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Lagrangian Multipliers and Superfluous Variables

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This paper develops new forms for the Lagrangian Multipliers used in studies of constrained systems, as well as variants of the Euler-Lagrange equations. These formulas facilitate the computation of the multipliers and solution of the Euler-Lagrange equations. In addition, the link between virtual displacement and the multipliers Euler-Lagrange is elucidated.

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CONTENTS

Lagrangian Multipliers and Superfluous Variables

1. Introduction

This paper uses superfluous coordinates and the "constraint jiggling" approach (usually introduced when performing displacements violating constraints in order to compute the forces of constraint) to develop a new form for Lagrangian Multipliers. As a part of the development, this paper generates unified equations combining a new variant of the Euler-Lagrange equations with a new closed form expression for these multipliers in terms of the superfluous coordinates and the boundary conditions. These results elucidate the mechanism by which the constraints determine the Lagrangian Multiplier, and their role in the Euler-Lagrange equations with constraints. This approach facilitates the computations for the following reasons:

(a) Finding the inverse functions requires the solution (at worst) of a system of m equations in $3n$ coordinates, where the $3n-m$ independent coordinates are treated as known constants and no derivatives are involved. Once done, the dependent coordinates $x_1, \ldots, x_m$ and their derivatives can be replaced in the constrained form of the Euler-Lagrange equations derived in the paper by $g_1, \ldots, g_m$ and their derivatives. so these equations will now contain only the independent $3n-m$ coordinates and their derivatives. The standard treatment requires the solution of a system of $3n$ differential equations and m algebraic equations in $3n+m$ variables.

(b) Once these Euler-Lagrange equations are solved, the Lagrange Multipliers are obtainable without further solution of equations.

(c) The inversion can often be performed by inspection, as is the case in the example cited in the paper.

(d) A good choice of coordinates may trivialize some of the m equations when computing the inverse functions.

2. Statement of the Problem

We assume that there are n particles traveling in 3-dimensional space, subject to m smooth independent holonomic constraints

$$f_\alpha(x_1, \ldots, x_{3n};t)=c_\alpha \quad \text{for } \alpha = 1, \ldots, m \tag{1}$$

We treat this problem in the $3n$-dimensional configuration space of the n particle system, by concatenating the n 3-dimensional position vectors into one vector in R^{3n}. Since the constraints are assumed to be smooth, the constraint force corresponding to constraint surface

$f_\alpha(x_1, \ldots, x_{3n};t)=c_\alpha$

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must be co-linear with

$$\nabla f_\alpha(x_1,\ldots,x_{3n};t)=\sum_{k=1}^{3n}\frac{\partial f_\alpha}{\partial x_k}(x_1,\ldots,x_{3n};t)\,\hat{i}_k,$$

where $\hat{i}_k$ is the k^{th} canonical unit vector in R^{3n}. Therefore, the total constraint force $\vec{R}$ must be a linear combination of the ∇f_α.

We also assume that the constraint forces (and gradients) are locally linearly independent along the trajectory of the system in configuration space.

Note that all vectors in this paper that do not involve time will be R^{3n} dimensional, whereas vectors that do involve time will be R^{3n+1} dimensional.

3. Results to be proved

The following discussion will:

(1) Derive a new expression for the Lagrangian multipliers

$$\lambda_\beta(t)=\sum_{\alpha=1}^{m}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\alpha}\right)-\frac{\partial L}{\partial x_\alpha}\right]\frac{\partial g_\alpha}{\partial y_\beta}\Big|_{y_1=c_1,\ldots,y_m=c_m}$$

where the $g_\alpha(y_1,\ldots,y_m,x_{m+1},\ldots,x_{3n};t)$ are the inverse functions of the m constraint functions $f_\alpha(x_1,\ldots,x_{3n};t)$, where $y_\alpha=f_\alpha(x_1,\ldots,x_{3n};t)$, for $\alpha=1,\ldots,m$.

(2) Derive a new variant of Lagrange's equations, using the superfluous coordinates.

(3) Show that the $\lambda_\beta(t)$ are components of a generalized constraint force in R^{3n}.

4. Finding the Dependent Coordinates and Relating them to the Independent Coordinates

Let $(x_1,\ldots,x_{3n};t)$ be an arbitrary point of R^{3n+1}. We next want to find the dependent coordinates, and show that we can relate them (in a neighborhood of $(x_1,\ldots,x_{3n};t)$) to the independent coordinates. The dependent coordinates are also referred to as " superfluous " in some of the physics literature. We have assumed that the m constraints are independent, and therefore have assured that there are $3n-m$ independent coordinates. We will intentionally remain vague about the exact meaning of "neighborhood" in R^{3n}, but a good discussion may be found in the first cited reference.

