	AFWAL/FIBG	F33615-86-0	C-3216		
AFWAL/FIBG PROGRAM PROJECT TASK W Wright-Patterson AFB 0H 45433-6553 11 TITLE (Include Security Chanfication) 00 62201F 2401 04 11 TITLE (Include Security Chanfication) 00 62201F 2401 04 12 PERSONAL AUTHORS) 00 62201F 2401 04 13 TYPE OF REPORT 13b. TIME COVERED 14. DATE OF REPORT //Y. Mo. Day/ 15. PAGE COUNT 14 TYPE OF REPORT 13b. TIME COVERED MAY 1987 15. PAGE COUNT 14 TYPE OF REPORT 13b. TIME COVERED MAY 1987 15. PAGE COUNT 17 TYPE OF REPORT 13b. TIME COVERED MAY 1987 15. PAGE COUNT 18 SUPLEMENTARY NOTATION 15. SUPCOMENT MAY 1987 24 18 SUPLEMENTARY NOTATION 15. PAGE COUNT Atwood's pendulum 19 COSATI CODES 18. SUBJECT TERMS (Continue on reverse (Increaser) and identify by block number/ ATWOO'S pendulum is defined as an Atwood's machine in which one of two masses allowed to swing as a pendulum while the other remains constrained to move only in vertical direction. The pendulum motion of the one mass induces a varying tension connecting wire; this, in turn, produces motion in the second mass. It is shown the supersociation of two motions. The first is mot a constant gravitational field where the effe	3c. ADDRESS (City, State and ZIP Code)	· · · · · · · · · · · · · · · · · · ·	10. SOURCE OF FUR	NDING NOS.		
Wright-Patterson AFB OH 45433-6553 11 TTLE (Include Security Claufication) On Periodic Solutions of an Atwood's Pendulum 62201F 2401 12 PERSONAL AUTHOR(S) Donald Mittleman 13. TYPE OF REFORT Interim 13. TIME COVERED FROM_AUC 1985 Interim FROM_AUC 1985 13. SUPPLEMENTARY NOTATION 14. SUPPLEMENTARY NOTATION 15. SUPPLEMENTARY NOTATION 16. SUPPLEMENTARY NOTATION 17. COSATI CODES 18. SUBJECT TERMS (Continue on reverse (Increasery and identify by block number) A Atwood's pendulum 22 01 An Atwood's pendulum is defined as an Atwood's machine in which one of two masses allowed to swing as a pendulum while the other remains constrained to move only in vertical direction. The pendulum motion of the one mass induces a varying tension connecting wire; this, in turn, produces motion in the second mass. It is shown th motion can be made periodic if the ratio of the two masses and the dependency of th on the initial conditions are chosen as prescribed in this report. If this conditi not met, the motion consists of the superposition of two motions. The first is mot a constant gravitational field where the effective "gravity" is kg; the factor k is determined explicitly. The second is the periodic motion that is the central theme this report. During the course of the analysis, the fundamental	AFWAL/FIBG		PROGRAM ELEMENT NO	PROJECT	TASK NO.	WORK U
11 TTLE (Include Security Classified 0n Periodic Solutions of an Atwood's Pendulum 62201F 2401 04 12 PERSONAL AUTHORIS) Donald Mittleman 13b. TIME COVERED FROM AUG 1985 To JUL 1986 14 DATE OF REPORT (Yr. Mo. Dev) 15. PAGE COUNT 24 13a TYPE OF REPORT Interim 13b. TIME COVERED FROM AUG 1985 To JUL 1986 14 DATE OF REPORT (Yr. Mo. Dev) 15. PAGE COUNT 24 13a SUPPLEMENTARY NOTATION 14. DATE OF REPORT (Yr. Mo. Dev) 15. PAGE COUNT 24 14. SUPPLEMENTARY NOTATION 14. SUBJECT TERMS (Continue on reverse (I necessary and identify by block number) Atwood's pendulum periodic motion dynamics 14. DATE OF REPORT (Yr. Mo. Dev) 15. PAGE COUNT 24 17. COSATI CODES 18. SUBJECT TERMS (Continue on reverse (I necessary and identify by block number) 24 18. SUBJECT TERMS (Continue on reverse (I necessary and identify by block number) Atwood's pendulum periodic motion 14. DATE OF REPORT (Yr. Mo. Dev) 24 19. Wet for second second second second second second mass induces a varying tension 15. PAGE COUNT Atwood's pendulum 14. DATE OF REPORT (Yr. Mo. Dev) 24 19. Wet for second is the pendulum 14. DATE OF REPORT (Yr. Mo. Dev) 14. Date OF REPORT (Yr. Mo. Dev) 24 19. Mastract Second is the periodic in the second mass. 15. FAGE COUNT (Yr. Mo. Dev) 26 27	Wright-Patterson AFB OH 45433	3-6553				
12. FERSINAL AUTHORISI Donald Mittleman 13b. TIME COVERED FROM AUG 1985To JUL 1986 14. DATE OF REPORT (Yr. Mo., Day) 15. PAGE COUNT MAY 1987 13. TYPE OF REPORT Interim 13b. TIME COVERED FROM AUG 1985To JUL 1986 14. DATE OF REPORT (Yr. Mo., Day) 15. PAGE COUNT MAY 1987 14. DATE OF REPORT Interim 13b. TIME COVERED FROM AUG 1985To JUL 1986 14. DATE OF REPORT (Yr. Mo., Day) 15. PAGE COUNT MAY 1987 17. COSATI CODES FIELD GROUP 18. SUBJECT TERMS (Continue on reverse () necessary and identify by block number) 24 17. COSATI CODES FIELD GROUP SUB.GR. AtWood's pendulum periodic motion dynamics 14. DATE OF REPORT (Yr. Mo., Day) 15. PAGE COUNT MAY 1987 17. COSATI CODES FIELD GROUP 18. SUBJECT TERMS (Continue on reverse () necessary and identify by block number) 24 17. COSATI CODES FIELD GROUP 18. SUBJECT TERMS (Continue on reverse () necessary and identify by block number) 24 17. MAY 1987 19. The formation of standard () and () standard () and () and () standard () and ()	On Periodic Solutions of an Atv	wood's Pendulum	62201F	2401	04	32
