

David W. Taylor Naval Ship Research and Development Center
Bethesda, MD 20884-5000

AD-A181 153

DTNSRDC-87/007 April 1987
Ship Performance Department
Research and Development Report

**Statistics of Nonlinear Response to
Short Crested Nonlinear Seas:
The Quadratic Case-Response**

by
John F. Dalzell

DTNSRDC-87/007 Statistics of Nonlinear Response to Short Crested
Nonlinear Seas: The Quadratic Case

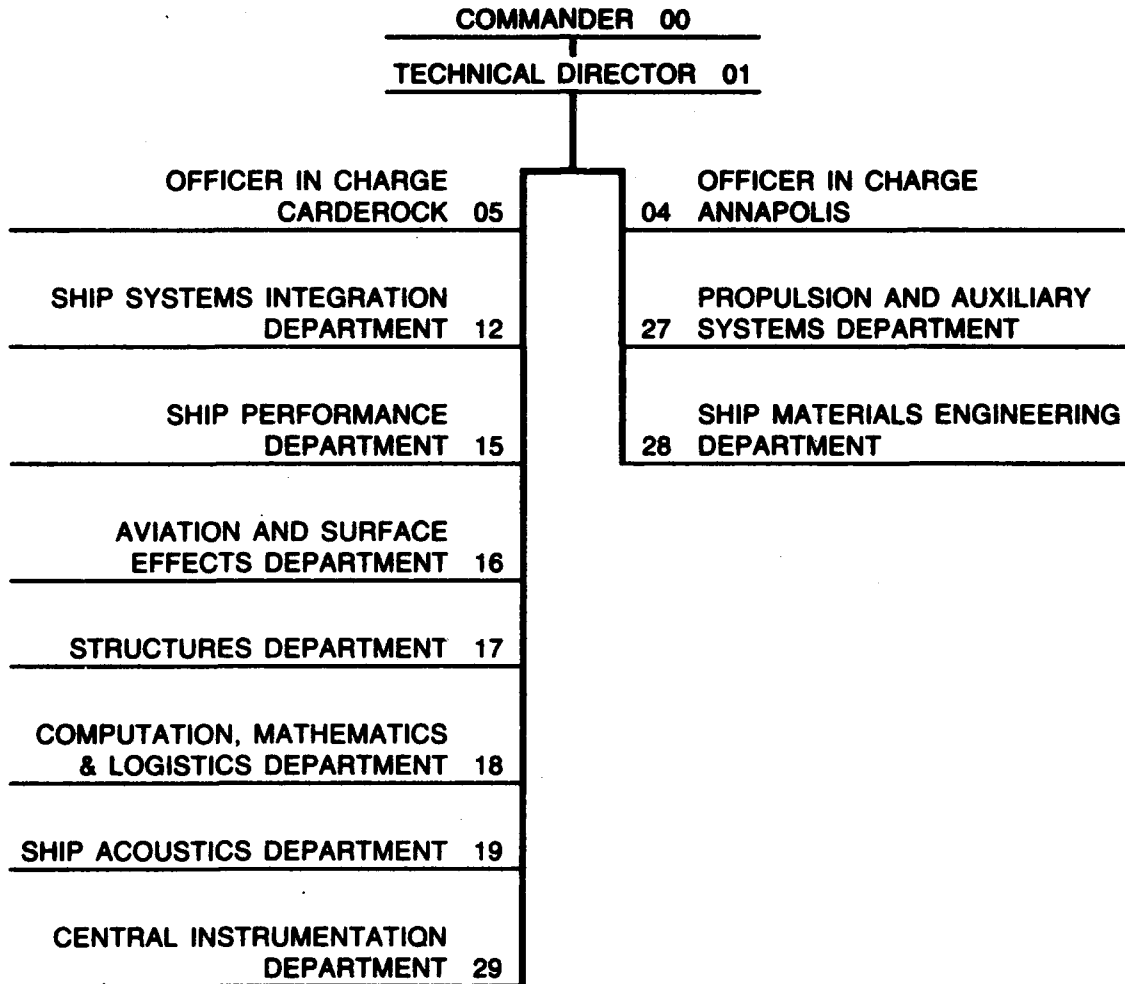
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SECURITY CLASSIFICATION OF THIS PAGE

AD-A181 153

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION Unclassified		1b. RESTRICTIVE MARKINGS	
2a. SECURITY CLASSIFICATION AUTHORITY		3. DISTRIBUTION / AVAILABILITY OF REPORT APPROVED FOR PUBLIC RELEASE: DISTRIBUTION IS UNLIMITED.	
2b. DECLASSIFICATION / DOWNGRADING SCHEDULE			
4. PERFORMING ORGANIZATION REPORT NUMBER(S) DTNSRDC-87/007		5. MONITORING ORGANIZATION REPORT NUMBER(S)	
6a. NAME OF PERFORMING ORGANIZATION David W. Taylor Naval Ship R&D Center	6b. OFFICE SYMBOL (if applicable) Code 1561	7a. NAME OF MONITORING ORGANIZATION	
6c. ADDRESS (City, State, and ZIP Code) Bethesda, MD 20084-5000		7b. ADDRESS (City, State, and ZIP Code)	
8a. NAME OF FUNDING / SPONSORING ORGANIZATION Naval Sea Systems Command	8b. OFFICE SYMBOL (if applicable) Code 05R	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER	
8c. ADDRESS (City, State, and ZIP Code) Washington, DC 20362-5101		10. SOURCE OF FUNDING NUMBERS	
		PROGRAM ELEMENT NO 61153N	PROJECT NO SR02301
		TASK NO SR0230101	WORK UNIT ACCESSION NO DN506011
11. TITLE (Include Security Classification) STATISTICS OF NONLINEAR RESPONSE TO SHORT CRESTED NONLINEAR SEAS: THE QUADRATIC RESPONSE			
12. PERSONAL AUTHOR(S) Dalzell, John F.			
13a. TYPE OF REPORT Final	13b. TIME COVERED FROM 85/10/1 TO 86/9/30	14. DATE OF REPORT (Year, Month, Day) 1987 April	15. PAGE COUNT 43
16. SUPPLEMENTARY NOTATION			
17. COSATI CODES		18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)	
FIELD	GROUP	SUB-GROUP	
13	10	Random processes	
12	01	Short-crested seas	
		Quadratic nonlinearities	
		Seakeeping	
19. ABSTRACT (Continue on reverse if necessary and identify by block number)			
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20. DISTRIBUTION / AVAILABILITY OF ABSTRACT <input type="checkbox"/> UNCLASSIFIED/UNLIMITED <input checked="" type="checkbox"/> SAME AS RPT <input type="checkbox"/> DTIC USERS		21. ABSTRACT SECURITY CLASSIFICATION Unclassified	
22a. NAME OF RESPONSIBLE INDIVIDUAL John F. Dalzell		22b. TELEPHONE (Include Area Code) (202) 227-1692	22c. OFFICE SYMBOL Code 1561

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(Block 19 continued)

response functions required by the previous work are modified by a term which amounts to the linear ship system response to those quadratic wave field components which are produced by the wave field nonlinearities. In practice the most important statistic involving quadratic response is the expectation, or mean value. In this special case an extension is not necessary. To second order, the mean response of a linear plus quadratic ship system to a short-crested wave field is not affected by wave field nonlinearities. The present development suggests that the mean value estimator may be robust through third order.

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DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
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Distribution/	
Availability Codes	
Dist	Avail and/or Special
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NOTATION

a_j	Amplitude of discrete wave component
b_j	Complex amplitude of discrete wave component
$F_N(\bar{\kappa})$	Vector wave number variance spectrum, nominal field
$F_T(\bar{\kappa})$	Vector wave number variance spectrum, true field to second order
$f_1(t_1, x_1, y_1)$	Linear time domain kernel
$g_2(t_1, x_1, y_1, t_2, x_2, y_2)$	Wave-wave interaction kernel (time domain)
$h_1(t_1, x_1, y_1)$	Linear time domain kernel relating nominal and true wave fields
$h_2(t_1, x_1, y_1, t_2, x_2, y_2)$	Quadratic time domain kernel relating nominal and true wave fields
$k_1(t_1, x_1, y_1)$	Linear time domain kernel relating true wave field and ship response
$k_2(t_1, x_1, y_1, t_2, x_2, y_2)$	Quadratic time domain kernel relating true wave field and ship response
$Q_f(\bar{\kappa}_1, \bar{\kappa}_2)$	Quadratic wave-wave interaction function, sum frequency
$Q_T(\bar{\kappa}_1, \bar{\kappa}_2)$	Quadratic response function relating nominal and true wave fields, sum frequency
$Q_f(\bar{\kappa}_1, \sigma_1, \bar{\kappa}_2, \sigma_2)$	Quadratic response function relating linear wave field and ship response, sum frequency
$\tilde{Q}(\bar{\kappa}_1, \sigma_1, \bar{\kappa}_2, \sigma_2)$	Quadratic response function relating nominal wave field and ship response to second order, sum frequency
$R_f(\bar{\kappa}_1, \bar{\kappa}_2)$	Quadratic wave-wave interaction function, difference frequency
$R_T(\bar{\kappa}_1, \bar{\kappa}_2)$	Quadratic response function relating nominal and true wave fields, difference frequency
$R_f(\bar{\kappa}_1, \sigma_1, \bar{\kappa}_2, \sigma_2)$	Quadratic response function relating linear wave field and ship response, difference frequency
$\tilde{R}(\bar{\kappa}_1, \sigma_1, \bar{\kappa}_2, \sigma_2)$	Quadratic response function relating nominal wave field and ship response to second order, difference frequency
$T_T(\bar{\kappa})$	Linear response function relating nominal and true wave fields
$T_f(\bar{\kappa}, \sigma)$	Linear response function relating linear wave field and ship response
$\tilde{T}(\bar{\kappa}, \sigma)$	Linear response function relating nominal wave field and ship response to second order
t	Time
U	Ship velocity

NOTATION (Continued)

$W(j)$	Complex random amplitudes
X, Y	Fixed horizontal coordinate system
\bar{X}	Vector position, fixed coordinates
x, y	Horizontal coordinate system moving with the ship
\bar{x}	Vector position relative to ship
x_j, y_j	Spatial variables of integration
ϵ_j	Phase of discrete wave component
$\zeta_L(\bar{X}, t)$	Linear wave field
$\zeta_T(\bar{X}, t)$	True wave field to second order
$\zeta_N(\bar{X}, t)$	Nominal wave field
$\bar{\kappa}$	Vector wave number
κ	Scalar wave number
κ_X, κ_Y	Component wave numbers
μ	Component wave direction relative to ship track
$\xi(t)$	Ship response
ρ	Vector position difference
σ	Encounter frequency, circular
τ	Time difference
ω	Wave frequency, circular

ABSTRACT

The present work is an extension of previous work on the statistics of quadratic nonlinear ship response to short-crested seas. In this previous work the basic response model was a truncated functional series; the short-crested exciting seaway was modeled as the superposition of linear component waves. To be mathematically consistent the excitation must be allowed the same degree of nonlinearity as the system being excited. A consistent approach is desirable, even if it merely yields indirect estimates of the magnitude of the error resulting from the inconsistency.

The objectives of the present work included the development of a model for the statistics of quadratic ship system response in short-crested seas which is consistent in that quadratic nonlinearities in the wave field are taken into account. It was found that, if the statistical representation of the response to the nonlinear short-crested deep water wave field is carried out strictly through contributions which may be considered second order in wave excitation, the statistical estimating formulae for the quadratic response to short-crested seas in which quadratic nonlinearities are included are identical to those previously developed when no wave nonlinearities were included. The difference is that the sum and difference frequency response functions required by the previous work are modified by a term which amounts to the linear ship system response to those quadratic wave field components which are produced by the wave field nonlinearities. In practice the most important statistic involving quadratic response is the expectation, or mean value. In this special case an extension is not necessary. To second order, the mean response of a linear plus quadratic ship system to a short-crested wave field is not affected by wave field nonlinearities. The present development suggests that the mean value estimator may be robust through third order.

ADMINISTRATIVE INFORMATION

This work was sponsored by the General Hydromechanics Research program under Task SO0230101, Program Element 61153N, and Work Unit 1561-112, and administered by NAVSEA Code 05R.

INTRODUCTION

Since the first introduction to the marine field, [1], [2] there have been developed a number of applications of the quadratic functional model to the prediction or simulation of nonlinear response of ships and platforms to random seas. The details of the various approaches to quadratic nonlinear response vary. However, all the state-of-the-art models now in use are essentially single input models, meaning that validity is confined to the long-crested case.

The basic formulation for the short-crested case was outlined in 1966 by Hasselmann [1]. Recently this formulation was extended and embellished to an extent by the present writer [3]. This short-crested model involves the assumption that the sea is a zero-mean stationary and

homogeneous Gaussian process, which excites a nonlinear system containing quadratic level nonlinearities and thus produces a response which is neither zero mean nor Gaussian. As in the linear case [4], the fundamental difference between the long- and the short-crested sea model is that a wave field rather than a wave profile must be dealt with. The result is that the linear and quadratic response functions which define the ship dynamics become functions of vectors rather than scalars. Given a directional wave spectrum and the linear and quadratic response functions corresponding to some response of interest, the model provides estimating formulae for a variety of statistics. When the nonlinearities are negligible, the estimates reduce to those long known [4]. Relative to the corresponding formulae for the long-crested quadratic model, there are no real surprises in the short-crested estimation formulae, only the greater complication that would be expected because of the increase in the dimensions of the problem. It was found [3] that in almost all cases there is no necessity to work in the encounter domain. This result implies that the previous restriction of the long-crested case to the encounter frequency domain may be largely removed. The formula for the mean value of the response in short-crested seas is especially simple, in that it was found to be hardly more computationally demanding than estimating the variance of a purely linear system in short-crested seas.

