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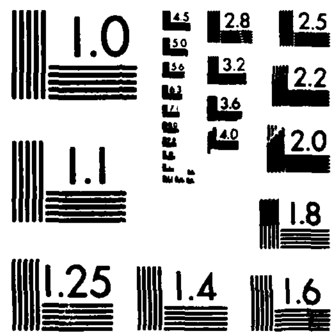

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FINAL REPORT

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Subitted By:
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ON NONLINEAR CONTROL FOR
BANK-TO-TURN MISSILE AUTOPILOTS *

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ABSTRACT

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1. INTRODUCTION

1.1 The Autopilot Design Problem

The design of an autopilot to control a short range air-to-air weapon is a complex process due to the nature of the plant, the control mechanism, and the available measurements. The plant can be considered to be the vehicle itself with its associated aerodynamic and inertial properties and the rocket motor. The control mechanism, an actuation system for controlling movable tail surfaces, is also considered part of the plant. The measurement devices are assumed to be three angular rate and three translational acceleration devices. The difficulty in controlling the plant can be summarized in terms of the following seven factors:

(i) The plant is time-varying. The rotational and translational motion of the vehicle can be described in terms of a set of nonlinear differential equations with time variable coefficients. These equations can be linearized about an operating point, but this operating point will vary with time. The aerodynamic forces and moments are functions of speed (V), angle-of-attack (α) and side-slip (β), fin deflections (δ), dynamic pressure (q_0), and body angular rates (p,q,r). Of these, only fin deflection and the rates are available directly from measurement. The other quantities are unmeasured states of the system (e.g, V, α, β or equivalently V_x, V_y, V_z in body coordinates) or functions of the states (e.g., $q_0 = 1/2\rho(h)\, V^2$, where h = altitude). Clearly the estimation of the system states is advantageous in developing a control strategy. Knowledge of the system states will aid in the estimation of other system parameters, most notably the aerodynamic coefficients. Another time variable aspect of the plant is the inertial properties. As the rocket motor burns, there are variations in mass, c.g. locations, and moments of inertia.

(ii) The plant is nonlinear in the sense that the forces and moments are nonlinear functions of the system states. For example, the dynamic pressure is proportional to the square of the velocity. A more troublesome nonlinearity for the autopilot designer is the nonlinear nature of the aerodynamics. The aerodynamic moment coefficients C_m and C_n are in general, functions of α and β, Mach number (M_n) and fin deflection (δ). Variations with angle-of-attack, e.g. $\partial C_m/\partial\alpha$, affect the airframe stability. If this slope is negative, the open loop airframe tends to be stable. Otherwise, the transform function will have poles in the right half-plane. The nonlinear nature of the C_m curve is shown by the example in Figure 1.1. Notice that at low angles-of-attack the vehicle is stable but becomes unstable between 5 and 10 degrees. Then it becomes stable again above 15 degrees.

(iii) The plant is uncertain. In particular, several aerodynamic parameters such as moment coefficients, dynamic derivatives, nonlinear derivatives based on wind-tunnel and aero prediction methods do not have high accuracy. The estimates of these coefficients can be improved using previous flight test data, but a certain amount of uncertainty will always be present in the parameter estimates. This requires use of either robust control design which can accommodate parameter uncertainties or adaptive control design which reduces uncertainties using on-line parameter estimation.

(iv) The plant has coupled inputs and outputs. Equivalently, the three autopilot channels (roll, pitch and yaw) or axes of motion are coupled. The coupling comes about through kinematic, aerodynamics, e.g., $C_{\ell\beta}(M_n,\alpha)$, bank-to-turn guidance law and actuator. In classical design, this coupling is ignored.

(v) The plant can be unstable. As discussed under the nonlinear nature of the aerodynamics, the vehicle may be stable or unstable, depending on the operating point. The stability of the plant affects the type of control strategy, the need for robustness in that strategy, and the required accuracy of the estimation. In addition to the possible unstable poles due to $C_{m\alpha}$, tail control yields non-minimum phase zeros in the aerodynamic transfer function due to the force on the tail, i.e., $C_{n\delta}$. This can be particularly troublesome when using classical design techniques, especially at low dynamic pressure flight conditions.

(vi) The system constraints are significant. These constraints include the finite bandwidth, slew rate, deflection and torque of the actuator and the quality of the measurements available from the sensors. For the sensors to be reasonable in cost, the quality of the output in terms of range, scale factor errors, and noise will be limited. The quality of the measurements directly affects the accuracy of the state and parameter estimates.

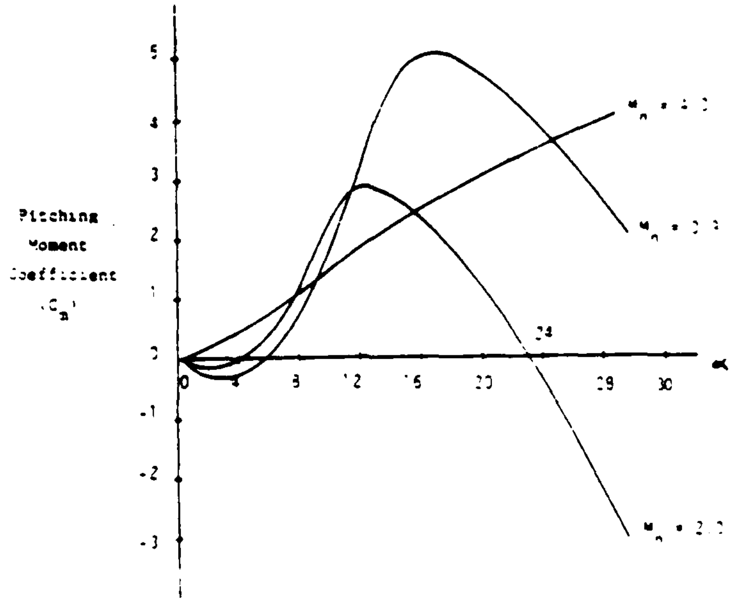

Figure 1.1 Nonlinear Nature of Airframe Aerodynamics

(vii) The design requirements are severe. Due to the short flight times involved with the short range air-to-air mission, the required weapon system bandwidths are high. These short flight times, coupled with highly maneuverable targets, yield requirements for high missile accelerations and angles-of-attack. These high α's aggravate the problem of nonlinear aero-dynamics and coupling.

In summary, the autopilot design problem can be expressed as a problem of state and parameter estimation, as well as the design of the control system for a nonlinear coupled mulivariable system with constraints.

1.2 Bank-to-Turn Steering

Current missiles using skid-to-turn (STT) steering have performance limitations; for example, angle of attack and turn maneuverability are restricted as a result of yaw/roll cross coupling which tends to rotate the airframe away from the desired maneuver plane. These undesirable torques increase with angle of attack and may exceed the trim capability (Trans...

1985). Large angles of attack in STT missiles can result in large sideslip angles which in turn cause large vortex wake asymmetries. These large wake asymmetries can result in unwanted roll and yaw moments which may be large enough to exceed the trim capability of the control system.

In order to alleviate these undersirable effects, bank-to-turn (BTT) steering has been developed and has recently received much attention. In this type of steering, the control system continuously banks the missile so as to minimize the angle of sideslip and, hence, the asymmetry of its vortex wake. The result of this is that the missile can be maneuvered at higher angles of attack, increasing its lift capability (see, e.g. Froning, 1985). BTT designs can reduce weight and drag while maintaining high maneuverability. Even though BTT steering appears desirable from these aspects, there are currently no high performance antiair missiles within the U.S. arsenal using this technology (Arrow, 1985). Flight tests and detailed simulation results on BTT steering have not achieved total response times currently available with existing STT configurations. The most significant characteristic which impacts the achievable response time is the dynamic coupling primarily due to nonzero missile body rates and accelerations (Arrow, 1985). A number of BTT configurations have been developed. Perhaps the most challenging one from a control point of view is the monoplanar single inlet configuration in which angle of attack is restricted to a small negative number or a large positive number. In this configuration, very large lift forces can be attained. However, the roll angle may have to be changed 180° in response to a command from the guidance system.

Current autopilot designs for BTT steering typically utilize linear or linearized control design techniques. Williams, et al (1985) countered the effects of nonlinearities by freezing roll rates and using a linear-quadratic-gaussian design methodology at each fixed roll rate. By including actuator dynamics, the resulting state vector dimension was 10 for the pitch/yaw channel and 2 for the roll channel. Coupling from pitch/yaw to roll was treated as a disturbance term in the roll dynamics. Hardy (1985) performed an autopilot design for a BTT missile using pole assignment. The equations of motion were linearized, resulting in a fourth order pitch axis model and an eighth order roll/yaw axis model. Dixon and Klabunde (1985) used linear design methods to develop a lateral autopilot using gains scheduled as a function of angle of attack. Chung and Shapiro (1982) used a linear modal synthesis approach.

Another example of the classical approach is the design study by Rockwell under Eglin AFB sponsored "Interlaboratory Air-to-Air Technology" (ILAAT). These included analytical studies with two different airframes (Emmert et. (1976, 1978). The initial studies, dating from 1976, employed a dither adaptive approach for state / parameter identification. By dithering the airframe in yaw at a frequency in the range of 10-13 Hz, an estimate of dynamic pressure was generated. This estimate was used to schedule the gains of the pitch and roll rate loops to maintain the desired bandwidth, independent of flight condition. This was successful, because the gain of the aerodynamic transfer function (r/δ_r) is roughly proportional to q_0. The acceleration loops were also scheduled as a function of q_0, although the aerodynamic gain was proportional to speed (V) rather than V^2. Scheduled limits on commanded acceleration were used to limit angles-of-attack and side-slip. The dither adaptive approach had the advantage of requiring no interface with the launch aircraft and only two accelerometers.

In a later design, (Emmert et.al. (1978)) the dither adaptive approach was again employed, but a velocity estimate was generated based on an added axial accelerometer. A rough estimate of α and β was also developed. The velocity estimate assured closer control of acceleration loop bandwidth and a more linear acceleration loop gain schedule. However, the dither adaptive design has a number of shortcomings which can be overcome with modern control approaches. First, the bandwidth of the system is limited by system nonlinearities and non-minimum phase zeros of the aerodynamic transfer function, particularly at low dynamic pressures. The zeros are also functions of flight condition, thrust level and c.g. location and can vary by a factor of 10 in frequency. Secondly, a significant amount of yaw/roll coupling exists in spite of a high bandwidth roll rate loop. It is desirable to lower the rate loop bandwidtns which affect actuator requirements, while minimizing coupling between channels. This cannot be achieved effectively with classical control design which relies on high gains to produce decoupling. Reduced rate loop bandwidths will also alleviate the computational burden on the digital processor. Thirdly, it is desirable to improve the robustness of the autopilot and provide some degree of fault tolerance.

Several authors have utilized nonlinear transformations in flight control and attitude control systems. Meyer, Su and Hunt (1984) found that a nonlinear helicoptor model could be linearized via a nonlinear transformation, since it was in the class of equivalent linear systems (Su, 1982). Dwyer (1984) derived linearizing transformations for the rigid body attitude control problem and solved several particular cases (rest-to-rest and detumbling) using linear quadratic control theory. One problem with these approaches is that control and state constraints become generally more complex in form and are not easily handled. Thus, if constraints are taken into account, these methods tend to conserve complexity.

Decoupling is another technique which leads to nonlinear controls. In BTT steering, the kinematic and inertial coupling of the roll and yaw systems during combined pitch and roll maneuvers is significant. The problem is to maintain a small sideslip angle, and one way to accomplish this is to use cross-feed signals between the axes to decouple the yaw and roll control axes. Froning and Gieseking (1973) used decoupling for a linear BTT model. Reed, et al, (1985) developed a noninteracting controller for a BTT missile which decoupled the roll, pitch and yaw axes and used simulations to study the performance. Their feedback law was based on a quadratic performance index. Unfortunately, decoupling also tends to conserve complexity.