Uonald Mittleman 13a TYPE OF REPORT Interim 13b. TIME COVERED FROM_AUG 1985To JUL 1986 14. DATE OF REPORT (Yr. Mo., Day) MAY 1987 15. PAGE COUNT 24 16. SUPPLEMENTARY NOTATION 16. SUBJECT TERMS (Continue on reverse (I necessary and identify by block number) Atwood's pendulum periodic motion dynamics 17. COSATI CODES 18. SUBJECT TERMS (Continue on reverse (I necessary and identify by block number) Atwood's pendulum periodic motion dynamics 17. COSATI CODES 18. SUBJECT TERMS (Continue on reverse (I necessary and identify by block number) Atwood's pendulum periodic motion dynamics 17. COSATI CODES 18. SUBJECT TERMS (Continue on reverse (I necessary and identify by block number) Atwood's pendulum periodic motion dynamics 18. SUBJECT TERMS (Continue on reverse (I necessary and identify by block number) Atwood's pendulum periodic motion dynamics 19. Atwood's pendulum of the one mass induces a varying tension connecting wire; this, in turn, produces motion in the second mass. It is shown to not ne initial conditions are chosen as prescribed in this report. If this conditi not met, the motion consists of the superposition of two motions. The first is mot a constant gravitational field where the effective "gravity" is kg; the factor k is determined explicitly. The second is the periodic motion that is the central theme this report. During the course of the analysis, the fundamental frequency of the p of comparable length swinging in the earth's gravitational field; the factor is giv explicitly. This work is restricted to the extent that small angle approximations introduced initially for trigonometr	2. PERSONAL AUTHOR (S)		1		l``	
Intervin Iso. The COVERED Iso. Date OF REPORT (Yr. Mo. Day) Is PAGE COUNT Intervin FROM_AUG 1985 To JUL 1986 MAY 1987 24 Is. SUPPLEMENTARY NOTATION Is. SUBJECT TERMS (Continue on reverse if necessary and identify by block number) 24 FIELD GROUP SUB_GR. Atwood's pendulum 21 Prielo GROUP SUB_GR. Atwood's pendulum 21 Atwood's pendulum is defined as an Atwood's machine in which one of two masses allowed to swing as a pendulum while the other remains constrained to move only in vertical direction. The pendulum motion of the one mass induces a varying tension connecting wire; this, in turn, produces motion in the second mass. It is shown th motion can be made periodic if the ratio of the two masses and the dependency of the motion consists of the superposition of two motions. The first is mot a constant gravitational field where the effective "gravity" is kg; the factor k is determined explicitly. The second is the periodic motion that is the central theme this report. During the course of the analysis, the fundamental frequency of the p motion is determined. It is shown to be slightly higher than the frequency of a pe of comparable length swinging in the earth's gravitational field; the factor is give explicitly. This work is restricted to the extent that small angle approximations introduced initially for trigonometric functions. 21 ABSTRACT SECUNITY CLASSIFICATION UNCLASSIFIED/UNLIMITED & SAME AS RET Cloric USERS 21 ABSTRACT SECUNITY CLASSIFICATION 22 OFFICE SYMBOL 22 NA	Donald Mittleman				40.0000	0.11NT
16. SUPPLEMENTARY NOTATION 17. COSATI CODES 18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number) FIELD GROUP 22 01 22 01 24 01 25 01 26 01 27 01 28 01 29 01 20 01 20 01 21 01 22 01 23 01 24 05 25 01 26 01 27 01 28 01 29 01 20 01 20 01 21 02 22 01 23 01 24 01 25 01 26 01 27 01 28 01 29 01 20 01 20 <td>Interim FROM AUC</td> <td>G 1985_{TO} JUL 1986</td> <td>MAY 19</td> <td>нт (тг., мо., Day) 987</td> <td>15. PAGE C</td> <td>24</td>	Interim FROM AUC	G 1985 _{TO} JUL 1986	MAY 19	нт (тг., мо., Day) 987	15. PAGE C	24
introduced initially for trigonometric functions 20 DISTRIBUTION/AVAILABILITY OF ABSTRACT 21 ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED/UNLIMITED & SAME AS RPT DITICUSERS 22. NAME OF RESPONSIBLE INDIVIDUAL Robert W. Gordon	An Atwood's pendulum is defin allowed to swing as a pendulum vertical direction. The pendu connecting wire; this, in turn motion can be made periodic if on the initial conditions are not met, the motion consists o a constant gravitational field determined explicitly. The se this report. During the cours motion is determined. It is s of comparable length swinging explicitly. This work is rest	dynamics identify by block number identify by block number indentify by block number indentify by block number identify by block number if the othe inthe ratio of t identify by block number if the othe if the othe if the ratio of t chosen as prese if the superposit where the effect is the per- ie of the analys hown to be slig in the earth's ricted to the effect inthe earth's	d's machine in er remains con the one mass i ion in the sec the two masses tribed in this ition of two m ective "gravit riodic motion sis, the funda gravitational extent that sm	which one strained to nduces a v ond mass. and the do report. otions. The that is the mental free han the free field; the all angle of	of two mass o move only arying tensi It is shown ependency of If this conc the first is the factor k e central th quency of the equency of a factor is approximatio	es is in the on in t that t this r lition i motion is eme of e perio pendul given ns are
UNCLASSIFIED/UNLIMITED & SAME AS RPT DITICUSERS DUNCLASSIFIED/UNLIMITED & SAME AS RPT DITICUSERS DUNCLASSIFIED 220. NAME OF RESPONSIBLE INDIVIDUAL Robert W. Gordon 220. Telephone NUMBER (Include Area Code) (513)255-5236 220. OFFICE SYMBOL (513)255-5236 220. OFFICE SYMBOL	O DISTRIBUTION/AVAILABILITY OF ABSTRACT	T	21 ABSTRACT SECU	JRITY CLASSIFI		
220 NAME OF RESPONSIBLE INDIVIDUAL 226 TELEPHONE NUMBER 22C OFFICE SYMBOL Robert W. Gordon (513)255-5236 AELIAL / ETDC	INCLASSIFIED/UNLIMITED 🕅 SAME AS APT [DTIC USERS	Unclassified			
Robert W. Gordon (513)255-5236 AEUNI /EIDC	22. NAME OF RESPONSIBLE INDIVIDUAL		226 TELEPHONE N	UMBER	22c OFFICE SYM	BOL
	Robert W. Gordon		(513)255-5	236	AFWAL/FI	BG
DD FORM 1473, 83 APR EDITION OF 1 JAN 73 IS OBSOLETE.	D FORM 1473 92 APP				ASSIFIED	_
SECURITY CLASSIFICATION OF		EDITION OF 1 JAN 73 I	S OBSULE IE.		73. J. J J J J J J J	