Note that while it is usually possible (discussion of "usually" deferred) to relate the dependent to the independent variables, this can only be done locally (therefore the "neighborhood"). Variables that are dependent in one area could in principle be independent elsewhere. In any case, the functional expressions relating the dependent to the independent variables may only have local validity. In the examples, it is shown that in some cases, this locality is not much of a problem, because the neighborhoods are really enormous. The paper will generally assume some neighborhood N in which the discussion takes place. The Inverse Function Theorem[1] provides the necessary conditions for the existence of the m dependency relations (the "usually" mentioned above) and for the existence of neighborhood N.

We start by defining a function that is a set of $3n$ coordinate transformations with time as an added coordinate. We next show that a certain determinant (its Jacobian) is non-zero in a neighborhood of $(x_1,\ldots,x_{3n};t)$. This condition is required by the Inverse Function Theorem. This theorem will then justify the existence of the inverse coordinate transformations and the appropriate neighborhood N.

We pick a point $(x_1,\ldots,x_{3n};t)$, and we assume that the m constraint functions $f_\alpha(x_1,\ldots,x_{3n};t)$ for $\alpha = 1,\ldots,m$ are independent, i.e. have gradients $\nabla f_1,\ldots,\nabla f_m$ which are linearly independent[2] at the point. Hence the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{3n}} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_{3n}} \end{pmatrix}$$

has m independent columns (or equivalently[3] has rank m).

By relabeling the variables if necessary, we may assume that the leftmost m columns are linearly independent, so that

$$\det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_m} \end{pmatrix} \neq 0 \text{ [4]}.$$

Since the determinant is a continuous function, we can find a neighborhood of $(x_1,\ldots,x_{3n};t)$ in which the determinant is non-zero. In other words, we have a neighborhood of $(x_1,\ldots,x_{3n};t)$ in which the the leftmost m columns of the matrix are linearly independent.

We now define

$$F(x_1,\ldots,x_{3n};t) = \begin{pmatrix} f_1(x_1,\ldots,x_{3n};t) \\ \cdots \\ f_m(x_1,\ldots,x_{3n};t) \\ f_{m+1}(x_1,\ldots,x_{3n};t) \\ \cdots \\ f_{3n}(x_1,\ldots,x_{3n};t) \\ t \end{pmatrix},$$

where

$$f_j(x_1,\ldots,x_{3n};t) = x_j, \quad \text{for } j = m+1,\ldots,3n.$$

The Jacobian of F is the $(3n+1)$ by $(3n+1)$ determinant

$$\det\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{3n}} & \frac{\partial f_1}{\partial t} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_{3n}}{\partial x_1} & \cdots & \frac{\partial f_{3n}}{\partial x_{3n}} & \frac{\partial f_{3n}}{\partial t} \\ 0 & \cdots & 0 & 1 \end{pmatrix} = \det\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_m} \end{pmatrix} \neq 0.$$

Therefore the Jacobian of F is non-zero in the aforementioned neighborhood of $(x_1, \ldots, x_{3n};t)$.

Define $y_\alpha = f_\alpha(x_1, \ldots, x_{3n};t)$, for $\alpha = 1, \ldots, m$.

The Inverse Function Theorem now allows us to conclude that there are $3n+1$ functions

$$g_s(y_1, \ldots, y_m, x_{m+1}, \ldots, x_{3n};t) \quad \text{for } s = 1, \ldots, 3n+1$$

1) Possessing continuous partial derivatives.
2) Defined on points $(y_1, \ldots, y_m, x_{m+1}, \ldots, x_{3n};t)$ in F[N], the image of some neighborhood N of $(x_1, \ldots, x_{3n};t)$.

In addition, these functions satisfy the important (inverse) condition

$$g_s(f_1(x_1, \ldots, x_{3n};t), \ldots, f_m(x_1, \ldots, x_{3n};t), x_{m+1}, \ldots, x_{3n};t) = x_s,$$

$$\text{for } s = 1, \ldots, 3n+1, \text{ where } x_{3n+1} = t.$$

These functions are the required inverse transformations, and will be used extensively in the following sections.

Recall that $f_\alpha(x_1, \ldots, x_{3n};t) = c_\alpha$ for $\alpha = 1, \ldots, m$, due to the constraints. Therefore

$$x_k = g_k(c_1, \ldots, c_m x_{m+1}, \ldots, x_{3n};t) = g_k(x_{m+1}, \ldots, x_{3n};t),$$

$$\text{so} \begin{cases} \text{for } k \leq m,\ x_k \text{ depends on } x_{m+1}, \ldots, x_{3n} \\ \text{for } k > m,\ g_k \text{ is the projection whose value is } x_k \end{cases}$$

These observations are crucial to the rest of the argument.

Note that since the g_k functions associated with the independent coordinates are projections, only the m functions associated with the dependent coordinates must be found. Note also that these functions may sometimes be found by inspection, as may be seen in the example cited later in the text. Even when these functions must be derived by solving equations, the $3n-m$ independent coordinates may be treated as known constants, so only an m x m system must be solved. Lastly, a good choice of coordinates may trivialize some of the remaining m equations.

5. Path Variation and Coordinate Independence

The next argument depends on Hamilton's Principle[5]. We will use Hamilton's Principle in N to obtain generalized Euler-Lagrange equations for the n particle system with the stated constraints.

