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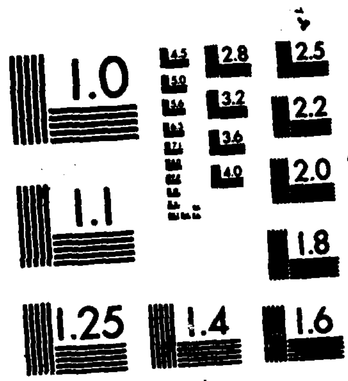

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Research Report CCS 559

CONE RATIO DATA ENVELOPMENT ANALYSIS AND MULTI-OBJECTIVE PROGRAMMING

by

A. Charnes
W.W. Cooper
Q.L. Wei*
Z.M. Huang

CENTER FOR CYBERNETIC STUDIES

The University of Texas
Austin, Texas 78712

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*The People's University of China in Beijing

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CENTER FOR CYBERNETIC STUDIES

A. Charnes, Director

College of Business Administration, 5.202
The University of Texas at Austin
Austin, Texas 78712-1177
(512) 471-1821

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ABSTRACT

A new "cone-ratio" Data Envelopment Analysis model which substantially generalizes the CCR model and the Charnes-Cooper Thrall approach characterizing its efficiency classes is herein developed and studied. It allows for infinitely many DMU's and arbitrary closed convex cones for the virtual multipliers as well as the cone of positivity of the vectors involved. Generalizations of linear programming and polar cone dualizations are the analytical vehicles employed.

KEY WORDS:

Data Envelopment Analysis
Multi-attribute Optimization;
Efficiency Analysis,
Cone-Ratio Models
Polar Cones

1. Introduction

We develop the following new "cone-ratio" DEA model which substantially generalizes the CCR model [3] as well as the approach of Charnes, Cooper and Thrall [8] to characterizing its efficiency classes:

$$(C^2WH)\begin{cases} \text{Max} \quad u^T y_{jo} / v^T x_{jo} \\ \text{s.t.} \quad v^T \bar{X} - u^T \bar{Y} \in K \\ \qquad v \in V, \quad u \in U, \quad (V \neq \emptyset, U \neq \emptyset) \end{cases}$$

where

$V \subset E_+^m$ is a closed convex cone, and Int $V \neq 0$.

$U \subset E_+^s$ is a closed convex cone, and Int $U \neq 0$.

$K \subset E^n$ is a closed convex cone, and

$$\delta_j = (0, \ldots, 0, \underbrace{1}_{j}, 0, \ldots, 0)^T \in -K^*, \quad j = 1, \ldots, n,$$

where $K^* = \{k \mid \hat{k}\, k \leq 0, \ \forall\, \hat{k} \in K\}$ is the "polar cone" of the set K.

$\bar{X} = [x_1, \ldots, x_n]$ is an m x n matrix.

$\bar{Y} = [y_1, \ldots, y_n]$ is an s x n matrix.

x_j is the input vector of DMU_j, $x_j \in \text{Int}(-V^*)$.

y_j is the output vector of DMU_j, $y_j \in \text{Int}(-U^*)$.

We shall require the following facts about acute cones. Cone U is said to be an "acute" cone if there exists an open half-space

$H = \{u: \ a^T u > 0\}$

such that $\bar{U} \subset H \cup \{0\}$, where $\bar{U}$ is the closure of U. It is easy to prove the following results:

(i) Int $U^* \neq 0$ if and only if U is an acute cone (See [13]).

(ii) When V is an acute cone, Int $V^* = \{v: \ v^T \hat{v} < 0, \forall \hat{v} \in V, \hat{v} \neq 0\}$ (See [13]).

(iii) When V is a closed convex cone and Int $V \neq \emptyset$, $V^* \cap (-V^*) = \{0\}$.