Foreword

This report documents an effort performed by Dr. Don Mittleman, Professor of Mathematics at Oberlin College, Oberlin, Ohio for the Structural Vibration and Acoustics Branch, Structures and Dynamics Division, Flight Dynamics Laboratory, Air Force Wright Aeronautical Laboratories, Wright Patterson Air Force Base, Ohio. The investigation was conducted under Flight Dynamics Laboratory Project 2401, Structural Mechanics; Task 240104, Vibration Prediction and Control, Measurement and Analysis; Work Unit 24010432, Large Space Structures Technology Program.

The work was performed by Dr. Mittleman as a visiting scientist to the Flight Dynamics Laboratory through a contract with Anamet Laboratories, Inc. during the period August 1985 through July 1986. This report is an interim result of Dr. Mittleman's work.



Table of Contents

Section		Page
ł	Introduction	1
II	The Mathematical Statement	3
111	Discussion	10
IV	Conclusion	14
	References	17

۷

1 × 1

Section I

Introduction

The Structural Vibration and Acoustics Branch has initiated a program to study the dynamics and control of Large Space Structures (LSS). This Large Space Structures Technology Program (LSSTP) is intended to enable the Flight Dynamics Laboratory to instrument, test, and analyze large space structures on the ground in order to predict their behavior in space. Since the testing is to be done in a ground based laboratory, i.e. under 1-g acceleration, the experiments must be designed so as to counteract this gravitational effect. One proposal is to use soft suspension systems. Long cables can provide pendulum support with low frequency for horizontal motion, but, for a pendulum, the restraint in the vertical direction is rigid. One way to provide soft restraint vertically, while maintaining the pendulum approach, is to counter balance the test model.

The simplest counter-balanced suspension system is the classic Atwood's machine.^[1] This consists of two masses m_1 and m_2 connected by an inextensible, flexible wire of negligible mass, draped over a frictionless, massless pulley, which in turn is rigidly suspended from an overhead support. The motion of the two masses is assumed to be constrained to the vertical direction: either there is no motion, the situation that occurs when the two masses are equal and there is no initial velocity, or, as one mass rises the other falls.

In previous reports ^{[2],[3]}, the Atwood's configuration was studied when the one mass, m_1 , remained constrained to move vertically but the other mass, m_2 , was swung as a pendulum. The results described in these two reports may be summarized as follows: after deriving the equations of motion, these were programmed and run on a digital computer using a Runga-Kutta-Fehlberg ^[4] routine. The first numerical experiments assumed the two masses m_1 and m_2 were equal. As one normally does with a

pendulum, the mass m_2 was displaced through an initial angle θ_0 , released with zero initial velocity, $\dot{\theta}_0(0) = 0$ and the subsequent motion studied. Several runs were made, taking successive values for θ_0 of 0.01, 0.10, 0.20, and 0.50 radians. In all four cases, the mass m_1 initially dropped ever so slightly and then rose monotonically. Obviously, this is equivalent to a lengthening of the pendulum arm. As the pendulum arm increased, the amplitude of swing decreased.

While not concerned per se with counter-balanced suspension systems, material of collateral interest may be found in [5],[6]. These two references treat problems whereby a swinging mass is being hoisted by a winch.