Recall that

$$x_\alpha = g_\alpha(x_{m+1}, \ldots, x_{3n};t) \quad \text{for } \alpha = 1, \ldots, m,$$

so $x_1, \ldots, x_m$ are functions of the independent coordinates $x_{m+1}, \ldots, x_{3n}$.
We define

$$L_1(x_{m+1}, \ldots, x_{3n}, \dot{x}_{m+1}, \ldots, \dot{x}_{3n};t) \equiv L(g_1, \ldots, g_m, x_{m+1}, \ldots, x_{3n}, \dot{g}_1, \ldots, \dot{g}_m, \dot{x}_{m+1}, \ldots, \dot{x}_{3n};t).$$

Also note that $\delta g_\alpha = \sum_{j=m+1}^{3n} \frac{\partial g_\alpha}{\partial x_j}\delta x_j$, and that the path variations in Hamilton's Principle satisfy the condition

$$\delta \dot{x}_j = \frac{d}{dt}\left(\delta x_j\right) \text{ for } j = m+1, \ldots, 3n$$

Analogously, we define $\delta \dot{g}_\alpha \equiv \frac{d}{dt}\left(\delta g_\alpha\right)$, so we can compute that

$$\delta \dot{g}_\alpha \equiv \frac{d}{dt}\left(\delta g_\alpha\right) = \sum_{j=m+1}^{3n} \frac{d}{dt}\left(\frac{\partial g_\alpha}{\partial x_j}\right)\delta x_j + \sum_{j=m+1}^{3n} \frac{\partial g_\alpha}{\partial x_j}\frac{d}{dt}\left(\delta x_j\right)$$

$$= \sum_{j=m+1}^{3n} \frac{d}{dt}\left(\frac{\partial g_\alpha}{\partial x_j}\right)\delta x_j + \sum_{j=m+1}^{3n} \frac{\partial g_\alpha}{\partial x_j}\delta \dot{x}_j$$

We now use Hamilton's Principle on L_1, for points $(x_{m+1}, \ldots, x_{3n};t)$, where $(x_1, \ldots, x_{3n};t)$ is in neighborhood N.

$$0 = \delta \int_{t_1}^{t_2} dt \; L_1(x_{m+1}, \ldots, x_{3n}, \dot{x}_{m+1}, \ldots, \dot{x}_{3n};t)$$

$$= \int_{t_1}^{t_2} dt \left\{ \sum_{\alpha=1}^{m} \frac{\partial L}{\partial x_\alpha}\delta g_\alpha + \sum_{j=m+1}^{3n} \frac{\partial L}{\partial x_j}\delta x_j + \sum_{\alpha=1}^{m} \frac{\partial L}{\partial \dot{x}_\alpha}\delta \dot{g}_\alpha + \sum_{j=m+1}^{3n} \frac{\partial L}{\partial \dot{x}_j}\delta \dot{x}_j \right\}$$

$$= \int_{t_1}^{t_2} dt \left\{ \sum_{\alpha=1}^{m} \frac{\partial L}{\partial x_\alpha} \sum_{j=m+1}^{3n} \frac{\partial g_\alpha}{\partial x_j}\delta x_j + \sum_{j=m+1}^{3n} \frac{\partial L}{\partial x_j}\delta x_j + \sum_{\alpha=1}^{m} \frac{\partial L}{\partial \dot{x}_\alpha} \sum_{j=m+1}^{3n} \frac{d}{dt}\left(\frac{\partial g_\alpha}{\partial x_j}\right)\delta x_j \right.$$

$$\left. + \sum_{\alpha=1}^{m} \frac{\partial L}{\partial \dot{x}_\alpha} \sum_{j=m+1}^{3n} \frac{\partial g_\alpha}{\partial x_j}\delta \dot{x}_j + \sum_{j=m+1}^{3n} \frac{\partial L}{\partial \dot{x}_j}\delta \dot{x}_j \right\}$$

$$= \int_{t_1}^{t_2} dt \left\{ \sum_{j=m+1}^{3n} \left(\sum_{\alpha=1}^{m} \frac{\partial L}{\partial x_\alpha}\frac{\partial g_\alpha}{\partial x_j} + \sum_{\alpha=1}^{m} \frac{\partial L}{\partial \dot{x}_\alpha}\frac{d}{dt}\left(\frac{\partial g_\alpha}{\partial x_j}\right) + \frac{\partial L}{\partial x_j} \right)\delta x_j \right.$$

$$\left. + \sum_{j=m+1}^{3n} \left(\sum_{\alpha=1}^{m} \frac{\partial L}{\partial \dot{x}_\alpha}\frac{\partial g_\alpha}{\partial x_j} + \frac{\partial L}{\partial \dot{x}_j} \right)\delta \dot{x}_j \right\}$$

We now recall the following observations:

for $j = m+1, \ldots, 3n$

1) $\delta \dot{x}_j = \frac{d}{dt}\left(\delta x_j\right)$
2) $\delta x_j(t_1) = 0$
3) $\delta x_j(t_2) = 0.$

