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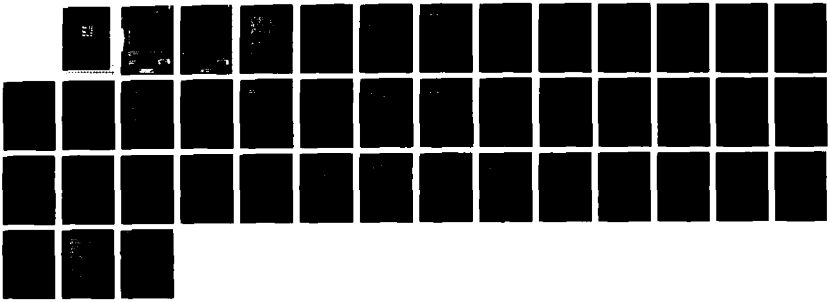

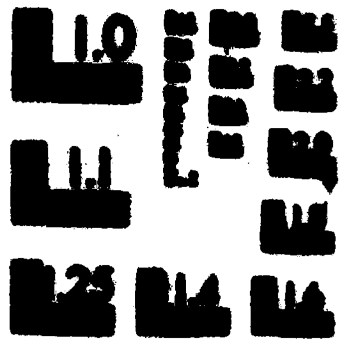
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15 May 1987

Richard A. Sandbothe*

and

Gerald L. Thompson**

Graduate School of Industrial Administration
William Larimer Mellon, Founder
Pittsburgh, PA 15213-3890

Carnegie Mellon

(W.P. No. 27-86-87)

Management Sciences Research Report No. MSRR 532

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Richard A. Sandbothe*

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Gerald L. Thompson**

* Clarkson University, Potsdam, N.Y., 13676

** Carnegie Mellon University, Pittsburgh, PA 15213

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Management Science Research Group
Graduate School of Industrial Administration
Carnegie Mellon University
Pittsburgh, PA 15213

ABSTRACT

~~In~~ this paper ~~we~~ consider the lot size model for the production and storage of a single commodity with limitations on production capacity and the possibility of not meeting demand, i.e., stockouts, at a penalty. The stockout option means that horizons can exist and permits the use of horizons to develop a forward algorithm for solving the problem. The forward algorithm is shown in the worst case to be asymptotically linear in computational requirements, in contrast to the case for the classical lot size model which has exponential computing requirements. Two versions of the model are considered: the first in which the upper bound on production is the same for every time period; and the second, in which the upper bound on production is permitted to vary each time period. In the first case the worst case computational difficulty increases in a cubic fashion initially, and then becomes linear. In the second case the initial increase is exponential before becoming linear. Besides the forward algorithm, a number of necessary conditions are derived which reduce the computational burden of solving the integer programming problem posed by the model.

1. INTRODUCTION

The classic lot size model [9] involves the production of a single product, storing it in a warehouse of unlimited capacity until needed, and requires the complete fulfillment of all (deterministic) demands. Various modifications have been made to that model which include the introduction of upper bounds on production and on inventory space, and the backlogging of orders. Each modification permits the derivation of special results relevant to that case. In the present paper we consider a new variation of the model, that of permitting stockouts.

One line of research in the area of production planning deals with capacitated lot size scheduling of a single item. Once the single item case is understood, knowledge from it is used to solve the multiple item situation. Research in this area was pioneered by the uncapacitated lot size model of Wagner and Whitin [9]. Others include Zabel [10], Zangwill [11] [12], and Lundin and Morton [7]. Each of these solutions involves a forward algorithm approach which solves successively longer finite horizon problems until a decision horizon is encountered. By definition, if optimal production decisions during the interval $[1,t^*]$ are completely independent of the demand data beyond $t^{**} \geq t^*$, then t^* is a decision horizon and t^{**} is a forecast horizon. From a practical standpoint, we are really only interested in determining the first decision horizon since the production problem is typically resolved periodically to incorporate improved forecast data. Single item capacitated lot size model research has been limited to static horizon length problems and assumptions as to the demand pattern, capacity pattern, and cost functions (see Florian and Klein [2], Jagannathan and Rao [3], Love [6], Swoveland [8], Louveaux [5], Korgaonker [4], and Baker, Dixon, Magazine, and Silver [1]). The drawback of the static horizon assumption is that information from beyond the end of the horizon could change the entire production plan; that is, no decision horizons are obtainable.

The purpose of this paper is to show how decision and forecast horizons may be obtained for single item capacitated lot size models. In order to obtain horizons, stockouts (lost sales) are permitted but no backorders. This situation has not been considered by previous authors. A concave production function consisting of a fixed cost component and a linear component is assumed. Production is bounded by capacity constraints which may or may not be constant from period to period.

The formulation of the model under study is presented in Section 2. Necessary conditions for a solution to be optimal are stated and proved in Section 3. Section 4 two types of horizon theorems are discussed. The first type involves the interaction of the stockout and holding costs, the second arises because of constraints on production capacity. Two forward solution algorithms that utilize both the optimality conditions and the horizon theorems are presented in Section 5. The first applies to problems for which the maximum production capacity is constant from period to period. It is shown that the worst case effort involved in solving this model is initially cubic but asymptotically becomes linear in the length of the problem. The second forward algorithm treats the case in which the maximum production capacity varies with time, and here it is found that the worst case effort is initially exponential but asymptotically becomes linear in the length of the problem. If decision horizons are detected frequently, so that the lengths of the subproblems are kept small, even this problem is not too computationally demanding. Computational results with the algorithms are given in Section 6.

2. MODEL FORMULATION

The capacitated lot size model with stockouts, which is an integer linear program, is given in Figure 2-1 and is called Problem P^T. In Problem P^T, Objective function 1 is to be minimized subject to Constraints (2) - (5).

$$\text{Min } \Sigma_{\tau=1}^{T}(\sigma\delta_\tau + pX_\tau + hI_\tau + sS_\tau) \tag{1}$$

Subject to

$$I_{\tau-1} + X_\tau - I_\tau + S_\tau = d_\tau \quad \text{for } \tau=1,\ldots,T \tag{2}$$

$$X_\tau - XMAX_\tau\delta_\tau \leq 0 \text{ for } \tau=1,\ldots,T \tag{3}$$

$$X_\tau, I_\tau, S_\tau \text{ nonnegative and integer} \quad \text{for } \tau=1,\ldots,T \tag{4}$$

$$\delta_\tau \text{ binary} \quad \text{for } \tau=1,\ldots,T \tag{5}$$

Figure 2-1: Problem P^T: Capacitated Lot Size Model with Stockouts

The variables of the model are: X_τ, the production quantity in period τ; I_τ, the inventory level at the end of period τ; S_τ, the number of stockouts incurred in period τ; and δ_τ, a binary variable which is 1 when $X_\tau > 0$ and 0 otherwise. The parameters of the model are: σ, the setup cost; p, the variable production cost per unit (assumed constant); h, the holding cost per unit per period; s, the stockout cost per unit; $XMAX_\tau$, the production capacity in period τ; and d_τ, the demand requirements in period τ.

The objective is to minimize the sum of total setup costs, total variable production costs, total holding costs, and total stockout costs as stated in Objective Function (1). Constraint (2) defines the production-inventory-stockout relationship with the demand requirements. It is assumed that $I_0 = I_T = 0$ without loss of generality and that $I_\tau \geq 0$ for all t so that backorders are prohibited. The upper bounding of production is accomplished by constraint (3). Constraint (3) also forces the setup cost to be incurred when production is positive. Constraint (4) requires the production, inventory, and stockout quantities in each period to be nonnegative and integer. Finally, constraint (5) imposes the binary {0,1} restriction on the δ_τ variables.