It is intuitively obvious that if the mass m_1 is made slightly greater than m_2 that this monotonic lengthening of the pendulum arm with time would be slowed and that if m_1 were made much greater than m_2 , the length of the pendulum arm would actually decrease. Thus, there must be some critical relation between m_1 and m_2 for which the length of the pendulum arm is either constant or a periodic function of time. Further numerical experiments ^[2] indicated that this relationship is:

 $m_1 = m_2 [1.000 + 7.451 \times 10^{-8} \Theta_0 + 0.250 \Theta_0^2]$

It was found using a least square fit to data points corresponding to $\theta_0=0.1$, 0.2, 0.3, 0.4, and 0.5 radians.

In this report, a theoretical derivation of the relationship between m_1 and m_2 is obtained under the added restriction, however, that the initial angular displacement θ_0 is small. The technique used is classical perturbation theory.^[7]

Section II

The Mathematical Statement

The geometry of the configuration studied is given in Figure 1. From this geometry, assuming that the masses of the pulley and the wire are negligible and that the radius of the pulley may be neglected also, the differential equations describing the motion of the mass m_2 are:

$$\rho \,\ddot{\theta} + 2\dot{\rho}\dot{\theta} + g\sin\theta = 0 \tag{1}$$

$$(m_1 + m_2) \dot{\rho} - m_2 \rho \dot{\theta}^2 + m_1 g - m_2 g \cos \theta = 0$$
(2)

where dots indicate differentiation with respect to the time *t*. Equations 1 and 2 may be simplified slightly if we observe that by letting $\rho = \rho g$, both equations contain g as a factor and we may divide through by it. This is equivalent to choosing units so that g = 1. Furthermore, by using the small angle approximation, these equations then become:

$$\rho \dot{\theta} + 2\dot{\rho}\dot{\theta} + \theta = 0 \tag{3}$$

$$(m_1 + m_2) \dot{\rho} - m_2 \rho \dot{\theta}^2 + (m_1 - m_2) + \frac{1}{2} m_2 \theta^2 = 0$$
(4)

Case i) Constant p.

We prove now that there is no solution of Equations 3 and 4 for which $\rho = \rho_0 = \text{constant}$. Actually, we shall show that assuming that $\rho = \text{constant}$ leads to a contradiction. Using this assumption, Equations 3 and 4 become

$$\rho_0 \ddot{\theta} + \theta = 0 \tag{3.1}$$

$$m_2 \rho_0 \dot{\theta}^2 - (m_1 - m_2) - \frac{1}{2} m_2 \theta^2 = 0$$
 (4.1)

Differentiate Equation 4.1 with respect to the time t and divide through by m_2 to get:

$$2\rho_0 \dot{\Theta}\dot{\Theta} - \Theta\dot{\Theta} = 0 \tag{5}$$

If $\dot{\theta} \equiv 0$, then Equation 3.1 implies that $\theta \equiv 0$ and Equation 4.1 then implies that $m_1 = m_2$, the classic Atwood's machine case with equal masses. For this classic example there is indeed a solution $\rho = \rho_0$, a constant. If $\dot{\theta} \neq 0$, then Equation 5 implies that

$$2\rho_0 \dot{\theta} = \theta \tag{6}$$

Equations 3.1 and 6 are incompatible. There is, therefore, no solution of Equations 3 and 4 for which ρ is constant.

We proceed now to discover the relationship between m_1 and m_2 for which $\rho(t)$ and $\theta(t)$ are periodic solutions of Equations 3 and 4.

Case ii) The General Case.

The perturbation technique requires that Equations 3 and 4 be modified and rewritten as follows:

$$\dot{\rho} = \in [\mathbf{m}_2 \rho \dot{\theta}^2 \cdot (\mathbf{m}_1 \cdot \mathbf{m}_2) \cdot {}_2 \mathbf{m}_2 \theta^2]$$
(7)

$$\rho \ddot{\theta} + \theta = -2\dot{\rho}\dot{\theta} \tag{8}$$

where we have introduced a parameter \in and written $\mathbf{m}_1 = \mathbf{m}_1/(\mathbf{m}_1 + \mathbf{m}_2)$ and $\mathbf{m}_2 = \mathbf{m}_2/(\mathbf{m}_1 + \mathbf{m}_2)$. In Equations 7 and 8, we change the time variable *t* to a pseudo-time **t** by means of the formula $t = \sqrt{\omega} t$ and get

$$\rho'' = \in [\mathbf{m}_2 \rho \theta'^2 + \omega(\mathbf{m}_2 - \mathbf{m}_1) - \frac{1}{2} \mathbf{m}_2 \omega \theta^2]$$
(9)

$$\rho \theta \,\widetilde{}\, + \omega \theta = -2\rho \,\widetilde{\theta} \,\widetilde{}\, \tag{10}$$

where the f indicates differentiation with respect to **t**. Assuming that ρ, θ, ω and \mathbf{m}_1 are analytic functions of \in :

 $\rho = \mathbf{r}_0 + \epsilon \mathbf{r}_1 + \epsilon^2 \mathbf{r}_2 + \dots$ $\theta = \theta_0 + \epsilon \theta_1 + \epsilon^2 \theta_2 + \dots$ $\omega = \omega_0 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \dots$ $\mathbf{m}_1 = \mu_0 + \epsilon \mu_1 + \epsilon^2 \mu_2 + \dots$