We use the above observations while integrating the $\delta \dot{x}_j$ term by parts in the equation preceding the observations to obtain:

$$0 = \int_{t_1}^{t_2} dt \left\{ \sum_{j=m+1}^{3n} \left(\sum_{\alpha=1}^{m} \frac{\partial L}{\partial x_\alpha} \frac{\partial g_\alpha}{\partial x_j} + \sum_{\alpha=1}^{m} \frac{\partial L}{\partial \dot{x}_\alpha} \frac{d}{dt}\left(\frac{\partial g_\alpha}{\partial x_j}\right) + \frac{\partial L}{\partial x_j} \right) \delta x_j \right.$$

$$\left. - \sum_{j=m+1}^{3n} \left(\sum_{\alpha=1}^{m} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\alpha}\right) \frac{\partial g_\alpha}{\partial x_j} + \sum_{\alpha=1}^{m} \frac{\partial L}{\partial \dot{x}_\alpha} \frac{d}{dt}\left(\frac{\partial g_\alpha}{\partial x_j}\right) + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) \right) \delta x_j \right\}$$

Note that the term $\sum_{\alpha=1}^{m} \frac{\partial L}{\partial \dot{x}_\alpha} \frac{d}{dt}\left(\frac{\partial g_\alpha}{\partial x_j}\right)$ cancels out, and we now have the equality

$$0 = \int_{t_1}^{t_2} dt \sum_{j=m+1}^{3n} \left\{ \sum_{\alpha=1}^{m} \left(\frac{\partial L}{\partial x_\alpha} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\alpha}\right) \right) \frac{\partial g_\alpha}{\partial x_j} + \left(\frac{\partial L}{\partial x_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) \right) \right\} \delta x_j$$

Now the δx_j for $j = m+1, \ldots, 3n$ are independent (while the $\delta x_1, \ldots, \delta x_{3n}$ were not), and we may conclude that for points in N

$$\frac{\partial L}{\partial x_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) + \sum_{\alpha=1}^{m} \left(\frac{\partial L}{\partial x_\alpha} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\alpha}\right) \right) \frac{\partial g_\alpha}{\partial x_j} = 0 \quad \text{for } j = m+1, \ldots, 3n.$$

We will now use this equation to motivate our formula for the Lagrangian Multipliers and to derive the usual expression for the Lagrangian Equations with constraints.

6. Deriving the Expression for the Lagrangian Multiplier

Let us rewrite the previous equations in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) - \frac{\partial L}{\partial x_j} + \sum_{\alpha=1}^{m} \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\alpha}\right) - \frac{\partial L}{\partial x_\alpha} \right) \frac{\partial g_\alpha}{\partial x_j} = 0 \qquad (2)$$

for $j = m+1, \ldots, 3n$ which is valid for points $(x_1, \ldots, x_{3n}; t)$ in N.
Now let

$G_\alpha(x_1, \ldots, x_{3n}; t)$
$\equiv g_\alpha(f_1(x_1, \ldots, x_{3n}; t), \ldots, f_m(x_1, \ldots, x_{3n}; t), x_{m+1}, \ldots, x_{3n}; t)$
$= x_\alpha$ for $\alpha = 1, \ldots, m$

Therefore $\frac{\partial G_\alpha}{\partial x_j} = 0$, for $j = m+1, \ldots, 3n$,
since no change in x_j affects the value of G_α.
Also, recall that $y_\alpha = f_\alpha(x_1, \ldots, x_{3n}; t)$, for $\alpha = 1, \ldots, m$.

It follows that

$$0 = \frac{\partial G_\alpha}{\partial x_j} = \sum_{\beta=1}^{m} \frac{\partial g_\alpha}{\partial y_\beta}\frac{\partial f_\beta}{\partial x_j} + \frac{\partial g_\alpha}{\partial x_j}$$

$$\text{or } \frac{\partial g_\alpha}{\partial x_j} = -\sum_{\beta=1}^{m} \frac{\partial g_\alpha}{\partial y_\beta}\frac{\partial f_\beta}{\partial x_j}$$

Combining this result with equation (2), we see that for $j = m+1, \ldots, 3n$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) - \frac{\partial L}{\partial x_j} = -\sum_{\alpha=1}^{m}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\alpha}\right) - \frac{\partial L}{\partial x_\alpha}\right]\frac{\partial g_\alpha}{\partial x_j}$$

$$= -\sum_{\alpha=1}^{m}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\alpha}\right) - \frac{\partial L}{\partial x_\alpha}\right]\left(-\sum_{\beta=1}^{m}\frac{\partial g_\alpha}{\partial y_\beta}\frac{\partial f_\beta}{\partial x_j}\right)$$

$$= \sum_{\beta=1}^{m}\left\{\sum_{\alpha=1}^{m}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\alpha}\right) - \frac{\partial L}{\partial x_\alpha}\right]\frac{\partial g_\alpha}{\partial y_\beta}\right\}\frac{\partial f_\beta}{\partial x_j} \tag{3}$$