Hasselmann [1] remarks that, since the basic model is a truncated functional series, for mathematical consistency the excitation must be allowed the same degree of nonlinearity as the system being excited, and thus the model of Dalzell [3] is inconsistent in this regard. Thus the overall objective of the present investigation was to remove the inconsistency by allowing the wave field to have quadratic nonlinearities. The detailed present objectives included clarification of how one would actually carry out a practical computation, what further assumptions are being made, and perhaps most importantly, whether the complication affects estimates of the mean nonlinear response which is the most important aspect of quadratic response in practical applications.

The inconsistency at issue involves *a priori* ideas about the magnitude of quadratic nonlinearities. In most if not all hydromechanical analyses of response the quadratic nonlinearities are "second order" effects. Similarly, quadratic nonlinearities in the wave field have long been known to be "second order" from the Stokes expansion. The question is whether "small" nonlinearities in wave field have a disproportionate influence on "small" nonlinearities of response. For engineering purposes the answer will depend on the details of the problem to a greater or lesser extent. Accordingly, a mathematically consistent approach is desirable, even if it merely yields indirect estimates of the magnitude of the error resulting from the inconsistency.

Hasselmann [1] suggested an approach to the consistency problem which is grounded in the Stokes expansion, dates at least to Tick [5], and was used by Hasselmann et al. [6] in the interpretation of wave bispectra. More recently, the approach has been used for the long-crested case by Hineno [7]. This approach centers on the assumption that there exists a stationary and homogeneous zero mean Gaussian and linear "wave field" which generates the true wave field, correct to second order, through quadratic wave-wave interactions. The true wave field in turn then excites the ship system. In the linear context the approach is conceptually equivalent to the often used model in which the response is visualized as the result of cascading white noise through a wave spectral filter and exciting the ship system with the result.

QUADRATIC WAVE-WAVE INTERACTIONS

The solution to the quadratic wave-wave interaction problem via the Stokes expansion has been available for some time, having been used by Tick [5], Phillips [8], Longuet-Higgins [9], and Hasselmann [6], [10] among others. Appendix A summarizes the results in notation close to that used by Longuet-Higgins.

For present purposes the important aspect of the hydrodynamic analysis is that the corrections to an assumed first order wave field due to second order wave-wave interactions assume the form of the quadratic part of the response of the linear plus quadratic functional model. To be specific, $\zeta_L(\bar{X}, t)$ will denote a deterministic wave field composed of N superimposed small amplitude, first order progressive gravity waves, where \bar{X} is a vector position relative to an X-Y coordinate system fixed in the mean water plane, and t is time. This same representation of the wave field was used previously [3] and may be written

$$\zeta_L(\bar{X}, t) = \sum_j^N \text{Re} \{ b_j \text{Exp}[i\bar{\kappa}_j \cdot \bar{X} - i\omega_j t] \} \quad (1)$$

where $b_j = a_j \text{Exp}[i\varepsilon_j]$.

In this representation a_j and ε_j are the amplitudes and phases of the discrete components of the small amplitude waves, $\bar{\kappa}_j$ is the corresponding vector wave number, and ω_j denotes wave frequency. The wave frequency, ω_j , is assumed to be connected with $|\bar{\kappa}_j|$ via a dispersion relation which will be taken to be the deep water relationship for present purposes.

To second order, wave-wave interactions will distort the assumed field of Eq. 1, and to this order the corrected or "true" wave field may be written:

$$\begin{aligned} \zeta_T(\bar{X}, t) = & \sum_j^N \text{Re} \{ b_j \text{Exp}[i\bar{\kappa}_j \cdot \bar{X} - i\omega_j t] \} \\ & + \frac{1}{2} \sum_j^N \sum_k^N \text{Re} \{ b_j b_k Q_\zeta(\bar{\kappa}_j, \bar{\kappa}_k) \\ & \quad \text{Exp}[i(\bar{\kappa}_j + \bar{\kappa}_k) \cdot \bar{X} - i(\omega_j + \omega_k)t] \} \\ & + \frac{1}{2} \sum_j^N \sum_k^N \text{Re} \{ b_j b_k^* R_\zeta(\bar{\kappa}_j, \bar{\kappa}_k) \\ & \quad \text{Exp}[i(\bar{\kappa}_j - \bar{\kappa}_k) \cdot \bar{X} - i(\omega_j - \omega_k)t] \} \end{aligned} \quad (2)$$

The detail of this transition is contained in Appendix A, and explicit formulae for the functions $Q_\zeta(\bar{\kappa}_j, \bar{\kappa}_k)$, and $R_\zeta(\bar{\kappa}_j, \bar{\kappa}_k)$ are given there for the deep water case. Except that they are purely real, these two functions have the same symmetries as the quadratic frequency response functions previously defined [3], which relate the quadratic sum and difference frequency response of a ship system to the amplitudes of component small amplitude waves in the excitation. In this case the functions relate the components of the true field, which correspond to sums and differences of vector wave numbers, to the interactions of the constituent small amplitude waves.

The analogy to the ship system response case would be much closer if the first term of Eq. 2 were different from Eq. 1 by a factor of some function of wave number inside the

summation. Such a factor would correspond to a linear frequency response function. In this case the hydrodynamic theory shows that there is no linear transformation of the small amplitude wave field, and thus the analogy must be completed by taking the linear frequency response function to be unity.

Although it has not been done explicitly in Appendix A, it appears that the potential of the true wave field may also be expressed in a form similar to that of Eq. 2. This result tends to justify the idea that the properties of the true wave field to second order may be represented as the response of a physical linear plus quadratic system to a fictive "linear" small amplitude field. It also strongly suggests that the theoretical response of the ship system to the true field may be approached as the result of one linear plus quadratic system (the ship) acting upon another (the wave-wave interactions).

REPRESENTATION OF THE WAVE FIELD

The hydrodynamic arguments suggest that the true wave field may be taken to be a functional of a "linear", small amplitude field, at least to second order. In accordance with the previous work involving system response [3], if the linear field is continuous and differentiable and vanishes outside some finite region in the space of all admissible linear fields, or if the field is square integrable, the true wave field may be represented as a functional Taylor expansion in the neighborhood of zero wave field. With further general assumptions noted in Appendix B, the Taylor expansion through the second order term may be written as follows:

$$\begin{aligned} \zeta_T(t, X, Y) = & \zeta_L(t, X, Y) \\ & + \int \dots \int g_2(t_1, x_1, y_1, t_2, x_2, y_2) \times \\ & \zeta_L(t-t_1, X-x_1, Y-y_1) \zeta_L(t-t_2, X-x_2, Y-y_2) \times \\ & dt_1 dx_1 dy_1 dt_2 dx_2 dy_2 \end{aligned} \quad (3)$$

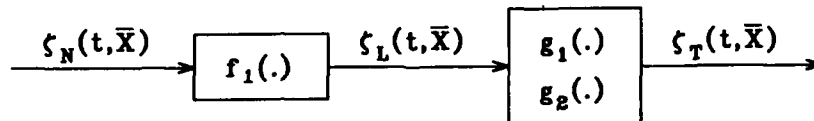
where the integrals are infinite. The kernel within the integral in Eq. 3 is a function of time and space differences and contains the physical relationship between the linear field and the second order components of the true wave field. The variables t , X , and Y denote absolute time and position in a fixed frame of reference. As in the previous work it has been assumed that the integrands differ from zero only in a finite range of the dummy arguments.

A difference between Eq. 3 and the corresponding representation [3] of the response should be noted. The response of the ship system is supposed invariant with absolute position, and in the representation of the response this restriction was taken into account in the formulation for the encounter domain. In effect, had the response been presumed a function of absolute position, the form of the final functional representation would have been similar to Eq. 3 except that a linear convolution would also have appeared.

The hydrodynamics of the problem noted in the last section shows that there is no linear transformation of the linear field. This fact accounts for the absence of a linear convolution in Eq. 3.

An immediate problem which arises with this representation is both theoretical and practical (from a computational point of view). The quadratic wave-wave interaction response functions are known in the wave number domain. The consequence of the representation of Eq. 3 is that the second degree kernel and the quadratic response functions must form special multi-dimensional Fourier transform pairs. Unfortunately the analysis of Appendix A shows that the quadratic response functions do not possess all the required properties [11] to make the required transforms valid. The problem with these functions is that they are unlimited as wave number tends to infinity.

A way out of the problem is detailed in Appendix B. Essentially, a wave number window is hypothesized such that the final expansion represents the interactions only between wave components having wave numbers less than some large but finite value. The window is represented by a linear transformation, and the revised formulation may be summarized in a cascade diagram:



Here the notation $\zeta_N(t, \bar{X})$ represents the nominal linear wave field. This field is presumed to be operated upon by a linear system $f_1(\cdot)$ in the time/space domain to produce a linear small amplitude wave field $\zeta_L(t, \bar{X})$ which is wave number limited but defined as before. The linear small amplitude field is acted upon by the quadratic wave interaction system with kernels $g_1(\cdot)$ and $g_2(\cdot)$ to result in the true wave field $\zeta_T(t, \bar{X})$. The last functional relation is the same as Eq. 3, with the missing linear kernel presumed to be a delta function.

The Taylor expansion of the true field in terms of the nominal field becomes

$$\begin{aligned} \zeta_T(t, X, Y) = & \int \int \int h_1(t_1, x_1, y_1) \zeta_N(t-t_1, X-x_1, Y-y_1) dt_1 dx_1 dy_1 \\ & + \int \dots \int h_2(t_1, x_1, y_1, t_2, x_2, y_2) \times \\ & \zeta_N(t-t_1, X-x_1, Y-y_1) \zeta_N(t-t_2, X-x_2, Y-y_2) \times \\ & dt_1 dx_1 dy_1 dt_2 dx_2 dy_2 \end{aligned} \quad (4)$$

and the response functions relating nominal and true wave fields may be written as follows:

$$T_T(\bar{\kappa}) = \int \dots \int h_1(t_1, x_1, y_1) \text{Exp}[-i(\kappa_X x_1 + \kappa_Y y_1 - \omega t_1)] dt_1 dx_1 dy_1 \quad (5)$$

$$\begin{aligned} Q_T(\bar{\kappa}_1, \bar{\kappa}_2) = & \int \dots \int h_2(t_1, x_1, y_1, t_2, x_2, y_2) \times \\ & \text{Exp}[-i(\kappa_{X1} x_1 + \kappa_{Y1} y_1 - \omega_1 t_1)] \times \\ & \text{Exp}[-i(\kappa_{X2} x_2 + \kappa_{Y2} y_2 - \omega_2 t_2)] dt_1 \dots dy_2 \end{aligned} \quad (6)$$

$$\begin{aligned}
R_T(\bar{\kappa}_1, \bar{\kappa}_2) = & \int \dots \int h_B(t_1, x_1, y_1, t_2, x_2, y_2) \times \\
& \text{Exp}[-i(\kappa_{x1}x_1 + \kappa_{y1}y_1 - \omega_1 t_1)] \times \\
& \text{Exp}[i(\kappa_{x2}x_2 + \kappa_{y2}y_2 - \omega_2 t_2)] dt_1 \dots dy_2 \quad (7)
\end{aligned}$$

where frequency has been omitted from the arguments of the response functions because wave frequency is a function of wave number via the dispersion relation.

From the cascade analysis of Appendix B the relationships between the functions defined by Eqs. 5 through 7, the quadratic wave-wave interaction functions defined previously, and the linear window response function $T_N(\bar{\kappa})$ are defined as follows:

$$T_T(\bar{\kappa}) = T_N(\bar{\kappa}) \quad (8)$$

$$Q_T(\bar{\kappa}_1, \bar{\kappa}_2) = Q_\zeta(\bar{\kappa}_1, \bar{\kappa}_2) T_N(\bar{\kappa}_1) T_N(\bar{\kappa}_2) \quad (9)$$

$$R_T(\bar{\kappa}_1, \bar{\kappa}_2) = R_\zeta(\bar{\kappa}_1, \bar{\kappa}_2) T_N(\bar{\kappa}_1) T_N^*(\bar{\kappa}_2) \quad (10)$$

The relation between the window response function and the space/time domain window $f_1(\cdot)$ is the same as Eq. 5 with $f_1(\cdot)$ substituted for $h_1(\cdot)$.

Except that it must approach zero at a sufficient rate to make the left hand sides of Eqs. 9 and 10 approach zero for large wave number, the window response function is quite arbitrary. In what follows it will be assumed to be analogous to a non-realizable low pass filter with unity gain up to some cutoff wave number and a rapid transition to zero above. Since the wave-wave interaction functions are real, this assumption makes Eqs. 8 through 10 purely real.

The interpretation of the windowed response functions just defined is identical to that of the last section. If the nominal wave field is represented by a sum of cosinusoids exactly as in Eq. 1 and this representation is substituted into Eq. 4, the resulting expression for the steady state true wave field becomes

$$\begin{aligned}
\zeta_T(t, \bar{X}) = & \sum_j^N T_T(\bar{\kappa}_j) \text{Re}\{b_j \text{Exp}[i\bar{\kappa}_j \cdot \bar{X} - i\omega_j t]\} \\
& + \frac{1}{2} \sum_j^N \sum_k^N Q_T(\bar{\kappa}_j, \bar{\kappa}_k) \times \\
& \quad \text{Re}\{b_j b_k \text{Exp}[i(\bar{\kappa}_j + \bar{\kappa}_k) \cdot \bar{X} - i(\omega_j + \omega_k)t]\} \\
& + \frac{1}{2} \sum_j^N \sum_k^N R_T(\bar{\kappa}_j, \bar{\kappa}_k) \times \\
& \quad \text{Re}\{b_j b_k^* \text{Exp}[i(\bar{\kappa}_j - \bar{\kappa}_k) \cdot \bar{X} - i(\omega_j - \omega_k)t]\} \quad (11)
\end{aligned}$$

As before, the windowed quadratic response functions $Q_T(\bar{\kappa}_j, \bar{\kappa}_k)$ and $R_T(\bar{\kappa}_j, \bar{\kappa}_k)$ relate those components of the nominal field which correspond to sums and differences of vector wave numbers, to the interactions of small amplitude waves which reside within the window. The range of wave numbers present in the linear part of the true field is determined by $T_T(\bar{\kappa}_j)$ in the first term of the expression.