We introduce these expansions into Equations 9 and 10 and collect terms in like powers of \in . Since \in is an arbitrary parameter, each coefficient of the powers of \in must be zero. This results in setting up an infinite system of pairs of second order differential equations which can be solved successively and recursively. The first set of pairs is obtained by setting \in =0. The two equations obtained are:

$$\mathbf{r}_{0} = \mathbf{0} \tag{11}$$

$$\mathbf{r}_0 \Theta_0 \stackrel{\prime \prime}{=} + \omega_0 \Theta_0 = -2\mathbf{r}_0 \stackrel{\prime }{\Theta}_0 \stackrel{\prime \prime}{=} \tag{12}$$

The solution of Equation 11 is $\mathbf{r}_0 = \mathbf{r}_0$, a constant, since we assume as an initial condition that \mathbf{r}_0 (0) = 0. The solution of Equation 12 then becomes

$$\theta_0 = \theta_0 \cos(kt) \tag{13}$$

where Θ_0 is the initial angular displacement, $k^2 = \omega_0/r_0$ and we have assumed that $\Theta_0'(0) = 0$. It will turn out that ω_0 plays no role in the final form of the formulas for $\rho(t)$ and $\theta(t)$. In terms of the physical variables, $k^2 = g/\rho(0)$.

If Equation 9 is differentiated with respect to the parameter \in and then \in is set equal to zero, we obtain:

$$\mathbf{r}_{1} = \mathbf{m}_{2} \mathbf{r}_{0} \theta_{0}^{2} + \mu_{0} \omega_{0} + \mathbf{m}_{2} \omega_{0} (1 - \frac{1}{2} \theta_{0}^{2})$$
(14)

When the values of θ_0 and θ_0 , as given from Equation 13 and its derivative, are substituted into Equation 14, and remembering that $\mathbf{r}_0 = \mathbf{r}_0$, a constant, then, after some algebraic manipulation, Equation 14 may be written as:

$$T_1 = \omega_0 [-\mu_0 + m_2 + (1/4)m_2 \theta_0^2 - (3/4)m_2 \theta_0^2 \cos(2kt)]$$
(15)

Since we are seeking a <u>periodic</u> solution for \mathbf{r} , the secular term in the general solution of Equation 15, the term that would give rise to a quadratic increase or decrease with time in \mathbf{r}_1 , will be eliminated if we set:

$$\mu_0 = \mathbf{m}_2 + (1/4)\mathbf{m}_2 \theta_0^2 \tag{16}$$

Otherwise, \mathbf{r}_1 will contain a term of the form $(-\mu_0 + \mathbf{m}_2 + (1/4)\mathbf{m}_2\theta_0^2)\mathbf{t}^2/2$ and, depending on the relative magnitude chosen for μ_0 and $(\mathbf{m}_2 + (1/4)\mathbf{m}_2\theta_0^2)$, \mathbf{r}_1 increases or decreses quadratically with time. The solution of Equation 15, for the initial conditions $\mathbf{r}_1(0) = \mathbf{r}_1(0) = 0$, is:

$$r_1 = (3/16)r_0 m_2 \theta_0^2 (\cos(2kt) - 1)$$
 (17)

When Equation 10 is differentiated with respect to \in and then \in is set equal to zero, we obtain:

$$\mathbf{r}_{0}\boldsymbol{\Theta}_{1}^{\prime} + \boldsymbol{\omega}_{0}\boldsymbol{\Theta}_{1} = -\mathbf{r}_{1}\boldsymbol{\Theta}_{0}^{\prime} - \boldsymbol{\omega}_{1}\boldsymbol{\Theta}_{0} - 2\mathbf{r}_{1}^{\prime}\boldsymbol{\Theta}_{0}^{\prime}$$
(18)

Since \mathbf{r}_0 , $\mathbf{\theta}_0$ and \mathbf{r}_1 are known, we substitute their respective values into the right-hand side of Equation 18 to obtain:

$$r_{0} \Theta_{1}^{\prime} + \omega_{0} \Theta_{1} = \Theta_{0} \{ -[\omega_{1} + (15/32)m_{2}\Theta_{0}^{2}\omega_{0}]\cos(kt) + [(15/32)m_{2}\Theta_{0}^{2}\omega_{0}]\cos(3kt) \}$$
(19)

We are interested in obtaining the particular solution of Equation 19 that excludes the secular term. This is done by choosing

$$\omega_{1} = -(15/32)\mathbf{m}_{2}\Theta_{0}^{2}\omega_{0} \tag{20}$$

With this done, the particular solution of Equation 20, with initial conditions $\theta_1(0) = \theta_1(0) = 0$, is:

$$\theta_{1} = (15/256) \theta_{0}(\mathbf{m}_{2} \theta_{0}^{2}) [\cos(kt) - \cos(3kt)]$$
(21)

Note that if ω_1 is not chosen as in Equation 20, then the solution for θ_1 will go to infinity with time. We interpret this by saying that the mathematical procedure fails. Thus, in order to retain mathematical viability, we must pick ω_1 as given in Equation 20.