Problem P^T can be formulated as a concave cost network flow problem as follows. Let node M be the master supply node that contains the sum of the supplies for all T periods. We define two transhipment nodes, P and S. Node P will tranship all the units that satisfy demands via production and node S will tranship all the units that satisfy demands via stockouts. Finally, nodes 1, 2, . . . , T are the period demand nodes such that the demand at node τ is d_τ. The arc set consists of the following four arc subsets:

1. Directed arcs (M,P) and (M,S) which ship flows at zero cost and have upper bounds of $+\infty$;

2. Directed arcs (P,τ) with upper bounds of $XMAX_\tau$ for $\tau = 1, . . . , T$ which each incur zero shipping cost for zero arc flow and a cost of $\sigma + (p * X_\tau)$ if the arc's flow, X_τ, is greater than zero (these constitute the production arcs);

3. Directed arcs (S,τ) with upper bounds of $+\infty$ for $\tau = 1, . . . , T$ which

ship flows at a cost of s per unit flow (these are the stockout arcs);

4. Directed arcs $(\tau,\tau+1)$ with upper bounds of ∞ for $\tau = 1, \ldots, T-1$ which ship flows at a cost of h per unit flow (the inventory arcs).

A pictorial representation of the Problem P^T flow network is given in Figure 2-2.

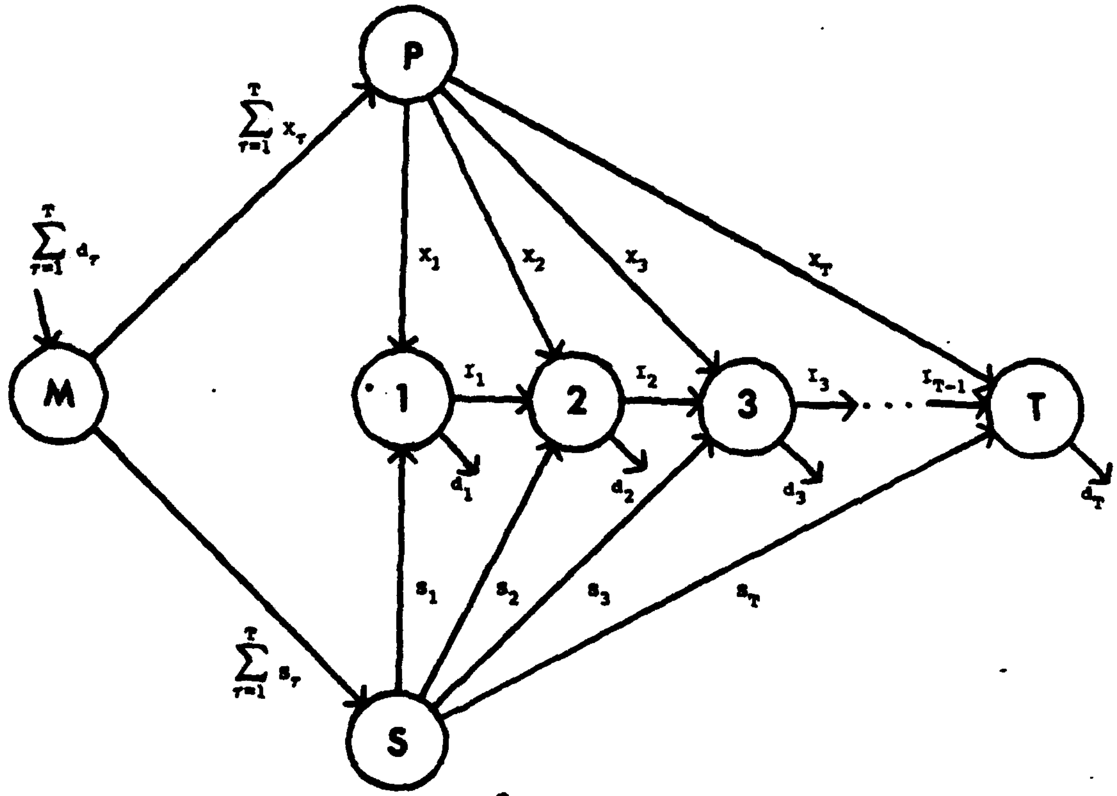

Figure 2-2: Problem P^T as a Network Flow Problem

An optimal solution to Problem P^T will be a spanning tree of the P^T network in Figure 2-2 and each unit travels from the supply node, M, to its respective demand node by the lowest cost available path.

3. NECESSARY CONDITIONS FOR A SOLUTION TO BE OPTIMAL

Necessary conditions on the optimal solution to Problem P^T are now stated and proved.

Definition 1: Define the *maximum holding period* to be $k = \lfloor (s-p)/h \rfloor$. The following result justifies this name.

Lemma 2: In an optimal solution, the maximum number of periods a unit will be held in inventory is the maximum holding period k.

Proof: Assume there exists an optimal solution with one or more units being held in inventory for more than k periods, say k + 1 periods. Then there will be a path of k + 1 basic inventory arcs from a period t to period t + k + 1. The total holding and variable production cost for one unit will be $p + h*(k+1)$. Now since $k = \lfloor (s-p)/h \rfloor$, which implies $[(s-p)/h] - 1 < k \le (s-p)/h$, or, solving for s,

$$p + (h*k) \le s < p + h*(k+1).$$

The rightmost inequality implies that the stockout cost is less than the sum of its total holding cost and variable production cost. Therefore the original solution could not have been optimal. ∎

The lemma which follows describes how units should be distributed from inventory in an optimal solution to Problem P^T. The FIFO rule, or first-in, first-out rule, distributes the oldest item first while the LIFO rule (last-in, first-out rule) distributes the newest item first. Both the FIFO and LIFO rules should be regarded as bookkeeping systems for inventory valuation. Rarely are these systems used as the actual method of distributing units from inventory. The main purpose for proving this lemma is to aid in the proof of an important theorem which follows it.

Lemma 3: In an optimal solution, the newest item in inventory should be distributed first (LIFO).

Proof: We begin with a plan that for at least two units, the first unit produced is distributed before the second (FIFO). It is then shown that switching the distribution of these two units from FIFO to LIFO will never lead to a higher cost solution.

Consider two units that are distributed on a FIFO basis:

Unit	When Produced	When Distributed
i_1	t_1	t'_1
i_2	t_2	t'_2

It is assumed that $t_1 \le t_2$ (which implies that $t'_1 \le t'_2$ for FIFO to hold) and that $t'_1 - t_1 \ge k$ and $t'_2 - t_2 \ge k$ so that both units *are* produced.

Case I: $t_1 = t_2$ or $t'_1 = t'_2$

Case II: $t_1 = t_2 = t'_1 = t'_2$

Case IIIa: $t_1 + k \ge t'_2$

Case IIIb: $t_1 + k < t'_2$

For Cases I and II, the total costs under FIFO and LIFO are equal.

For Case IIIa and IIIb, $t_1 < t_2 \le t'_1 < t'_2$. The FIFO total variable production and holding cost for the two units is: $2*p + h*[(t'_1-t_1)+(t'_2-t_2)]$.

If we switch to LIFO, the distribution pattern will be as follows:

Unit	When Produced	When Distributed
i_1	t_1	t'_2
i_2	t_2	t'_1

If $t_1 + k \ge t'_2$ (Case IIIa), then unit i_1 will still be produced under LIFO. The LIFO total variable production and holding cost for the two units is: $2*p + h*[(t'_2-t_1)+(t'_1-t_2)]$ which is equal to the FIFO holding cost.

If $t_1 + k < t'_2$ (Case IIIb), then unit i_1 will not be produced under LIFO. The LIFO total variable production, holding, and stockout cost is $p + h*(t'_1-t_2) + s$. The proof is complete when it is shown that $p + h*(t'_1-t_2) + s \le 2*p + h*[(t'_1-t_1)+(t'_2-t_2)]$.

Assume not, i.e., $p + h*(t'_1-t_2) + s > 2*p + h*[(t'_1-t_1)+(t'_2-t_2)]$, which reduces to:

$$h * (t'_2-t_1) < s - p \tag{6}$$

By definition, $k = \lfloor (s-p)/h \rfloor$, which implies that

$$h * k \le s - p < h * (k+1). \tag{7}$$

Recall that it was assumed that $t_1 + k < t'_2$ or

$$k < t'_2 - t_1. \tag{8}$$

Combining inequalities (6) and (7) yields $h * (t'_2-t_1) < h * (k+1)$ and since it was assumed that $h > 0$,

$$t'_2 - t_1 < k + 1. \tag{9}$$

Combining inequalities 8 and 9 leads to

$$k < t'_2 - t_1 < k + 1$$

which is impossible since k, t'_2, and t_1 are integers. ∎

Theorem 4: In an optimal solution, every k + 1 consecutive periods forms a regeneration set, i.e., contains one or more regeneration points at which the ending inventory is zero.