So if we define

$$\lambda_\beta(t) = \sum_{\alpha=1}^{m}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\alpha}\right) - \frac{\partial L}{\partial x_\alpha}\right]\frac{\partial g_\alpha}{\partial y_\beta}\bigg|_{y_1=c_1, \ldots, y_m=c_m} \tag{4}$$

we note that $\lambda_\beta(t)$ is the y_β component of the generalized force in coordinates $y_1, \ldots, y_m$ corresponding to the m-dimensional force $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\alpha}\right) - \frac{\partial L}{\partial x_\alpha}$ in the $x_1, \ldots, x_m$ coordinates. Recall that $x_1, \ldots, x_m$ are the dependent coordinates i.e. the coordinates that are functions of $x_{m+1}, \ldots, x_{3n}$ due to the constraints).

We now summarize the results of the last two sections by combining the equations for the Euler-Lagrange equations and the multiplier equations:

$$\sum_{k=1}^{3n}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_k}\right) - \frac{\partial L}{\partial x_k}\right]\frac{\partial g_k}{\partial y_l}\bigg|_{y_1=c_1, \ldots, y_m=c_m} = \begin{cases}\lambda_l(t) & \text{for } l = 1, \ldots, m \\ 0 & \text{for } l = m+1, \ldots, 3n\end{cases} \tag{5}$$

7. Interpreting the Lagrangian Multiplier

We can interpret the meaning of $\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_k}\right] - \frac{\partial L}{\partial x_k}$ by appealing to a basic result in Lagrangian Mechanics, if we assume smooth constraints and conservative forces.

Recall that if $Q_{3(q-1)+1}, Q_{3(q-1)+2}, Q_{3(q-1)+3}$ are the components of the total force on particle q, then $\vec{Q}$ is the total force in configuration space, and we have[6]

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_k}\right] - \frac{\partial T}{\partial x_k} = Q_k, \quad \text{for } k = 1, \ldots, 3n, \tag{6}$$

where T is the kinetic energy.

Now $Q_k = F_k + R_k$, where F_k is the k^{th} component of the $3n$-dimensional external force and R_k the k^{th} component of the $3n$-dimensional smooth constraint force. We assumed conservative forces, therefore there is a potential V such that

$$F_k = -\frac{\partial V}{\partial x_k}.$$

We proceed to define $L = T - V$ as usual, so by (6),

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_k}\right) - \frac{\partial L}{\partial x_k} = R_k, \quad \text{for } k = 1, \ldots, 3n.$$

Utilizing this result for the first m variables only, we see that

$$\lambda_\beta(t) = \sum_{\alpha=1}^{m} R_\alpha \frac{\partial g_\alpha}{\partial y_\beta},$$

so the Lagrangian Multiplier appears to be the portion of the total generalized constraint force due to the dependent coordinates. However, since the constraints are smooth, the constraint force has zero components in the directions corresponding to the independent coordinates, and therefore the components corresponding to the dependent coordinates in fact determine the total force of constraint. Note that the equations (5) combining the multipliers and the Euler-Lagrange equations have the constraint force on the right hand side.

8. Showing Multipliers Satisfy the Usual Constrained Euler-Lagrange Equations

Now for $j = m+1, \ldots, 3n$ and points in N, we immediately have by (3) and the definition of $\lambda_\beta(t)$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) - \frac{\partial L}{\partial x_j} - \sum_{\beta=1}^{m} \lambda_\beta(t) \frac{\partial f_\beta}{\partial x_j} = 0.$$

Now for $\gamma = 1, \ldots, m$:

Again, for $\alpha = 1, \ldots, m$

$$g_\alpha(f_1(x_1, \ldots, x_{3n};t), \ldots, f_m(x_1, \ldots, x_{3n};t), x_{m+1}, \ldots, x_{3n};t) = x_\alpha,$$

so if $\gamma \neq \alpha$, x_γ does not affect the value of g_α. It follows that:

$$\sum_{\beta=1}^{m} \frac{\partial g_\alpha}{\partial y_\beta} \frac{\partial f_\beta}{\partial x_\gamma} = \frac{\partial x_\alpha}{\partial x_\gamma}$$

$$= \delta_{\alpha\gamma}, \text{ the Kronecker delta.}$$

Therefore we see that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\gamma}\right) - \frac{\partial L}{\partial x_\gamma} - \sum_{\beta=1}^{m} \lambda_\beta(t) \cdot \frac{\partial f_\beta}{\partial x_\gamma}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\gamma}\right) - \frac{\partial L}{\partial x_\gamma} - \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\gamma}\right) - \frac{\partial L}{\partial x_\gamma}\right] = 0.$$

Combining the results for j and γ, we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_k}\right) - \frac{\partial L}{\partial x_k} - \sum_{\beta=1}^{m} \lambda_\beta(t) \frac{\partial f_\beta}{\partial x_k} = 0, \quad \text{for } k = 1, \ldots, 3n.$$

This is the usual form of the constrained Euler-Lagrange equations.