In Fact, since $(V^*)^* = V$ and Int $V \neq 0$, V^* is an acute cone. Hence there exists an open half-space $H = \{u: a^T u > 0\}$ such that

$$V^* \subset H \cup \{0\}$$

Namely

$$a^T v^* > 0 \text{ for all nonzero } v^* \in V^*, \tag{1}$$

So

$$a^T \mu^* < 0 \text{ for all nonzero } \mu^* \in -V^*. \tag{2}$$

Combining (1) and (2), we have

$$V^* \cap (-V^*) = \{0\}.$$

We can get $v^T x_{jo} > 0$ from $x_{jo} \in \text{Int}(-V^*)$ and $v \in V$, $v \neq 0$.

Employing the Charnes-Cooper transformation of fractional programming [2],

$$w = tv, \quad \mu = tu, \quad tv^T x_{jo} = 1$$

we obtain the following pair of dual convex programs as in Ben-Israel, Charnes and Kortanek [12]:

$$V_P = \max \ \mu^T y_{jo}$$

$$(P) \quad \text{s.t.} \quad w^T \bar{X} - \mu^T \bar{Y} \in K,$$

$$w^T x_{jo} = 1,$$

$$w \in V, \ \mu \in U.$$

and

$$V_D = \min \theta$$

$$(D) \quad \text{s.t.} \quad \bar{X}\lambda - \theta x_{jo} \in V^*,$$

$$-\bar{Y}\lambda + y_{jo} \in U^*,$$

$$\lambda \in -K^*.$$

Since $\delta_j \in -K^*$, we can get $K \subset E_+^n$. Therefore

$$V_P = \max \ \mu^T y_{jo} \leq w^T x_{jo} = 1.$$

<u>Definition 1:</u> DMU_{jo} is said to be "DEA-efficient" if there exists an optimal solution (w^0, μ^0) of program (P) such that

$$\mu^{oT} y_{jo} = 1$$

and

$$w^o \in \text{Int } V, \ \mu^o \in \text{Int } U.$$

Definition 2: DMU_{jo} is said to be "weak DEA-efficient" if there exists an optimal solution (w^o, μ^o) of program (P) such that

$$\mu^{oT} y_{jo} = 1.$$

The pair of dual programming problems (P) and (D) constitute a model in which convex cones are used to measure the efficiency of DMU's (In the appendix, we present the dual theorem concerning the dual programming problems (P) and (D).) In this paper, we establish the equivalence of DEA efficient solutions and nondominated solutions of multiobjective programming (VP) (see section 2). We also discuss the "projection" of decision making units onto the efficiency surface and the existence of DEA efficiency of DMUs (see section 3).

Let $V = E_+^m$, $U = E_+^s$ and $K = E_+^n$. The pair (P) and (D) is then the CCR model [3]

$$(P1)\begin{cases} V_{P1} = \max \mu^T y_{jo} \\ \text{s.t.} \quad w^T \bar{X} - \mu^T \bar{Y} \geq 0, \\ \qquad\qquad w^T x_{jo} = 1. \\ \qquad\qquad w, \mu \geq 0. \end{cases}$$

and

$$(D1)\begin{cases} V_{D1} = \min \theta \\ \text{s.t.} \quad \bar{X}\lambda - \theta x_{jo} \leq 0, \\ \qquad -\bar{Y}\lambda + y_{jo} \leq 0, \\ \qquad \lambda \geq 0. \end{cases}$$

If we set $K = E_+^n$ the pair (P) and (D) becomes

$$
(P2)\begin{cases} V_{P2} = \max \mu^T y_{jo} \\ \text{s.t.} \quad w^T\bar{X} - \mu^T\bar{Y} \geq 0, \\ \qquad\qquad w^T x_{jo} = 1 \\ \quad w \in V, \mu \in U. \end{cases}
$$

and

$$
(D2)\begin{cases} V_{D2} = \min \theta \\ \text{s.t.} \quad \bar{X}\lambda - \theta x_{jo} \in V^*, \\ \quad -\bar{Y}\lambda + y_{jo} \in U^*, \\ \qquad \lambda \geq 0. \end{cases}
$$

In (P2), the more general conditions $w \in V$, $\mu \in U$ replace the non-negativity conditions of the CCR model.