Solution of the general optimal control problem in missile autopilot design seems intractable. The inclusion of Euler angles to include body-fixed constraints in an inertial formulation generates a time-varying, nonlinear two-point-boundary-value problem which appears to make a real-time solution impossible with current computer technology (Gupta, et al, 1981).

1.3 Nonlinear Control Decoupling via Feedback

The missile dynamics are nonlinear and it is very desirable to consider control designs that apply to nonlinear problems. Pointwise linearization followed by gain scheduling is the most common technique for controlling nonlinear systems, but it is not very effective for highly maneuverable systems which rarely operate around a steady state condition. One of the most promising techniques for nonlinear control design is based on the use of nonlinear feedback in such a way that the closed loop system has the desired behavior. For missile autopilot design one would like to achieve fast decoupled response between guidance commands and acceleration outputs so that the autopilot would follow the guidance commands accurately. Notice that with 3 control inputs, one can decouple only 3 output channels. The most appropriate ones for missile autopilot design will be angle of attack and sideslip (α, β) and roll (ϕ), which are closely related to pitch and yaw accelerations for a BTT missile.

The problem of control decoupling via feedback together with the closely related problems of disturbance decoupling and invariance has arisen in many engineering applications, including missile and aircraft control problems. Since Rozenoer's (1963) initial work, the subject of control decoupling via feedback has been extensively developed, and a reasonably large body of literature currently exists. Some papers which have been important milestones are Wonham and Morse (1970), Tokamaru and Iwai (1972), Majumdar and Chaudhury (1972), and Isidori et al. (1981). In addition to this theoretical work, several applications to problems of aircraft control have been studied including Singh and Schy (1978) and work by G. Meyer (1981) on the design of an autopilot system for the Bell UH-IH helicopter.

The basic idea behind the theory of control decoupling is quite simple: Suppose there is given a nonlinear control system of the form

$$\dot{x} = f(x) + u_1\, g_1(x) + u_2\, g_2(x)$$

$$y_i = h_i(x) \qquad (i=1,2)$$

We wish to consider modifications of the system dynamics using feedback controls $u = a(x) + B(x)v$ such that

$$\dot{x} = \tilde{f}(x) + v_1\, \tilde{g}_1(x) + v_2\, \tilde{g}_2(x) \tag{1.1}$$

$$y_i = h_i(x) \qquad (i=1,2) \tag{1.2}$$

where

$$\tilde{f} = f + \sum_{j=1}^{2} a_j\, g_j$$

$$\tilde{g}_i = \sum_{j=1}^{2} B_{ji}\, g_j$$

The decoupling problem is to find a and B such that v_i controls y_i and only y_i. (That is, we want v_1 to have no influence on the output y_2 and vice-versa).

Techniques for finding a and B in the case in which f and the h_i's are linear and the g_i's are independent of x are well known and may be found in Wonham (1970). For nonlinear systems, considerably less is known and although the beginnings of the theory date back to 1962, many aspects remain to be understood.

L. Rozonoer (1963) obtained conditions necessary for invariance (i.e. independence of the output upon the input) in nonlinear system, by using a variational method similar to the Pontryagin's optimality priciple.

H. Tokumaru and Z. Iwai (1972), and A. Majumdar and A. Chaudry (1972), independently have applied the variational method used by Rozonoer to obtain necessary conditions for non-interacting control, for linear time-variant systems (1968) and for nonlinear systems (1971). A concrete design for control decoupling feedback for the automatic piloting of a mine hunter boat was given by E. Daclin (1980).

All theoretical work mentioned above neglected the existence of constraints on controls in virtually every real-life situation (e.g., a limit on control deflections, acceleration rates, etc). The effects of such contraints have been studied by D. Hanson and F. Stengel (1981), particularly for system with two degrees of freedom.

A different approach to the problem was used by R. Su, G. Meyer and L. Hunt (1983), who used nonlinear transformations to reduce the nonlinear problem to the linear one, which can be treated by standard methods, as in Wonham (1970). The main drawback here is that it is not generally possible to carry out this type of linearization; there are both algebraic and topological obstructions which are essentially the same ones that are encounted if one were to try to do nonlinear decoupling directly.

An important application of control decoupling theory to the problem of aircraft dynamics was given by S. Singh and A. Schy, (1980). These authors derive a feedback for a simplified aircraft model in which certain aerodynamic forces had been ignored.

1.4 Summary of Report

In this report we consider the general problem of developing efficient attitude transfer maneuvers in response to guidance commands, in order to meet desired translational accelerations. Throughout, we consider a very general missile configuration and develop techniques which should be applicable to a wide class of BTT missiles. The problem formulation includes nonlinearities which are significant at high body rates and at high angles of attack. In addition, constraints on angle-of-attack, control surface deflections and body rates are included. Using several reasonable assumptions, we study the problem analytically and develop several nonlinear control strategies using both decoupling control and time optimal control methods. Decoupling controllers are presented using two control modes; (1) body rates as controls, (2) control surface deflections as controls. In all cases, the controller is nonsingular if the missile is not in a stalled flight condition and not on a constraint boundary. Explicit conditions for avoidance of constraint boundaries are derived, which may be used to set key autopilot design parameters such as time constants. The controllers are shown to be quite robust to estimation errors. A minimum-time controller which includes constraints on both controls and angle-of-attack is developed and an example is given.

2. EQUATIONS OF MOTION:

We make several assumptions in order to obtain the equations of motion:

(i) the missile is a rigid body

(ii) the body axes are the principal axes of inertia

(iii) the principal moments I_{yy} and I_{zz} are equal

(iv) gravitational acceleration is neglected.

The assumed inertial properties simplify the analysis in the sequel. The design methodology holds for arbitrary inertial properties, however.

With these assumptions we have:

Translation

$$m\,(\dot{u} + qw - rv) = F_x \tag{2.1}$$

$$m\,(\dot{v} + ru - pw) = F_y \tag{2.2}$$

$$m\,(\dot{w} + pv - qu) = F_z \tag{2.3}$$

where:

F_x, F_y, F_z are the external forces, due to thrust and aerodynamics, resolved into body axes

u, v, w are the inertial velocity components, resolved into body axes.

p, q, r are the body angular rates, in body coordinates

m is the vehicle mass, assumed constant

$\dot{u}$, $\dot{v}$, $\dot{w}$ are the time derivatives of u, v, w

Rotation

$$L = I_{xx}\,\dot{p} \tag{2.4}$$

$$M = I_{yy}\,\dot{q} - (I_{yy} - I_{xx})\,r\,p \tag{2.5}$$

$$N = I_{yy}\,\dot{r} + (I_{yy} - I_{xx})\,q\,p \tag{2.6}$$

where:

L, M, N are the resultant moments about the vehicle center of gravity, resolved along body axes

I_{xx}, I_{yy} are the principal moments of inertia ($I_{xx} < I_{yy}$)

$\dot{p}$, $\dot{q}$, $\dot{r}$ are the time derivatives of p, q, r measured in body coordinates

Since we will be principally concerned with controlling the angle of attack (α) and angle of sideslip (β), it is more convenient to work directly with them. Using the definitions

$$\tan\alpha = w/u \tag{2.7}$$

$$\tan\beta = v/u \tag{2.8}$$

we get

$$\dot{\alpha} = q + \frac{F_z\cos\alpha - F_x\sin\alpha}{mV\,\cos\beta} - (p\,\cos\alpha + r\,\sin\alpha)\,\cos\alpha\,\tan\beta \tag{2.9}$$

$$\dot{\beta} = -r + \frac{F_y\cos\beta - F_x\sin\beta}{mV\cos\alpha} + (p\cos\beta + q\sin\beta)\tan\alpha\cos\beta \qquad (2.10)$$

where V is the velocity:

$$V = (u^2 + v^2 + w^2)^{1/2}$$

The terms involving F_x, F_y, F_z are the rates of changes of flight path angle which will be much smaller that the body angular rates. Therefore, we will neglect them; the resulting equations are

$$\dot{\alpha} = q - (p\cos\alpha + r\sin\alpha)\cos\alpha\tan\beta \qquad (2.11)$$

$$\dot{\beta} = -r + (p\cos\beta + q\sin\beta)\tan\alpha\cos\beta. \qquad (2.12)$$

The angles α and β are shown in figure 2.1.

This figure suggests that if the missile rolls while maintaining zero pitch and yaw rates ($q = r = 0$) the velocity vector rotates in the missile frame, around the x-axis, thus describing a cone. This intuition is confimed by the following calculation. Setting $q = r = 0$ in (2.11) and (2.12) we obtain

$$\begin{bmatrix} \tan\alpha \\ \tan\beta \end{bmatrix} = \begin{bmatrix} \cos\phi(t) & -\sin\phi(t) \\ \sin\phi(t) & \cos\phi(t) \end{bmatrix} \begin{bmatrix} \tan\alpha(0) \\ \tan\beta(0) \end{bmatrix},$$

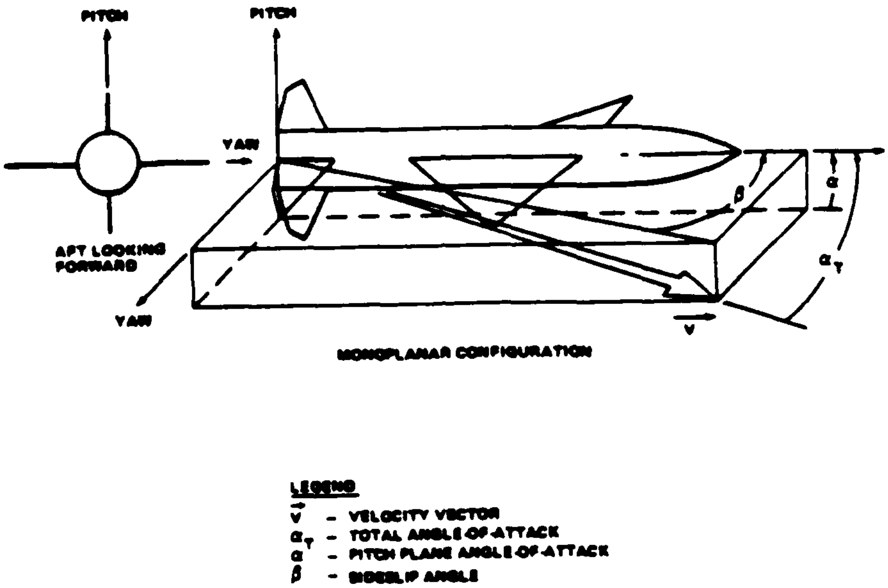

Figure 2.1 Definition of Angles for Monoplanar Missile Configuration (from Arrow (1985)).

where $\phi(t) = \int_0^t p\,dt$ is the roll in time t.

The moments L, M, N are assumed to be generated solely by aerodynamic forces. Thus we will neglect, for example, thrust misalignment effects. The assumed aerodynamic moment equations are:

$$L = L_p\ p + L_\beta\ \beta + L_\delta\ \delta_p \tag{2.13}$$

$$M = M_q\ q + M_\alpha\ \alpha + M_\delta\ \delta_q \tag{2.14}$$

$$N = N_r\ r + N_\beta\ \beta + N_\delta\ \delta_r \tag{2.15}$$

where:

L_p, M_q, N_r are the aerodynamic damping coefficients

L_β, N_β are angular acceleration derivatives with respect to β

M_α is the pitch moment coefficient

L_δ, M_δ, N_δ are the control moment coefficients

These aerodynamic coefficients are, in general, complex functions of Mach number, α, β, δ_p, δ_q and δ_r. In this paper, we will generally assume that they are known functions of their arguments and that the arguments are known.