Proof: Assume the contrary, that there exists an optimal solution with $k + 1$ consecutive periods that do not form a regeneration set, say periods $t' - k, t' - k + 1, \ldots, t' + 1$, so that there are $k + 1$ periods of positive ending inventory. Since an optimal plan can be found using LIFO distribution (Lemma 3), a unit produced in period $t' - k$ will be held until at least period $t' + 1$, that is, at least $k + 1$ periods, which contradicts Lemma 2. ■

Theorem 4 states that in an optimal solution, for a sequence of any $k + 1$ consecutive periods, the inventory level at the end of at least one of these periods must be zero (i.e., a regeneration point). The determination of a regeneration point makes it possible to partition a large problem into smaller subproblems, each of which can be solved easily.

Lemma 5: In an optimal solution, $I_t * S_t = 0$ for all t.

Proof: If in an optimal solution, $I_{t'} * S_{t'} > 0$ for some t', then there is a path from node M to node t + 1 via nodes S and t at cost s + h. The path via node S only, using arc (S,t+1), has cost s, so that a lower cost solution is obtainable by using this alternate route, contradicting the assumption that the original solution was optimal. ■

Definition 6: Define the *minimum positive flow on a production arc* to be XMIN = $\lceil \sigma/(s-p) \rceil$, where $\lceil x \rceil$ is defined to be the smallest integer >= x. The next theorem justifies this name.

Theorem 7: In an optimal solution, if $X_t > 0$, then XMIN $\le X_t \le$ $XMAX_t$.

Proof: Clearly $X_t \le XMAX_t$ for all t. Now assume there is an optimal solution such that on a production arc (P,t') there is a positive flow

$X_{t'} < \sigma/(s-p)$. (The ceiling brackets may be removed since X_t must be integer.) Rearranging, we have $s * X_{t'} < \sigma + (p*X_{t'})$. This means it is cheaper to stock out of the $X_{t'}$ units than it is to produce them, a contradiction. ∎

Corollary 8: If $XMAX_{t'} < XMIN$ for some t', then $X_{t'} = 0$.

A theorem similar to the next one is proved in Baker, Dixon, Magazine, and Silver [1].

Theorem 9: In an optimal solution, $I_{t-1} * (XMAX_t - X_t) * X_t = 0$ for all t.

Proof: Assume the contrary, that there exists a period t' in an optimal solution with $I_{t'-1} * (XMAX_{t'} - X_{t'}) * X_{t'} > 0$. This means that arcs (t'−1,t') and (P,t') are both basic (refer to Figure 2–2). Since arc (t'−1,t') is basic, there must be an arc (P,t") (t"<t') with a positive flow. This positive flow must be at $XMAX_{t''}$, otherwise a cycle will have formed, violating the characterization of optimal flows as spanning trees. However, a lower cost flow is obtainable by increasing the flow on (P,t') until it becomes nonbasic and decreasing the flow on (P,t") by the amount of the increase on (P,t'). Production costs remain the same but total holding costs are reduced. Since a lower cost flow is possible, this contradicts the initial assumption. ∎

Theorem 10: In an optimal solution, if $k = \lfloor (s-p)/h \rfloor < (s-p)/h$ and for some period t, $S_t > 0$, then for each t' in the interval $(t-k \le t' \le t)$, if $X_t > 0$ then $X_{t'} = XMAX_{t'}$.

Proof: Assume an optimal solution exists with a period t'' such that $S_{t''} > 0$ and a period t''' $(t''-k \le t''' \le t'')$ with $0 < X_{t'''} < XMAX_{t'''}$.

If the inventory arcs $(t''',t'''+1)$, $(t'''+1,t'''+2)$, ... , $(t''-1,t'')$ are all basic with positive flows or $t'' = t'''$, then a cycle exists and the solution is not optimal.

Now assume that at least one of the inventory arcs between periods t''' and t'' $(t''' < t'')$ is nonbasic with zero flow. Since $S_{t''} > 0$ and $0 < X_{t'''} < XMAX_{t'''}$, it must be cheaper to stock out in period t'' than to increase production in period t''' to satisfy demands in period t''. That is,

$$p + h*(t''-t''') > s$$

Now since $k = \lfloor (s-p)/h \rfloor < (s-p)/h$ and $t''-k \le t'''$, we can obtain

$$p + h*(t''-t''') < s$$

which is a contradiction. Therefore our original solution could not have been optimal. ∎

The next theorem is a variation on one due to Baker, Dixon, Magazine, and Silver [1].

Theorem 11: In an optimal solution, if $t' = \max\{ t \mid X_t > 0\}$, then $X_{t'} = \min(XMAX_{t'}, \Sigma_{j=t'}^{t'+k} d_j) \geq XMIN$ provided $t' + k \leq T$. If $t' + k > T$, then $X_{t'} = \min(XMAX_{t'}, \Sigma_{j=t'}^{T} d_j) \geq XMIN$.

Proof: Let $t' = \max\{ t \mid X_t > 0 \}$, i.e., t' is the last period of positive production and assume that $t' + k \leq T$. (The proof can be easily extended to the case where $t' + k > T$). Clearly $XMAX_{t'} \geq X_{t'} \geq 0$. If $XMAX_{t'} < \Sigma_{j=t'}^{t'+k} d_j$, then $X_{t'}$ can not be $\Sigma_{j=t'}^{t'+k} d_j$ and period $t' + k$ must have stockouts. If, however, $X_{t'}$ were set to some quantity less than $XMAX_{t'}$, this will violate Theorem 10. If $\Sigma_{j=t'}^{t'+k} d_j < XMAX_{t'}$, then setting $X_{t'}$ equal to anything greater than $\Sigma_{j=t'}^{t'+k} d_j$ will give a solution having at least $k + 1$ periods of positive inventory, contradicting Theorem 4. If $X_{t'}$ is set to some quantity less than $\Sigma_{j=t'}^{t'+k} d_j$, the solution will again have stockouts in period $t' + k$, which contradicts Theorem 10. ∎

Theorem 12: In an optimal solution, if for some period t, $d_t \geq XMAX_t$ and $X_t > 0$, then $X_t = XMAX_t$.

Proof: Assume there exists an optimal solution having a period t' with $d_{t'} \geq XMAX_{t'}$ and $0 < X_{t'} < XMAX_{t'}$. Then, either arcs $(t'-1,t')$ and (P,t') are both basic (which contradicts Theorem 9), or arcs (S,t') and (P,t') are both basic (which contradicts Theorem 10). Hence the theorem is true. ∎

As seen above, it is possible to generate many necessary conditions an optimal solution must satisfy. A solution that satisfies these conditions is not necessarily optimal, i.e., these conditions are not sufficient for optimality. However use of the necessary conditions reduces the size of the solution space that must be considered in searching for an optimal solution.

4. THE DETECTION OF DECISION AND FORECAST HORIZONS

In this section three theorems are presented that sometimes permit the determination of decision and forecast horizons for the problem under consideration. When they exist, these horizons are used in the forward algorithms described in Section 5.

A forward algorithm begins by solving the initial one period problem and proceeds by solving successively longer finite horizon problems. To be useful, there must be a stopping rule in order to know when to terminate the forward algorithm. When decision and forecast horizons can be detected the algorithm can stop, since the solution up to the decision horizon is guaranteed to be part of the optimal solution to any problem longer than the forecast horizon. Specifically, the solution for the first period will be part of the optimal solution to any problem longer than the decision horizon. Since production planning models are usually re-solved every few periods to make use of updated, more accurate forecast information, only the first few decisions are needed.

Definition 13: If X_j^t is the optimal period j production quantity in the t-period problem P^t, then the sequence $\{X_1^t, \ldots, X_t^t\}$ is called an *optimal production sequence* for Problem P^t.

Definition 14: If the sequence $\{X_1^t, \ldots, X_t^t\}$ is an optimal production sequence for Problem P^t, then the sequence $\{X_1^t, \ldots, X_{t'}^t\}$ is an *optimal production subsequence* for Problem P^t, provided $t' \leq t$.