If we were to proceed no further, we would have the following approximate solutions to Equations 3 and 4 (after setting $\epsilon = 1$):

$$\rho = r_0 [1 + (3/16) m_2 \theta_0^2 (\cos (2kt) - 1)]$$
(22)

$$\Theta = \Theta_0 \{ \cos(kt) + (15/256) (m_2 \Theta_0^2) [\cos(kt) - \cos(3kt)] \}$$
(23)

where

$$\mathbf{m}_1 = \mathbf{m}_2 + (1/4)\mathbf{m}_2 \Theta_0^2$$
 (24)

$$\omega = \omega_0 [1 - (15/32)\mathbf{m}_2 \Theta_0^2]$$
(25)

$$k^2 = \omega_0 / r_0 \tag{26}$$

We proceed to get the next higher order terms in the expansions for $\rho, \theta, \mathbf{m}_1$, and ω .

By equating the coefficients of \in^2 from both sides of Equation 9, we get

$$\mathbf{r_{2}}^{\prime} = -[\omega_{0}\mu_{1} + \omega_{1}(\mu_{0} - \mathbf{m_{2}})] + \mathbf{m_{2}}[\mathbf{r_{1}}\theta_{0}^{\prime 2} + 2\mathbf{r_{0}}\theta_{0}^{\prime}\theta_{1}^{\prime}] - \frac{1}{2}\mathbf{m_{2}}[\omega_{1}\theta_{0}^{\prime 2} + 2\omega_{0}\theta_{0}\theta_{1}]$$
(27)

After substituiting the several quantities already found for their respective values as given in the right-hand side of Equation 27, it may be rewritten as:

$$r_2 = (9/512)m_2^2 \Theta_0^4 \omega_0 [4 \cos (2kt) + 9 \cos (4kt)]$$
 (28)

where, in order to preclude the introduction of a secular term in ${\bf r_2}$, we had set:

$$\mu_1 = (63/512)\mathbf{m}_2^2 \Theta_0^4 \tag{29}$$

The solution of Equation 28, subject to the initial conditions $\mathbf{r}_2(0) = \mathbf{r}_2(0) = 0$, is:

$$\mathbf{r}_{2} = (9/2^{13})(\mathbf{m}_{2} \Theta_{0}^{2})^{2} \mathbf{r}_{0} [25 - 16 \cos(2kt) - 9 \cos(4kt)]$$
(30)

By equating the coefficients of ϵ^2 from both sides of Equation 10, we get:

$$\mathbf{r}_{0}\boldsymbol{\Theta}_{2}^{\prime\prime} + \boldsymbol{\omega}_{0}\boldsymbol{\Theta}_{2} + [\mathbf{r}_{1}\boldsymbol{\Theta}_{1}^{\prime\prime} + \boldsymbol{\omega}_{1}\boldsymbol{\Theta}_{1} + 2\mathbf{r}_{1}^{\prime\prime}\boldsymbol{\Theta}_{1}^{\prime\prime}] + [\mathbf{r}_{2}\boldsymbol{\Theta}_{0}^{\prime\prime} + \boldsymbol{\omega}_{2}\boldsymbol{\Theta}_{0} + 2\mathbf{r}_{2}^{\prime\prime}\boldsymbol{\Theta}_{0}^{\prime\prime}] = \mathbf{0}$$
(31)

After substituting the values already found for $\mathbf{r}_0, \theta_0, \mathbf{r}_1, \theta_1, \omega_0, \omega_1$ and their derivatives where appropriate, we get:

$$\mathbf{r}_{0}\theta_{2} + \omega_{0}\theta_{2} = (9/2^{14})\mathbf{m}_{2}^{2}\theta_{0}^{5}\omega_{0}[163\cos(3kt) - 291\cos(5kt)]$$
(32)

and where, in order to preclude the introduction of a secular term in θ_2 , we have set:

$$\omega_2 = (9/128) \mathbf{m}_2^2 \theta_0^4 \omega_0 \tag{33}$$

The value of θ_2 , subject to the initial conditions $\theta_2(0) = \theta_2(0) = 0$, found by integrating Equation (32) is:

$$\Theta_2 = (9/2^{17}) \mathbf{m}_2^2 \Theta_2^5 [66 \cos(kt) -163 \cos(3kt) + 97 \cos(5kt)]$$
(34)

Summarizing the results obtained thus far, we see that after setting \in = 1, we get the following approximate solutions for $\rho(t)$ and $\theta(t)$:

$$\rho(t) = r_0 [1 + (3/16)m_2 \Theta_2^2(\cos (2kt) - 1) + (9/2^{13})m_2^2 \Theta_0^4(25 - 16\cos (2kt) - 9\cos (4kt)]$$
(35)

$$\Theta(t) = \Theta_0 \{ \cos (kt) + (15/256) m_2 \Theta_0^2 [\cos (kt) - \cos (3kt)] + (9/2^{17}) m_2^2 \Theta_2^4 [66 \cos (kt) - 163 \cos (3kt) + 97 \cos (5kt)] \},$$
(36)

where

$$\mathbf{m}_{1} = \mathbf{m}_{2} + (1/4)\mathbf{m}_{2}\theta_{0}^{2} + (63/512)\mathbf{m}_{2}^{2}\theta_{0}^{4}$$
(37)

$$\omega = \omega_0 [1 - (15/32) \mathbf{m}_2 \Theta_0^2 + (9/128) \mathbf{m}_2^2 \Theta_0^4]$$
(38)