In order to simplify the following analysis we will use the following notation

$$\Theta = [\phi \; x_1 \; x_2]^T$$

$$\omega = [p \; q \; r]^T$$

$$u = [\delta_p \; \delta_q \; \delta_r]^T$$

where $x_1 = \tan \alpha$, $x_2 = \tan \beta$.

Then the equations of motion may be written in the form

$$\dot{\Theta} = K_\Theta \, \omega \tag{2.16}$$

$$\dot{\omega} = F_\Theta \, \Theta + F_\omega \, \omega + Bu \tag{2.17}$$

where

$$K_\Theta = \begin{bmatrix} 1 & 0 & 0 \\ -x_2 & 1+x_1{}^2 & -x_1 \, x_2 \\ x_1 & x_1 \, x_2 & -(1 + x_2^2) \end{bmatrix} \tag{2.18}$$

$$F_\Theta = \begin{bmatrix} 0 & 0 & \tilde{L}_\beta \\ 0 & \tilde{M}_\alpha & 0 \\ 0 & 0 & \tilde{N}_\beta \end{bmatrix} \tag{2.19}$$

$$F_\omega = \begin{bmatrix} \tilde{L}_p & 0 & 0 \\ 0 & \tilde{M}_q & -kp \\ 0 & -kp & \tilde{N}_r \end{bmatrix} \tag{2.20}$$

where

$\tilde{M}_\alpha = M_\alpha I_{yy}\,\alpha/\tan\alpha$, $\tilde{L}_\beta = L_\beta I_{xx}\,\beta/\tan\beta$,

$\tilde{N}_\beta = N_\beta I_{yy}\,\beta/\tan\beta$, $\tilde{L}_p = L_p I_{xx}$,

$\tilde{M}_q = M_q I_{yy}$, $\tilde{N}_r = N_r I_{yy}$

$$B = \begin{bmatrix} \tilde{L}_\delta & 0 & 0 \\ 0 & \tilde{M}_\delta & 0 \\ 0 & 0 & \tilde{N}_\delta \end{bmatrix} \tag{2.21}$$

where $\tilde{L}_\delta = L_\delta I_{xx}$, $\tilde{M}_\delta = M_\delta I_{yy}$, $\tilde{N}_\delta = N_\delta I_{yy}$, and $k = 1 - I_{xx}/I_{yy}$

In designing control strategies it will sometimes be useful to keep in mind the overall behavior of the attitude, in particular the overall angle-of-attack, for stability reasons. Define (see Figure 2.1)

$$\alpha_T^2 = \tan^2\alpha + \tan^2\beta \tag{2.22}$$

which is the magnitude squared of the overall angle-of-attack, since α and β are measured in orthogonal directions. Then, from (2.16) and (2.18).

$$\frac{d}{dt}(\alpha_T^2) = 2(1+\tan^2\alpha + \tan^2\beta)(q\tan\alpha - r\tan\beta) \tag{2.23}$$

so that

$$\frac{d}{dt}(\alpha_T^2) \begin{cases} >0 \; ; & q/r > \tan\beta/\tan\alpha \\ =0 \; ; & q/r = \tan\beta/\tan\alpha \\ <0 \; ; & q/r < \tan\beta/\tan\alpha \end{cases} \tag{2.24}$$

3. AUTOPILOT OBJECTIVES

The objectives of the autopilot are to attain commanded accelerations along the body y and z axes in a minimum time, without violating a set of constraints. This is accomplished via an attitude transfer maneuver, subject to certain constraints:

(C1) the control surface deflections and deflection rates are subject to hard constraints, due to mechanical limitations

(C2) the angle of sideslip (β) must be kept suitably small; this is a soft constraint

(C3) body attitude rates are subject to soft constraints, dependent on actuator dynamics and control effectiveness.

In addition, autopilot design includes the following requirements:

(i) provide dynamic stability for the airframe

(ii) body motion should be minimized

(iii) maintain sufficient bandwidth to respond to low-frequency guidance commands but not so high as to respond to high-frequency noise

(iv) avoid resonance with other missile components, including airframe bending modes, actuators and instruments

(v) achieve time response rapid enough to maintain guidance loop stability in the presence of noise and errors.

These functions must be performed over a wide range of altitudes, Mach No., dynamic pressure and roll angles. In this study we have concentrated on the generic constraints (C1) - (C3), but have also considered (i), (ii) and (v). The requirements (iii) and (iv) require a specific missile configuration, which was beyond the scope of this study.

The autopilot maneuver is sometimes divided into two separate maneuvers:

1) A roll/yaw maneuver to null the component of commanded acceleration along the body y axis. This places the commanded acceleration in the body x - z plane.

2) A pitch maneuver to achieve the desired acceleration.

The roll/yaw maneuver uses a commanded roll angle ϕ_c which satisfies

$$\tan \phi_c = \frac{a_{yc}}{a_{zc}} \tag{3.1}$$

where a_{yc}, a_{zc} are the commanded accelerations. The pitch maneuver is performed to meet the total commanded acceleration:

$$Z_\alpha \alpha + Z_\delta \delta_q = (a_{yc}^2 + a_{zc}^2)^{1/2} \tag{3.2}$$

where Z_α, Z_δ are the z axis acceleration coefficients.

Eq (3.2) may be solved for α, given a_{yc}, a_{zc} and employing (2.14) to meet desired terminal conditions (e.g., M=0). The division into two separate maneuvers is often done for simplicity of design. It is more efficient, however, to combine maneuvers, since this enables us to attain the desired attitude in less time in most cases. Throughout this study we have considered combined maneuvers only.

4. ATTITUDE MANEUVERS USING BODY RATES AS CONTROL

In typical missile systems the actuator bandwidths are much higher than the desired autopilot bandwidths. Generally, autopilot bandwidths are 5-10 Hz while actuator bandwidths are 30 Hz or more. Bandwidth separation is usually greatest for the roll channel. Although actuator dynamics need to be included in detailed simulations, we can neglect them to do an approximate analysis. Total maneuver times may be greater than one second; under these conditions the rate control loop will be much faster than the attitude control loop so that the actuator dynamics can be neglected. With this assumption, the vehicle dynamics are given by (2.16) only, viewing ω as the control. The control dynamics are governed by (2.17). By using a feedback law we can easily achieve a stable rate loop. The feedback signals for the rate loop are the body rates measured, for example, by a set of strapdown gyros. This design is straightforward.

To illustrate a possible design approach, consider the rate dynamics given by (2.17) with the control

$$u = u_0 + G(\omega_c - \omega) \tag{4.1}$$

where ω_c is the commanded rate and G is a control gain matrix to be determined. By defining $\delta\omega = \omega - \omega_c$, with ω_c assumed constant, we obtain

$$\frac{d}{dt}(\delta\omega) = F_\Theta\, \Theta + F_\omega\, \omega_c + Bu_0 + (F_\omega - BG)\, \delta\omega$$

If we set

$$u_0 = - B^{-1} (F_\Theta\, \Theta + F_\omega\, \omega_c) \tag{4.2}$$

then

$$\frac{d}{dt}(\delta\omega) = (F_\omega - BG)\, \delta\omega \tag{4.3}$$

so that the closed - loop rate controller dynamics are given by $F_\omega - BG$. The rate loop may be uncoupled by setting

$$F_\omega - BG = \Lambda_\omega \tag{4.4}$$

where

$$\Lambda_\omega = \text{diag}\,(1/\tau_p,\ 1/\tau_q,\ 1/\tau_r)$$

is the desired actuator dynamic matrix. Λ_ω must be selected to be within the bandwidth allowed by the actual lags in the actuator response.

This gives, for G,

$$G = B^{-1}\,(F_\omega - \Lambda_\omega)$$

$$= \begin{bmatrix} \dfrac{\tilde{L}_p - \dfrac{1}{\tau_p}}{\tilde{L}_\delta} & 0 & 0 \\[2em] 0 & \dfrac{\tilde{M}_\delta - \dfrac{1}{\tau_q}}{\tilde{M}_\delta} & -\dfrac{kp}{\tilde{M}_\delta} \\[2em] 0 & -\dfrac{kp}{\tilde{N}_\delta} & \dfrac{\tilde{N}_r - \dfrac{1}{\tau_r}}{\tilde{N}_\delta} \end{bmatrix} \qquad (4.5)$$

The control then becomes

$$u = -B^{-1}\,[F_\theta\,\theta + F_\omega\,\omega + \Lambda_\omega\,(\omega_c - \omega)] \qquad (4.6)$$

which is nonlinear in both θ and ω.

4.1 Decoupling Control

Now consider the problem of attaining a desired attitude $\theta(t_f) = \theta_c$ at an unspecified terminal time t_f starting from the initial state $\theta(0) = \theta_0$ at $t=0$. We can meet the attitude constraint by picking a time history $\dot{\theta}(t)$ such that

$$\int_0^{t_f} \dot{\theta}(t)\, dt = \theta_c - \theta_0 \tag{4.7}$$

Once we have chosen $\dot{\theta}(t)$, the control is calculated as

$$\omega = K_\theta^{-1}\,\dot{\theta} \tag{4.8}$$

4.1.1 Control Using Constant Rates

If we assume a constant value of $\dot{\theta}$ during the maneuver interval, then

$$\dot{\theta} = \frac{1}{t_f}\,(\theta_c - \theta_0) \tag{4.9}$$

and the resulting control is

$$\omega = \frac{1}{t_f}\,K_\theta^{-1}\,(\theta_c - \theta_0) \tag{4.10}$$

If we let $\dot{\theta} = [v_\phi,\ v_\alpha,\ v_\beta]^T$ then the resulting rates are

$$p = v_\phi \tag{4.11}$$

$$q = x_2\,v_\phi + \frac{1}{D}\left[(1+x_2^2)\,v_\alpha - x_1\,x_2\,v_\beta\right] \tag{4.12}$$

$$r = x_1\,v_\phi + \frac{1}{D}\left[x_1\,x_2\,v_\alpha - (1+x_1^2)\,v_\beta\right] \tag{4.13}$$

where $D = 1 + x_1^2 + x_2^2$.

The body rates will produce the desired values $\dot{\phi} = v_\phi$, $\frac{d}{dt}(\tan\alpha) = v_\alpha$, $\frac{d}{dt}(\tan\beta) = v_\beta$ or, equivalently, $\dot{\alpha} = v_\alpha / (1 + x_1^2)$, $\dot{\beta} = v_\beta / (1 + x_2^2)$. Clearly, for t_f sufficiently small, p, q, or r will exceed the capability of the missile, so t_f needs to be selected so that none of the body rate constraints are violated. If we assume that $|\phi_c - \phi| \leq \pi$, $|\tan\alpha_c - \tan\alpha| \leq 1$, $|\tan\beta_c - \tan\beta| \leq \varepsilon$ then it follows that the rate constraints are met if

$$\left.\begin{aligned} &\text{(i)} \quad \pi \leq t_f\, p_{max} \\ &\text{(ii)} \quad \pi\,\varepsilon + 1 + 2\,\varepsilon^2 \leq t_f\, q_{max} \\ &\text{(iii)} \quad \pi + 3\,\varepsilon \leq t_f\, r_{max} \end{aligned}\right\} \quad (4.14)$$

Since p_{max} is generally larger than q_{max} or r_{max} and ε is generally less than 0.25, condition (iii) dominates, with the result that the constant - rate control can be met if $t_f\, r_{max} \geq 4$. For example, if $r_{max} = 2$ rad/sec, then $t_f \geq 2$ sec is required.