If we set $V = E_+^m$, $U = E_+^s$, we get the pair (P) and (D) as

$$
(P3)\begin{cases} V_{P3} = \max \mu^T y_{jo} \\ \text{s.t.} \quad w^T\bar{X} - \mu^T\bar{Y} \in K, \\ \qquad\qquad w^T x_{jo} = 1, \\ \qquad\qquad w, \mu \geq 0. \end{cases}
$$

and

$$
(D3)\begin{cases} V_{D3} = \min \theta \\ \text{s.t.} \quad \bar{X}\lambda - \theta x_{jo} \leq 0, \\ \quad -\bar{Y}\lambda + y_{jo} \leq 0, \\ \qquad \lambda \in -K^*. \end{cases}
$$

In (D3), we have $\lambda \in -K^*$ which replaces and generalizes the conical hull conditions about the production possibility set in the CCR model [6].

2. DEA Efficiency (or Weak DEA Efficiency) and Nondominated Solutions of Multiobjective Programming Problems

Consider the multiobjective programming problem

$$(V_P)\begin{cases} v-\min\,(f_1(x,y),\ldots,f_m(x,y),f_{m+1}(x,y),\ldots,f_{m+s}(x,y)) \\ \text{s.t.}\quad (x,y)\in T \end{cases}$$

where

$$T=\{(x,y):(x,y)\in(\bar{X}\lambda,\ \bar{Y}\lambda)+(-V^*,U^*),\ \lambda\in -K^*\}$$

is the production possibility set (it is easy to show that T is a convex cone). Also

$$f_k(x,y)=\begin{cases} x_k, & 1\le k\le m, \\ -y_{k-m}, & m+1\le k\le m+s \end{cases}$$

as in C^2GS^2, where

$$x=(x_1,\ldots,x_k,\ldots,x_m)^T,$$

$$y=(y_1,\ldots,y_r,\ldots,y_s)^T.$$

Since $\delta_j\in -K^*$, we have the input-output vector pairs $(x_j,y_j)\in T$, $j=1,\ldots,n$.

Let

$$f(x,y)=(f_1(x,y),\ldots,f_{m+s}(x,y))^T.$$

Definition 3: $(x_{jo},y_{jo})\in T$ is said to be a nondominated solution of the (VP) associated with $V^*\times U^*$ if there exists no $(x,y)\in T$ such that

$$f(x,y)\in f(x_{jo},y_{jo})+(V^*,U^*),\ (x,y)\ne(x_{jo},y_{jo})$$

Namely, there exists no $(x,y)\in T$ such that

$$(x,-y)\in(x_{jo},-y_{jo})+(V^*,U^*),\ (x,y)\ne(x_{jo},y_{jo})$$

Definition 4: $(x_{jo},y_{jo})\in T$ is said to be a nondominated solution of (VP) associated with $\operatorname{Int}V^*\times\operatorname{Int}U^*$ if there exists no $(x,y)\in T$ such that

$$f(x,y)\in f(x_{jo},y_{jo})+(\operatorname{Int}V^*,\operatorname{Int}U^*)$$

Namely, there exists no $(x,y)\in T$ such that

$$(x,-y)\in(x_{jo},-y_{jo})+(\operatorname{Int}V^*,\operatorname{Int}U^*)$$

In this section, we will study the relations between DEA efficiency (or weak DEA efficiency) of DMU's and nondominated solutions of (VP) associated with $V^* \times U^*$ (or Int $V^* \times$ Int U^*).