9. Relating the Forces of Constraint and the Lagrangian Multipliers

We showed that $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_k}\right) - \frac{\partial L}{\partial x_k} = R_k$, the k^{th} component of the constraint force when the external forces are conservative and when the constraints are smooth. Therefore, $R_k = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_k}\right) - \frac{\partial L}{\partial x_k} = \sum_{\beta=1}^{m} \lambda_\beta(t) \frac{\partial f_\beta}{\partial x_k}$. We can rewrite this in $3n$ dimensional space as $\vec{R} = \sum_{\beta=1}^{m} \lambda_\beta(t) \vec{\nabla} f_\beta$, which coincides with another popular expression for the constraint force.

This expression makes physical sense because smooth constraints must be co-linear with the gradients of the constraint equations.

10. Example for Constrained Time-Independent Two-Dimensional Motion

Now for an example of single particle motion in two dimensions. Let a bead slide frictionlessly along a wire whose shape is given by the equation $y=x^2$. Let the constraint equation (in the article's notation) be

$$0 \equiv c_1 = f(x_1,x_2) = x_2 - x_1^2.$$

Define

$$F(x_1,x_2) = \begin{pmatrix} x_2 - x_1^2 \\ x_2 \end{pmatrix}.$$

The Jacobian of F is $2x_1$, so the Inverse Function Theorem applies when $x_1 \neq 0$. The Inverse Function Theorem only guarantees the existence of an inverse function but doesn't provide its form. We need a function g satisfying $g(f(x_1,x_2),x_2) = x_1$. This is equivalent to finding a function g satifying $g(x_2 - x_1^2, x_2) = x_1$. Note that

$$sgn(x_2-(x_2-x_1^2))\sqrt{|x_2-(x_2-x_1^2)|} = x_1 \text{ if } x_1 > 0$$
$$sgn((x_2-x_1^2)-x_2)\sqrt{|x_2-(x_2-x_1^2)|} = x_1 \text{ if } x_1 < 0$$

where

$$sgn(x) = \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -1 & \text{if } x<0 \end{cases}$$

So we define two functions

$$g_1(y,x_2) \equiv sgn(x_2-y)\sqrt{|x_2-y|}$$
$$g_2(y,x_2) \equiv sgn(y-x_2)\sqrt{|x_2-y|}$$

where $y \equiv f(x_1,x_2)$.

The reader can easily show that if $x_1 > 0$ then $g_1(f(x_1,x_2),x_2) = x_1$.

We define the neighborhood N_1 by $N_1 \equiv \{(x_1,x_2) | \; x_1 > 0 \}$, where g_1 is defined on $F[N_1] \equiv \{(f(x_1,x_2),x_2) | (x_1,x_2) \in N_1\}$. Note also that g_1 is differentiable on $F[N_1]$.

Correspondingly, if $x_1 < 0$ then $g_2(f(x_1,x_2),x_2) = x_1$.

We define the neighborhood N_2 by $N_2 \equiv \{(x_1,x_2) | \; x_1 < 0 \}$, where g_2 is defined on $F[N_2] \equiv \{(f(x_1,x_2),x_2) | (x_1,x_2) \in N_2\}$. Again, g_2 is appropriately differentiable on $F[N_2]$.

Note that both of these neighborhoods are in fact quite large. In the rest of this section, we will use g interchangeably for either g_1 or g_2, depending on whether $(y,x_2) \in F[N_1]$ or $(y,x_2) \in F[N_2]$.

We see that

$$L = \frac{1}{2}m\left(\dot{x}_1^2+\dot{x}_2^2\right)-m\bar{g}x_2.$$

The reader can show that since x_1 is the dependent variable

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\frac{\partial L}{\partial x_1} = m\ddot{x}_1$$

and

$$\lambda = \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\frac{\partial L}{\partial x_1}\right)\frac{\partial q}{\partial y}\Big|_{y_1=c_1}$$

$$= -\frac{m\ddot{x}_1}{2x_1}$$

This is the same value for λ as would be obtained from the usual constrained Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\frac{\partial L}{\partial x_1}-\lambda\frac{\partial f}{\partial x_1} = 0$$

with the constraint equation $x_2-x_1^2 = 0$.

We see that for $x_1 = 0$, λ is undefined, as should have been expected from the necessity for the exclusion of such points in the definition of N_1 and N_2.

11. *Example of Constrained Time-Dependent* Two Dimensional Motion (No Gravity)

Let an infinite rod with a pivoted end at the origin of a plane be rotating in the plane with frequency ω about the origin. Let a bead slide frictionlessly on that rod.

The constraint equations are

$x-r\cos\omega t = 0 \equiv c_1$

$y-r\sin\omega t = 0 \equiv c_2$.

We translate to the article's notation as follows:

$y_1 \equiv f_1(x_1,x_2,x_3;t) = x_1-x_3\cos\omega t$

$y_2 \equiv f_2(x_1,x_2,x_3;t) = x_2-x_3\sin\omega t$.