We can now make a further simplifying assumption, namely, that $\beta = 0$ at the beginning of the maneuver and the commanded value is zero. With this assumption $x_2 = 0$, $v_\beta = 0$ and the pitch and yaw rates simplify to

$$q = \frac{v_\alpha}{1 + x_1^2}$$

$$r = x_1\, v_\phi$$

which yields, from (16), $\dot{x}_1 = v_\alpha$, or $x_1 = \tan\alpha_0 + v_\alpha t$. We have, finally

$$q = \frac{v_\alpha}{1 + (\tan\alpha_0 + v_\alpha t)^2} \qquad (4.15)$$

$$r = (\tan\alpha_0 + v_\alpha t)\, v_\phi$$

These rates are well-behaved. Yaw rate r is a linear function of time. Pitch rate q is always of the same sign. In addition q obeys the differential equation

$$\dot{q} = -\,2 \tan\alpha\, q^2$$

For a BTT maneuver, we would typically find that $\tan\alpha > 0$ during the entire maneuver, so that q would be monotonically decreasing during the maneuver.

4.1.2 <u>Control Using Exponential Rates</u>

If we set

$$\dot{\Theta} = \Lambda_\Theta\, (\Theta_c - \Theta) \qquad (4.16)$$

where $\Lambda_\Theta = \mathrm{diag}\,(1/\tau_\phi,\ 1/\tau_\alpha,\ 1/\tau_\beta)$ then the body rates (assuming $\tau_\phi >> \tau_p$, $\tau_\alpha >> \tau_q$, $\tau_\beta >> \tau_r$) are

$$\omega = K_\Theta^{-1}\, \Lambda_\Theta\, (\Theta_c - \Theta) \qquad (4.17)$$

and the solution of (4.11) is

$$\Theta(t) = e^{-\Lambda_\Theta t}\, \Theta(0) + (I - e^{-\Lambda_\Theta t})\, \Theta_c \qquad (4.18)$$

Assuming $\Theta_c = [\phi_c,\ \tan\alpha_c,\ 0]^T$ we obtain

$$p = \frac{\phi_c - \phi}{\tau_\phi} \qquad (4.19)$$

$$q = p \tan\beta + \frac{1}{1 + \left(\frac{\tan\alpha}{\sec\beta}\right)^2} \frac{\tan\alpha_c - \tan\alpha}{\tau_\alpha} \tag{4.20}$$

$$+ \frac{1}{1 + \left(\frac{\sec\alpha}{\tan\beta}\right)^2} \frac{\tan\alpha}{\tau_\beta}$$

$$r = p \tan\alpha + \frac{\tan\alpha \quad \tan\beta}{\sec^2 \alpha + \tan^2 \beta} \frac{\tan\alpha_c - \tan\alpha}{\tau_\alpha} \tag{4.21}$$

$$+ \frac{1}{1 + \left(\frac{\tan\beta}{\sec\alpha}\right)^2} \frac{\tan\beta}{\tau_\beta}$$

In order to achieve these conditions, it is necessary that the body rate limits are not exceeded. By imposing these rate limits, we actually set lower bounds on the time constants. For example, since $|\phi_c - \phi| \leq \pi$, the roll rate constraint $|p| < p_{max}$ will be met if $\tau_\phi > \frac{\pi}{p_{max}}$. Let us assume that during the autopilot maneuver the following constraints are met

$|\tan\alpha| \leq 1$

$|\tan\alpha_c - \tan\alpha| \leq 1$

$|\tan\beta| \leq \varepsilon$

Then the pitch rate constraint will be met if

$$\frac{\pi\varepsilon}{\tau_\phi} + \frac{2}{\tau} < q_{max}$$

and the yaw rate constraint will be met if

$$\frac{\pi}{\tau_\phi} + \frac{2\varepsilon}{\tau} < r_{max}$$

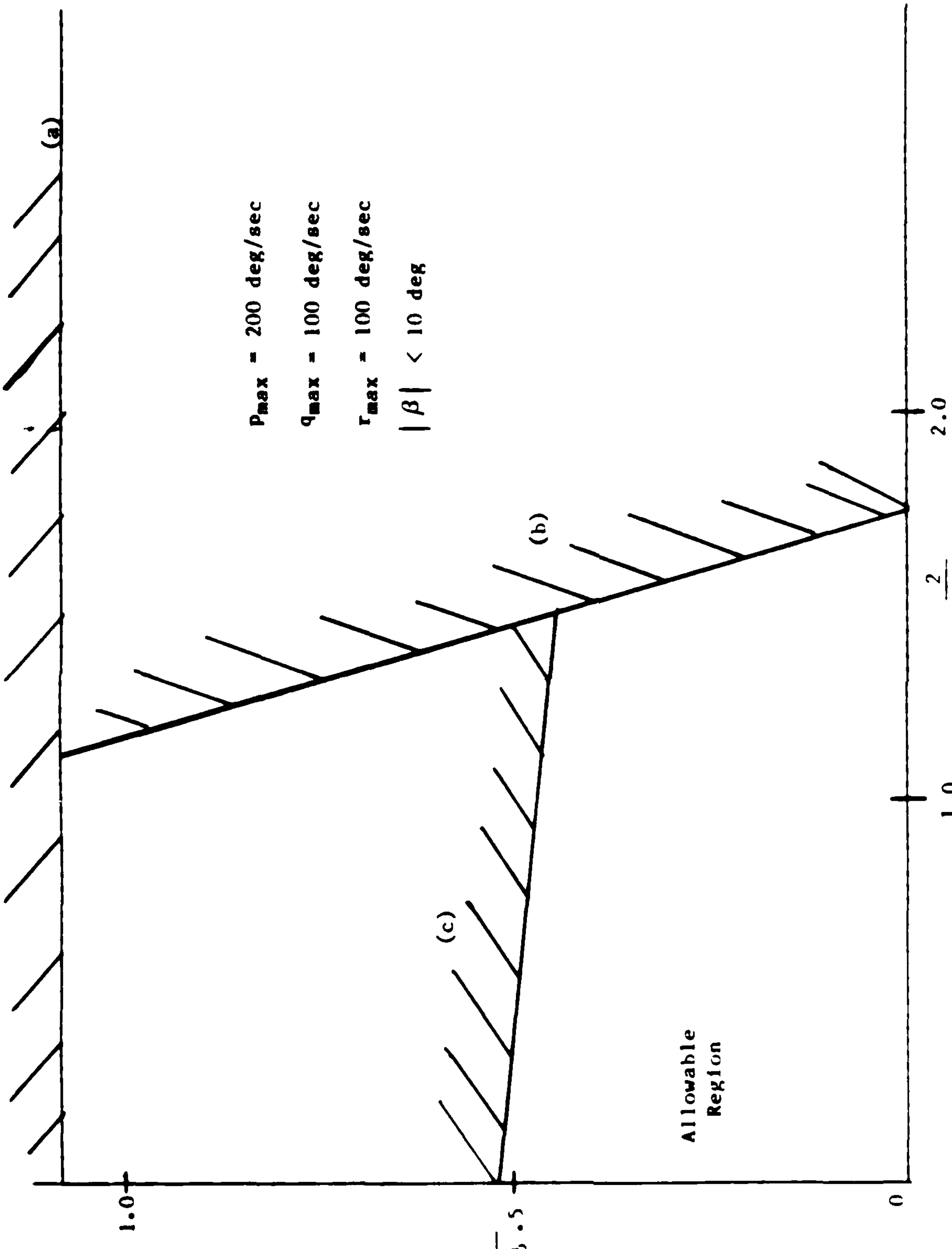

Figure 4.1 Allowable Time Constants to Meet Body Rate Constraints In Exponential Decoupled Controller

where

$$\frac{2}{\tau} = \frac{1}{\tau_\alpha} + \frac{1}{\tau_\beta}$$

Note that τ_α and τ_β have the same effect on meeting the constraints. As an example, suppose $p_{max} = 200°/sec$, $q_{max} = r_{max} = 100°/sec$, $\varepsilon = 0.1763$ ($\beta \leq 10°$). Then only the pitch and yaw constraints are active and the allowable region for ($1/\tau_\phi$, $2/\tau$) is shown in Figure 4.1. It appears for this case that a good compromise solution is at the upper right corner of the allowable region ($\tau_\phi = 2.17$ sec, $\tau/2 = 0.676$ sec).

A set of less conservative conditions for determining whether the body rate constraints are met is to use the closed - form solution (4.18) in the rate equations (4.19) - (4.21). If we assume that θ_c is fixed and start the maneuver at $t = 0$ with $\theta(0)$ given, then, assuming $\beta_c = 0$

$$\tan \alpha(t) = e^{-t/\tau_\alpha} (\tan\alpha(0) - \tan\alpha_c) + \tan\alpha_c$$

$$\tan \beta(t) = e^{-t/\tau_\beta} \tan\beta(0)$$

Substituting these into (4.19) - (4.21) yield explicit solutions for $p(t)$, $q(t)$, $r(t)$, $t \geq 0$. Depending on the initial conditions and commanded angles, $q(t)$ and $r(t)$ may attain maximum absolute values after the start of the maneuver. However, $p(t)$, $q(t)$, $r(t)$ eventually go to zero exponentially.

This control concept is essentially equivalent to a <u>predictive controller</u> in which a specified output trajectory (from the current output ϕ, α, β to the desired final output) is followed as closely as possible, subject to the system and control constraints. Since the trajectory and control are known in the future (for the deterministic case), it is straightforward to pick the trajectory (i.e. Λ_θ) so as to meet all of the constraints. In a more general setting, an exponential path may not be adequate for simultaneous avoidance of constraints and minimum-time maneuvers so that a more complex path satisfying (4.7) may need to be constructed. In this case, some ideas from robotics on minimum-time trajectory planning under path constraints (see, e.g., Rajan (1985), Sahar and Hollerback (1985), Sontag and Sussman (1985)) may be relevant. An alternate approach is to limit the controls according to the given constraints; this yields piecewise nonlinear controls which must be determined using an algorithmic approach such as Model Algorithmic Control (Rouhani and Mehra (1982)).

4.1.3 Inclusion of Rate Loop Dynamics

The nonlinear control law of (4.19) - (4.21) assumes that the desired body rates are met instantaneously or, at least, that the rate control loop time constants are much smaller than the autopilot time constants. In an actual missile this may not always be true. In particular, since a rapid autopilot response is desired it may be necessary to use time constants which approach those of the rate loop dynamics.

We assume a first-order lag model for the rate loop dynamics in the form

$$\dot{\omega} = \Lambda_a (\omega_c - \omega) \tag{4.22}$$

(cf (4.5)), where ω_c is the input to the rate loop. Here, for simplicity, we will use $\Lambda_a = (1/\tau_a)\, I$, where τ_a is the rate loop time constant, assumed identical in all three channels.