Theorem 15: Consider problems $P^{t-k}, \ldots, P^t$ where k is the maximum holding period $(k + 1 \leq t \leq T)$. If $X_j^{t-k} = X_j^{t-k+1} = \ldots = X_j^t$ for

$j = 1, \ldots, t'$ $(1 \le t' \le t - k)$, then period t is a forecast horizon and period t' is a decision horizon for any Problem P^{t^*}, where $t \le t^* \le T$.

Proof: From Theorem 4, it is known that for any sequence of $k + 1$ consecutive periods, the ending inventory of at least one of these periods must be zero in an optimal solution. When solving problems $P^{t-k}, P^{t-k+1}, \ldots, P^t$, the inventory at the end of each problem has to be zero in any optimal solution. Therefore, the production sequence in at least one of these problems will be part of the optimal solution to any problem of length t^*, where $t \le t^* \le T$. If $X_j^{t-k} = X_j^{t-k+1} = \ldots = X_j^t$ for $j = 1, \ldots, t'$ $(1 \le t' \le t - k)$, this means that each problem $P^{t-k}, \ldots, P^t$ has the same optimal production subsequence for the first t' periods and therefore this subsequence must be part of the optimal solution to any problem of length t^*, where $t \le t^* \le T$. Therefore period t' is a decision horizon. Since t periods of demand information are needed to determine this decision horizon, period t is a forecast horizon. ∎

This theorem is similar to the Lundin-Morton decision horizon theorem [7]. We know from Theorem 4 that every $k + 1$ consecutive periods form a regeneration set. If each of the $k + 1$ consecutive problems has the same optimal solution for the first t' periods, then this solution will be part of the optimal solution to any longer problem.

It is also possible to find decision horizons by using another procedure when the demand sequence is at or near a peak that is greater than the available production capacity. Theorem 16 applies to problems of length less than or equal to k and Theorem 17 applies to problems of length greater than k.

Theorem 16: Consider the optimal solution to a t-period problem P^t where $t \le k$. If $X_j^t = XMAX_j$ for $j = 1, \ldots, t$, then period t is a forecast and decision horizon for any Problem P^{t^*}, where $t \le t^* \le T$.

Proof: If in the optimal solution to P^t, there are k consecutive periods with production at the upper bound, then for problem P^{t^*}, where $t \le t^* \le T$, any demands occurring after period t must be satisfied from production after period t. The result is that $I_t = 0$ in any optimal solution and the optimal production sequence $\{X_1^t, \ldots, X_t^t\}$ will be unchanged by demands occurring after period t. Therefore period t is a decision horizon and since only t periods of demand information are needed to determine it, period t is also a forecast horizon. ∎

Theorem 17: Consider the optimal solution to a t-period problem P^t where $t > k$. If $X_j^t = XMAX_j$ for $j = t - k + 1, \ldots, t$, then period t is a forecast and decision horizon for any Problem P^{t^*}, where $t \le t^* \le T$.

Proof: If in the optimal solution to P^t, there are t consecutive periods having production at the upper bound, then for problem P^{t^*} where $t \le t^* \le T$, any demands occurring after period t must be satisfied from production after period t or prior to period $t - k + 1$. If satisfied from production prior to period $t - k + 1$, these units will be in inventory for at least $k + 1$ periods, violating Lemma 2. Therefore the demands occurring after period t must be satisfied from production after period t. As in the previous proof, this implies that $I_t = 0$ in any optimal solution and the optimal production sequence $\{X_1^t, \ldots, X_t^t\}$ will be unchanged by demands occurring after period t. Therefore period t is a decision horizon and since only t periods of demand information are needed to determine it, period t is also a forecast horizon. ■

5. THE FORWARD ALGORITHM

In this section, two forward algorithms are described and used to solve Problem P^T. The first algorithm is used when production capacity is constant, the second when it is not.

We begin by making two definitions:

Definition 18: Let V(t) be the optimal value of the t-period problem, P^t, where $t \le T$.

Definition 19: Let C(q,t) be the optimal value of the (t-q+1)-period subproblem, P(q,t), that begins at period q, ends at period t, with the restrictions that $I_{q-1} = 0$, $I_t = 0$, and $I_j > 0$ for $j = q, \ldots, t-1$.

The requirement that the P(q,t) subproblems have positive inventory in every period except the last is made in order to remove as much duplication of computational effort as possible (more on this shortly). In effect, period t is forced to be the only regeneration point in Subproblem P(q,t). The definition of Subproblem P(q,t) is similar to that of Florian and Klein's Capacity Constrained Sequences [2]. Subproblem P(q,t) can be modeled as an integer linear program as in Figure 5-1.

Subproblem P(q,t) is modeled similarly to Problem P^T as described in Figure 2-1 except that: 1) the only stockout variable is the one in the last period, S_t, and 2) a penalty variable Π_τ is included in both the objective function and in Constraint 14. Let M be a very large number so that if a solution has $I_\tau = 0$ for any $\tau = q, \ldots, t-1$, Π_τ is equal to $2 \cdot M$ and the objective function attains a very high value for that solution.

$$\text{Min } \Sigma_{\tau=q}^{t}(\sigma\delta_\tau + pX_\tau + hI_\tau + \Pi_\tau) + sS_t \tag{10}$$

Such that

$$I_{\tau-1} + X_\tau - I_\tau = d_\tau \quad \text{for } \tau=q,\ldots,t-1 \tag{11}$$

$$I_{t-1} + X_t - I_t + S_t = d_t \tag{12}$$

$$X_\tau - XMAX_\tau \delta_\tau \le 0 \text{ for } \tau=q,\ldots,t \tag{13}$$

$$\Pi_\tau - M\bullet[\,|I_\tau - 1| - (I_\tau - 1)] = 0 \text{ for } \tau=q,\ldots,t-1 \tag{14}$$

$$\Pi_t = 0 \tag{15}$$

$$I_{q-1} = I_t = 0 \tag{16}$$

$$X_\tau, I_\tau \text{ integer} \quad \text{for } \tau=q,\ldots,t \tag{17}$$

$$S_t \text{ integer} \tag{18}$$

$$\delta_\tau \text{ binary} \quad \text{for } \tau=q,\ldots,t \tag{19}$$

Figure 5-1: Subproblem P(q,t)

Definition 20: A solution to Subproblem P(q,t) with $\Pi_\tau = 0$ for $\tau = q, \ldots, t-1$ is said to be *admissible*. A solution to subproblem P(q,t) with at least one $\Pi_\tau > 0$ for $\tau = q, \ldots, t-1$ is said to be *inadmissible*.

In order to model the absolute value condition in Constraint 14, two new sets of constraints are added:

$$I_\tau - (y^+_\tau - y^-_\tau) = 1 \quad \text{for } \tau = q,\ldots,t-1 \tag{20}$$

$$y^+_\tau,\ y^-_\tau \geq 0 \quad \text{for } \tau = q,\ldots,t-1 \tag{21}$$

We then replace "$|I_\tau - 1|$" in Constraint 14 by "$(y^+_\tau + y^-_\tau)$".

The nature of the P(q,t) subproblems allows statement of several necessary conditions given in the previous section. In particular:

1. There will not be any stockouts in periods q through $t - 1$ since all the inventory arcs are basic (Lemma 5).

2. In periods $q + 1$ through t, production is either at zero or at the upper bound for that period (Theorem 9).

3. If $S_t > 0$, then $X_q = XMAX_q$ (Theorem 10 plus the fact that X_q must be positive in order for inventory arc (q,q+1) to be basic and have positive flow).

4. If $d_q \geq XMAX_q$, then there is no solution to the P(q,t) subproblem, since inventory arc (q,q+1) must be basic.

5. If the last positive production occurs in period $q' < t$, then $\Sigma_{j=q'}^{t-1} d_j < X_{q'} \leq \Sigma_{j=q'}^{t} d_j$ (Theorem 11 plus the fact that inventory arc (t-1,t) must be basic with positive flow).

6. In each period j, $j = q, \ldots, t-1$, $I_j \le \Sigma^{t}_{i=j+1} d_i$ (or else $I_t > 0$).

The value V(t) satisfies the following functional equation:

$$V(t) = \min_{r \le q \le t} \{ V(q-1) + C(q,t) \} \tag{22}$$
$$\text{where } r = \max \{ t - k, 1 \} \text{ and } V(0) = 0.$$

We now illustrate how this functional equation solves Problem P^T for a small nonnumerical example.