Since $\mathbf{t} = (\omega)^{-\frac{1}{2}} t$, $\rho(\mathbf{t}) \Rightarrow \rho(\omega^{-\frac{1}{2}} t)$, $\theta(\mathbf{t}) \Rightarrow \theta(\omega^{-\frac{1}{2}} t)$, i.e. in formulas (35) and (36) replace \mathbf{t} by $\omega^{-\frac{1}{2}} t$. Finally,

$$\rho(t) = g\rho(\omega^{-\frac{1}{2}}t)$$

and

$$\theta(t) = \theta(\omega^{-\frac{1}{2}}t)$$

Section III

Discussion

An understanding of the results presented in the previous section may be enhanced by a numerical example. Before doing this, however, we want to recall the relationships:

 $\rho(t) = g\rho(t)$ and $t = \sqrt{\omega t}$

These imply that $\dot{\rho} = g \dot{\rho} = g \rho''/\omega$. Using these we can calculate the acceleration acting on the mass m₁. We do this first using the approximate solutions given by Equations 22 and 23. In particular, we examine Equation 22 and take the second derivative with respect to t to get:

$$\rho''(t) = -(3/4)\mathbf{m_2}\Theta_0^2\omega_0 \cos(2kt)$$
(39)

and where we have used $k^2 = \omega_0/r_0$. Then, using the value of ω as given in Equation 25 :

$$\dot{\rho} = -\{(3/4)[\mathbf{m}_2 \theta_0^2] / [1 - (15/32)\mathbf{m}_2 \theta_0^2] \} \cos(2kt)$$
(40)

Remembering that $\mathbf{m}_1 = 1 - \mathbf{m}_2$, we solve Equation 24 for \mathbf{m}_2 and substitute that value into Equation (40) to get:

0.222.22

$$\ddot{\rho} = -[24\theta_0^2/(64 - 7\theta_0^2)] \cos(2kt)$$
(41)

For values of θ_0^2 of interest, say not to exceed 0.1 radians, the expression within the [...] in Equation 41 is a monotonically increasing function of θ_0 and will therefore attain its maximum value at the end-point of the interval of

interest. If we take that to be 0 < $\theta_0 \le 0.1$ radians, then for $\theta_0 = 0.1$,

$$\rho(t) = -.0037541 \text{ g cos}(2\text{kt})$$
 (42)

Thus, the acceleration of the mass m_1 is only, at most, 0.00375 times the acceleration of gravity, a very small quantity. While the mass m_1 does not move as if it were in free space, it is subject to a gravitational influence that is only .00375 times the effect it would normally sense on earth.

The description given above is based on using only two terms in the expansion for $\rho(t)$. Some change in the calculated value of the acceleration of the mass m_1 should be expected if three terms are used in the expansion for $\rho(t)$. While it is possible to carry out these computations algebraically, no new insight into the motion is gained. Accordingly, so as not to get lost in a maze of algebraic manipulation and symbolism, we limit our discussion to a numerical example, indicating along the way how other numerical examples may be handled.

We start with Equation 35 and calculate:

$$\rho''(t) = \omega_0 (m_2 \Theta_0^2) [A_1 \cos(2kt) + A_2 \cos(4kt)]$$
(43)

where:

$$A_1 = [-(3/4) + (9/128) m_2 \Theta_0^2]$$

$$A_2 = (81/512) m_2 \Theta_0^2$$

and again we have used $k^2 {=} \omega_0 / r_0$. Then:

$$\dot{\rho}(t) = A_0[A_1 \cos(2kt) + A_2 \cos(4kt)]$$
(44)

where:

$$\mathsf{A}_{0} = (\mathsf{m}_{2} \Theta_{0}^{2} \omega_{0}) / \omega$$

or:

$$\mathsf{A}_0 = \mathsf{m}_2 \Theta_0^2 / [1 - (15/32) \mathsf{m}_2 \Theta_0^2 + (9/128) \mathsf{m}_2^2 \Theta_0^4]$$

where the value of ω is taken from Equation 38 .

The acceleration is stationary when $2kt_1 = 0 + n\pi$ and when $\cos(2kt_2) = -A_1/(4A_2)$. For these values of t_1 , the maximum value of the acceleration cannot exceed $|A_0|g[|A_1| + |A_2|]$; for values of θ_0 of interest, the magnitude of $A_1/(4A_2)$ is greater than 1 and there is no real value for t_2 .

In order to calculate values for A_0 , A_1 , and A_2 , we need to know $m_2 \theta_0^2$. For a given value of θ_0 , say $\theta_0 = 0.1$ radians, and remembering that $m_1 + m_2 = 1$, we find, using Equation 37, that $m_2 = .4993742479515557$. We then find that $A_0 = 5.0054505 \times 10^{-3}$, $A_1 = -.74964888$, $A_2 = 7.9002567 \times 10^{-4}$ and the maximum value of the acceleration is at most 3.7483759 $\times 10^{-3}$ g.

A comparison of the results when 2 or 3 terms are used in the determination of the acceleration of m_1 indicates that the difference is so small, of the order of 5.7×10^{-6} g, as to be negligible. For all practical purposes, the solution to the problem is given by the Equations 22 - 26 and Equation 41.

When the fundamental frequency of the system is compared to the frequency of a pendulum of length ρ (0), there is a change in frequency given by the multiplicative factor $[(64 + 8\theta_0^2)/(64 - 7\theta_0^2)]^{1/2}$. For $\theta_0 = 0.1$ radians, this factor is 1.0011725.

The solution to Equations 1 and 2 as given by Equations 22-26 or, if you prefer, by Equations 35-38, are *periodic* solutions. As already noted, to obtain a periodic solution for ρ it was necessary to carefully adjust the ratio of the two masses m_1 and m_2 . If values of m_1 and m_2 different from those determined by means of Equation 22 (or Equation 37) are used, then the solution for the pendulun arm $\rho(t)$ will contain an additional term involving

 t^2 . The case for $m_1 = m_2$ was studied in reference ^[8].