Then

$$\ddot{\theta} = \frac{\partial}{\partial \theta} (K_\theta \omega)\, \dot{\theta} + K_\theta\, \Lambda_a\, (\omega_c - \omega) \tag{4.23}$$

Now we <u>require</u> that the dynamics satisfy

$$\ddot{\theta} + \Lambda\, \dot{\theta} + M\, (\theta - \theta_c) = 0 \tag{4.24}$$

where Λ and M are specified positive - definite matrices picked to yield the desired closed-loop response. Here, for simplicity we choose

$$\Lambda = 2\zeta\omega_0\, I \tag{4.25}$$

$$M = \omega_0^2\, I \tag{4.26}$$

which yield identical damped, second-order responses for ϕ, $\tan\alpha$ and $\tan\beta$ with frequency ω_0 and damping ratio ζ. Substituting (4.25), (4.26), (4.23) and (2.16) into (4.24) yields, for the input ω_c:

$$\begin{aligned} \omega_c = {} & \omega_0^2\, \tau_a\, K_\theta^{-1} (\theta_c - \theta) \\ & + [\, I - 2\zeta\omega_0\tau_a\, I - \tau_a\, \psi\,]\, \omega \end{aligned} \tag{4.27}$$

where

$$\psi = K_\theta^{-1}\, \frac{\partial}{\partial \theta} (K_\theta \omega)\, K_\theta \tag{4.28}$$

As $\omega_0 \to \infty$, $\dot{\theta} \to \frac{\omega_0}{2\zeta} (\theta_c - \theta)$, which is a first-order response equivalent to (4.16) with $\Lambda_\theta = \frac{\omega_0}{2\zeta} I$. As $\tau_a \to 0$, $\omega \to \omega_c$.

The elements of ψ are

$$\psi_{11} = \psi_{12} = \psi_{13} = 0$$

$$\begin{aligned}\psi_{21} = {} & p[-x_1] / D \\ & + q[x_1 x_2 (D + 1 + x_2^2)] / D \\ & + r[1-D + x_2^2 (x_1 x_2 + x_1^2 - x_2^2)] / D\end{aligned} \tag{4.29}$$

$$\begin{aligned}\psi_{22} = {} & p[-x_1 x_2 (D+1)] / D \\ & + q[x_1 D + (1+x_1^2)(1+x_2^2) x_1] / D \\ & + r[-D - x_1^2 (1+x_1 x_2)] x_2 / D\end{aligned} \tag{4.30}$$

$$\begin{aligned}\psi_{23} = {} & p[1+x_2^2 (1+D)] / D \\ & + q[x_1 x_2 (1+x_2^2)] / D \\ & + r[1+x_2^2 (1+x_1 x_2)] x_1 / D\end{aligned} \tag{4.31}$$

$$\begin{aligned}\psi_{31} = {} & p[x_2] / D \\ & - q[x_1^2 (D+x_2^2)] / D \\ & + r[(x_1+x_2) 1+x_1^2) + x_1 (1-x_2^2)] x_2 / D\end{aligned} \tag{4.32}$$

$$\begin{aligned}\psi_{32} = {} & p[-x_1^2 (D+1) -1] / D \\ & + q[x_1^2 x_2 (1+x_1^2)] / D \\ & + r[x_1 x_2 (-x_1^2 - x_1^3 + x_2^2)] / D\end{aligned} \tag{4.33}$$

$$\begin{aligned}\psi_{33} = {} & p[x_1 x_2 (D+1)] / D \\ & + q[x_1 (D - x_1^2 x_2^2)] / D \\ & + r[-2D + x_1^2 (1+x_1 x_2) + x_1 x_2] x_2 / D\end{aligned} \tag{4.34}$$

Two types of physical constraints must be taken into account here. First, the body rate constraints $|p| < p_{max}$, $|q| < q_{max}$, $|r| < r_{max}$ must be met.

Second, the autopilot dynamics will meet the linear constraint equation (4.22) only up to a certain value of $\|\omega_c - \omega\|$; beyond this, saturation effects will start to occur. In order to study the effects of constraints more closely assume that (4.22) holds for $|p_c - p| \leq d_p$, $|q_c - q| \leq d_q$, $|r_c - r| \leq d_r$. In addition we will assume, for simplicity, that $x_2 = 0$, which is approximately true for a BTT missile. Then:

$$p_c - p = \omega_0^2 \tau_a (\phi_c - \phi) - 2\zeta\omega_0 \tau_a p \qquad (4.35)$$

$$q_c - q = (\omega_0^2 \tau_a \cos^2\alpha)(\tan\alpha_c - \tan\alpha) - 2\zeta\omega_0\tau_a q - \tau_a [rp + (2q^2 - p^2 + r^2)\sin\alpha\cos\alpha] \qquad (4.36)$$

$$r_c - r = (\omega_0^2 \tau_a \tan\alpha)(\phi_c - \phi) - 2\zeta\omega_0\tau_a r + \tau_a [pq\sec^2\alpha + (p\sin\alpha - r\cos\alpha) q\sin\alpha] \qquad (4.37)$$

Note that the body rate errors ($p_c - p$, $q_c - q$, $r_c - r$) depend linearly on τ_a. Now we make the following reasonable assumptions which should be always met in practice: $|\phi_c - \phi| \leq \pi$, $0 < \alpha < 45°$, $0 < \alpha_c < 45°$. Then the desired response of (4.24) is guaranteed if the following three conditions are simultaneously satisfied:

(i) $$\pi\omega_0^2 + 2\zeta\omega_0 p_{max} < d_p / \tau_a \qquad (4.38)$$

(ii) $$\omega_0^2 + 2\zeta\omega_0 q_{max} + (p_{max} + r_{max})^2 / 2 + q_{max}^2 < d_q / \tau_a \qquad (4.39)$$

(iii) $$\pi\omega_0^2 + 2\zeta\omega_0 r_{max} + 2.5\, p_{max} q_{max} + 0.5\, q_{max} r_{max} < d_r / \tau_a \qquad (4.40)$$

These equations can be used in autopilot system design to set upper limits on ω_0 and ζ, given specified values of the rate loop dynamic limits (d_p, d_q,

d_r), rate loop time constant (τ_a) and body rate limits (p_{max}, q_{max}, r_{max}). As a numerical example, a typical missile might have the following parameter values: $d_p = d_q = d_r = 10$ rad/sec, $\tau_a = 0.10$ sec, $p_{max} = 5$ rad/sec, $q_{max} = r_{max} = 2$ rad/sec. In addition, assume a damping ratio of 0.50. Then we get the following conditions:

(i) $\pi\omega_0^2 + 5\,\omega_0 < 100$

(ii) $\omega_0^2 + 2\,\omega_0 < 71.5$

(iii) $\pi\omega_0^2 + 2\omega_0 < 73$

condition (iii) dominates, giving $\omega_0 < 4.51$ rad/sec

4.1.4 Stability Analysis

The decoupling control of (4.8) always exists, since K_Θ is nonsingular. Thus as long as $\dot{\Theta}$ is finite, ω is finite. If the constraints are not violated, then the desired path $\{\dot{\Theta}(t)\; t\varepsilon\; (0,\, t_\varepsilon)\}$ satisfying (4.7) will be followed with no error in the deterministic case. This is one advantage of the decoupling control concept and the use of a desired reference path.

If, however, the state Θ is not perfectly known, the control will be in error and an instability in closed loop may result. In order to study this possibility, define the state error as

$$e = \hat{\Theta} - \Theta \qquad (4.41)$$

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where $\hat{\theta}$ is the state estimate used by the controller. Consider the exponential rate case (4.16). The resulting control is

$$\omega = K_\Theta^{-1}(\hat{\theta})\,\Lambda_\Theta\,(\Theta_c - \hat{\theta}) \tag{4.42}$$

Then, to first-order, the closed-loop dynamics are

$$\dot{\Theta} = [\,I + K_\Theta(\theta)\,E_K(\theta, e)]\,\Lambda_\theta\,(\theta_c - \theta - e) \tag{4.43}$$

where

$$E_K(\Theta, e) = \left[\frac{\partial}{\partial x_1} K_\Theta^{-1}(\theta)\right] e_1 + \left[\frac{\partial}{\partial x_2} K_\Theta^{-1}(\theta)\right] e_2 \tag{4.44}$$

and $e_1 = \hat{x}_1 - x_1$, $e_2 = \hat{x}_2 - x_2$. The roll error e_ϕ does not appear since K_θ is independent of ϕ. The closed-loop stability now depends on the behavior of the time-varying matrix

$$M(\Theta, e) = K_\Theta(\Theta)\,E_K(\theta, e) \tag{4.45}$$

If all eigenvalues of $I+M(\theta,e)$ are in the right half plane, then stability is assured, in the sense that any deviations from the desired path will tend to decay to zero exponentially in the absence of future disturbances or errors. The driving term $-\Lambda_\theta e$ can cause an offset. If e is a constant then at steady-state ($\dot{\theta} = 0$), $\theta = \theta_c - e$.

If we write $M(\theta,e)$ in the form

$$M(\theta, e) = M_1(\theta)e_1 + M_2(\theta)e_2 \tag{4.46}$$

then

$$M_1(\theta) = \begin{bmatrix} 0 & 0 & 0 \\ -x_1x_2 & -\dfrac{x_1(2+x_2^2)}{D} & -\dfrac{x_2(1-x_1^2)}{D} \\ -(1+x_2^2) & -\dfrac{x_2(1+x_2^2)}{D} & \dfrac{x_1x_2^2}{D} \end{bmatrix} \tag{4.47}$$

$$M_2(\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 1+x_1^2 & \dfrac{x_1^2x_2}{D} & -\dfrac{x_1(1+x_1^2)}{D} \\ x_1x_2 & -\dfrac{x_1(1-x_2^2)}{D} & -\dfrac{x_2(2+x_1^2)}{D} \end{bmatrix} \tag{4.48}$$

In order to analyze the stability of (4.43), assume $\Lambda_\theta = 1/_\tau\ I$. Then one stable pole is at $s = -1/_\tau$, corresponding to the roll channel. The other two poles are stable if the following two conditions are simultaneously met:

$$\text{(i)}\ e^Tx < 1 + x^Tx \tag{4.49}$$

$$\text{(ii)}\ (e_1x_2 - e_2x_1)^2 + 2e^Tx < 1 + x^Tx \tag{4.50}$$

where $e^T = (e_1\ e_2)$, $x^T = (x_1\ x_2)$.

These conditions can be given a geometrical interpretation by viewing e and x as vectors in R^2 (note that they have orthogonal components spatially). If θ_e is the angle between e and x then we have equivalent conditions:

(i) $\|e\| \|x\| \cos \theta_e < 1 + \|x\|^2$ (4.51)

(ii) $\|e\|^2 \|x\|^2 \sin^2\theta_e + 2 \|e\| \|x\| \cos\theta_e < 1 + \|x\|^2$ (4.52)

where $\|e\|^2 = e^Te$, etc. The most conservative bound on $\|e\|$ for stability may be found by maximizing the left hand sides of (4.51) and (4.52) with respect to θ_e. We find that $\theta_e = 0$ maximizes in either case and that (4.52) dominates yielding

$$\|e\| < \frac{1 + \|x\|^2}{2 \|x\|} \tag{4.53}$$

The right hand side of (4.53) is bounded below by 1. Thus, the final result is that stability is guaranteed if $\|e\| < 1$. This is equivalent to saying that the sum of the squared errors in angle-of-attack and angle of-sideslip estimates must be less than 1 rad^2, a condition which should always be met in practice. Recall, however, that (4.53) is the result of a linearized analysis and holds only for $\|e\|$ sufficiently small.