Let

$$S = \{(x_j, y_j), \quad j = 1, \ldots, n\}$$

$$\tilde{S} = \{(\bar{X}\lambda, \bar{Y}\lambda): \quad \lambda \in -K^*\}$$

$$T = \{(x, y) : (x, y) \in (\bar{X}\lambda, \bar{Y}\lambda) + (-V^*, U^*), \quad \lambda \in -K^*\}$$

Lemma 1. Let (w^o, μ^o) be an optimal solution of (P), and $\mu^{oT} y_{jo} = 1$. Then for an arbitrary $(x, y) \in T$ we have

$$w^{oT}x - \mu^{oT}y \geq 0 = w^{oT}x_{jo} - \mu^{oT}y_{jo}.$$

Proof: Since $\mu^{oT}y_{jo} = 1$, we have

$$w^{oT}x_{jo} - \mu^{oT}y_{jo} = 0$$

For an arbitrary $(x, y) \in \tilde{S}$ there exists $\lambda \in -K^*$ such that

$$(x, y) = (\bar{X}\lambda, \bar{Y}\lambda)$$

Since $w^{oT}\bar{X} - \mu^{oT}\bar{Y} \in K$, then we get

$$w^{oT}x - \mu^{oT}y = w^{oT}\bar{X}\lambda - \mu^{oT}\bar{Y}\lambda = (w^{oT}\bar{X} - \mu^{oT}\bar{Y})\lambda \geq 0.$$

For an arbitrary $(x, y) \in T$, there exists $\lambda \in -K^*$, $v^* \in -V^*$, $u^* \in -U^*$ such that

$$(x, y) = (\bar{X}\lambda + v^*, \bar{Y}\lambda - u^*)$$

So

$$w^{oT}x - \mu^{oT}y = w^{oT}(\bar{X}\lambda + v^*) - \mu^{oT}(\bar{Y}\lambda - u^*)$$

$$= (w^{oT}\bar{X} - \mu^{oT}Y)\lambda + w^{oT}v^* + \mu^{oT}u^* \geq 0.$$

Q.E.D.

Theorem 1 Let DMU_{jo} be DEA efficient. Then (x_{jo}, y_{jo}) is a nondominated solution of (VP) associated with $V^* \times U^*$.

Proof: If (x_{j0}, y_{j0}) is not a nondominated solution of (VP) associated with $V^* \times U^*$, then there exists $(\bar{x}, \bar{y}) \in T$ such that

$$(\bar{x}, -\bar{y}) \in (x_{j0}, -y_{j0}) + (V^*, U^*),\ (\bar{x}, \bar{y}) \neq (x_{j0}, y_{j0})$$

that is, there exists $(v^*, u^*) \in (V^*, U^*)$, $(v^*, u^*) \neq 0$ such that

$$(\bar{x}, -\bar{y}) = (x_{j0}, -y_{j0}) + (v^*, u^*)$$

Since DMU_{j0} is DEA efficient, there exists an optimal solution $(w^0, \mu^0) \in \text{Int } V \times \text{Int } U$ such that

$$\mu^{0T} y_{j0} = 1.$$

We have

$$\begin{aligned} & w^{0T}\bar{x} - \mu^{0T}\bar{y} \\ &= (w^{0T}x_{j0} - \mu^{0T}y_{j0}) + (w^{0T}v^* + \mu^{0T}u^*) \\ &< w^{0T}x_{j0} - \mu^{0T}y_{j0} \end{aligned}$$

as we shall see. For consides $(v^{*T}, u^{*T}) \neq 0$ and without loss of generality, suppose $v^* \neq 0$. Since $w^0 \in \text{Int } V$, $v^* \in V^*$ and V is acute, we have $w^{0T}v^* < 0$, $\mu^{0T}u^* \leq 0$, which suffices.

But by Lemma 1, we have

$$w^{0T}\bar{x} - \mu^{0T}\bar{y} \geq w^{0T}x_{j0} - \mu^{0T}y_{j0}$$

. a contradiction.

Q.E.D.

Theorem 2. Let (x_{j0}, y_{j0}) be a nondominated solution of (VP) associated with $V^* \times U^*$ and let Assumption (A) hold (see Appendix). Then DMU_{j0} is DEA efficient.