SPECTRAL REPRESENTATION OF THE TRUE WAVE FIELD

The next step in the development involves the introduction of randomness. This step may be approached in the same manner as in the case of response to a linear wave field [3]. For this purpose, it is assumed that the nominal linear wave field $\zeta_N(X, Y, t)$ is dispersive, homogeneous in space, stationary in time, and Gaussian. It is further assumed that the elevation variance of this nominal wave field may be represented by

$$\text{Var}(\zeta_N) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_N(\kappa_X, \kappa_Y) d\kappa_X d\kappa_Y \quad (12)$$

where $F_N(\kappa_X, \kappa_Y)$ is a vector wave number spectrum.

These assumptions together imply that the field has a "spectral representation" [12], [13], [1]. Although this representation is often written in the form of a Fourier-Stieltjes integral, it is convenient to work with the corresponding approximating Fourier sum. Accordingly, the approximating Fourier sum of the spectral representation of the nominal field is taken to be

$$\zeta_N(t, \bar{X}) = \sum_j \text{Re} \{ W(j) \text{Exp}[i\kappa_{Xj}X + i\kappa_{Yj}Y - i\omega_j t] \} \quad (13)$$

where $W(j)$ is a complex random function of the interval $\Delta x_{Xj}, \Delta x_{Yj}$. The initial assumptions that the nominal field is stationary and homogeneous and that the process is Gaussian imply that the random functions are Gaussian, zero mean, and statistically independent for different wave numbers and directions of propagation. Under these assumptions the ensemble expected values of the function and its various order moments may be defined and become either zero or relatively simple functions of the variance spectrum. Appendix C summarizes the results through sixth order moments.

As in the deterministic case of the previous sections, a substitution of Eq. 13 into Eq. 4 yields the statistically steady state spectral representation of the true wave field in the form of an approximating Fourier sum. Because the random functions are analogous to the complex amplitudes of Eq. 11, the algebra of the substitution and subsequent reduction is identical to that of the deterministic case. The resulting spectral representation of the true wave field is identical in form to Eq. 11 and may be written

$$\begin{aligned} \zeta_T(t, \bar{X}) = & \sum_j T_T(\bar{\kappa}_j) \text{Re} \{ W(j) \text{Exp}[i\bar{\kappa}_j \cdot \bar{X} - i\omega_j t] \} \\ & + \frac{1}{2} \sum_j \sum_k Q_T(\bar{\kappa}_j, \bar{\kappa}_k) \times \\ & \quad \text{Re} \{ W(j) W(k) \text{Exp}[i(\bar{\kappa}_j + \bar{\kappa}_k) \cdot \bar{X} - i(\omega_j + \omega_k)t] \} \\ & + \frac{1}{2} \sum_j \sum_k R_T(\bar{\kappa}_j, \bar{\kappa}_k) \times \\ & \quad \text{Re} \{ W(j) W^*(k) \text{Exp}[i(\bar{\kappa}_j - \bar{\kappa}_k) \cdot \bar{X} - i(\omega_j - \omega_k)t] \} \end{aligned} \quad (14)$$

where it should be noted that, as in all the development thus far, the spatial coordinate system is fixed, and the frequencies are wave frequencies related to scalar wave number through a dispersion relation.

STATISTICAL PROPERTIES OF THE TRUE WAVE FIELD

The true and nominal wave fields differ by the windowing function and the effects of wave-wave interaction. The spectral representation, Eq. 14, and the properties of the complex random functions noted in Appendix C allow some of the important statistical properties of the true field to be inferred.

First among the important statistics is the statistical mean value. When the ensemble mean value of Eq. 14 is taken, almost all the contributing terms drop out by virtue of Eqs. C.3, C.4, and C.6. The value of the single term remaining is determined by Eq. C.5, and the resulting mean value of the true wave field may be written

$$\langle \zeta_T(t, \bar{X}) \rangle = \sum_j R_T(\bar{\kappa}_j, \bar{\kappa}_j) F_N(\kappa_{Xj}, \kappa_{Yj}) \Delta \kappa_{Xj} \Delta \kappa_{Yj} = 0 \quad (15)$$

where the notation $\langle \dots \rangle$ denotes the ensemble expected value, and the expression is zero finally because the wave-wave interaction component of $R_T(\bar{\kappa}_j, \bar{\kappa}_j)$ is zero (Appendix A). Thus Eq. 15 expresses the physically expected result that the true wave field is a zero-mean process.

The other statistics of primary interest are the true wave variance and the true wave number spectrum. Both may be approached by first developing the space-time auto covariance function $\langle \zeta_T(t, \bar{X}) \zeta_T(t + \tau, \bar{X} + \rho) \rangle$ in which τ denotes a time difference and ρ a vector position difference. Using the spectral representation, Eq. 14, for each of the factors inside the expectation operator, expanding the real part operation into a sum of the function and its complex conjugate, and forming the required product results in thirty-six terms involving multiple summations of complex exponentials, the response functions, and the complex random amplitudes. Once the expectation is taken, all but twelve of these products drop out by virtue of Eqs. C.7 and C.8, and the remainder are evaluated and simplified by Eqs. C.4, C.5, C.9, and C.10. After a further algebraic reduction, which parallels that described for the response [3], the approximating Fourier sum representation of the space-time auto covariance of the true wave field becomes

$$\begin{aligned} \langle \zeta_T(t, \bar{X}) \zeta_T(t + \tau, \bar{X} + \rho) \rangle = & \\ & \sum_j T_T^R(\bar{\kappa}_j) F_N(\kappa_{Xj}, \kappa_{Yj}) \operatorname{Re} \left\{ \operatorname{Exp}[i \bar{\kappa}_j \cdot \rho - i \omega_j \tau] \right\} \Delta \kappa_{Xj} \Delta \kappa_{Yj} \\ & + \sum_j \sum_k Q_T^R(\bar{\kappa}_j, \bar{\kappa}_k) F_N(\kappa_{Xj}, \kappa_{Yj}) F_N(\kappa_{Xk}, \kappa_{Yk}) \times \\ & \quad \operatorname{Re} \left\{ \operatorname{Exp}[i(\bar{\kappa}_j + \bar{\kappa}_k) \cdot \rho - i(\omega_j + \omega_k) \tau] \right\} \Delta \kappa_{Xj} \Delta \kappa_{Yj} \Delta \kappa_{Xk} \Delta \kappa_{Yk} \\ & + \sum_j \sum_k R_T^R(\bar{\kappa}_j, \bar{\kappa}_k) F_N(\kappa_{Xj}, \kappa_{Yj}) F_N(\kappa_{Xk}, \kappa_{Yk}) \times \\ & \quad \operatorname{Re} \left\{ \operatorname{Exp}[i(\bar{\kappa}_j - \bar{\kappa}_k) \cdot \rho - i(\omega_j - \omega_k) \tau] \right\} \Delta \kappa_{Xj} \Delta \kappa_{Yj} \Delta \kappa_{Xk} \Delta \kappa_{Yk} \end{aligned} \quad (16)$$

At this stage all the complex random amplitudes have been eliminated and with them the necessity for the Fourier sum approximations to Fourier-Stieltjes integrals. Thus when the range of the summation is allowed to increase and the wave number intervals to decrease, the limiting form involves ordinary integrals

$$\begin{aligned}
\langle \zeta_{\mathbf{T}}(t, \bar{\mathbf{X}}) \zeta_{\mathbf{T}}(t+\tau, \bar{\mathbf{X}}+\rho) \rangle &= \int \dots \int \mathbf{T}_{\mathbf{T}}^{\mathbf{R}}(\bar{\kappa}) \mathbf{F}_{\mathbf{N}}(\bar{\kappa}) \operatorname{Re} \left\{ \operatorname{Exp}[i\bar{\kappa} \cdot \rho - i\omega\tau] \right\} d\bar{\kappa} \\
&+ \int \dots \int \mathbf{Q}_{\mathbf{T}}^{\mathbf{R}}(\bar{\kappa}_1, \bar{\kappa}_2) \mathbf{F}_{\mathbf{N}}(\bar{\kappa}_1) \mathbf{F}_{\mathbf{N}}(\bar{\kappa}_2) \times \\
&\quad \operatorname{Re} \left\{ \operatorname{Exp}[i(\bar{\kappa}_1 + \bar{\kappa}_2) \cdot \rho - i(\omega_1 + \omega_2)\tau] \right\} d\bar{\kappa}_1 d\bar{\kappa}_2 \\
&+ \int \dots \int \mathbf{R}_{\mathbf{T}}^{\mathbf{R}}(\bar{\kappa}_1, \bar{\kappa}_2) \mathbf{F}_{\mathbf{N}}(\bar{\kappa}_1) \mathbf{F}_{\mathbf{N}}(\bar{\kappa}_2) \times \\
&\quad \operatorname{Re} \left\{ \operatorname{Exp}[i(\bar{\kappa}_1 - \bar{\kappa}_2) \cdot \rho - i(\omega_1 - \omega_2)\tau] \right\} d\bar{\kappa}_1 d\bar{\kappa}_2 \tag{17}
\end{aligned}$$

Now if the time and vector space differences are set to zero, Eq. 17 represents the true wave variance

$$\begin{aligned}
\operatorname{Var}(\zeta_{\mathbf{T}}) &= \int \dots \int \mathbf{T}_{\mathbf{T}}^{\mathbf{R}}(\bar{\kappa}) \mathbf{F}_{\mathbf{N}}(\bar{\kappa}) d\bar{\kappa} \\
&+ \int \dots \int \left\{ \mathbf{Q}_{\mathbf{T}}^{\mathbf{R}}(\bar{\kappa}_1, \bar{\kappa}_2) + \mathbf{R}_{\mathbf{T}}^{\mathbf{R}}(\bar{\kappa}_1, \bar{\kappa}_2) \right\} \mathbf{F}_{\mathbf{N}}(\bar{\kappa}_1) \mathbf{F}_{\mathbf{N}}(\bar{\kappa}_2) d\bar{\kappa}_1 d\bar{\kappa}_2 \tag{18}
\end{aligned}$$

Since the response functions defined by Eqs. 8 through 10 are here assumed to be real valued, this result is exactly that which would be expected from the previous development [3].

In accordance with convention [8] the vector wave number spectrum of the true wave field $\mathbf{F}_{\mathbf{T}}(\bar{\kappa})$ may be defined as follows:

$$\langle \zeta_{\mathbf{T}}(\bar{\mathbf{X}}) \zeta_{\mathbf{T}}(\bar{\mathbf{X}}+\rho) \rangle = \int \dots \int \mathbf{F}_{\mathbf{T}}(\bar{\kappa}) \operatorname{Exp}[i\bar{\kappa} \cdot \rho] d\bar{\kappa} \tag{19}$$

where the wave field is assumed to be homogeneous. An expression for the wave number spectrum of the true wave field in terms of that of the nominal wave field is obtained by setting the time difference in Eq. 17 to zero and making changes of variable so as to put the result in the form of Eq. 19. By this process the expression for the true wave number spectrum becomes

$$\begin{aligned}
\mathbf{F}_{\mathbf{T}}(\bar{\kappa}) &= \mathbf{T}_{\mathbf{T}}^{\mathbf{R}}(\bar{\kappa}) \mathbf{F}_{\mathbf{N}}(\bar{\kappa}) \\
&+ \frac{1}{4} \int \dots \int \mathbf{Q}_{\mathbf{T}}^{\mathbf{R}}[(\bar{\kappa}+\bar{\nu})/2, (\bar{\kappa}-\bar{\nu})/2] \mathbf{F}_{\mathbf{N}}[(\bar{\kappa}+\bar{\nu})/2] \mathbf{F}_{\mathbf{N}}[(\bar{\kappa}-\bar{\nu})/2] d\bar{\nu} \\
&+ \frac{1}{4} \int \dots \int \mathbf{R}_{\mathbf{T}}^{\mathbf{R}}[(\bar{\kappa}+\bar{\nu})/2, (\bar{\nu}-\bar{\kappa})/2] \mathbf{F}_{\mathbf{N}}[(\bar{\kappa}+\bar{\nu})/2] \mathbf{F}_{\mathbf{N}}[(\bar{\nu}-\bar{\kappa})/2] d\bar{\nu} \tag{20}
\end{aligned}$$

The convolutions in this expression are of the general form which would be expected from the time-only theory for linear plus quadratic systems. In the practical case one might want to analyze an observed wave number spectrum by deriving the nominal spectrum. The results just

shown illustrate the difficulties in such an endeavor — the nominal vector wave number spectrum is defined by integral equation of greater complexity than that which obtains in the time-only case. The present model is oriented more toward synthesis than analysis, although some analysis steps have been made [6].

Because in marine practice the directional variance spectra of waves are usually considered in the wave frequency/direction form, it was of interest to see whether an equivalent or simpler form for the true wave spectrum would be obtained in the wave frequency/direction domain. The required transformations were carried through far enough to see that the equivalent expression would be far more complicated. Accordingly, if the computations equivalent to Eqs. 18 or 20 need to be done in this domain, it appears less troublesome to transform the nominal frequency/direction spectrum to the vector wave number domain, carry out the convolutions, and then transform the true vector wave number spectrum back to the wave frequency/direction domain.

REPRESENTATION OF THE RESPONSE

The representation of the ship response to the true wave field is essentially the same as in the previous work [3]. It is assumed that the ship is proceeding at mean speed U along the X axis in a fixed X - Y frame of reference. Component wave directions will be denoted by μ , and the resulting conventions are indicated in the figure.