SCREET ALLENS

If a value of ω different from the ones given in Equations 25 or 38 is used, then the mathematical process fails because secular terms, which physically and experimentally we know do not have meaning, are introduced into the equation for θ , Equations 23 or 36.

R).

Section IV

Conclusions

A theoretical description of an Atwood's pendulum machine is given. Implicit in these calculations were the assumptions that the pulley is frictionless and that its radius could be neglected. Also, the wire connecting the two masses is assumed to be perfectly flexible, inextensible and of negligible mass. The following results were deduced:

(1) If the ratio of the two masses are chosen as prescribed, then the vertical motion of the mass m_1 , which is the only motion possible for this mass, is periodic;

(2) Equivalently, the change in length of the pendulum arm associated with the swinging of the mass m_2 is exactly the same as the change in length of the wire supporting m_1 except that as the one increases the other decreases;

(3) If the two masses are not picked as prescribed so as to insure periodic motion, then the motion of m_1 consists of the superimposition of two motions:

(a) an accelerated motion as if the mass m_1 were in a gravitational field of magnitude < g, the magnitude of this deviation from g depending on just how much the two masses differ from the ratio required for the periodic motion, and

(b) the periodic motion superimposed (just added onto) the motion described in (a);

(4) While the conclusions (1),(2), and (3) given above were obtained by considering only two terms in the expansion of the relevant functions, the effect of using three terms in these expansions was shown to modify the conclusions only quantitatively, not qualitatively, and the change was of an

order less than 1%.

Constant in the start

00000

In mathematical terms, this periodic solution is referred to as a limit cycle; it is, however, unstable. Any set of conditions that differ ever so **slightly** from the ones needed to establish the limit cycle will produce a trajectory that increasingly diverges from the limit cycle and this translates into the statement that the pendulum arm goes either to zero or to infinity^{[9],[10]}.



Figure 1 A Schematic for an Atwood's Pendulum

References

- [1] Goldstein, Herbert. <u>Classical Mechanics.</u> Addison-Wesley Press, Inc., Cambridge, Mass., 1950, p.15
- [2] Mittleman, Don. Large Space Structures Dynamic Testing, a report submitted to: Southeastern Center for Electrical Engineering Education under contract to the Air Force Office of Scientific Research, Flight Dynamics Laboratory, Wright-Patterson AFB, OH, 45433, 20 July 1984.
- [3] Zeigler, Michael L. <u>Low-Restraint Suspensin Space Structure Dynamic</u> <u>Testing</u>. AFWAL-TM-85-196-FIBG, Flight Dynamics Laboratory, Wright-Patterson AFB, OH, 45433, April 1985.
- [4] Burden, R.L., Faires, J.D., Reynolds, A.C. <u>Numerical Analysis</u>, 2nd Edition, Prindle, Weber & Schmidt, Boston, Mass. p. 216.
- [5] Maurer, O.F. and Hall, K.S. <u>Evaluation of High-Speed Rescue Hoist</u> <u>Dynamics.</u> AFFDL-TR-78-140, Flight Dynamics Laboratory, Wright-Patterson AFB, OH, 45433, October 1978.
- [6] Wells, Dare A. Lagrangian Mechancs. Schaum's Outline Series, MaGraw-Hill Book Co. 1967. Problem 3.20, p. 55.
- [7] Nayfeh, A.H. Introduction to Perturbation Techniques, John Wiley & Sons, New York, 1981.
- [8] Lee, Jon. <u>Counter-Balanced Pendulum.</u> AFWAL-TR-86-3014, Wright-Patterson AFB, Ohio 45433.
- [9] Arrowsmith, D.K. & Place, C.M., <u>Ordinary Differential Equations</u>. Chapman and Hall, 1982, p.108.
- [10] Arnold, V.I., <u>Ordinary Differential Equations</u>, The MIT Press, 1985, p.93.

List of Symbols

 $m_1 = mass 1$

50 A 30

2.2.2.2.3.2.2.2

- $m_p = mass 2$
- $\mathbf{m}_1 = \text{reduced mass1} = \mathbf{m}_1/(\mathbf{m}_1 + \mathbf{m}_2)$
- \mathbf{m}_2 = reduced mass 2 = $\mathbf{m}_2/(\mathbf{m}_1 + \mathbf{m}_2)$
- t = real time, seconds
- t = pseudo-time
- θ_0 = initial angular displacement
- $\dot{\Theta}_{n}$ = initial angular velocity
- ρ = length of pendulum arm, meters
- $\theta = \theta$ = angular displacement of pendulum arm; two symbols are used to make the notation uniform
- $\rho = \rho/g$ = length of pendulum arm in a system of units in which g = 1
- ω = a parameter relating t and t; eventually shown to be a factor specifying frequency
- ϵ = a parameter used to establish asymptotic expansion
- \mathbf{r}_{i} = a term in the expansion of ρ
- θ_i = a term in the expansion of θ
- ω_i = a term in the expansion of ω
- μ_i = a term in the expansion of \mathbf{m}_1
- $r_0 = initial length of pendulum arm in system of units for which g = 1$
- k = a parameter relating to frequency in the system in which g = 1; $k^2 = \omega_0 / r_0$