Proof: Since $\tilde{S} \subset T$, the following system (I) is inconsistent:

$$(I)\ \begin{cases} (\bar{X}\lambda,\ -\bar{Y}\lambda) \in (x_{j0}, -y_{j0}) + (V^*, U^*),\ (\bar{X}\lambda, \bar{Y}\lambda) \neq (x_{j0}, y_{j0}) \\ \lambda \in -K^* \end{cases}$$

Now let us consider the pair of dual programming problems

$$(\bar{P})\begin{cases} V_{\bar{P}} = \min\,(w^T x_{jo} - \mu^T y_{jo}) \\ \text{s.t.} \quad w^T\bar{X} - \mu^T\bar{Y} \in K, \\ \qquad w - \tau \in V, \\ \qquad \mu - \hat{\tau} \in U. \end{cases}$$

and

$$(\bar{D})\begin{cases} V_{\bar{D}} = \max\,(\tau^T s^- + \hat{\tau}^T s^+) \\ \text{s.t.} \quad \bar{X}\lambda - x_{jo} + s^- = 0, \\ \qquad -\bar{Y}\lambda + y_{jo} + s^+ = 0, \\ \qquad \lambda \in -K^*,\ s^- \in -V^*,\ s^+ \in -U^*. \end{cases}$$

where $\tau \in \text{Int } V$, $\hat{\tau} \in \text{Int } U$.

First, we want to show $V_{\bar{D}} = 0$. For an arbitrary feasible solution (λ, s^-, s^+) of $(\bar{D})$, since $s^- \in -V^*$, $\tau \in \text{Int } V$, $s^+ \in -U^*$, $\hat{\tau} \in \text{Int } U$, then

$$\tau^T s^- \geq 0, \quad \hat{\tau}^T s^+ \geq 0,$$

so $V_{\bar{D}} \geq 0$. If $V_{\bar{D}} > 0$, namely there exists an optimal solution $(\lambda^0, s^{0-}, s^{0+})$ of $(\bar{D})$, such that

$$V_{\bar{D}} = \tau^T s^{0-} + \hat{\tau}^T s^{0+} > 0,$$

then we have

$$(X\lambda^0,\ -\bar{Y}\lambda^0) = (x_{jo},\ -y_{jo}) + (-s^{0-}, -s^{0+}),\ (-s^{0-}, -s^{0+}) \in (V^*,\ U^*),\ (s^{0-},\ s^{0+}) \neq 0$$

This yields a contradiction because (I) is inconsistent.

By the dual theorem (see Appendix, Th. 3), we have $V_{\bar{P}} = 0$.

Secondly, let $(\tilde{w}, \tilde{\mu})$ be an optimal solution of $(\bar{P})$, and let

$$w^0 = \tilde{w} / \tilde{w}^T x_{jo} \quad , \quad \mu^0 = \tilde{\mu} / \tilde{w}^T x_{jo}$$

Then we have

$$w^{oT}x_{jo} = \mu^{oT}y_{jo} = 1,$$

$$w^{oT}\bar{X} - \mu^{oT}\bar{Y} \in K$$

$$w^{0} \in \tau / \tilde{w}^{T}x_{jo} + V \subset \text{Int } V \quad (\text{since } \tau \in \text{Int } V)$$

$$\mu^{0} \in \hat{\tau} / \tilde{w}^{T}x_{jo} + U \subset \text{Int } U \quad (\text{since } \hat{\tau} \in \text{Int } U)$$

Namely,

$$\max \mu^{T}y_{jo} = \mu^{oT}y_{jo} = 1,$$

$$w^{oT}X - \mu^{oT}Y \in K,$$

$$w^{oT}x_{jo} = 1$$

$$w^{0} \in \text{Int } V,\ \mu^{0} \in \text{Int } U$$

So DMU_{jo} is DEA efficient.

Q.E.D.

Theorem 3 Let DMU_{jo} be weak DEA efficient. Then (x_{jo}, y_{jo}) is a nondominated solution of (VP) associated with Int V^* x Int U^*.

Its proof is similar to Theorem 1.

Theorem 4 Let (x_{jo}, y_{jo}) be a nondominated solution of (VP) associated with Int V^* x Int U^*, and Assumption (B) hold (see Appendix). Then DMU_{jo} is weak DEA efficient.