The overall conclusion is that the decoupling control law (4.42) appears to be extremely robust for the dynamical model of (2.16). Now we consider the stability of the controller of section 4.1.3, which included the rate loop dynamics. In this case (4.22) holds but the commanded rate of (4.27) becomes

$$\omega_c = \omega_0^2 \, \tau_a \, K_\Theta^{-1}(\hat{\theta})(\theta_c - \hat{\theta})$$

$$+ [\, I - 2\zeta\omega_0 \, \tau_a \, I - \tau_a \psi(\hat{\theta})] \, \hat{\omega} \qquad (4.54)$$

Inserting (4.54) into (4.23) and keep only the first-order error terms yields,

$$\ddot{\theta} + [\, 2\zeta\omega_0 I + MQ + E_Q + QK_\theta \, MK_\theta^{-1}] \, \dot{\theta}$$

$$+ \omega_0^2 \, (I + M) \, (\theta - \theta_c)$$

$$= \left(\frac{1}{\tau_a} I - 2\zeta\omega_0 I - Q \right) K_\theta e_\omega - \omega_0^2 \, e \qquad (4.55)$$

where

$$Q = \frac{\partial}{\partial\Theta} (K_\Theta \omega)$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2x_1 q - x_2 r & -p - x_1 r \\ 0 & p + x_2 q & x_1 q - 2x_2 r \end{bmatrix} \qquad (4.56)$$

$$E_Q = \frac{\partial Q}{\partial x_1} e_1 + \frac{\partial Q}{\partial x_2} e_2$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2qe_1 - re_2 & -re_1 \\ 0 & qe_2 & qe_1 - 2re_2 \end{bmatrix} \qquad (4.57)$$

M has been defined in (4.45), and $e_\omega = \hat{\omega} - \omega$. The rate estimation error e_ω could be caused by rate gyro errors, for example. We find that the attitude dynamics are driven by both the attitude error e and the rate error e_ω. Further, the stability of the attitude maneuver now depends on the body rates as well as attitude. However, it is interesting to note that the matrix Q, which contains the body rates, appears only multiplied by M in the dynamics. Thus if rates are small, the system is more stable. It is also interesting that the rate error e_ω does not effect stability properties; it affects only the magnitude of the offset from the desired path linearly.

The steady-state attitude error is found by setting $\ddot{\theta} = \dot{\theta} = 0$ in (4.55), yielding

$$\theta - \theta_c = \frac{1}{\omega_0{}^2} (I-M) \left(\frac{1}{\tau_a} I - 2\zeta\omega_0 I - Q\right) K_\theta e_\omega - (I-M)e \qquad (4.58)$$

where we have assumed $M << I$, or $(I+M)^{-1} \approx I-M$. The effect of rate error e_ω decreases as ω_0 increases and as τ_a increases. In normal operation, body rate errors will probably have a smaller effect on attitude offset than attitude error, since body rates will be measured by relatively accurate rate gyros.

The analysis of stability of (4.55) becomes complex algebraically and difficult to interpret. We can, however, study a special case in detail. Assume that $\|\theta\| << 1$ and define

$$\varepsilon = qe_1 - re_2 \qquad (4.59)$$

If we write (4.55) in the form

$$\ddot{\theta} + \Lambda_e \dot{\theta} + W_e \theta = d \tag{4.60}$$

then the characteristic equation

$$\begin{vmatrix} sI & -I \\ W_e & sI+\Lambda_e \end{vmatrix} = 0$$

factors to the form

$$(s^2 + 2\zeta\omega_0 s+\omega_0^2)\ (s^4 + a_3\ s^3 + a_2 s^2 + a_1\ s + a_0) = 0$$

where

$$a_3 = 4\zeta\omega_0 + 3\varepsilon$$

$$a_2 = 2(2\zeta\omega_0 + \varepsilon)\ (\zeta\omega_0 + \varepsilon) + 2\omega_0^2$$

$$a_1 = \omega_0{}^2\ a_3$$

$$a_0 = \omega_0^4$$

It follows that (4.55) is stable if $\zeta > 0$, which is true by design, and if

$$qe_1 - re_2 + \zeta\omega_0 > 0 \tag{4.61}$$

so that stability is enhanced by increasing ζ and ω_0, so long as our model (4.55) remains valid. Note that high body rates will tend to destabilize the closed-loop system, although this depends on the signs of the body rates and the errors in estimates of α and β. This result suggests that one could enhance stabilization at high body rates by biasing the estimates of α and β so as to keep the left hand side of 4.61 positive.

4.2 Control Accounting for Rate Constraints Directly

In practice, the attainable body rates will in many cases be the limiting factor in quickly attaining the desired attitude. Here we discuss a simple suboptimal control strategy which explicitly takes the body rate constraints into account. We will assume that $|p| \leq p_{max}$, $|q| \leq q_{max}$, $|r| \leq r_{max}$ and consider the same attitude transfer problem as before. A key consideration in control design is how to keep β sufficiently small throughout the maneuver. If a time-optimal control problem is set up, using constraints on β, the resulting optimality conditions are quite complex, since they involve state constraints (see, e.g. Maurer, 1974) and a two-point boundary-value problem must be solved on-line. We can eliminate this complexity by assuming that $\beta = 0$ throughout the attitude transfer maneuver. This assumption will be approximately true in most cases. With this assumption, the dynamics become

$$\dot{x}_1 = (1 + x_1^2)\, q \tag{4.61}$$

$$r = x_1 p \qquad (\beta = 0) \tag{4.62}$$

The minimum - time pitch maneuver is

$$q = q_{max}\, \mathrm{sgn}(\alpha_c - \alpha_0)$$

which yields $\tan\alpha(t) = \tan(\alpha_0 + qt)$ and a transfer time $t_f = |\alpha_c - \alpha_0| / q_{max}$. The roll/yaw maneuver involves checking the constraints at each time point, since x_1 changes with time. At each time point during the maneuver either the roll constraint or the yaw constraint (or both) will be active. The resulting rates are

$$p = \mathrm{sgn}(\phi_c - \phi) \min (p_{max}, r_{max} / |x_1|) \tag{4.63}$$

$$r = \mathrm{sgn}(x_1 \, p) \min (r_{max}, |x_1| \, p_{max}) \tag{4.64}$$

Roll rate p is set to zero when ϕ reaches ϕ_c; pitch rate q is set to zero when α reaches α_c. These will generally occur at different times. Inspection of this maneuver suggests that further improvement may be possible in cases where large roll maneuvers but only small pitch maneuvers are required. In this case, allowable roll rate magnitudes can be increased if α is first decreased in magnitude and then subsequently increased in magnitude to its final value. This can be seen more explicitly by formulating an appropriate minimum-time control problem.

4.2.1 A Minimum-Time Control Law

Consider the dynamical system comprised of (4.61) and $\dot{\phi} = r / x_1$, i.e., $\beta = 0$. If we view q and r as controls the variational Hamiltonian for the minimum-time problem is (e.g., Athans and Falb, 1966):

$$H = 1 + \lambda_1 \, q \, (1 + x_1^2) + \lambda_2 \, r/x_1 \tag{4.65}$$

The minimizing controls are

$$q^* = - q_{max} \, \mathrm{sgn} \, (\lambda_1^*) \tag{4.66}$$

$$r^* = - r_{max} \, \mathrm{sgn} \, (\lambda_2^*/x_1^*) \tag{4.67}$$

where the optimal costates λ_1^* and λ_2^* satisfy the Euler-Lagrange equations

$$\dot{\lambda}_1^* = -2\, q^*\, x_1^*\, \lambda_1^* + r^*\, \lambda_2^* \,/\, x_1^{*2}\ , \quad \lambda_1^*(t_f) = c_1 \tag{4.68}$$

$$\lambda_2^* = 0\ ; \quad \lambda_2^*(t_f) = c_2 \tag{4.69}$$

where sgn(.) is +1 for a postive argument, -1 for a negative argument and 0 otherwise, and c_1, c_2 are constants to be determined to meet the required terminal conditions and to satisfy $H(t) = 0$ for all t. The time-minimizing path is characterized by pitch angle ϕ^* and angle-of-attack α^*. We will assume at the moment that $\alpha^*(t) > 0$ along the optimal path. Then

$$\phi^*(t) = \int_0^t (r^* \,/\, x_1^*)\, dt + \phi_0 \tag{4.70}$$

$$\alpha^*(t) = \int_0^t q^*\, dt + \alpha_0$$

with $\phi^*(t_f) = \phi_c$, $\alpha^*(t_f) = \alpha_c$ and $x_1^* = \tan \alpha^*(t)$. If we integrate backward along the optimal path, assuming q^* and r^* are constant, we find

$$\phi^*(t) = \phi_c - \frac{r^*}{q^*} \log \left[\frac{\sin\alpha_c}{\sin\alpha(t)} \right] \tag{4.71}$$

$$\lambda_1^*(t) = \left[\,|c_2|\, r_{max} \cot \alpha^*(t) - 1 \right] \frac{\cos^2 \alpha^*(t)}{q^*} \tag{4.72}$$

We can now characterize the solution of the minimum-time problem. Since we assume that $\alpha^*(t) > 0$ for $t \in (0, t_f)$, r^* is constant for $t \in (0, t_f)$. However, q^* may change sign. However q^* can only switch from $-q_{max}$ to $+q_{max}$, not vice versa. Thus, if time permits, α^* is first reduced to increase roll rate, and then is increased to its final value.

We have assumed here that the optimal path does not violate the constraint $\alpha^*(t) > 0$, but the constraint was not explicitly included in the problem formulation. If $\alpha^*(t)$ becomes too small the roll rate constraint will be violated. The constraint $|p| < p_{max}$ can be included via the state constraint $\alpha > \alpha_{min}$ where

$$\alpha_{min} = \tan^{-1}\left(\frac{r_{max}}{p_{max}} \right) \qquad (4.73)$$

The optimal paths may contain arcs along the constraint boundary, depending on the constraint values and desired terminal state. If the state is on the constraint boundary, it comes off at the time which allows the terminal angle-of-attack constraint to met.

4.2.2 Inclusion of Roll Rate Constraint

If we include the roll rate constraint $|r/x_1| < p_{max}$ explicitly, the problem formulation changes. Writing the constraint as $s(x_1, r) \leq 0$, where $s = |r/x_1| - p_{max}$, the Hamiltonian becomes (Bryson & Ho, 1975):

$$H = 1 + \lambda_1 q\,(1 + x_1^2) + \lambda_2\, r/x_1 + \mu\, s \qquad (4.74)$$

where $\mu > 0$ for $s = 0$ (on the constraint boundary) and $\mu = 0$ for $s < 0$ (off the boundary). Off the boundary, the optimality conditions are the same as before. On the boundary we have

$$|r^*| = |x_1^*|\, p_{max} \qquad (4.75)$$

In addition, the condition

$$\left.\frac{\partial H}{\partial r}\right|^{*} = \frac{\lambda_2^{*}}{x_1^{*}} + \frac{\mu^{*}}{x_1^{*}} \operatorname{sgn}\left(\frac{r^{*}}{x_1^{*}}\right) \tag{4.76}$$

$$= 0$$

must be satisfied, yielding

$$\mu^{*} = - \lambda_2^{*} \operatorname{sgn}\left(\frac{r^{*}}{x_1^{*}}\right) \tag{4.77}$$

$$> 0$$

where $\lambda_2^{*}(t) = c_2$, as before. The resulting equation for λ_1^{*} is

$$\dot{\lambda}_1^{*} = - \left.\frac{\partial H}{\partial x_1}\right|^{*}$$

$$= - 2\, q^{*}\, x_1^{*}\, \lambda_1^{*};\ \lambda_1^{*}(t_f) = c_1 \tag{4.78}$$

The optimal controls must minimize the Hamiltonian

$$H = 1 + \lambda_1^{*}\, q(1 + x_1^{*2}) + c_2\, \rho_{max} \operatorname{sgn}\left(\frac{r}{x_1}\right) \tag{4.79}$$

Thus

$$q^{*} = - q_{max} \operatorname{sgn}(\lambda_1^{*}) \tag{4.80}$$

as before, and

$$r^{*} = - x_1^{*}\, \rho_{max} \operatorname{sgn}(c_2) \tag{4.81}$$

which yields, on the boundary

$$p^* = - p_{max}\ sgn\ (c_2) \qquad (4.82)$$

Integrating (4.67) backward from terminal time t_f using constant q^* gives, on the boundary

$$\lambda_1^* (t) = c_1 \frac{\cos^2 \alpha^*(t)}{\cos^2 \alpha_c} \qquad (4.83)$$

where $\alpha_c = \alpha^* (t_f)$.