Assume that $k = 2$ and $T = 5$. We begin by solving P^1 with optimal value V(1). From the functional equation, $r = 1$ and

$$V(1) = V(0) + C(1,1).$$

We then proceed to solve the 2-period problem P^2. Again, $r = 1$ and

$$V(2) = \min \{ V(1) + C(2,2), V(0) + C(1,2) \}.$$

Notice that the value of V(1) is already known and that C(2,2) is the optimal solution to the 1-period subproblem P(2,2). The value V(1) + C(2,2) immediately gives an upper bound on the optimal solution to P^2.

The 3-period problem is solved next. The parameter r is still 1 and

$$V(3) = \min \{ V(2) + C(3,3), V(1) + C(2,3), V(0) + C(1,3) \}.$$

The number of possible completions to V(3) is 3, which is equal to $k + 1$. This will always be the maximum number of completions to be evaluated.

When V(4) is computed, r is now equal to 2 and

$$V(4) = \min \{ V(3) + C(4,4), V(2) + C(3,4), V(1) + C(2,4) \}.$$

We do not evaluate V(0) + C(1,4) since C(1,4) would be the optimal value to the subproblem with positive ending inventory in periods 1, 2, and 3. It is known that this will never be part of an optimal solution to P^4 since the maximum number of consecutive periods with positive ending inventory is $\bar{k}$ which is 2.

Finally, V(5) is determined with r = 3 so that

$$V(5) = \min \{ V(4) + C(5,5),\ V(3) + C(4,5),\ V(2) + C(3,5) \}.$$

For large values of k, the computation of C(q,t) can be quite formidable since it is possible that $(XMAX_q - XMIN) * 2^{t-q}$ different production sequences would have to be enumerated. However, in the next section, a case is treated in which the P(q,t) subproblems can be solved in polynomial time.

One last comment concerns the decision horizon techniques and computational savings. Assume that when we evaluated V(4), we obtained a decision horizon by Theorem 15 which covered periods 1 and 2 (i.e., t' = 2). This means that when we evaluate V(5), we need not include periods 1 or 2. Notice in the above that when we evaluate V(5), we don't include these periods anyway, since r = 3. The point is that even if the longest possible decision horizon is detected by Theorem 15, the same amount of effort must go into evaluating V(5). The value of the horizon technique given by Theorem 15, therefore, is not that it saves computational effort, but that it provides a stopping rule. That is, if we only need to determine the production decision for period 1, we can stop the forward algorithm after evaluating V(4) and we need not evaluate V(5).

This is not the case for the decision horizons found by Theorems 16 and 17. If period t = 4 were found to be a decision (and forecast) horizon by these methods, we would evaluate V(5) as a single period problem consisting only of period 5.

For these reasons, the forward algorithms in this paper searched for decision horizons via Theorems 16 and 17 first. If this was not found to be fruitful, Theorem 15 was used to search for decision horizons.

5.1. Computing C(q,t) When Production Capacity is Constant

When production capacity is constant from period to period, then $XMAX_j = XMAX$ for all j. We continue to assume that $t - q \leq k$ or else Subproblem P(q,t) will not be part of the optimal solution. The number of positive productions in periods q through t can immediately be determined as follows:[1]

Definition 21: Let $\beta = \Sigma_{j=q}^{t} d_j$ (mod XMAX).

Definition 22: Let $N = (\Sigma_{j=q}^{t} d_j - \beta)/XMAX$.

The value of β is the remainder of the total demand in periods q through t that cannot be satisfied by production at XMAX each time period. Periods in which production is at XMAX are called *upper bounded production periods*. The number of upper bounded production periods in periods q through t will be equal to N. Again recall from the previous section that if $d_q \geq XMAX$, the P(q,t) subproblem does not have an admissible solution.

Theorem 23: If $\beta \geq \max\{XMIN, d_q+1\}$, then $X_q = \beta$, otherwise $S_t = \beta$.

Proof: The only positive production at a quantity less than the upper bound XMAX can occur in period q (Theorem 9 plus the fact that

[1] Both of these definitions assume that $\sum_{j=q}^{t} d_j \leq XMAX * (t-q+1)$. If this assumption is false, then set $X_j = XMAX$ for $j = q, \ldots, t$, set S_t equal to the remaining unfilled demand, and check that each inventory arc (q,q+1), . . . , (t+1,t) has positive flow.

each inventory arc is basic). It is necessary to produce β units in period q in order to avoid any stockouts in period t, as long as $\beta \geq$ XMIN (Theorem 11).

If $\beta < d_q + 1$, however, inventory arc (q,q+1) will be nonbasic and stockout arc (S,q) may be basic, violating the definition of the P(q,t) subproblem. Therefore $X_q = \beta$ only if $\beta > d_q$. If X_q is not set to β, the first upper bounded production will be in period q and the β units will not be produced at all, but will be stocked out, these stockouts coming in period t. ∎

The periods in which to place the N upper bounded productions must now be determined. Their determination is made possible by the fact that the production cost and holding cost functions are constant over time.

The procedure for finding them is to begin in period q, determining X_q and I_q. Once this is done, S_t can immediately be determined (see above discussion). We then move to period t and work back to period q + 1, placing each XMAX production as late as possible while still keeping every inventory arc basic. The procedure is illustrated with a numerical example.

We wish to solve P(1,4) where $\sigma = 5$, $p = 1$, $h = .3$, $s = 2$, XMAX = 8, $d_1 = 3$, $d_2 = 2$, $d_3 = 6$, and $d_4 = 6$.

Period 1 $\beta = 17 \pmod 8 = 1$ and $N = 2$. Since $\beta < 5 =$ XMIN, $X_1 = 8$, $I_1 = 5$, and $S_1 = 0$.

Period 4 $I_4 = 0$ and $S_4 = \beta = 1$. Recalculate d_4 to be $d_4 - S_4 + I_4 = 5$ (the total inflow of production plus inventory into period 4). Since $d_4 \le$ XMAX, set $X_4 = 0$.

Period 3 $I_3 = d_4 - X_4 = 5$ and $S_3 = 0$. Now d_3 is redetermined to be $d_3 - S_3 + I_3 = 11$. Because $d_3 >$ XMAX, set $X_3 =$ XMAX $= 8$.

Period 2 $I_2 = d_3 - X_3 = 3$ and $S_2 = 0$. The demand d_2 will now be replaced by $d_2 - S_2 + I_2 = 5$. Set $X_2 = 0$ because $d_2 \le$ XMAX.

Finish Since $I_1 = d_2 - X_2$, we have found the optimal P(1,4) subproblem solution.

The solution to the example follows:

j	1	2	3	4
d_j	3	2	6	6
X_j	8	0	8	0
I_j	5	3	5	0
S_j	0	0	0	1

C(1,4) = 31.9

Theorem 24: The procedure described above will find the optimal solution to the P(q,t) subproblems when production capacity is constant.

Proof: If $d_q \ge$ XMAX, then Subproblem P(q,t) will have no admissible solution since inventory arc (q,q+1) will be nonbasic with zero flow in any such subproblem.

We first compute β, the number of units not produced in upper bounded production periods, and N, the number of upper bounded production periods. The reasoning behind the placement of the production (or stockout) of the β units has been previously presented.

Working back from period t, production in a period j is set to XMAX only if d_j > XMAX where d_j is recalculated to be the unfilled demand from period j through t. This guarantees that inventory arc (j−1,j) will be basic with positive flow so that the structure of the P(q,t) subproblem is preserved while placing each production as late as possible, thus keeping inventory levels as low as possible and minimizing holding costs. (Once N and β have been determined, total setup costs, total variable production costs, and total stockout costs can be computed. The only remaining objective is to minimize total holding costs.)