Proof. Since (x_{jo}, y_{jo}) is a nondominated solution of (VP) associated with Int V^* x Int U^*, then the following system (II) is inconsistent.

$$(\text{II}) \begin{cases} (X\lambda, -\bar{Y}\lambda) \in (x_{jo}, -y_{jo}) + (\text{Int } V^*,\ \text{Int } U^*) \\ \lambda \in -K^* \end{cases}$$

Consider the pair of dual programming problems

$$
(\hat{P})\begin{cases}
V_{\hat{P}} = \min\,(w^T x_{j_0} - \mu^T y_{j_0}) \\
\text{s.t.}\quad w^T\bar{X} - \mu^T\bar{Y} \in K, \\
\qquad w - v \in V, \\
\qquad \mu - u \in U, \\
\qquad \tau^T v + \hat{\tau}^T u = 1, \\
\qquad v \in V, \quad u \in U.
\end{cases}
$$

and

$$
(\hat{D})\begin{cases}
V_{\hat{D}} = \max z \\
\text{s.t.}\quad \bar{X}\lambda - x_{j_0} + s^- = 0, \\
\qquad -\bar{Y}\lambda + y_{j_0} + s^+ = 0, \\
\qquad z\tau - s^- \in V^*, \\
\qquad z\hat{\tau} - s^+ \in U^*, \\
\qquad \lambda \in -K, \quad s^- \in -V^*, \quad s^+ \in -U^*
\end{cases}
$$

where $\tau \in \operatorname{Int} V$, $\hat{\tau} \in \operatorname{Int} U$.

Since $\delta_j \in -K^*$, $j = 1, \dots, n$, then

$$(\bar{\lambda}, \bar{s}^-, \bar{s}^+, \bar{z}) = (\delta_{j_0}, 0, 0, 0)$$

is a feasible solution of $(\hat{D})$, and

$$V_{\hat{D}} = \max z \geq 0.$$

First, we have to show $V_{\hat{D}} = 0$. If $V_{\hat{D}} > 0$, there exists an optimal solution $(\lambda^0, s^{0-}, s^{0+}, z^0)$ of $(\hat{D})$ such that

$$V_{\hat{D}} = \max z = z^0 > 0.$$

Since $V \subset E_+{}^m$, then

$$\operatorname{Int} V^* = \{w\colon\ w^T v < 0,\ \forall v \in V \text{ and } v \neq 0\}.$$

Because of $z^0\tau > 0$, we have

$$(-z^0\tau)^T v < 0, \quad \text{for all } v \in V \text{ and } v \neq 0.$$

So

$$-z^0\tau \in \text{Int } V^*.$$

Similarly we can show

$$-z^0\hat{\tau} \in \text{Int } U^*.$$

Hence we have

$$-s^{0-} \in V^* - z^0\tau \subset \text{Int } V^*,$$

$$-s^{0+} \in U^* - z^0\hat{\tau} \subset \text{Int } U^*.$$

This yields a contradiction because (II) is inconsistent.

By the dual theorem (see Appendix, Th. 4), we have $V_{\hat{P}} = V_{\hat{D}} = 0$.

Secondly, let $(\bar{w}, \bar{\mu}, \bar{v}, \bar{u})$ be an optimal solution of $(\hat{P})$, then we have

$$\bar{w} \in \bar{v} + V \subset V,$$

$$\mu \in u + U \subset U.$$

Since

$$\bar{w} = \bar{v} + v^{**}, \quad v^{**} \in V$$

$$\bar{\mu} = \bar{u} + u^{**}, \quad u^{**} \in U$$

we have

$$\tau^T\bar{w} + \hat{\tau}^T\bar{\mu} = (\tau^T\bar{v} + \hat{\tau}^T\bar{u}) + (\tau^T v^{**} + \hat{\tau}^T u^{**}) \geq 1.$$