Note that x_1 and x_2 may be chosen the dependent variables, upon examination of the Jacobian.

We next define

$$F(x_1,x_2,x_3;t) = \begin{pmatrix} f_1(x_1,x_2,x_3;t) \\ f_2(x_1,x_2,x_3;t) \\ x_3 \\ t \end{pmatrix}$$

$$= \begin{pmatrix} x_1-x_3\cos\omega t \\ x_2-x_3\sin\omega t \\ x_3 \\ t \end{pmatrix}$$

The Jacobian of F is equal to 1 , for all points $(x_1,x_2,x_3;t)$, so we may choose $N \equiv \{(x_1,x_2,x_3;t)\}$.

We need to find functions $g_1(y_1,y_2,x_3;t)$ and $g_2(y_1,y_2,x_3;t)$ satisfying $g_1(f_1(x_1,x_2,x_3;t),f_2(x_1,x_2,x_3;t),x_3;t) = g_1(x_1-x_3\cos\omega t,x_2-x_3\sin\omega t,x_3;t) = x_1$ and $g_2(f_1(x_1,x_2,x_3;t),f_2(x_1,x_2,x_3;t),x_3;t) = g_2(x_1-x_3\cos\omega t,x_2-x_3\sin\omega t,x_3;t) = x_2$.

The obvious choices are

$g_1(y_1,y_2,x_3;t) = y_1+x_3\cos\omega t$ and

$g_2(y_1,y_2,x_3;t) = y_2+x_3\sin\omega t$.

It is easily seen that g_1 and g_2 are appropriately differentiable and that the required conditions

$$g_1(f_1(x_1,x_2,x_3;t),f_2(x_1,x_2,x_3;t),x_3;t) = x_1$$
$$g_2(f_1(x_1,x_2,x_3;t),f_2(x_1,x_2,x_3;t),x_3;t) = x_2.$$

are satisfied.

Since $L = \frac{1}{2}m\left(\dot{x}_1^2+\dot{x}_2^2\right)$, we see that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_k}\right)-\frac{\partial L}{\partial x_k} = m\ddot{x}_k$$

for $k = 1,2$.

Consequently,

$$\lambda_1(t) = \left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\frac{\partial L}{\partial x_1}\right\}\frac{\partial g_1}{\partial y_1}\Big|_{y_1=c_1,y_2=c_2}$$
$$+\left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right)-\frac{\partial L}{\partial x_2}\right\}\frac{\partial g_2}{\partial y_1}\Big|_{y_1=c_1,y_2=c_2}$$
$$= m\ddot{x}_1\Big|_{y_1=c_1,y_2=c_2}.$$

Using the fact that $y_1 = c_1 \equiv 0$, we see that $x_1 = x_3\cos\omega t$. We may use this fact to compute $\ddot{x}_1$ and obtain $\lambda_1(t) = m\ddot{x}_3\cos\omega t-2m\omega\dot{x}_3\sin\omega t-m\omega^2x_3\cos\omega t$.

Analogously,

$$\lambda_2(t) = \left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\frac{\partial L}{\partial x_1}\right\}\frac{\partial g_1}{\partial y_2}\Big|_{y_1=c_1,y_2=c_2}$$
$$+\left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right)-\frac{\partial L}{\partial x_2}\right\}\frac{\partial g_2}{\partial y_2}\Big|_{y_1=c_1,y_2=c_2}.$$

Proceeding as before, we see that $\lambda_2(t) = m\ddot{x}_2|_{y_1=c_1,y_2=c_2}$, so since $y_2 = c_2 \equiv 0$, we have that $x_2 = x_3\sin\omega t$.

We can now show that

$$\lambda_2(t) = m\ddot{x}_3+2m\omega\dot{x}_3\cos\omega t$$
$$-m\omega^2x_3\sin\omega t.$$

Now the usual constrained Euler-Lagrange equations are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\frac{\partial L}{\partial x_1}-\lambda_1(t)\frac{\partial f_1}{\partial x_1}-\lambda_2(t)\frac{\partial f_2}{\partial x_1} = 0$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right)-\frac{\partial L}{\partial x_2}-\lambda_1(t)\frac{\partial f_1}{\partial x_2}-\lambda_2(t)\frac{\partial f_2}{\partial x_2} = 0$$

with constraints

$0 = x_1-x_3\cos\omega t$ and

$0 = x_2-x_3\sin\omega t$.

The reader can show that we get the same equations for $\lambda_1(t)$ and $\lambda_2(t)$, so the two methods are equivalent.

Note that the functions g_1, g_2 were obtained by inspection, so that the multiplier values were obtained essentially without solving equations.