The final step is to make sure that $I_q = d_{q+1} - X_{q+1}$. If this is not the case, fewer than the required N upper bounded productions have occurred. This could happen when a recomputed d_j is equal to XMAX and there is not enough production capacity in periods q through j − 1 to satisfy the demands occurring in these periods plus the recomputed d_j. Another instance where this can happen is when the total demand in periods q through t' (t' < t) is greater than total production capacity in periods q through t'. In these situations, the P(q,t) subproblem has no admissible solution. ▪

A single pass is made in solving the P(q,t) subproblems by the above method. Therefore the computational effort is linear in the number of periods for each of these subproblems. Since C(q,t) is not computed for t − q > k, the longest subproblem solved will be k + 1 periods long.

Two types of problems are considered, the first where $T \leq k + 1$, the second where $T > k + 1$.

For a problem P^T with $T \leq k + 1$, we solve for V(1), V(2), . . . , up to V(T). Given the solution to P^{t-1}, $\Sigma^t_{q=1}(q) = [t(t+1)/2]$ additional computations are performed to solve P^t. Therefore, the amount of effort in solving for V(T) will be $\Sigma^T_{t=1}[t(t+1)]/2 = (2T^3+6T^2+4T)/12$ or $O(T^3)$.

For a problem P^T with $T > k + 1$, we again solve for V(1), V(2), . . . , up to V(T), but for P^t with $t > k + 1$, we only solve $k + 1$ of the P(q,t) subproblems. Therefore, we must distinguish between the effort involved in solving P^1 through P^{k+1} and the effort involved in solving P^{k+2} through P^T. The effort involved in solving P^1 through P^{k+1} will be $\Sigma^{k+1}_{t=1}[t(t+1)]/2 = (2k^3+12k^2+22k+12)/12$. The effort involved in solving P^{k+2} through P^T will be $[(k+1)(k+2)/2] * [T-(k+1)]$. The total effort is then $O(Tk^2)$. This is a very interesting result. If $k + 1 \geq T$, the solution algorithm grows at a cubic rate. Otherwise, after a period of cubic growth, the solution algorithm becomes asymptotically linear in T.

5.2. Computing C(q,t) When Production Capacity is not Constant

When the production capacities are not constant from period to period, the previous method can not be used. Instead, the P(q,t) subproblems are solved via a branch-and-bound technique. There are up to $(XMAX_q - XMIN) * 2^{t-q}$ different production sequences that may solve Subproblem P(q,t), each of which must be considered explicitly or bounded away from consideration.

Several procedures are employed to save computations so that this maximum number of production sequences need not be enumerated. First, if $d_q \geq XMAX_q$, then

P(q,t) will not have an admissible solution.

Second, only those production sequences for which X_q = $XMAX_q$ need be considered. If a production sequence for this reduced search has $I_j > 0$ for j = q, . . . , t-1 and $I_t = 0$, then it is an admissible solution to Subproblem P(q,t). If instead a production sequence is found for this reduced search with $I_j > 0$ for j = q, . . . , t, we check to see: 1) if every ending inventory I_j, for j = q, . . . , t can be reduced by I_t units without any of these inventories becoming zero (except I_t itself) or negative, and 2) if X_q can be reduced by I_t units and still have X_q greater than or equal to XMIN. If both of these conditions are satisfied, the solution is modified by reducing X_q by I_t units and I_j for j = q, . . . ,t by I_t units. The result is an admissible solution to the P(q,t) subproblem with X_q < $XMAX_q$. The lowest cost admissible solution will be optimal.

Theorem 25: A reduced search on the part of the branch-and-bound tree with X_q = $XMAX_q$ along with the solution modification procedure just described will generate the optimal solution to a P(q,t) subproblem.

Proof: Clearly the optimal solution to Subproblem P(q,t) will be generated if the solution has X_q = $XMAX_q$.

Assume there exists an optimal solution V^* to Subproblem P(q,t) with first production X^*_q < $XMAX_q$. It follows that this solution will have no stockouts (Theorem 10), production in periods q + 1 through t at either zero or the upper bound (Theorem 9), and $I^*_t = 0$. This solution will not be in the reduced branch-and-bound tree because the first production quantity is not equal to the upper bound.

In the reduced branch-and-bound tree however, there will exist a solution V' with $X'_j = XMAX_j$, all other production quantities equal to those in V^*, no stockouts, and ending inventory levels that are $XMAX_q - X^*_j$ higher than those in solution V^*.

If we let $\Delta = XMAX_q - X^*_j$, solution V^* can be obtained from solution V' by reducing the first production quantity X'_j and every ending inventory by Δ units. ▪

The definition of Subproblem P(q,t) can be used to remove from further consideration any solution in which $I_j \leq 0$ for $j < t$. Solutions whose ending inventories are too high can also be removed. It is known that in an optimal solution, the ending inventory in a period j, called I_j, will be no greater than the sum of the demands in periods j + 1 through t, called $IMAX_j$ or else I_t will be positive. However, these ending inventories may eventually be reduced by I_t units if I_t is positive in order to construct an admissible solution. The maximum reduction that can occur to these inventories and still have an admissible solution is $XMAX_q - XMIN$ units, or else X_q will be less than XMIN. Furthermore, for a period $j \leq t$, if $I_{t^*} = \min_{q \leq t' \leq j} \{I_{t'}\}$, then the maximum reduction that can occur to the inventory in period j will be the $\min\{I_{t^*}-1, XMAX_q-XMIN\}$. If we reduce all inventories between periods q and t by I_{t^*} or more units, the resulting inventory level in period t^* will be zero or negative, thus violating the definition of Subproblem P(q,t). Therefore, if $I_j - \min\{I_{t^*}-1, XMAX_q-XMIN\} > IMAX_j$ for some j, the computation of the solution to this subproblem can be terminated because it is certain to be bounded.

The calculation of solutions to some subproblems can also be eliminated by using cost bounds. As V(t) is computed using different values of q (see Equation 22), an upper bound on each new P(q,t) subproblem is made available. For example, if in the solution of V(5), it is found that V(4) + C(5,5) = 29.7, then if V(3) = 26, for C(4,5) to be a candidate for part of the optimal solution to V(5), C(4,5) must be less than the current best value of V(5) minus V(3), or 3.7. In determining C(4,5), the calculation of any solution can be terminated that will have a final value greater than or equal to this upper bound. Once again, however, the value of the solution may be overstated if it will be reduced due to an $I_t > 0$. The maximum reduction will be by ($XMAX_q$ - XMIN) units. The actual reduction will lower the total production cost in period q and the inventory holding costs in every period q through t and will be no greater than ($XMAX_q$ - XMIN) * (p + h*[t-q+1]). If after making this cost adjustment, the value of the solution is still greater than or equal to the upper bound, its calculation is terminated due to cost bounds.

The branch-and-bound technique is illustrated with an example. Assume we wish to determine C(1,5) with σ = 5, p = 1, h = .3, and s = 2.7, so that k = 5 and XMIN = 3. The demands, capacities, and maximum inventory levels are given in the following table:

j	1	2	3	4	5
d_j	6	9	4	4	6
$XMAX_j$	9	8	7	6	7
$IMAX_j$	23	14	10	6	0

Besides the maximum inventory levels, it is also known that $7 \le X_1 \le 9$, $X_j \in \{0, XMAX_j\}$ for j = 2, 3, 4, and that $X_5 = 0$ (since $XMAX_5 \ge d_5$).

The branch-and-bound tree for this problem is constructed in Figure 5-2. The values on each arc are the production value, ending inventory, number of stockouts, and cost to that point in the tree, respectively. Several branches of the tree are bounded due to infeasibility (inventory going to zero or stockouts occurring before period t.

```
PERIOD(LEVEL)              |
                           |
  1                        | (9/3/0/14.9)
                          / \
                         /   \
  2          (8/2/0/28.5) /     \ (0/0/6/+oo)
                        /\      ===
                       /  \
  3          (7/5/0/42) /    \ (0/0/2/+oo)
                      /\     ===
                     /  \
  4     (6/7/0/55.1) /    \ (0/1/0/42.3)
                     |    |
                     |    |
  5  (0/1/0/55.4) "a" |    | (0/0/5/55.8) "b"
```

Figure 5-2: Branch-and-bound solution of a P(q,t) subproblem

Two candidate solutions are left at the end of the search, solution "a" with the production sequence {9,8,7,6,0} and solution "b" with production sequence {9,8,7,0,0}.