Two examples of minimum time maneuvers are shown in Figures 4.2 and 4.3. In each case, the maximum roll rate was 4 rad/sec, the maximum pitch and yaw rates were 1.0 rad/sec and the final desired angle-of-attack was 20 degrees. The desired roll maneuver was 90 degrees. In the first case the initial angle-of-attack was assumed to be zero, which yielded an initial boundary arc (p = 4 rad/sec), followed by a control-constrained path (r = 1 rad/sec) starting at tgo = 0.196 sec. The (minimum) elapsed time was 0.445 sec. The pitch rate was 1 rad/sec, starting at tgo = 0.436 sec (the possibility of switching the sign of q over the initial 0.009 seconds was neglected in view of physical lags in a real system).

In the second example, the initial angle-of-attack was 25 degrees; the remaining parameters were unchanged. In this case, the minimum - time path includes two arcs off the boundary and an interior boundary arc for the roll channel. The pitch channel includes a pitch-down maneuver (q = -1 rad/sec) for the first half and a pitchup maneuver (q = 1) for the second half. The reduction in angle-of-attack allowed the roll rate to be increased to its maximum possible value, thus reducing transfer time. Elapsed maneuver time was 0.498 sec. Note that a maneuver made without reducing angle-of-attck would require 0.733 sec.

Figure 4.2 Time Optimal Control - Eample 1

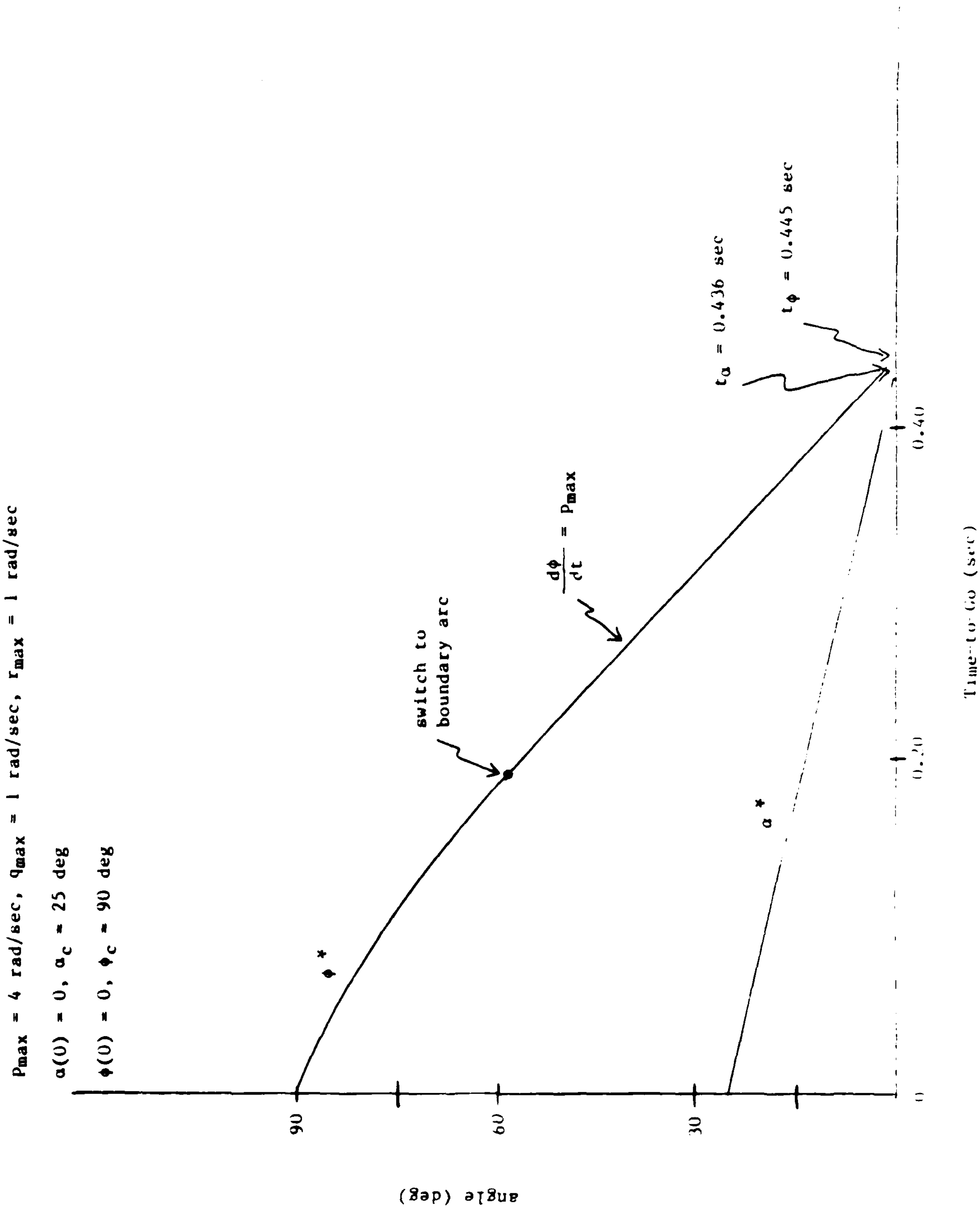
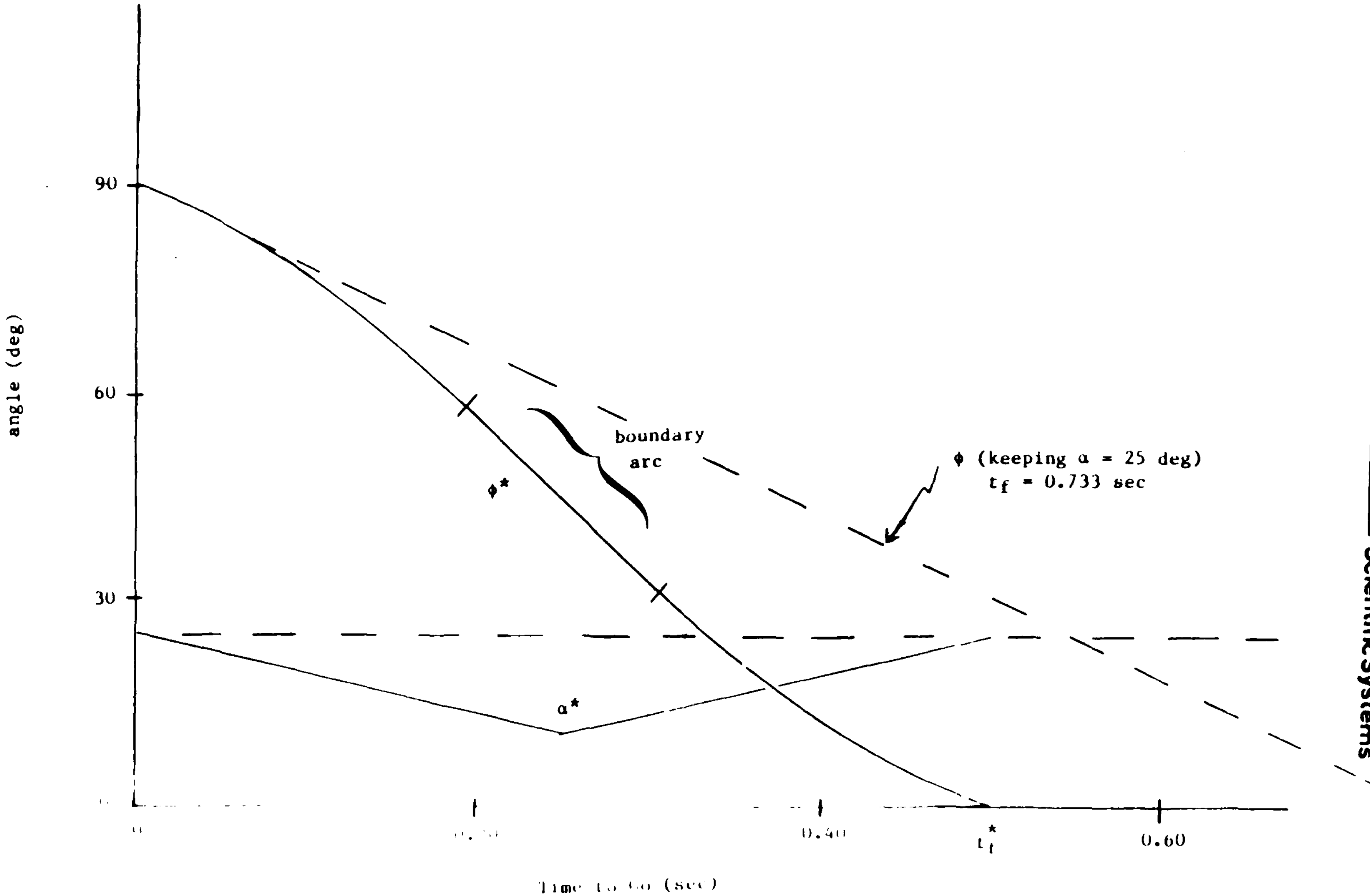


Figure 4.3 Time Optimal Control - Example 2

p_{max} = 4 rad/sec, q_{max} = 1 rad/sec, r_{max} = 1 rad/sec

$\alpha(0)$ = 25 deg, α_c = 25 deg

$\phi(0)$ = 0, ϕ_c = 90 deg



5. FULL DECOUPLING CONTROL

Our objective is to control the components of Θ independently. Introducing the output $y = \Theta$, our objective is to attain the commanded output $y(t_f) = y_d$ at the unspecified final time t_f. In addition, we wish to achieve the condition

$$\dot{y}(t) = 0;\ t \geq t_f \tag{5.1}$$

i.e., the angles ϕ, α, β are held constant from the end of the present maneuver until the beginning of the next maneuver. From (2.16):

$$\dot{y} = K_\Theta\ \omega \tag{5.2}$$

so that (5.1) is satisfied if $\omega(t) = 0;\ t \geq t_f$. From (2.17), this requires

$$u(t) = -B^{-1}\ F_\Theta\ \Theta;\ t \geq t_f$$

Prior to the terminal time t_f, we can develop a decoupling controller as follows.

In order to introduce the control u, we need to take another derivative of y which yields

$$\ddot{y} = \tilde{F}_\Theta\ \Theta + \tilde{F}_\omega\ \omega + \tilde{B}u \tag{5.3}$$

where

$$\tilde{F}_\Theta = K_\Theta\ F_\Theta \tag{5.4}$$

$$\tilde{F}_\omega = K_\Theta F_\omega + \frac{\partial}{\partial\Theta}(K_\Theta \omega) K_\Theta \quad (5.5)$$

$$\tilde{B} = K_\Theta B \quad (5.6)$$

It is clear that we can control $\ddot{y}$ explicitly via u if $\tilde{B}$ is non-singular. The determinant of $\tilde{B}$ is

$$|\tilde{B}| = -|B|(1 + \tan^2\alpha + \tan^2\beta) \quad (5.7)$$

$$= -|B| D$$

Thus, we can decouple the control problem if $|B| \neq 0$. From (2.21) $|B| = 0$ only if one or more of the control derivatives is zero, that is, if one or more of the body axis torques is not controllable through control surface deflections. The conclusion is that a nonlinear decoupling control exists if the missile is physically controllable, a condition which must be met for any steering law. The condition of physical controllability is equivalent to the condition that non-zero torques can be applied independently to all three axes, which will hold if the missile is not in a stalled flight condition and actuator torque limits are not exceeded.