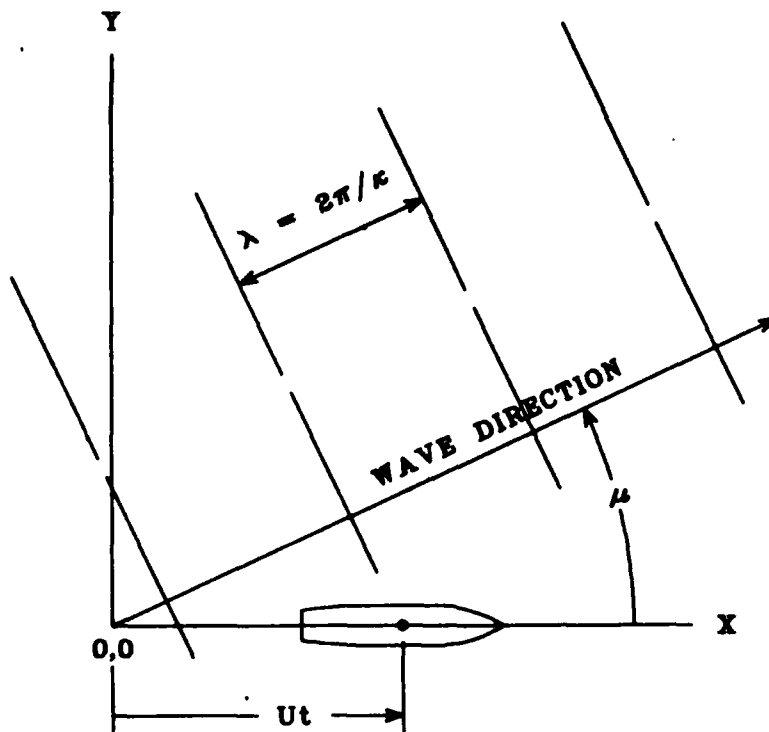


Fig. 1. Component wave direction.

An (x-y) coordinate system moving with the ship is conventionally defined as follows:

$$x = X - Ut$$

$$y = Y$$

and accordingly the encounter frequency corresponding to a given wave component becomes

$$\sigma = \omega - \kappa_x U$$

$$= \text{encounter frequency}$$

As in the previous work, the ship response $\xi(t)$ may be represented by a functional Taylor expansion in the neighborhood of zero true wave field as encountered by the ship. The resulting model is written as follows:

$$\begin{aligned} \xi(t) = & \int \int \int k_1(t_1, x_1, y_1) \zeta_T(t-t_1, x_1, y_1) dt_1 dx_1 dy_1 \\ & + \int \dots \int k_2(t_1, x_1, y_1, t_2, x_2, y_2) \times \\ & \zeta_T(t-t_1, x_1, y_1) \zeta_T(t-t_2, x_2, y_2) dt_1 dx_1 dy_1 dt_2 dx_2 dy_2 \end{aligned} \quad (21)$$

The first (constant) term in the expansion has been dropped on the presumption that no response of current interest occurs in the absence of waves, and the series has been truncated at the quadratic term. The convolutions are with respect to time and space differences. Since this is an encounter domain model, the space differences are with respect to a reference point on the moving ship. The linear ship dynamics are assumed to be represented by the linear kernel $k_1(t_1, x_1, y_1)$ and the quadratic nonlinearities by the kernel $k_2(t_1, x_1, y_1, t_2, x_2, y_2)$. Apart from the explicit note that the local wave field is the true field, Eq. 21 is identical to Eq. 3 of the previous work [3].

The identification of linear and quadratic response functions corresponding to the kernels of Eq. 21 is identical to that of the previous work. The steady state response to assumed "true" wave fields composed of one and two superimposed linear regular waves is obtained, and in this process the relationships between the kernels and response functions may be defined in the following form:

$$T_\xi(\bar{\kappa}, \sigma) = \int \dots \int k_1(t_1, x_1, y_1) \text{Exp}[-i(\kappa_x x_1 + \kappa_y y_1 + \sigma t_1)] dt_1 dx_1 dy_1 \quad (22)$$

$$\begin{aligned} Q_\xi(\bar{\kappa}_1, \sigma_1, \bar{\kappa}_2, \sigma_2) = & \int \dots \int k_2(t_1, x_1, y_1, t_2, x_2, y_2) \times \\ & \text{Exp}[-i(\kappa_{x1} x_1 + \kappa_{y1} y_1 + \sigma_1 t_1)] \times \\ & \text{Exp}[-i(\kappa_{x2} x_2 + \kappa_{y2} y_2 + \sigma_2 t_2)] dt_1 \dots dy_2 \end{aligned} \quad (23)$$

$$R_f(\bar{\kappa}_1, \sigma_1, \bar{\kappa}_2, \sigma_2) = \int \int k_g(t_1, x_1, y_1, t_2, x_2, y_2) \times \\ \text{Exp}[-i(\kappa_{x1}x_1 + \kappa_{y1}y_1 + \sigma_1 t_1)] \times \\ \text{Exp}[i(\kappa_{x2}x_2 + \kappa_{y2}y_2 + \sigma_2 t_2)] dt_1 \dots dy_2 \quad (24)$$

As before [3], Eq. 22 defines the linear response function, Eq. 23 the quadratic sum frequency response function, and Eq. 24 the quadratic difference frequency response function. The existence of these transforms requires that the kernels be square integrable — in effect that the time/space “memory” of the physical ship system be finite (or that some artifice to make this so can be introduced if necessary). The notation departs from that used previously [3] in that the encounter frequency corresponding to each vector wave number argument is noted explicitly in the argument of each function. In the previous work if a dispersion relation and the ship speed are assumed, the encounter frequency is determined, and accordingly the encounter frequency argument was dropped on this basis. In portions of the present work this understanding is not necessarily valid, and the present more cumbersome notation is of advantage. However, the meaning of the response functions is identical to that previously given. The linear response function is the normalized amplitude of response in the encounter frequency to a linear regular wave component. The sum (difference) frequency response function is the normalized amplitude of response at an encounter frequency equal to the sum (difference) of the encounter frequencies corresponding to each of two linear regular wave components.

From the definitions two symmetry relations arise:

$$Q_f(\bar{\kappa}_1, \sigma_1, \bar{\kappa}_2, \sigma_2) = Q_f(\bar{\kappa}_2, \sigma_2, \bar{\kappa}_1, \sigma_1) \\ R_f(\bar{\kappa}_1, \sigma_1, \bar{\kappa}_2, \sigma_2) = R_f^*(\bar{\kappa}_2, \sigma_2, \bar{\kappa}_1, \sigma_1) \quad (25)$$

as well as the formal relation

$$Q_f(\bar{\kappa}_1, \sigma_1, -\bar{\kappa}_2, -\sigma_2) = R_f(\bar{\kappa}_1, \sigma_1, \bar{\kappa}_2, \sigma_2) \quad (26)$$

SPECTRAL REPRESENTATION OF THE RESPONSE

Because the response model, Eq. 21, is for the encounter domain, and the spectral representation of the true field, Eq. 14, involves the fixed spatial coordinates of Figure 1, a conversion is necessary. Transformation of Eq. 14 from the fixed (X-Y) coordinates to the moving (x-y) system results in the following spectral representation of the true wave field:

$$\zeta_T(t, \bar{X}) = \sum_j T_T(\bar{\kappa}_j) R_o\{W(j) \text{Exp}[i\bar{\kappa}_j \cdot \bar{X} - i\sigma_j t]\} \\ + \frac{1}{2} \sum_j \sum_k Q_T(\bar{\kappa}_j, \bar{\kappa}_k) \times \\ R_o\{W(j)W(k) \text{Exp}[i(\bar{\kappa}_j + \bar{\kappa}_k) \cdot \bar{X} - i(\sigma_j + \sigma_k)t]\}$$

$$+ \frac{1}{8} \sum_j \sum_k R_T(\bar{\kappa}_j, \bar{\kappa}_k) \times \operatorname{Re} \left\{ W(j) W^*(k) \exp[i(\bar{\kappa}_j - \bar{\kappa}_k) \cdot \bar{x} - i(\sigma_j - \sigma_k)t] \right\} \quad (27)$$

The form of Eqs. 14 and 27 is the same. Wave frequencies in Eq. 14 are replaced by encounter frequencies, and the fixed position vectors become relative to the mean ship position. The only difference between this stage and the corresponding stage in the previous work [3] is the presence of wave-wave interaction terms. Equation 27 is the Fourier sum approximating the local random wave field as it would be sensed aboard the ship.

The interest here is in the statistically steady state ship response to the true wave field. As in the previous work, the substitution of the Fourier sum spectral representation of the true wave field, Eq. 27, into the time domain model, Eq. 21, yields a spectral representation of the steady state response in the form of an approximating Fourier sum. It was anticipated from the form of the two equations that this direct substitution would yield a result which in part would involve contributions corresponding to nonlinearities of higher degree than those which are of interest here. The approach taken was to ignore the potential inconsistency pending the complete result.

The mechanics of this substitution involves replacing the real parts of the various expressions in Eq. 27 by half the sum of the respective expression and its complex conjugate, substituting the result in Eq. 21, and carrying out the integrations with respect to the time and space difference variables by applying the response function definitions, Eqs. 22 through 24. The raw result involves 6 summations from the first integral of Eq. 21 and 36 summations from the second. These summations occur in conjugate pairs so that the final results may be expressed in real part notation. The form of the summations allows some terms to be combined, and the symmetry relations, Eqs. 25 and 26 serve to combine others. The net result is 12 terms of different form which may be expressed as 8 summations, and the final result is given as Eq. 28 on the next page.

It will be remembered that the function $T_N(\bar{\kappa}_j)$ is a wave number window which has been assumed to be purely real and whose purpose is to make the wave-wave interaction model valid. This window function has been combined with the wave-wave interaction response functions $Q_\zeta(\bar{\kappa}_j, \bar{\kappa}_m)$ and $R_\zeta(\bar{\kappa}_j, \bar{\kappa}_m)$ in Eqs. 9 and 10. The cascaded response functions defined by Eqs. 9 and 10 have been used in the immediately previous development to compress notation. In the case of Eq. 28 the window functions have been factored outside of the real part operator. All the summations of Eq. 28 are over wave number, and the net result is as would be expected; that is, in any of the summations, any term which involves a wave number above the cutoff determined by the window function will be attenuated or zeroed entirely.

Each of the summations of Eq. 28 has a similar form: first, the product of as many window functions as there are summations, and then the real part of a complex expression made up of factors of three kinds. Within the complex expression in each case is the product of as many complex random amplitudes $W(j)$ as there are summations. These factors are followed by an exponential factor containing time and sums and differences of encounter frequencies. Finally, the last factor involves products and sums of wave-wave interaction and system response functions.

$$\begin{aligned}
f(t) = & \sum_j T_N(\bar{k}_j) \operatorname{Re} \{ W^*(j) \operatorname{Exp}[i\sigma_j t] T_f(\bar{k}_j, \sigma_j) \} \\
& + \frac{1}{2} \sum_j \sum_m T_N(\bar{k}_j) T_N(\bar{k}_m) \operatorname{Re} \{ W^*(j) W^*(m) \operatorname{Exp}[i(\sigma_j + \sigma_m)t] \times \\
& \quad [Q_f(\bar{k}_j, \bar{k}_m) T_f(\bar{k}_j + \bar{k}_m, \sigma_j + \sigma_m) + Q_f(\bar{k}_j, \sigma_j, \bar{k}_m, \sigma_m)] \} \\
& + \frac{1}{2} \sum_j \sum_m T_N(\bar{k}_j) T_N(\bar{k}_m) \operatorname{Re} \{ W^*(j) W(m) \operatorname{Exp}[i(\sigma_j - \sigma_m)t] \times \\
& \quad [R_f(\bar{k}_j, \bar{k}_m) T_f(\bar{k}_j - \bar{k}_m, \sigma_j - \sigma_m) + R_f(\bar{k}_j, \sigma_j, \bar{k}_m, \sigma_m)] \} \\
& + \frac{1}{2} \sum_j \sum_m \sum_n T_N(\bar{k}_j) T_N(\bar{k}_m) T_N(\bar{k}_n) \operatorname{Re} \{ W^*(j) W^*(m) W^*(n) \times \\
& \quad \operatorname{Exp}[i(\sigma_j + \sigma_m + \sigma_n)t] Q_f(\bar{k}_j, \bar{k}_m) Q_f(\bar{k}_j + \bar{k}_m, \sigma_j + \sigma_m, \bar{k}_n, \sigma_n) \} \\
& + \frac{1}{2} \sum_j \sum_m \sum_n T_N(\bar{k}_j) T_N(\bar{k}_m) T_N(\bar{k}_n) \operatorname{Re} \{ W^*(j) W^*(m) W(n) \times \\
& \quad \operatorname{Exp}[i(\sigma_j + \sigma_m - \sigma_n)t] [Q_f(\bar{k}_j, \bar{k}_m) R_f(\bar{k}_j + \bar{k}_m, \sigma_j + \sigma_m, \bar{k}_n, \sigma_n) \\
& \quad + R_f(\bar{k}_j, \bar{k}_n) Q_f(\bar{k}_j - \bar{k}_n, \sigma_j - \sigma_n, \bar{k}_m, \sigma_m) \\
& \quad + R_f(\bar{k}_n, \bar{k}_m) R_f^*(\bar{k}_n - \bar{k}_m, \sigma_n - \sigma_m, \bar{k}_j, \sigma_j)] \} \\
& + \frac{1}{8} \sum_j \sum_m \sum_n \sum_p T_N(\bar{k}_j) T_N(\bar{k}_m) T_N(\bar{k}_n) T_N(\bar{k}_p) \times \\
& \quad \operatorname{Re} \{ W^*(j) W^*(m) W^*(n) W^*(p) \operatorname{Exp}[i(\sigma_j + \sigma_m + \sigma_n + \sigma_p)t] \times \\
& \quad Q_f(\bar{k}_j, \bar{k}_m) Q_f(\bar{k}_n, \bar{k}_p) Q_f(\bar{k}_j + \bar{k}_m, \sigma_j + \sigma_m, \bar{k}_n + \bar{k}_p, \sigma_n + \sigma_p) \} \\
& + \frac{1}{2} \sum_j \sum_m \sum_n \sum_p T_N(\bar{k}_j) T_N(\bar{k}_m) T_N(\bar{k}_n) T_N(\bar{k}_p) \times \\
& \quad \operatorname{Re} \{ W^*(j) W^*(m) W^*(n) W(p) \operatorname{Exp}[i(\sigma_j + \sigma_m + \sigma_n - \sigma_p)t] \times \\
& \quad Q_f(\bar{k}_j, \bar{k}_m) R_f(\bar{k}_n, \bar{k}_p) Q_f(\bar{k}_j + \bar{k}_m, \sigma_j + \sigma_m, \bar{k}_n - \bar{k}_p, \sigma_n - \sigma_p) \} \\
& + \frac{1}{8} \sum_j \sum_m \sum_n \sum_p T_N(\bar{k}_j) T_N(\bar{k}_m) T_N(\bar{k}_n) T_N(\bar{k}_p) \times \\
& \quad \operatorname{Re} \{ W^*(j) W^*(m) W(n) W(p) \operatorname{Exp}[i(\sigma_j + \sigma_m - \sigma_n - \sigma_p)t] \times \\
& \quad [Q_f(\bar{k}_j, \bar{k}_m) Q_f(\bar{k}_n, \bar{k}_p) R_f(\bar{k}_j + \bar{k}_m, \sigma_j + \sigma_m, \bar{k}_n + \bar{k}_p, \sigma_n + \sigma_p) \\
& \quad + R_f(\bar{k}_j, \bar{k}_n) R_f(\bar{k}_m, \bar{k}_p) Q_f(\bar{k}_j - \bar{k}_n, \sigma_j - \sigma_n, \bar{k}_m - \bar{k}_p, \sigma_m - \sigma_p) \\
& \quad + R_f(\bar{k}_j, \bar{k}_p) R_f(\bar{k}_n, \bar{k}_m) R_f(\bar{k}_j - \bar{k}_p, \sigma_j - \sigma_p, \bar{k}_n - \bar{k}_m, \sigma_n - \sigma_m)] \} \\
\end{aligned} \tag{28}$$