Solution "a" has an extra unit in inventory however and thus does not satisfy the form of Subproblem P(q,t). This solution is modified by reducing each ending inventory and the first period's production by 1 unit with cost savings of 2.5. A new solution is found with production sequence {8,8,7,6,0}. This solution has a cost of 55.4 − 2.5 = 52.9, which is optimal.

The optimal solution, derived from solution "a" in Figure 5-2, is given in the following table:

j	1	2	3	4	5
d_j	6	9	4	4	6
$xmax_j$	9	8	7	6	7
X_j	8	8	7	6	0
I_j	2	1	4	6	0
S_j	0	0	0	0	0

$C(1,5) = 52.9$

Let us now look at the computational complexity question for the variable upper bound capacitated lot size model. In solving $P(q,t)$, the branch-and-bound tree will have $2^{t-q+1}-1$ branches, thus requiring at most that many computations. Again, the value of $C(q,t)$ is not computed for $t - q > k$.

Two types of problems are considered, the first where $T \leq k + 1$, the second where $T > k + 1$.

For a problem P^T with $T \leq k + 1$, we solve for $V(1)$, $V(2)$, . . . , up to $V(T)$. Given the solution to P^{t-1}, $\Sigma_{q=1}^{t} (2^{t-q+1}-1) = 2^{t+1} - t - 2$ additional computations are performed to solve P^t. Therefore, the amount of effort in solving for $V(T)$ will be $\Sigma_{t=1}^{T} (2^{t+1} - t - 2) = 2^{T+2} - 4 - [T(T+1)]/2 - 2T$ or $O(2^T)$.

For a problem P^T with $T > k + 1$, we again solve for $V(1)$, $V(2)$, . . . , up to $V(T)$, but for P^t with $t > k + 1$, only $k + 1$ of the $P(q,t)$ subproblems are solved. Therefore, we must distinguish between the effort involved in solving P^1 through P^{k+1} and the effort involved in solving P^{k+2} through P^T. The effort involved in solving P^1 through P^{k+1} will be $\Sigma_{t=1}^{k+1} (2^{t+1} - t - 2) = 2^{k+3} - 4 - [(k+1)(k+2)]/2 - 2(k+1)$. The effort involved in solving P^{k+2} through P^T will be $[2^{k+2} - k - 3]*[T-(k+1)]$. The total effort is $O(T2^k)$. In a fashion similar to the constant production capacity case, if $k + 1$

$\geq$ T, the solution algorithm grows at an exponential rate. Otherwise, after a period of exponential growth, the solution algorithm becomes asymptotically linear in T.

6. COMPUTATIONAL RESULTS

It has already been shown that if $k + 1 \geq T$, then the worst case solution effort for the capacitated lot size model is $O(T^3)$ when production capacity is constant and $O(2^T)$ when production capacity varies from period to period. The average performance of most algorithms is usually much better than their worst case bounds and this is true for the forward algorithms just described. In this section the details of how the average computational performances were determined for both the constant production capacity and variable production capacity algorithms are given.

6.1. Variable Production Capacity Algorithm Performance

A group of 48 test problems was created to study the performance of the variable production capacity algorithm. Each test problem was 50 periods in length. The values of the variable production cost per unit (p) and the stockout cost per lost sale (s) were kept fixed at \$1 and \$5, respectively. The setup cost (σ) was varied between \$40 and \$160. This generated minimum positive production quantities (XMINs) of 10 units and 40 units, respectively. By altering the holding cost per unit per period (h), different values of the maximum holding period (k) could be generated. Holding costs of 0.26 ($k = 15$), 0.20 ($k = 20$), and 0.16 ($k = 25$) were used.

Demands were generated randomly from two uniform distributions, U(20,60) and U(0,80). (We use the convention that the uniform distribution U(x,y) has minimum value x and maximum value y.) Cyclical demand patterns (1 cycle = 24 periods) were also constructed using the same ranges as the random demand patterns. Production upper bounds were generated randomly from two uniform distributions as well, U(60,80) and U(50,90).

Each test problem was actually solved three times. The first run (RUN I) used both the decision horizon techniques and Subproblem P(q,t) fathoming techniques (from the previous section). The second run (RUN II) used only the fathoming techniques. The third run (RUN III) used neither the decision horizon techniques nor the fathoming techniques.

The test problems were solved on an IBM 4341 computer using a forward algorithm written in VS FORTRAN at Clarkson University.

It was found that the maximum holding period (k) was the single most factor affecting the solution times. Considering only RUN I, for k = 15 the average solution time was 20.6 seconds (CPU), for k = 20 the average solution time was 251.1 seconds, and for k = 25 the average solution time was 1882.4 seconds.

Generally, the test problems with XMIN = 40 took longer to solve than the problems with XMIN = 10. This occurred only in RUN I and RUN II, which used the subproblem fathoming techniques. The differences in solution times ranged from 10% for k = 15 to 35% for k = 25. This is explained by the fact that the higher XMIN value is caused by the larger setup cost (σ). The larger setup cost results in fewer setups and higher inventory levels, thus lessening the effect of fathoming by inventory lower bounds. On the other hand, this should have increased the likelyhood of fathoming by inventory upper bounds. The evidence, however, does not bear this out.

As expected, the decision horizon techniques did not reduce the solution times of the test problems, except in those cases where horizons were found by Theorems 16 and 17. The horizon checking procedures actually increased the solution times by an average of 2.6%.

The Subproblem P(q,t) fathoming techniques had a significant effect on the solution times. Although RUN III was not carried out for k = 25, for k = 15 and k = 20, the average solution time was 81.4% less when using the fathoming techniques (RUN II).

		Average Solution Times in CPU Seconds		
k	XMIN	RUN I	RUN II	RUN III
15	10	19.489	19.143	89.981
20	10	220.476	216.192	1632.869
25	10	1598.864	1561.156	---
15	40	21.646	20.788	90.645
20	40	281.667	275.014	1629.617
25	40	2165.842	2104.239	---

Table 6-1: Computational Results: Variable Production Capacities

Finally, the solution times for RUN II for k = 25 were regressed in order to estimate the average growth rate. The exponential curve of best fit was $Y(T) = 3.73*(1.41)^T$ with $r^2 = 76.8\%$, where T is the length of the problem and Y(T) is the expected time needed to solve a problem of length T, measured in CPU milliseconds. Although this estimate is better than the worst case bound of $O(2^T)$, it is still hopelessly exponential.

6.2. Constant Production Capacity Algorithm Performance

In a fashion similar to the above, a group of 24 test problems was created to study the performance of the constant production capacity algorithm. Half as many test problems than were used previously were generated since the production bound was taken to be the largest possible period demand. All other factors, including 3 runs for each test problem, were kept identical to those in the previous section.

		Average Solution Times in CPU Seconds		
k	XMIN	RUN I	RUN II	RUN III
15	10	0.607	0.595	0.671
20	10	0.789	0.773	0.908
25	10	0.934	0.931	1.098
15	40	0.685	0.694	0.711
20	40	0.884	0.888	0.923
25	40	1.037	1.042	1.113

Table 6-2: Computational Results: Constant Production Capacities

Again, the maximum holding period (k) was the single most factor affecting the solution time. The solution times were, however, much lower than in the variable capacity case. Considering only RUN I, for k = 15 the average solution time was 0.65 seconds, for k = 20 the average solution time was 0.84 seconds, and for k = 25 the average solution time was 0.99 seconds.

Because of the very low solution times, any attempt to draw significant conclusions must be prefaced by the remark that a relatively large part of the variation in solution times could be caused by the varying load on the computer used. However, the low solution times did make it possible to study longer test problems. A new group of 6 test problems was created that were each 100 periods in length. A random demand pattern was used. Demands were generated from the uniform distribution

U(20,60). The production capacity (XMAX) was taken as 60. The solution results are presented in Table 6-3. Only the fathoming techniques were used in these problems (no horizon checking).

k	XMIN	Solution Times in CPU Seconds
25	10	2.431
50	10	5.839
100	10	9.667
25	40	3.044
50	40	7.305
100	40	11.628

Table 6-3: Computational Results: 100 Period Test Problems

Plots of the growth in solution time are given in Figures 6-1 for XMIN = 10 and in Figure 6-2 for XMIN = 20. Notice that in each case, the solution times are near equal for the first k + 1 periods, after which the growth rates become linear.