Now suppose we wish to achieve

$$\ddot{y}(t) = g(t) \quad (5.8)$$

From (5.3), we find that the decoupling control is

$$u = \tilde{B}^{-1}[g - \tilde{F}_\Theta \Theta - \tilde{F}_\omega \omega] \quad (5.9)$$

Substituting (5.9) into (2.16) and (2.17) gives the dynamics of the closed-loop system under decoupling control:

$$\dot{\theta} = K_\theta \, \omega \tag{5.10}$$

$$\dot{\omega} = K_\theta^{-1} g - K_\theta^{-1} \left[\frac{\partial}{\partial \theta} (K_\theta \, \omega) \right] K_\theta \, \omega \tag{5.11}$$

$$= K_\theta^{-1} g - \psi \, \omega$$

where the elements of ψ are given in (4.29) - (4.34).

This controller is closely related to the <u>linear - equivalent</u> control methodology (see, e.g., Su (1982), Meyer, et al (1984)), in which a nonlinear dynamical system is converted into a linear system via nonlinear transformations. If we define

$$x_1(t) = \int_0^t K_\theta(\theta(\tau))\omega(\tau) \, d\tau$$

$$x_2(t) = K_\theta(\Theta(t))\omega(t)$$

or, equivalently, $x_1 = \theta$, $x_2 = \dot{\theta}$, then the dynamics are in the linear, time-invariant canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} v$$

where

$$v = K_\theta \, [Bu + F_\theta \, \theta + F_\omega \, \omega]$$

$$+ \frac{\partial}{\partial \theta} (K_\theta \omega) \quad K_\theta \omega$$

$$= g(t) \quad (\text{cf}(5.8))$$

and the actual control u can be found as a function of the linear - equivalent control v and the states x_1, x_2 via known memoryless nonlinear transformations. This is essentially the result of Su (1982) applied to the missile autopilot problem. Reed, et al (1985) obtained essentially the same result and then solved a quadratic optimal control problem. Dwyer (1984) used the same approach to solve large - angle rigid body rotational maneuver problems. Although these approaches lead to attractive dynamical models, the problem of control under constraints is not resolved. In essence, the nonlinearities are pushed into the control and it is not at all clear how (actual) constraints on u map into constraints on v. Thus, while these methods simplify the dynamics, they tend to preserve complexity in control constrained problems. However, there is an apparent advantage to these approaches, namely, that the complexity now resides in a single expression, relating the original and linear - equivalent controls via a nonlinear, memoryless transformation. Another potential disadvantage is that any state constraints have to be nonlinearly transformed to be adequately handled.

5.1 A Decoupling Controller With Exponential Response

We can use the results so far to design a decoupling controller to give a desired response characteristic. For example, suppose we wish to obtain a response with the following dynamics

$$\ddot{\theta} + \Lambda \dot{\theta} + W (\theta - \theta_c) = 0 \qquad (5.12)$$

where

$$\Lambda = \mathrm{diag}\,(2\zeta_\phi\omega_\phi,\ 2\zeta_\alpha\omega_\alpha,\ 2\zeta_\beta\omega_\beta) \qquad (5.13)$$

$$W = \mathrm{diag}\,(\omega_\phi^2,\ \omega_\alpha^2,\ \omega_\beta^2) \qquad (5.14)$$

with ζ_ϕ, ζ_α, ζ_β the desired damping ratios for the roll, pitch and yaw channels, respectively and ω_ϕ, ω_α, ω_β the desired frequencies. The solution of (5.12) is a set of uncoupled damped second-order systems of the form

$$z(t) = e^{-\zeta\omega t}\left[\psi(0)\cos\tilde{\omega}t + \frac{\zeta\omega\psi(0) + \dot{\psi}(0)}{\tilde{\omega}}\sin\tilde{\omega}t\right] \tag{5.15}$$

$$\tilde{\omega} = \omega\sqrt{1-\zeta^2}$$

which has poles at $-\zeta\omega \mp j\,\tilde{\omega}$.

Then

$$g(t) = -\Lambda\dot{\Theta} - W(\Theta - \Theta_c)$$

and the decoupling control satisfies

$$K_\Theta B u = W\Theta_c - (W + K_\Theta F_\Theta)\Theta - \left[\Lambda + K_\Theta F_\omega K_\Theta^{-1} + \frac{\partial}{\partial\Theta}(K_\Theta\omega)\right]K_\Theta\omega \tag{5.16}$$

In order to interpret this control law, we write out the control components explicitly, assuming $\beta_c = 0$:

$$\delta_p = \frac{1}{\tilde{L}_\delta}\left[\omega_\phi^2(\phi_c - \phi) - \tilde{L}_\beta\tan\beta - (2\zeta_\phi\omega_\phi + \tilde{L}_p)p\right] \tag{5.17}$$

$$\begin{aligned}\delta_q = \frac{1}{\tilde{M}_\delta}\Big[&\omega_\alpha^2\cos^2\alpha\,(\tan\alpha_c - \tan\alpha) - \tilde{M}_\alpha\tan\alpha \\ &- (k_p - (p + r\tan\alpha)\cos^2\alpha)(p\tan\alpha - r) \\ &- (2\zeta_\alpha\omega_\alpha + \tilde{M}_q + 2q\tan\alpha)q\Big]\end{aligned} \tag{5.18}$$

$$\delta_r = \frac{1}{\tilde{N}_\delta}\Big[\omega_\phi^2 \tan\alpha\,(\phi_c - \phi) + (\omega_\beta^2 - \tilde{N}_\beta)\tan\beta - (2\zeta_\phi\,\omega_\phi + \tilde{N}_r)\,p\tan\alpha$$

$$+ (k + \sec^2\alpha)\,p\,q + (2\zeta_\beta\,\omega_\beta + \tilde{N}_r + q\tan\alpha)\,(p\tan\alpha - r)\Big] \qquad (5.19)$$

In order to insure that the amplitude and rate constraints on δ_p, δ_q and δ_r are met, simulations using actual numerical values of the stability derivatives are required.

In order to analyze this case further we will assume that $\beta = 0$. Then $p\tan\alpha = r$ and we have:

roll

$$\tilde{L}_\delta\,\delta_p = \omega_\phi^2\,(\phi_c - \phi) - (2\zeta_\phi\omega_\phi + \tilde{L}_p)\,p$$

pitch

$$\tilde{M}_\delta\,\delta_q = \omega_\alpha^2\cos^2\alpha\,(\tan\alpha_c - \tan\alpha) - \tilde{M}_\alpha\tan\alpha$$

$$- (2\zeta_\alpha\omega_\alpha + \tilde{M}_q + 2q\tan\alpha)\,q$$

yaw

$$\tilde{N}_\delta\,\delta_r = \omega_\phi^2\tan\alpha\,(\phi_c - \phi) - (2\zeta_\phi\omega_\phi + \tilde{N}_r)\,p\tan\alpha$$
$$+ (k + \sec^2\alpha)\,p\,q$$

where all terms are torques in rad/sec^2.

We can derive a set of sufficient conditions for full application of decoupling control under constraints with several assumptions. Assume the following constraints:

(i) control deflections

$$|\delta_p| < \delta_{pmax}, \; |\delta_q| < \delta_{qmax}, \; |\delta_r| < \delta_{rmax}$$

(ii) control effectiveness

$$|\tilde{L}_\delta| > \tilde{L}_{\delta min}, \; |\tilde{M}_\delta| > \tilde{M}_{\delta min}, \; |\tilde{N}_\delta| > \tilde{N}_{\delta min}$$

(iii) rate damping

$$|\tilde{L}_p| < \tilde{L}_{pmax}, \; |\tilde{M}_q| < \tilde{M}_{qmax}, \; |\tilde{N}_r| < \tilde{N}_{rmax}$$

(iv) moment coefficients

$$|\tilde{M}_\alpha| < \tilde{M}_{\alpha max}$$

(v) body rates

$$|p| < p_{max}, \; |q| < q_{max}, \; |r| < r_{max}$$

(vi) attitude

$$|\phi_c - \phi| < \pi, \; |\tan\alpha_c - \tan\alpha| < 1 \;, \; \beta = 0,$$

$$0 < \alpha < \pi/4$$

This leads to the following sufficient conditions for full decoupling control:

(a) $\pi\,\omega_\phi^2 + p_{max}\max(2\zeta_\phi\omega_\phi, \; -\tilde{L}_{pmax} - 2\zeta_\phi\omega_\phi) < \tilde{L}_{\delta min}\,\delta_{pmax}$

(b) $\omega_\alpha^2 + \tilde{M}_{\alpha max} + q_{max}\max(2\zeta_\alpha\omega_\alpha, \; -\tilde{M}_{qmax} - 2\zeta_\alpha\omega_\alpha) < \tilde{M}_{\delta min}\,\delta_{qmax}$

(c) $\pi\omega_\phi^2 + (k + 2)\,p_{max}\,q_{max} + p_{max}\max\,(2\zeta_\phi\omega_\phi, \; -\tilde{N}_r - 2\zeta_\phi\omega_\phi)$

$< \tilde{N}_{\delta min}\,\delta_{rmax}$

Here we have assumed that the missile is statically stable, which requires $L_p < 0$, $M_q < 0$ $N_r < 0$. If conditions (a), (b), (c) are met, then it is guaranted that the decoupling controller will not cause any of the constraints (i) - (vi) to be exceeded, so that the roll, pitch and yaw channels will be fully decoupled.

5.2 Stability

We can analyze the stability of the exponential decoupling controller in a similar fashion as used in Section 4.1.4. Define

$$E_{F\omega} = \hat{F}_\omega - F_\omega \tag{5.21}$$

$$E_{F\Theta} = \hat{F}_\Theta - F_\Theta \tag{5.22}$$

$$R_K = (\hat{K}_\Theta - K_\Theta)\ K_\Theta^{-1} \tag{5.23}$$

$$R_B = (\hat{B} - B)\ B^{-1} \tag{5.24}$$

Then, analagous, to (5.12) we can write the closed-loop dynamics as

$$\ddot{\Theta} + \Lambda_e\, \dot{\Theta} + W_e\, \Theta = d \tag{5.25}$$

where

$$\begin{aligned} \Lambda_e = {} & \Lambda + K_\Theta\ E_{F\omega}\ K_\Theta^{-1} + (\Lambda+Q)\ R_K + E_Q - R_K(\Lambda+Q) \\ & - K_\Theta\ R_B\ [K_\Theta^{-1}\ (\Lambda+Q) + F_\omega K_\Theta^{-1}] \end{aligned} \tag{5.26}$$

$$W_e = W + K_\Theta\ E_{F\Theta} - R_K W - K_\Theta\ R_B\ (F_\Theta + K_\Theta^{-1}\ W) \tag{5.27}$$

$$d = (I - K_\Theta R_B K_\Theta^{-1} - R_K) W\Theta_c - (W + K_\Theta F_\Theta) e$$
$$- [K_\Theta F_\omega + (\Lambda + Q) K_\Theta] e_\omega \qquad (5.28)$$

Note that the matrix Q, which is linear in the body rates q and r, can have a significant effect on stability for high rates. Note that R_K and R_B are composed of terms which are essentially error ratios, and thus are less than one in magnitude for moderate percentage errors. Also, the elements of K_Θ and K_Θ^{-1} are less than one in magnitude, except for the (1,1) elements which are equal to one. We can qualitatively conclude that this controller appears robust to estimation errors, and that the most critical error terms are those involving Q in Λ_e. Note also that stability should be enchanced by increasing ζ and ω_0 up to a point, since this increases the positive-definitiness of Λ_e and ω_e. If the values of ω_0 and ζ are too low, then the error terms are likely to dominate the dynamics, leading to instability.

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