12. Example Comparing the Standard and New Approaches

By way of illustration, let's compare the solution processes required to obtain the equations of motion and forces of constraint for a particle of mass m moving on a surface defined by the constraints

$$f_1(x_1,x_2,x_3) \equiv x_1x_2x_3 = c_1 \neq 0$$

and

$$f_2(x_1,x_2,x_3) \equiv x_1+x_2 = c_2$$

subject to the potential

$$V(x_1,x_2,x_3) = \frac{1}{2}k\left(x_1^2+x_2^2+x_3^2\right).$$

We define new coordinates by $y_1 = f_1(x_1,x_2,x_3)$ and $y_2 = f_2(x_1,x_2,x_3)$. The Lagrangian now has the form

$$L(x_1,x_2,x_3,\dot{x}_1,\dot{x}_2,\dot{x}_3) = \frac{1}{2}m\left(\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2\right)-\frac{1}{2}k\left(x_1^2+x_2^2+x_3^2\right).$$

It may easily be shown that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\frac{\partial L}{\partial x_1} = m\ddot{x}_1+kx_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right)-\frac{\partial L}{\partial x_2} = m\ddot{x}_2+kx_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_3}\right)-\frac{\partial L}{\partial x_3} = m\ddot{x}_3+kx_3$$

The standard formulation provides the equations

$$0 = \left(m\ddot{x}_1+kx_1\right)-\lambda_1x_2x_3-\lambda_2$$

$$0 = \left(m\ddot{x}_2+kx_2\right)-\lambda_1x_1x_3-\lambda_2$$

$$0 = \left(m\ddot{x}_3+kx_3\right)-\lambda_1x_1x_2$$

$$x_1x_2x_3 = c_1$$

$$x_1+x_2 = c_2,$$

which requires solving 5 equations in 5 unknowns.

The formulation in this paper requires the computation of the inverse functions in (say) $N_1 \equiv \{(x_1,x_2,x_3) | x_1>x_2\}$, the case for $N_2 \equiv \{(x_1,x_2,x_3) | x_1<x_2\}$ is identical except for interchanging the definitions of the functions described below.

The inverse functions, obtained by solving two equations in two unknowns are:

$$g_1(y_1,y_2,x_3) = \frac{y_2+\left[y_2^2-4y_1/x_3\right]^{\frac{1}{2}}}{2}$$

and

$$g_2(y_1,y_2,x_3) = \frac{y_2 - \left[y_2^2 - 4y_1/x_3\right]^{\frac{1}{2}}}{2}.$$

Note that $x_1 = g_1(y_1,y_2,x_3)$, $x_2 = g_2(y_1,y_2,x_3)$, and $x_3 = g_3(y_1,y_2,x_3)$.

We now compute all first order partial derivatives of g_1, g_2, and g_3 evaluated at (c_1,c_2,x_3) and substitute these values in the new equations, immediately obtaining the following system of equations:

$$\lambda_1 = -\left(m\ddot{x}_1 + kx_1\right)\frac{\left[c_2^2 - 4c_1/x_3\right]^{-\frac{1}{2}}}{x_3} + \left(m\ddot{x}_2 + kx_2\right)\frac{\left[c_2^2 - 4c_1/x_3\right]^{-\frac{1}{2}}}{x_3}$$

$$\lambda_2 = \frac{1}{2}\left(m\ddot{x}_1 + kx_1\right)\left(1 + c_2\left[c_2^2 - 4c_1/x_3\right]^{-\frac{1}{2}}\right) + \frac{1}{2}\left(m\ddot{x}_2 + kx_2\right)\left(1 - c_2\left[c_2^2 - 4c_1/x_3\right]^{-\frac{1}{2}}\right)$$

$$0 = \left(m\ddot{x}_1 + kx_1\right)\left(\frac{c_1\left[c_2^2 - 4c_1/x_3\right]^{-\frac{1}{2}}}{x_3^2}\right) - \left(m\ddot{x}_2 + kx_2\right)\left(\frac{c_1\left[c_2^2 - 4c_1/x_3\right]^{-\frac{1}{2}}}{x_3^2}\right) + m\ddot{x}_3 + kx_3.$$

We see that this approach involved finding two inverse functions by solving a 2 by 2 system of equations, and then computing somewhat harder derivatives than was required in the standard case. On the other hand, these equations contain only x_3 and its derivatives once x_1 and x_2 and their derivatives are replaced by $g_1(c_1,c_2,x_3)$ and $g_2(c_1,c_2,x_3)$ and their derivatives. In addition, the Lagrange Multipliers were available immediately. Therefore, we see that solving the equations to obtain inverses and substitution of the inverse functions and their derivatives for the dependent coordinates and their derivatives replaces solving a more complex system of equations.

13. Conclusion

This paper has related the Lagrangian multipliers to superfluous coordinates, and has derived a new equation combining a variant of the Euler-Lagrange equation with a new closed form for the multipliers in the process. Since the new multiplier equation lead to the usual form of the Euler-Lagrange equations, the new form for the multiplier must indeed yield the same value as the usual Lagrangian multiplier.

The author feels that this approach improves one's understanding of the meaning of the multipliers and the role of (and relationship to) superfluous coordinates. In addition, this approach facilitates the computation of the Lagrange Multipliers by reducing the complexity of the equations that must be solved, as was the case in the example discussed.

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