A regression was performed on the two test problems with k = 100. It was found that the polynomial curve of best fit was $Y(T) = 0.29T^{2.2}$ with $r^2 = 97.5\%$, slightly better than the worst case bound of $O(T^3)$. As before, T is the length of the problem and Y(T) is the expected time needed to solve a problem of length T, measured in CPU milliseconds.

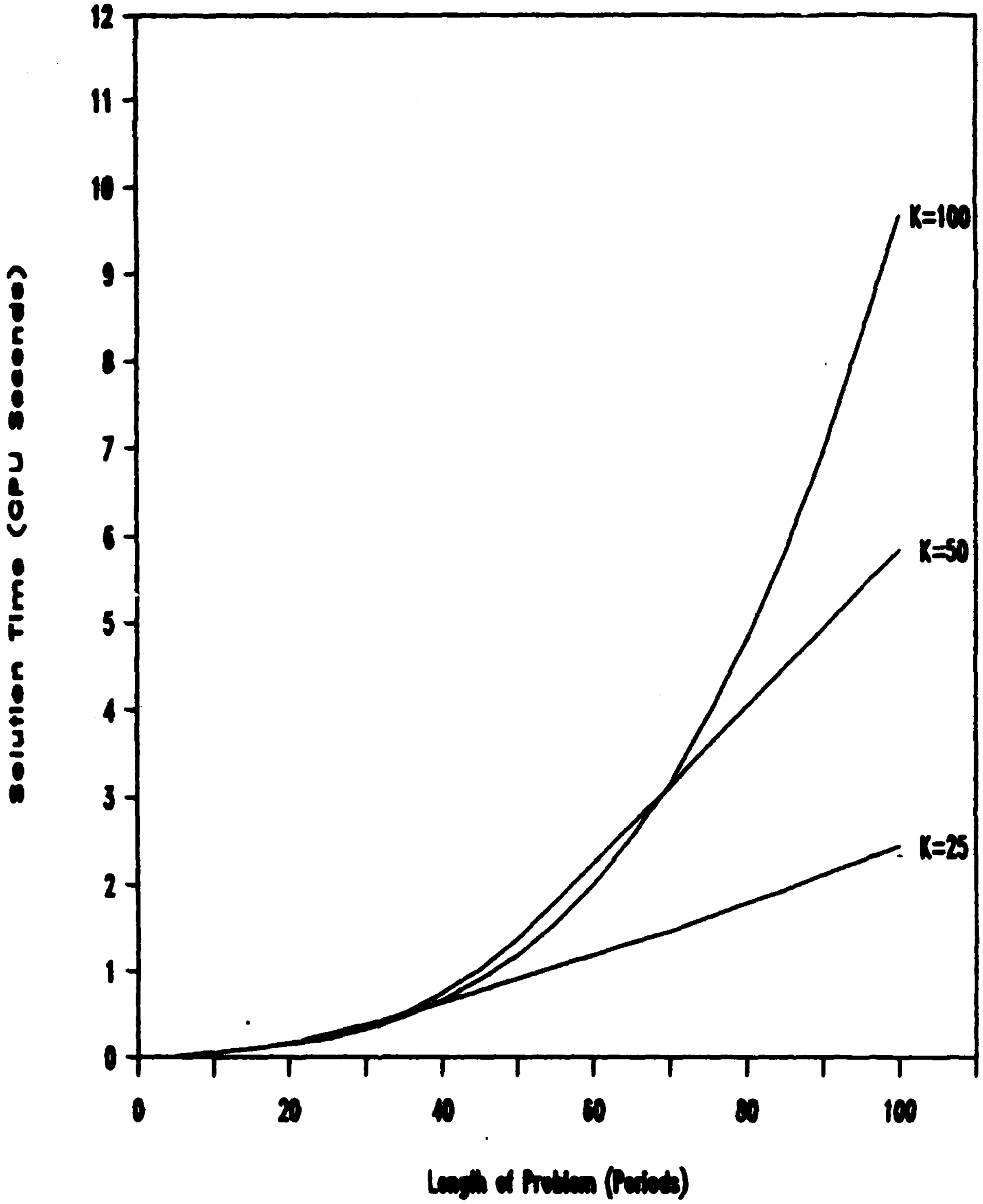

Figure 6-1: Solution Time Growth: XMIN = 10

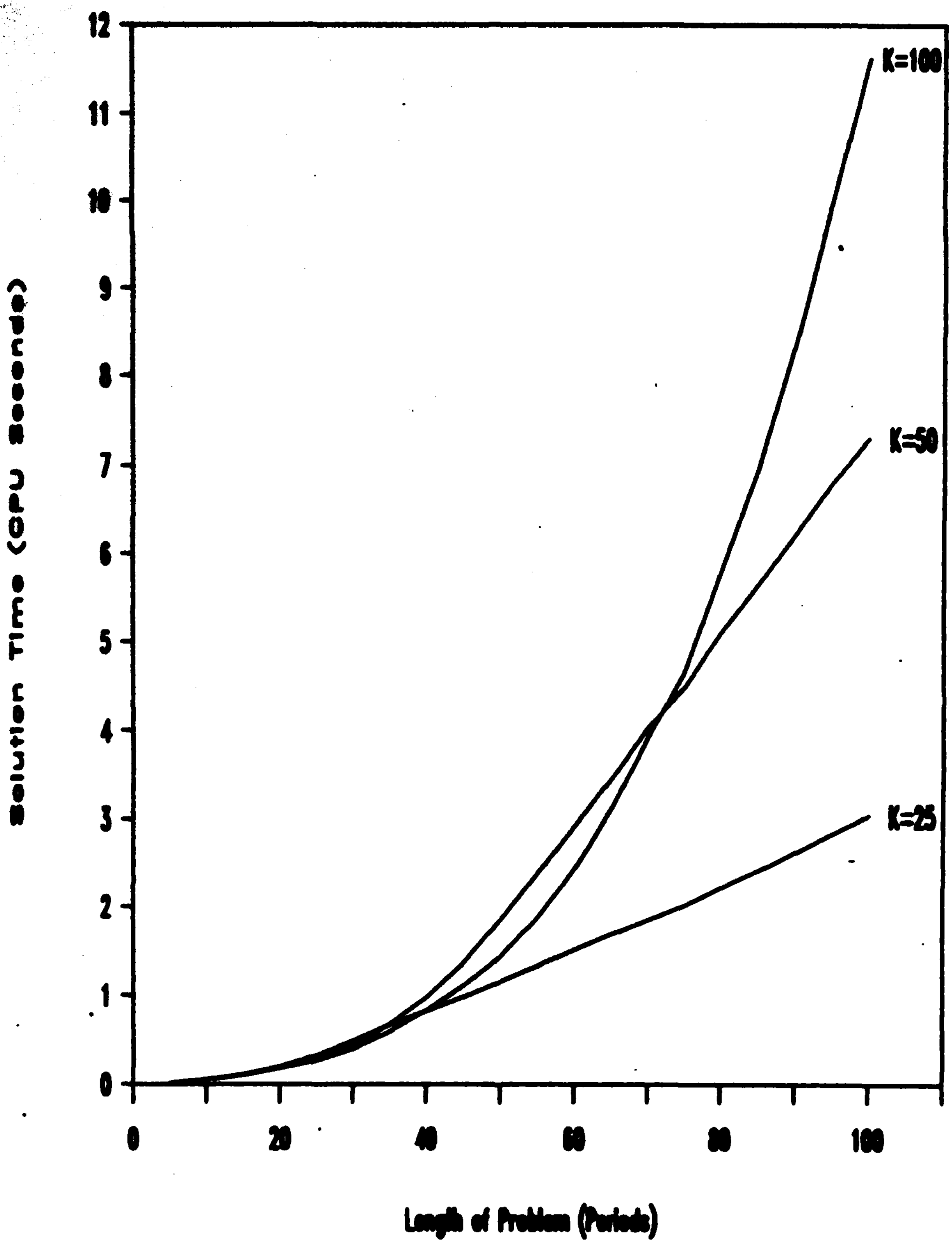

Figure 6-2: Solution Time Growth: XMIN = 40

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