The degree of nonlinearity of each term of the expression is easily determined by a count of the number of summations. In each term the number of wave number window functions in the product equals the number of summations. The number of complex amplitude functions in the product is the same, as is the number of encounter frequencies summed in the exponential. Had the Taylor expansion of the response, Eq. 21, been truncated after the fourth instead of after the second functional, the corresponding approximating Fourier sum would have been of very similar form to Eq. 28, with the response function corresponding to the N'th degree kernel appearing in an N-fold summation which also involves an N-fold summation of encounter frequencies.

The first term of Eq. 28 is the only one of first degree; it represents the linear ship system response to the nominal wave field and is the same as the corresponding term of the response representation of [3] except for the window function. The next two terms are second degree and represent the quadratic response in sum and difference encounter frequencies. Since the final factor in these complex expressions is a sum of two terms of the form $Q_{\xi}(\bar{\kappa}_j, \bar{\kappa}_m) \times T_{\xi}(\bar{\kappa}_j + \bar{\kappa}_m, \sigma_j + \sigma_m)$ and $Q_{\xi}(\bar{\kappa}_j, \sigma_j, \bar{\kappa}_m, \sigma_m)$, the response of second degree has two qualitative origins. One is the quadratic ship system response to pairs of linear wave field components (identical to the results in the previous work [3]). The other is the linear ship system response to all the wave-wave interaction sum or difference frequency components present in the true wave field. This last contribution is the first of the modifications due to the difference between true and nominal wave fields.

None of the remaining terms in Eq. 28 have counterparts in the earlier work [3]. The fourth and fifth terms in Eq. 28 are of third degree and involve response in an encounter frequency equal to the sum as well as a partial difference of three linear encounter frequencies. These terms represent all the possible ship system quadratic responses to all the pairs of true wave field components which may be formed with one linear wave field component and one of the components of the wave field produced by quadratic wave-wave interactions.

The sixth through eighth summations in Eq. 28 correspond to fourth degree nonlinearities. This part of the response occurs at sums and differences of four linear encounter frequencies. These terms represent all the possible ship system quadratic responses to pairs of the components in the true wave field which are produced by wave-wave interactions.

The form of the arguments of some of the response functions in Eq. 28 has come about purely as a result of the definitions, Eqs. 22 through 24. The reason for the addition of the encounter frequency argument in the definition of these functions may now be apparent. For example, in the linear function $T_{\xi}(\bar{\kappa}_j + \bar{\kappa}_m, \sigma_j + \sigma_m)$ the argument $\bar{\kappa}_j + \bar{\kappa}_m$ defines a wave-wave interaction component which may have a propagation direction different from those of the two linear wave field components involved and almost certainly a different wave length. The wave frequency which would be inferred by applying the deep water dispersion relationship to the absolute value of the vector sum is not necessarily the same as the sum of the wave frequencies of the components, nor is the corresponding encounter frequency. Since many ship motion responses involve a mixture of dynamic resonances with a geometric matching of wave shape and vessel shape (wave matching), it is important to keep track of such differences.

As has been noted, the result, Eq. 28, has been obtained under assumptions which are mathematically inconsistent in the context of series expansions. In particular, it has been assumed that the wave-wave interactions are strictly limited to the quadratic ("second order")

type; that is, cubic and nonlinear interactions of higher degree are assumed to be identically zero. Similarly, nonlinear ship system interactions of degree higher than two are assumed not to exist. Equation 28 is a Fourier sum approximating the spectral representation of the response under these assumptions.

It is helpful to define an "order" (in the magnitude sense) of the various terms of Eq. 28. The most convenient way is to assume that the magnitude of response is most closely associated with the magnitude of the wave components. If, as is necessary when dealing with weakly nonlinear systems, it is assumed that the components of the nominal (linear) wave field are small, then products of components are smaller yet. In the present context the complex amplitudes $W(j)$ are a measure of the magnitude of each wave component. If these amplitudes are small and assumed to be of "first order", then N-fold products of the complex amplitudes are of N'th order. On this ordering assumption it may be seen that the order of the various terms in Eq. 28 is the same as the degree of nonlinearity. Thus, if it seems permissible to disregard response of third and higher order, only the first three summations of Eq. 28 will remain.

EXPECTED VALUE OF EQUATION 28

The form of the statistical mean value of Eq. 28 as it stands is instructive. The required algebra is not overpowering, and thus the order of the various terms will be temporarily ignored.

The ensemble average of six of the eight summations of Eq. 28 may be immediately seen to be zero from the relations of Appendix C. In particular, the mean of the first summation is zero by Eq. C.3, and the mean of the second by Eq. C.6. The mean of the fourth and fifth summations is zero by Eq. C.7. The mean of the sixth and seventh summations is zero by Eq. C.8. Most of the terms in the expansion of the third summation are zero by Eq. C.4, and most of the terms of the eighth by Eq. C.9. The non-zero portion of the third summation is determined by the application of Eq. C.5, and the non-zero portion of the eighth summation by Eq. C.10. The application of Eqs. C.5 and C.10 to these terms has the effect of converting the third summation of Eq. 28 from a double to a single sum, converting the eighth summation from four-fold to two-fold, and eliminating all the exponential factors. Combining some of the terms remaining algebraically, and using Eqs. 25 and 26 as well as the result that the wave-wave interaction function $R_{\xi}(\bar{\kappa}_j, \bar{\kappa}_j)$ is zero (appendix A), results in the following Fourier sum approximation to the statistical mean value of Eq. 28:

$$\begin{aligned}
 \langle \xi(t) \rangle &= \sum_j R_{\xi}(\bar{\kappa}_j, \sigma_j, \bar{\kappa}_j, \sigma_j) T_T^2(\bar{\kappa}_j) F_N(\bar{\kappa}_j) \Delta \bar{\kappa}_j \\
 &+ \sum_j \sum_m R_{\xi}(\bar{\kappa}_j + \bar{\kappa}_m, \sigma_j + \sigma_m, \bar{\kappa}_j + \bar{\kappa}_m, \sigma_j + \sigma_m) Q_T^2(\bar{\kappa}_j, \bar{\kappa}_m) F_N(\bar{\kappa}_j) F_N(\bar{\kappa}_m) \Delta \bar{\kappa}_j \Delta \bar{\kappa}_m \\
 &+ \sum_j \sum_m R_{\xi}(\bar{\kappa}_j - \bar{\kappa}_m, \sigma_j - \sigma_m, \bar{\kappa}_j - \bar{\kappa}_m, \sigma_j - \sigma_m) R_T^2(\bar{\kappa}_j, \bar{\kappa}_m) F_N(\bar{\kappa}_j) F_N(\bar{\kappa}_m) \Delta \bar{\kappa}_j \Delta \bar{\kappa}_m
 \end{aligned}
 \tag{29}$$

The compress notation in this result the wave number window functions have been combined with the wave-wave interaction functions in accordance with Eqs. 8 through 10, and component wave number intervals are represented by vector wave number intervals. Equation 25 and previous definitions of the wave-wave response functions make the expression purely real.

Because all the complex random amplitudes have been eliminated, the range of the summations may be allowed to increase and the wave number intervals allowed to decrease so that the limiting form of Eq. 29 involves ordinary integrals as follows:

$$\begin{aligned} \langle \xi(t) \rangle = & \int \dots \int R_{\xi}(\bar{\kappa}_1, \sigma_1, \bar{\kappa}_1, \sigma_1) T_T^2(\bar{\kappa}_1) F_N(\bar{\kappa}_1) d\bar{\kappa}_1 \\ & + \int \dots \int R_{\xi}(\bar{\kappa}_1 + \bar{\kappa}_2, \sigma_1 + \sigma_2, \bar{\kappa}_1 + \bar{\kappa}_2, \sigma_1 + \sigma_2) Q_T^2(\bar{\kappa}_1, \bar{\kappa}_2) F_N(\bar{\kappa}_1) F_N(\bar{\kappa}_2) d\bar{\kappa}_1 d\bar{\kappa}_2 \\ & + \int \dots \int R_{\xi}(\bar{\kappa}_1 - \bar{\kappa}_2, \sigma_1 - \sigma_2, \bar{\kappa}_1 - \bar{\kappa}_2, \sigma_1 - \sigma_2) R_T^2(\bar{\kappa}_1, \bar{\kappa}_2) F_N(\bar{\kappa}_1) F_N(\bar{\kappa}_2) d\bar{\kappa}_1 d\bar{\kappa}_2 \quad (30) \end{aligned}$$

Apart from the wave number window $T_T(\bar{\kappa})$ the first term of Eq. 30 is exactly the same as the estimator for the mean developed earlier [3] for the linear wave field case. Since the wave number window has been assumed to be unity up to some large wave number, its inclusion in the first term makes no practical change in the estimator because the nominal wave number variance spectrum $F_N(\bar{\kappa})$ generally tends toward zero for large wave number.

Comparison with Eq. 20 and the discussion of Eq. 28 show that the last two terms of Eq. 30 are the mean quadratic ship system response to the true wave field components produced by wave-wave interaction. The form is very like that of the true variance, Eq. 18. Were it not for the sum and difference frequency arguments in the ship system response function, it would be possible to change variables in Eq. 30 so as to compact the expression.

On the assumption that the order of magnitude of the terms in Eq. 30 is determined by the magnitude of the wave excitation, the first term is second order and the last two terms are fourth order. In effect, if the representation, Eq. 28, is assumed to be correct through fourth order, the modification of the nominal wave field by wave-wave interactions influences estimates of the mean response only by the addition of terms which are two orders of magnitude smaller than the leading term.

STATISTICS OF RESPONSE TO SECOND ORDER

In the discussion of the spectral representation, Eq. 28, it was noted that this representation has been developed on the assumption that the truncated functional expansions of both the true wave field and the response are exact. Had the functional expansions been truncated at third degree nonlinearities, for example, two additional third degree contributions would appear in Eq. 28. These contributions would correspond to the linear response to third order wave-wave interactions and a response of third degree to the nominal linear wave field. There would also appear contributions to the terms of fourth degree, as well as new fifth and sixth degree terms. If the order of magnitude of the various contributions is assumed to be based on the wave components as outlined previously, Eq. 28 includes third and fourth order contributions which are quite possibly incomplete, and thus inconsistent.

Inconsistency through nonlinearities of second degree (order) [3] led to the present work. By exactly the same arguments the present results through Eq. 30 are inconsistent with respect to third and higher order responses.

Thus, to develop response statistics which are consistent to second order it is necessary to disregard the terms of Eq. 28 of third and higher degrees. Accepting this premise, and defining $\tilde{\xi}(t)$ as the Fourier sum approximation to the spectral representation of the response through second order results in

$$\begin{aligned} \tilde{\xi}(t) = & \sum_j \operatorname{Re}\{W^*(j) \operatorname{Exp}[i\sigma_j t] \tilde{T}(\bar{\kappa}_j, \sigma_j)\} \\ & + \frac{1}{2} \sum_j \sum_m \operatorname{Re}\{W^*(j) W^*(m) \operatorname{Exp}[i(\sigma_j + \sigma_m)t] \tilde{Q}(\bar{\kappa}_j, \sigma_j, \bar{\kappa}_m, \sigma_m)\} \\ & + \frac{1}{2} \sum_j \sum_m \operatorname{Re}\{W^*(j) W(m) \operatorname{Exp}[i(\sigma_j - \sigma_m)t] \tilde{R}(\bar{\kappa}_j, \sigma_j, \bar{\kappa}_m, \sigma_m)\} \end{aligned} \quad (31)$$

where the products of wave number windows and response functions have been consolidated as follows:

$$\tilde{T}(\bar{\kappa}_j, \sigma_j) = T_N(\bar{\kappa}_j) T_\xi(\bar{\kappa}_j, \sigma_j) \quad (32)$$

$$\begin{aligned} \tilde{Q}(\bar{\kappa}_j, \sigma_j, \bar{\kappa}_m, \sigma_m) = & T_N(\bar{\kappa}_j) T_N(\bar{\kappa}_m) \times \\ & [Q_\xi(\bar{\kappa}_j, \bar{\kappa}_m) T_\xi(\bar{\kappa}_j + \bar{\kappa}_m, \sigma_j + \sigma_m) + Q_\xi(\bar{\kappa}_j, \sigma_j, \bar{\kappa}_m, \sigma_m)] \end{aligned} \quad (33)$$

$$\begin{aligned} \tilde{R}(\bar{\kappa}_j, \sigma_j, \bar{\kappa}_m, \sigma_m) = & T_N(\bar{\kappa}_j) T_N(\bar{\kappa}_m) \times \\ & [R_\xi(\bar{\kappa}_j, \bar{\kappa}_m) T_\xi(\bar{\kappa}_j - \bar{\kappa}_m, \sigma_j - \sigma_m) + R_\xi(\bar{\kappa}_j, \sigma_j, \bar{\kappa}_m, \sigma_m)] \end{aligned} \quad (34)$$

It will be noted that the functions defined in Eqs. 32 through 34 are effective linear and quadratic ship system response functions relating the response to the nominal linear wave field components. When the wave number window is assumed real and equal to unity over the range of wave number of interest, the effective linear function, Eq. 32, is the same as the linear response function defined in Eq. 22. Apart from notation it is the same as the linear response function previously developed [3]. The quadratic response functions, Eqs. 33 and 34, have the same symmetries as the response functions, Eq. 25, by virtue of the symmetry of the wave-wave interaction functions and the definition, Eq. 22. In addition, because of the zero mean property of the wave-wave difference frequency response function,

$$\tilde{R}(\bar{\kappa}_j, \sigma_j, \bar{\kappa}_j, \sigma_j) = T_N^2(\bar{\kappa}_j) R_\xi(\bar{\kappa}_j, \sigma_j, \bar{\kappa}_j, \sigma_j) \quad (35)$$

In effect, the function determining the mean value of the effective response to a true wave component is the same as the corresponding function for the response to a linear wave component.

As would be expected, the spectral representation, Eq. 31, is analogous to the second order system expansions of Papanikolaou [14], in which the contributions of second order nonlinearities include both pure second order hydrodynamic response and interactions of second order. In the present case the grouping of the origins of the response is less specific. (No attempt has been made here to analyze the hydrodynamic response problem other than to note that the "true wave field" could just as well have been represented by "true wave potential".)

Equations 33 and 34 explain what Hasselmann [1] meant by the assertion that the corrections to the response due to wave-wave interactions could be absorbed into the quadratic response functions. In a consistent expansion correct to second order in the true wave field the wave-wave interaction functions can be combined with the ship system response functions with advantage. In a higher order expansion the second order wave-wave interactions also contribute to the higher order response functions, so that the details of the "absorption" are relative to the order of the expansion.

As just noted, the symmetries of the effective functions, Eqs. 32 through 34, are the same as the ship system response functions, Eqs. 22 through 24. Both these sets of functions are the same as the linear and quadratic response functions developed earlier [3] in the sense that they represent the normalized complex response of the system to linear components of the wave field. The spectral representation through second order of the response to the true wave field, Eq. 31, is identical in form to the earlier spectral representation of the response to the linear field (Eq. 34 of reference [3]). The difference is that the linear and quadratic response functions of Eq. 31 represent the response to both the linear field and the second order components produced by wave-wave interactions. The subsequent derivations of statistics of response in the earlier work [3] involved only the symmetries of the response functions. Thus it is pointless to continue here with derivations of the various statistics. The results will be the same as those of the earlier work [3] when the effective response functions, Eqs. 32 through 34 are substituted for the corresponding response functions of reference [3].

With respect to the most practically important statistic, the expected value of response, the above result, with Eq. 35, means that to second order the estimator for the mean response to the true wave field is exactly the same as the estimator for the mean response to the linear field. To this order the mean value of response is not affected by wave-wave interactions.

To the extent that the effective response functions, Eqs. 32 through 34, are obtainable, the other statistics derived earlier [3] are also. Because the wave number windows in Eqs. 32 through 34 have consistently been assumed to be unity in the wave number range of importance, they constitute no particular problem — in fact, they formally represent only the finite truncation of wave number interval which would be necessary in a practical calculation. On this basis Eq. 32 and the second terms of Eqs. 33 and 34 also constitute no new problems.

The wave-wave interaction functions $Q_{\zeta}(\bar{\kappa}_j, \bar{\kappa}_m)$ and $R_{\zeta}(\bar{\kappa}_j, \bar{\kappa}_m)$ appearing in Eqs. 33 and 34 have been defined explicitly for deep water. The linear response functions of the form $T_{\xi}(\bar{\kappa}_j \pm \bar{\kappa}_m, \sigma_j \pm \sigma_m)$ which appear in Eqs. 33 and 34 present a slightly different problem from the ship motions perspective. The argument $\bar{\kappa}_j \pm \bar{\kappa}_m$ defines a wave which is in general of different length and direction than either of the superimposed linear waves. However, the argument $\sigma_j \pm \sigma_m$ denotes that the response is at a frequency which is the sum or difference of the encounter frequencies of the two linear waves. The phase speed of the second order

interaction component is not determined by the free wave dispersion relationship. This result is consistent with the result that the second order component of a Stokes wave has the same phase speed as the first order component, although the wave number is twice that of the first order component. Obtaining the required linear response function of the ship motion system, if it is needed, will thus require the treatment of a somewhat different potential flow situation than is ordinary. The level of effort, however, is likely to be much smaller than the effort required for the quadratic responses to the linear waves.

CONCLUSIONS

The present objectives included a) the development of a model for the statistics of quadratic ship system response in short-crested seas which is consistent in that quadratic nonlinearities in the wave field are taken into account, b) a clarification of the assumptions being made, c) an assessment of what is involved in carrying out computations which include the wave field nonlinearities, and d) an assessment of the influence of the wave field nonlinearities upon the most important statistic associated with quadratic response, the mean value.

In this development the lead of Hasselmann was followed and amplified. If the statistical representation of the response to the linear part as well as to the quadratic nonlinearities of a deep-water wave field is carried out strictly through contributions which may be considered second order in wave excitation, and if higher order response is neglected, the statistical estimating formulae for the quadratic response to short-crested seas in which quadratic nonlinearities are included are identical to those previously developed for the case in which no wave nonlinearities are taken into account.

The only difference is that the quadratic sum and difference frequency response functions required by the previous work are modified by a term which amounts to the linear ship system response to those quadratic wave field components produced by the wave field nonlinearities. The main assumption producing this result is that the magnitude of the nonlinear ship system response follows the magnitude of the wave excitation, and thus that nonlinear ship system response to quadratic wave field components is of higher order and may reasonably be neglected. The alternative to this assumption involves the development of a consistent higher order model which has not been attempted here.

In so far as practical computation of response functions is concerned, the conceptual idea is the same as in previous work. The linear response function is exactly the same. The present development requires that the response through second order terms in excitation be computed for pairs of waves, just as in the previous work. The difference in the context of potential flow hydrodynamics is that the assumed wave potential for the wave pair must include the potential due to the quadratic wave-wave interactions. The net effect, after the computed responses in sum and difference encounter frequencies have been identified, will be to add the linear response to the wave field nonlinearities noted above. This additional contribution is a slightly different potential flow problem because the wave-wave interaction components are not free waves. On the whole, however, if the addition of this effect is required, it could easily involve less work than is involved in the computation of quadratic response to the linear components of the wave field.

The most important statistic in practice involving quadratic response is the expectation, or mean value. In this special case the extension just outlined is not necessary. To second order, the mean response of a linear plus quadratic ship system to a short-crested wave field is not affected by wave field nonlinearities. The previously developed approach appears to apply equally well when quadratic wave-wave interactions are taken into account. The present development suggests that the mean value estimator may be robust through third order, although a consistent development through this order must be made for confirmation.

APPENDIX A QUADRATIC WAVE-WAVE INTERACTION THEORY

The theory of quadratic wave-wave interactions has been developed in a number of ways. The treatment here summarizes that of Longuet-Higgins [9], and will be specialized to the deep water case. To first order, the potential solutions for small amplitude gravity waves superimpose. The waves propagate independently and without interaction. To second order the waves interact, and the interaction produces a small, bounded modification to the wave motion. It is the second order modifications which are of interest here, and to study these it is sufficient to consider the interactions between pairs of waves.

To fix notation the parameters describing a single small-amplitude gravity wave will be taken as follows:

$$\begin{aligned}
 \lambda &= \text{wave length} \\
 \omega &= \text{wave frequency} \\
 \bar{\kappa} &= \text{vector wave number} \\
 &\quad \text{with components;} \\
 \kappa_X &= \kappa \cos(\mu) \\
 \kappa_Y &= \kappa \sin(\mu) \\
 \mu &= \text{wave direction} \\
 &= \tan^{-1}(\kappa_Y/\kappa_X) \\
 \kappa &= \text{scalar wave number} \\
 &= 2\pi/\lambda = |\bar{\kappa}| \\
 &= \omega^2/g \text{ for deep water}
 \end{aligned}$$

where the deep-water dispersion relationship just expressed will be assumed hereafter, and a fixed X-Y coordinate system in the undisturbed water surface with the Z coordinate positive upward is implied.

The derivation of Longuet-Higgins involves the postulation of a velocity potential satisfying the Laplace equation and the free surface boundary conditions. The result is three equations in the unknown potential, the velocity, and the free surface elevation. These equations are expanded in a Taylor series about the mean water position, and formal perturbation series are assumed for the potential, velocity, and surface elevation of two intersecting trains of regular waves. Substitution of the series, gathering terms of first and second order, with the requirement that the potential satisfy the Laplace equation results in a solution for the total velocity potential of the two intersecting waves (correct to second order for deep water) in the form

$$\varphi = \varphi_{01} + \varphi_{10} + \varphi_{02} + \varphi_{20} + \varphi_{11} \tag{A.1}$$

In this result the first and second terms are the well-known potentials for two first-order regular waves. The third and fourth terms represent quadratic interactions of each first-order wave with itself. Both are functions of time only, corresponding to the vanishing of second-

order velocities in a single first-order wave. The last term represents the potential resulting from second-order interactions.

To be specific, the two intersecting first-order deep-water wave trains may be assumed in the form

$$\begin{aligned}\zeta_{10} &= a_1 \cos(\bar{\kappa}_1 \cdot \bar{X} - \omega_1 t + \epsilon_1) \\ \zeta_{01} &= a_2 \cos(\bar{\kappa}_2 \cdot \bar{X} - \omega_2 t + \epsilon_2)\end{aligned}\quad (\text{A.2})$$

where \bar{X} represents the vector position in fixed coordinates and ϵ_n is an arbitrary phase angle.

The corresponding first-order potentials may be taken in the form

$$\begin{aligned}\varphi_{10} &= a_1 \omega_1 \kappa_1^{-1} \text{Exp}[\kappa_1 Z] \sin(\bar{\kappa}_1 \cdot \bar{X} - \omega_1 t + \epsilon_1) \\ \varphi_{01} &= a_2 \omega_2 \kappa_2^{-1} \text{Exp}[\kappa_2 Z] \sin(\bar{\kappa}_2 \cdot \bar{X} - \omega_2 t + \epsilon_2)\end{aligned}\quad (\text{A.3})$$

With these representations and the result for the interaction potential from Longuet-Higgins [9] the total velocity potential for deep water of two intersecting wave trains may be represented by

$$\begin{aligned}\varphi &= a_1 \omega_1 \kappa_1^{-1} \text{Exp}[\kappa_1 Z] \sin(\bar{\kappa}_1 \cdot \bar{X} - \omega_1 t + \epsilon_1) \\ &+ a_2 \omega_2 \kappa_2^{-1} \text{Exp}[\kappa_2 Z] \sin(\bar{\kappa}_2 \cdot \bar{X} - \omega_2 t + \epsilon_2) \\ &+ a_1 a_2 A(\bar{\kappa}_1, \bar{\kappa}_2) \text{Exp}[|\bar{\kappa}_1 - \bar{\kappa}_2| Z] \times \\ &\quad \sin\{(\bar{\kappa}_1 - \bar{\kappa}_2) \cdot \bar{X} - (\omega_1 - \omega_2)t + (\epsilon_1 - \epsilon_2)\} \\ &- a_1 a_2 B(\bar{\kappa}_1, \bar{\kappa}_2) \text{Exp}[|\bar{\kappa}_1 + \bar{\kappa}_2| Z] \times \\ &\quad \sin\{(\bar{\kappa}_1 + \bar{\kappa}_2) \cdot \bar{X} - (\omega_1 + \omega_2)t + (\epsilon_1 + \epsilon_2)\}\end{aligned}\quad (\text{A.4})$$

The third and fourth terms of Eq. A.4 are the second-order wave-wave interaction potential, and the as yet undefined functions in these terms may be written

$$A(\bar{\kappa}_1, \bar{\kappa}_2) = \frac{2\omega_1 \omega_2 (\omega_1 - \omega_2) \cos^2[(\mu_1 - \mu_2)/2]}{(\omega_1 - \omega_2)^2 - g|\bar{\kappa}_1 - \bar{\kappa}_2|}\quad (\text{A.5})$$

$$B(\bar{\kappa}_1, \bar{\kappa}_2) = \frac{2\omega_1 \omega_2 (\omega_1 + \omega_2) \sin^2[(\mu_1 - \mu_2)/2]}{(\omega_1 + \omega_2)^2 - g|\bar{\kappa}_1 + \bar{\kappa}_2|}\quad (\text{A.6})$$

With the velocity potential defined the total surface elevations of the two intersecting wave trains may be derived from the Longuet-Higgins analysis [9] and are written as follows:

$$\begin{aligned}
\tilde{\zeta}(\bar{X}, t) = & a_1 \cos(\bar{\kappa}_1 \cdot \bar{X} - \omega_1 t + \epsilon_1) \\
& + a_2 \cos(\bar{\kappa}_2 \cdot \bar{X} - \omega_2 t + \epsilon_2) \\
& + a_1^2 \frac{\omega_1^2}{2g} \cos\{2(\bar{\kappa}_1 \cdot \bar{X}) - 2\omega_1 t + 2\epsilon_1\} \\
& + a_2^2 \frac{\omega_2^2}{2g} \cos\{2(\bar{\kappa}_2 \cdot \bar{X}) - 2\omega_2 t + 2\epsilon_2\} \\
& + a_1 a_2 Q_f(\bar{\kappa}_1, \bar{\kappa}_2) \cos\{(\bar{\kappa}_1 + \bar{\kappa}_2) \cdot \bar{X} - (\omega_1 + \omega_2)t + (\epsilon_1 + \epsilon_2)\} \\
& + a_1 a_2 R_f(\bar{\kappa}_1, \bar{\kappa}_2) \cos\{(\bar{\kappa}_1 - \bar{\kappa}_2) \cdot \bar{X} - (\omega_1 - \omega_2)t + (\epsilon_1 - \epsilon_2)\}
\end{aligned} \tag{A.7}$$

As in the case of the potential, two new functions appear and are defined as follows:

$$Q_f(\bar{\kappa}_1, \bar{\kappa}_2) = \frac{\omega_1 \omega_2}{g} \left\{ \gamma - \frac{\gamma + 1 + \sqrt{\gamma^2 - \sin^2 \Gamma}}{\gamma + 1 - \sqrt{\gamma^2 - \sin^2 \Gamma}} \sin^2 \Gamma \right\} \tag{A.8}$$

$$R_f(\bar{\kappa}_1, \bar{\kappa}_2) = \frac{\omega_1 \omega_2}{g} \left\{ \gamma + \frac{\gamma - 1 + \sqrt{\gamma^2 - \cos^2 \Gamma}}{\gamma - 1 - \sqrt{\gamma^2 - \cos^2 \Gamma}} \cos^2 \Gamma \right\} \tag{A.9}$$

where:

$$\begin{aligned}
\omega_n &= \sqrt{|\bar{\kappa}_n|g} \\
\gamma &= \frac{1}{2} \left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} \right) \\
\Gamma &= (\mu_1 - \mu_2)/2
\end{aligned} \tag{A.10}$$

Special limiting cases of Eqs. A.8 and A.9 are as follows:

$$\begin{aligned}
Q_f(\bar{\kappa}_1, \bar{\kappa}_1) &= \omega_1^2/g \\
R_f(\bar{\kappa}_1, \bar{\kappa}_1) &= 0
\end{aligned} \tag{A.11}$$

The result for the corrected wave field, Eq. A.7, is very close in form to that obtained [3] for the response of a linear plus quadratic system to two superimposed sinusoids. If the position in the wave field is taken to be fixed, the expression involves six frequencies: those of the two small amplitude waves, their second harmonics, and the sum and difference frequencies. Missing is the shift in the mean obtained for the linear plus quadratic system. However, terms corresponding to a shift in the mean would involve the special case of $R_f(\bar{\kappa}_1, \bar{\kappa}_2)$ indicated in the second of Eqs. A.11, so that Eq. A.7 could be put in the form of

the response of a linear plus quadratic system. Essentially, the analysis says that there is no shift in the mean of the corrected wave field (as might be expected from continuity considerations for incompressible flow). The functions, Eqs. A.8 and A.9, are real and symmetric in their arguments, and thus they satisfy the requirements for quadratic frequency response functions noted in Dalzell [3]. These considerations along with the position of the functions in Eq. A.7, suggest that they may be regarded as quadratic frequency response functions which relate the sum and difference frequency components of the true wave field to the interactions of the two superimposed small amplitude waves. The first of Eqs. A.11 serves to put the coefficients of the second harmonics of Eq. A.7 into the form of the response of the linear plus quadratic system. Eq. A.7 contains no explicit notation for the linear frequency response function, which thus may be regarded as unity.

The preceding arguments indicate that the true wave field (to second order) may be regarded as the response of a linear plus quadratic system to a fictive "linear" small amplitude wave field. The quadratic system is defined by a linear frequency response function of unity and quadratic frequency response functions defined by Eqs. A.8 and A.9. Although it has not been done explicitly here, the wave potential, Eq. A.4, may also be interpreted in this way. These observations form the hydrodynamic background for Hasselmann's approach [1] to the consistency problem mentioned in the main body of the report.

Had the Longuet-Higgins derivation been for the true wave potential and elevations (to second order) corresponding to the superposition of N independent small amplitude waves, the perturbation expansion would have involved the interactions of all possible pairs of waves. These interactions would have given rise to a total potential of the form

$$\varphi = \sum_j^N (\varphi_{0j} + \varphi_{j0}) + \sum_j^N \sum_k^N \varphi_{jk} \quad (\text{A.12})$$

The summations on pairs of waves carries through to the elevations. With a manipulation of Eq. A.7, and a use of Eqs. A.11, the true wave field corresponding to the summation of N small amplitude waves may be put in the form

$$\begin{aligned} \zeta(\bar{X}, t) = & \sum_j^N \text{Re} \{ b_j \text{Exp}[i\bar{\kappa}_j \cdot \bar{X} - i\omega_j t] \} \\ & + \frac{1}{2} \sum_j^N \sum_k^N \text{Re} \{ b_j b_k Q_\zeta(\bar{\kappa}_j, \bar{\kappa}_k) \\ & \quad \text{Exp}[i(\bar{\kappa}_j + \bar{\kappa}_k) \cdot \bar{X} - i(\omega_j + \omega_k)t] \} \\ & + \frac{1}{2} \sum_j^N \sum_k^N \text{Re} \{ b_j b_k^* R_\zeta(\bar{\kappa}_j, \bar{\kappa}_k) \\ & \quad \text{Exp}[i(\bar{\kappa}_j - \bar{\kappa}_k) \cdot \bar{X} - i(\omega_j - \omega_k)t] \} \end{aligned} \quad (\text{A.13})$$

where $b_j = a_j \text{Exp}[i\epsilon_j]$

With the position vector fixed, this result is nearly identical to that for the response of a linear plus quadratic system to the sum of N small amplitude waves.

APPENDIX B FUNCTIONAL REPRESENTATION OF THE WAVE FIELD

The hydrodynamic arguments suggest that the true wave field may be taken to be a functional of the "linear", small amplitude field, at least to second order. In accordance with the previous work involving system response [3], if the linear field is continuous and differentiable and vanishes outside some finite region in the space of all admissible linear fields, or if the field is square integrable, the true wave field may be represented as a functional Taylor expansion in the neighborhood of zero wave field. With a further assumption to be noted, the Taylor expansion through the second order term may be written as follows:

$$\begin{aligned}
 \zeta_T(t, X, Y) = & g_0 \\
 & + \int \int \int g_1(t_1, x_1, y_1) \zeta_L(t-t_1, X-x_1, Y-y_1) dt_1 dx_1 dy_1 \\
 & + \int \dots \int g_2(t_1, x_1, y_1, t_2, x_2, y_2) \times \\
 & \quad \zeta_L(t-t_1, X-x_1, Y-y_1) \zeta_L(t-t_2, X-x_2, Y-y_2) \times \\
 & \quad dt_1 dx_1 dy_1 dt_2 dx_2 dy_2
 \end{aligned} \tag{B.1}$$

where the integrals are infinite. The kernels within the integrals in Eq. B.1 are functions of time and space differences and contain the physical relationships between the linear and true wave fields. The variables t , X , and Y denote absolute time and position in a fixed frame of reference. As in the previous work it has been assumed that the integrands differ from zero only in a finite range of the dummy arguments.

The difference between Eq. B.1 and the corresponding representation [3] of the response should be noted. The response of the ship system is supposed invariant with absolute position, and in the representation of the response this restriction was taken into account in the formulation for the encounter domain. In effect, had the response been presumed a function of absolute position, the form of the final functional representation would have been of exactly the same form as Eq. B.1

The hydrodynamics of the problem noted in Appendix A allows an immediate simplification of the representation. First, note that in Eq. B.1, when the linear field approaches zero, there is left the zero'th order term g_0 , and this term may be taken equal to zero on the basis that the true wave field must approach the linear field as the latter approaches zero. The hydrodynamic analysis also shows that there is no linear transformation of the linear field. Thus the kernel $g_1(t_1, x_1, y_1)$ of the second term of Eq. B.1 may be regarded as a delta function in the dummy variables. The net result of these considerations is that the functional representation of the true wave field becomes

$$\begin{aligned}
\zeta_T(t, X, Y) &= \zeta_L(t, X, Y) \\
&+ \int \dots \int g_2(t_1, x_1, y_1, t_2, x_2, y_2) \times \\
&\quad \zeta_L(t-t_1, X-x_1, Y-y_1) \zeta_L(t-t_2, X-x_2, Y-y_2) \times \\
&\quad dt_1 dx_1 dy_1 dt_2 dx_2 dy_2
\end{aligned} \tag{B.2}$$

As in the development of the interpretation of the time domain kernels of response, the quadratic kernel of Eq. B.2 may be interpreted by assuming that the linear wave field is the superposition of two small amplitude regular waves, which is the same as Eq. 1 with $N=2$. The steady state true wave field is obtained by substitution into Eq. B.2. The result of this operation is the same as Eq. 2, or Eq. A.13, with $N=2$ if the following transforms hold:

$$\begin{aligned}
Q_\zeta(\bar{\kappa}_1, \bar{\kappa}_2) &= \int \dots \int g_2(t_1, x_1, y_1, t_2, x_2, y_2) \times \\
&\quad \text{Exp}[-i(\kappa_{X1}x_1 + \kappa_{Y1}y_1 - \omega_1 t_1)] \times \\
&\quad \text{Exp}[-i(\kappa_{X2}x_2 + \kappa_{Y2}y_2 - \omega_2 t_2)] dt_1 \dots dy_2
\end{aligned} \tag{B.3}$$

$$\begin{aligned}
R_\zeta(\bar{\kappa}_1, \bar{\kappa}_2) &= \int \dots \int g_2(t_1, x_1, y_1, t_2, x_2, y_2) \times \\
&\quad \text{Exp}[-i(\kappa_{X1}x_1 + \kappa_{Y1}y_1 - \omega_1 t_1)] \times \\
&\quad \text{Exp}[i(\kappa_{X2}x_2 + \kappa_{Y2}y_2 - \omega_2 t_2)] dt_1 \dots dy_2
\end{aligned} \tag{B.4}$$

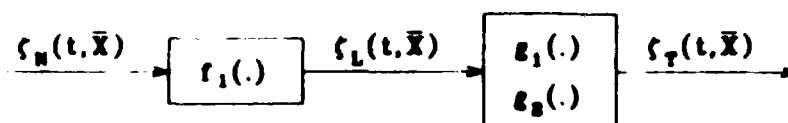
In these transforms it is implicitly assumed that frequency and wave number are connected by a dispersion relation. Thus the sum and difference response functions are denoted only as functions of vector wave numbers, although this is not mathematically essential.

To ensure that Eqs. B.3 and B.4 are valid representations the integral of the absolute value of the time-space kernel in the transforms must exist, and the kernel must be continuous [11]. Additionally, if the inverse transforms are valid, the kernel must be square integrable, and the same continuity and integrability conditions must hold for the inverse functions of wave number.

These last considerations pose a serious practical problem in view of the results of the hydrodynamic theory, Appendix A. The hydrodynamic functions corresponding to the left-hand sides of Eqs. B.3 and B.4 are defined in Eqs. A.8 through A.11. As the magnitude of the vector wave number arguments tends to infinity, the value of the functions does also. Thus the functions defined by Eqs. A.8 and A.9 do not satisfy the conditions required for the validity of Eqs. B.3 and B.4. As a practical example, were it required to generate the time-space kernel by transforming the response functions, there would be an immediate problem because the limits of the integrals of Eqs. B.3 and B.4 as well as those of their inverses are infinite. Hineno [7] ran into this problem in conjunction with a simulation of long-crested waves and solved it numerically with a truncation technique which is mathematically similar to that which will be developed next.

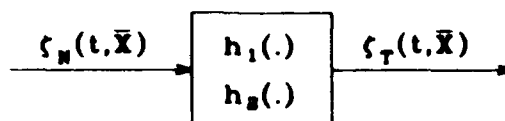
One way out of such a problem is analogous to the "white noise" excitation approach often used to mitigate similar problems in the time-only case. Essentially, in the time-only case white noise is operated upon with a linear transformation to produce an excitation to the system of interest, and the response functions connecting the noise to the system response are considered in lieu of those connecting the actual excitation to the response. In the time-only case the implementation is through well-known relationships developed for the results of cascading Volterra series.

Thus to pursue this direction in the present case it is necessary to hypothesize a noise or nominal field $\zeta_N(t, \bar{X})$ in which time and position vectors have the same meaning as before. The cascade of fields may be diagrammed as follows:



The nominal field is presumed to be operated upon by a linear system $f_1(.)$ in the time/space domain to produce the linear small amplitude wave field $\zeta_L(t, \bar{X})$ which is defined as before. The linear small amplitude field is acted upon by a linear plus quadratic system with kernels $g_1(.)$ and $g_2(.)$ to result in the true wave field $\zeta_T(t, \bar{X})$. The last functional relation is the same as Eq. B.1 with the zero'th term omitted.

The relationship desired between the nominal field and the true wave field may be diagrammed as follows:



where the kernels $h_1(.)$ and $h_2(.)$ incorporate the relationships between the nominal and linear fields and those between the linear and true wave fields.

If the zero'th order term is omitted, the corresponding time/space domain Taylor expansion of the true wave field in terms of the nominal field has the same form as Eq. B.1,

$$\begin{aligned} \zeta_T(t, X, Y) = & \int \int \int h_1(t_1, x_1, y_1) \zeta_N(t-t_1, X-x_1, Y-y_1) dt_1 dx_1 dy_1 \\ & + \int \dots \int h_2(t_1, x_1, y_1, t_2, x_2, y_2) \times \\ & \zeta_N(t-t_1, X-x_1, Y-y_1) \zeta_N(t-t_2, X-x_2, Y-y_2) \times \\ & dt_1 dx_1 dy_1 dt_2 dx_2 dy_2 \end{aligned} \quad (B.5)$$

On the assumption that the kernels in Eq. B.5 are sufficiently well behaved, the corresponding linear and quadratic response functions relating nominal and true wave fields may be written as follows:

$$T_T(\bar{\kappa}) = \int \int h_1(t_1, x_1, y_1) \text{Exp}[-i(\kappa_X x_1 + \kappa_Y y_1 - \omega t_1)] dt_1 dx_1 dy_1 \quad (\text{B.6})$$

$$Q_T(\bar{\kappa}_1, \bar{\kappa}_2) = \int \int h_2(t_1, x_1, y_1, t_2, x_2, y_2) \times \\ \text{Exp}[-i(\kappa_{X1} x_1 + \kappa_{Y1} y_1 - \omega_1 t_1)] \times \\ \text{Exp}[-i(\kappa_{X2} x_2 + \kappa_{Y2} y_2 - \omega_2 t_2)] dt_1 \dots dy_2 \quad (\text{B.7})$$

$$R_T(\bar{\kappa}_1, \bar{\kappa}_2) = \int \int h_2(t_1, x_1, y_1, t_2, x_2, y_2) \times \\ \text{Exp}[-i(\kappa_{X1} x_1 + \kappa_{Y1} y_1 - \omega_1 t_1)] \times \\ \text{Exp}[i(\kappa_{X2} x_2 + \kappa_{Y2} y_2 - \omega_2 t_2)] dt_1 \dots dy_2 \quad (\text{B.8})$$

To complete the definitions required for the cascade problem the assumed linear transformation between noise and linear wave fields must be defined and may be written in the time/space domain as follows:

$$\zeta_L(t, X, Y) = \int \int \int f_1(t_1, x_1, y_1) \zeta_N(t-t_1, X-x_1, Y-y_1) dt_1 dx_1 dy_1 \quad (\text{B.9})$$

The linear response function corresponding to the kernel of Eq. B.9 will be written

$$T_N(\bar{\kappa}) = \int \int f_1(t_1, x_1, y_1) \text{Exp}[-i(\kappa_X x_1 + \kappa_Y y_1 - \omega t_1)] dt_1 dx_1 dy_1 \quad (\text{B.10})$$

Now if the model, Eq. B.9, for the linear field is substituted into the initial expansion of the true field, Eq. B.2, the dummy time and space variables are renumbered, and a simple variable transformation is made (noting that all integrals are infinite), the result is sufficiently similar to the expansion, Eq. B.5, that the relations between the kernels of Eqs. B.2, B.9, and B.5 are developed. These turn out to be

$$h_1(t_1, x_1, y_1) = f_1(t_1, x_1, y_1) \quad (\text{B.11})$$

$$h_2(t_1, x_1, y_1, t_2, x_2, y_2) = \\ \int \int g_2(t_1-t_3, x_1-x_3, y_1-y_3, t_2-t_4, x_2-x_4, y_2-y_4) \times \\ f_1(t_3, x_3, y_3) f_1(t_4, x_4, y_4) dt_3 dx_3 \dots dy_4 \quad (\text{B.12})$$

The relations between the response functions of the cascade and those relating the nominal and true wave fields are developed by substituting Eqs. B.11 and B.12 into the definitions, Eqs. B.6 through B.9, with the following results:

$$T_T(\bar{\kappa}) = T_N(\bar{\kappa}) \quad (\text{B.13})$$

$$Q_T(\bar{\kappa}_1, \bar{\kappa}_2) = Q_f(\bar{\kappa}_1, \bar{\kappa}_2) T_N(\bar{\kappa}_1) T_N(\bar{\kappa}_2) \quad (B.14)$$

$$R_T(\bar{\kappa}_1, \bar{\kappa}_2) = R_f(\bar{\kappa}_1, \bar{\kappa}_2) T_N(\bar{\kappa}_1) T_N^*(\bar{\kappa}_2) \quad (B.15)$$

If the functions defined in Eqs. B.14 and B.15 can be made to satisfy the square integrability condition, the representation, Eq. B.5, will be valid, as will be the inverse transforms of Eqs. B.7 and B.8. For this to be true, the kernel of Eq. B.9 and its transform must first be square integrable. Inspection of Eqs. A.8 and A.9 discloses that under the worst circumstance (one wave number very much greater than the other) the values of the wave-wave interaction response functions are nearly proportional to the greater of the two scalar wave numbers. Thus if $T_N(\bar{\kappa})$ approaches zero at a rate greater than $1/|\bar{\kappa}|$ for large wave number, the left-hand sides of Eqs. B.14 and B.15 should approach zero sufficiently rapidly to satisfy the integrability condition.

It is possible to interpret the linear transformation hypothesized here in a number of ways. If the vector wave number spectrum of the linear field may be put in the form $T_N(\bar{\kappa}_j)T_N(\bar{\kappa}_k)$, then the linear transformation could be called a "spectral filter" in direct analogy to the common practice in the time-only case. (Scalar wave number spectra of ocean waves decay more rapidly with wave number than is required to ensure validity.) Alternatively, the transformation might be viewed as an analytical "window" which limits the range of possible wave interaction. In this interpretation $T_N(\bar{\kappa})$ could be chosen, for instance, to be purely real and equal to unity up to some cutoff wave number higher than the range of interest, after which a relatively arbitrary transition to zero could be made. The present development is essentially an artifice to overcome the existence problems brought about by the hydrodynamic development for the wave-wave interactions.

As usual, the functions defined by Eqs. B.6 and B.7 may be interpreted by assuming that the nominal field $\zeta_N(\bar{X}, t)$ is a summation of cosinusoidal waves in the form

$$\zeta_N(t, \bar{X}) = \sum_j^N \text{Re} \{ c_j \text{Exp}[i\bar{\kappa}_j \cdot \bar{X} - i\omega_j t] \} \quad (B.16)$$

where $c_j = a_j \text{Exp}[i\epsilon_j]$

and a_j and ϵ_j are the arbitrary amplitudes and phases of the N cosinusoids. When Eq. B.16 is substituted into the model, Eq. B.5, and the transform definitions are taken into account, there results the following expression for the true wave field:

$$\begin{aligned}
\zeta_T(t, \bar{X}) = & \sum_j^N \text{Re} \{ c_j T_N(\bar{\kappa}_j) \text{Exp}[i\bar{\kappa}_j \cdot \bar{X} - i\omega_j t] \} \\
& + \frac{1}{2} \sum_j^N \sum_k^N \text{Re} \{ c_j c_k Q_T(\bar{\kappa}_j, \bar{\kappa}_k) T_N(\bar{\kappa}_j) T_N(\bar{\kappa}_k) \\
& \quad \text{Exp}[i(\bar{\kappa}_j + \bar{\kappa}_k) \cdot \bar{X} - i(\omega_j + \omega_k)t] \} \\
& + \frac{1}{2} \sum_j^N \sum_k^N \text{Re} \{ c_j c_k^* R_T(\bar{\kappa}_j, \bar{\kappa}_k) T_N(\bar{\kappa}_j) T_N^*(\bar{\kappa}_k) \\
& \quad \text{Exp}[i(\bar{\kappa}_j - \bar{\kappa}_k) \cdot \bar{X} - i(\omega_j - \omega_k)t] \} \tag{B.17}
\end{aligned}$$

The difference between the results of the hydrodynamic analysis, Appendix A, and those just completed is that the functions defining the sum and difference wave interactions are modified by the linear response function presumed to relate the nominal and linear fields. Additionally, the linear components of the true field involve an arbitrary linear transformation not present in the development of Appendix A.

APPENDIX C
SPECTRAL REPRESENTATION OF THE NOMINAL WAVE FIELD

For present purposes the existence of a nominal underlying linear wave field $\zeta_N(X, Y, t)$ is assumed. It is further assumed that the elevation variance of the nominal wave field may be represented by

$$\text{Var}(\zeta_N) = \iint_{-\infty-\infty}^{\infty-\infty} F_N(\kappa_X, \kappa_Y) d\kappa_X d\kappa_Y \quad (\text{C.1})$$

where $F_N(\kappa_X, \kappa_Y)$ is a vector wave number spectrum.

It is also assumed that the nominal field is dispersive, homogeneous in space, stationary in time, and Gaussian. These assumptions together imply that the field has a "spectral representation". Although this representation is often written in the form of a Fourier-Stieltjes integral, it is convenient to work with the corresponding approximating Fourier sum. Accordingly, the approximating Fourier sum of the spectral representation of the nominal field may be written

$$\zeta_N(X, Y, t) = \sum_j \text{Re}\{W(j) \text{Exp}[i\kappa_{Xj}X + i\kappa_{Yj}Y - i\omega_j t]\} \quad (\text{C.2})$$

where $W(j)$ is a complex random function of the interval $\Delta\kappa_{Xj}, \Delta\kappa_{Yj}$. This function has the following properties: ($\langle \cdot \rangle$ denotes the ensemble expected value.)

$$\langle W(j) \rangle \equiv 0 \quad (\text{C.3})$$

$$\langle W(j) W^*(k) \rangle = 0 \quad (\text{for } j \neq k) \quad (\text{C.4})$$

$$\langle W(j) W^*(j) \rangle = 2F_N(\kappa_{Xj}, \kappa_{Yj}) \Delta\kappa_{Xj} \Delta\kappa_{Yj} \quad (\text{C.5})$$

$$\langle W(j) W(k) \rangle \equiv 0 \quad (\text{C.6})$$

Now the initial assumption on the nominal wave field of stationarity in time, with the Gaussian process assumption, as well as that of homogeneity imply that the random functions $W(j)$ are Gaussian and statistically independent for different wave numbers and directions of propagation. Thus from the properties of the third and fourth moments of Gaussian variables the following additional relations arise:

$$\langle W(j) W(k) W(m) \rangle = \langle W(j) W(k) W^*(m) \rangle \equiv 0 \quad (\text{C.7})$$

$$\langle W(j) W(k) W(m) W(n) \rangle = \langle W(j) W(k) W(m) W^*(n) \rangle \equiv 0 \quad (\text{C.8})$$

$$\langle W(j) W(k) W^*(m) W^*(n) \rangle = 0 \quad (\text{C.9})$$

unless (j,k = m,n or j,k = n,m) in which case

$$\begin{aligned}
 & \langle W(j)W(k)W^*(j)W^*(k) \rangle \\
 &= 4F_N(\kappa_{Xj}, \kappa_{Yj})F_N(\kappa_{Xk}, \kappa_{Yk}) \Delta\kappa_{Xj}\Delta\kappa_{Yj} \Delta\kappa_{Xk}\Delta\kappa_{Yk} \\
 &= 8[F_N(\kappa_{Xj}, \kappa_{Yj}) \Delta\kappa_{Xj}\Delta\kappa_{Yj}]^2 \quad (\text{if } k = j) \tag{C.10}
 \end{aligned}$$

The complex random amplitudes $W(j)$ are assumed to be Gaussian and zero mean. Eqs. C.3 through C.10 define the expectations of first through fourth order moments. Under this assumption the expectations of fifth and sixth order moments are also known. The fifth order expectations are identically zero because of the Gaussian assumption.

The sixth order expectation of Gaussian variables may be written as the sum of fifteen products of three second order expectations. The pairing of variables in the second order expectation products corresponds to all the ways in which the six variables may be grouped into three pairs. Noting Eqs. C.1 through C.3, the only way the second order expectation can be other than zero is when one of the terms in the product is a complex conjugate. Thus the only way the product of three second order expectations can be other than zero is that each second order expectation must be of the form of Eq. C.5. Accordingly, exactly three of the six variables in the sixth order expectation must be complex conjugates. Otherwise the process of grouping the six variables into pairs will result in at least one of the terms in each of the fifteen triple products of second order expectations being of the form of Eq. C.6, and thus the entire expectation will be zero.

Writing out the expansion of the sixth order expectation in which exactly three of the variables are complex conjugates, and applying Eq. C.2 results in the following definition of the only ways in which this expectation can be other than zero:

$$\begin{aligned}
 & \langle W(j)W(k)W(m)W^*(n)W^*(p)W^*(q) \rangle \\
 &= 8F_N(\kappa_{Xj}, \kappa_{Yj})F_N(\kappa_{Xk}, \kappa_{Yk})F_N(\kappa_{Xm}, \kappa_{Ym}) \Delta\kappa_{Xj}\Delta\kappa_{Yj} \Delta\kappa_{Xk}\Delta\kappa_{Yk} \Delta\kappa_{Xm}\Delta\kappa_{Ym} \\
 & \quad \text{If } n = j \text{ , } p = k \text{ , and } q = m \text{ ;} \\
 & \quad \text{or } n = j \text{ , } p = m \text{ , and } q = k \text{ ;} \\
 & \quad \text{or } n = k \text{ , } p = m \text{ , and } q = j \text{ ;} \\
 & \quad \text{or } n = k \text{ , } p = j \text{ , and } q = m \text{ ;} \\
 & \quad \text{or } n = m \text{ , } p = j \text{ , and } q = k \text{ ;} \\
 & \quad \text{or } n = m \text{ , } p = k \text{ , and } q = j \text{ .} \tag{C.11}
 \end{aligned}$$

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