So $(\bar{w}, \bar{\mu}) \neq 0$. Since $V_{\hat{P}} = V_{\hat{D}} = 0$, then we get

$$\bar{w}^T x_{jo} = \bar{\mu}^T y_{jo}.$$

Therefore $\bar{w} \neq 0, \bar{\mu} \neq 0$. Let

$$w^0 = \bar{w} / \bar{w}^T x_{jo} \quad , \qquad \mu^0 = \bar{\mu} / \bar{w}^T x_{jo}$$

we have

$$\mu^{oT} y_{jo} = w^{oT} x_{jo} = 1,$$

$$w^{oT}\bar{X} - \mu^{oT}\bar{Y} \in K,$$

$$w^0 \in \bar{v} / \bar{w}^T x_{jo} + V \subset V$$

$$\mu^0 \in \bar{u} / \bar{w}^T x_{jo} + U \subset U$$

Namely,

$$\begin{cases} \max & \mu^T y_{jo} = \mu^{oT} y_{jo} = 1 \\ \text{s.t.} & w^T \bar{X} - \mu^T \bar{Y} \in K, \\ & w^T x_{jo} = 1, \\ & w \in V, \quad \mu \in U \end{cases}$$

and $w^0 \in V$, $\mu^0 \in U$. So DMU_{jo} is weak DEA efficient.

Q.E.D.

3. Efficiency Surface "Projection" and Existence of DEA Efficiency

For an arbitrary $(x_{jo}, y_{jo}) \in S = \{(x_j, y_j), \quad j = 1, \ldots, n\}$, we consider the following programming problem:

$$(P^{jo}) \begin{cases} \max & (\tau^T s^- + \hat{\tau}^T s^+) \\ \text{s.t.} & \bar{X}\lambda - x_{jo} + s^- = 0, \\ & -\bar{Y}\lambda + y_{jo} + s^+ = 0, \\ & \lambda \in -K^*, \; s^- \in -V^*, \; s^+ \in -U^* \end{cases}$$

where $\tau \in \text{Int } V$, $\hat{\tau} \in \text{Int } U$.

Suppose $(\lambda^0, s^{0-}, s^{0+})$ is an optimal solution of (P^{jo}). Let

$$\hat{x} = \bar{X}\lambda^0 = x_{jo} - s^{0-},$$

$$\hat{y} = \bar{Y}\lambda^0 = y_{jo} + s^{0+}.$$

We call $(\hat{x}, \hat{y})$ the "projection" of DMU_{jo} onto the efficiency "surface" of the production function (see [4], p 70).

It is obvious that $(\hat{x}, \hat{y}) \in T$. Since $y_{jo} \in \text{Int}(-U^*)$, $s^{0+} \in -U^*$, we have

$$\hat{y} = y_{jo} + s^{0+} \in \text{Int}(-U^*).$$

Because $0 \bar{\in} \text{Int}(-U^*)$, then we get $\hat{y} \neq 0$. Therefore $(\hat{x}, \hat{y}) \neq 0$

<u>Theorem 5.</u> The projection $(\hat{x}, \hat{y})$ of DMU_{jo} is a nondominated solution of the (VP) associated with $V^* \times U^*$.

<u>Proof.</u> Suppose $(\hat{x}, \hat{y})$ is not a nondominated solution of (VP) associated with $V^* \times U^*$. Then there exists $(\bar{x}, \bar{y}) \in T$ and $(\hat{v}, \hat{u}) \in (V^*, U^*)$ such that

$$(\bar{x}, \bar{y}) = (\hat{x}, \hat{y}) + (\hat{v}, \hat{u}), \quad (\hat{v}, \hat{u}) \neq 0$$

Since $(\bar{x}, \bar{y}) \in T$, there exists $\bar{\lambda} \in -K^*$ and $(\bar{v}, \bar{u}) \in (V^*, U^*)$ such that

$$(\bar{x}, \bar{y}) = (\bar{X}\bar{\lambda}, \ \bar{Y}\bar{\lambda}) + (-\bar{v}, \bar{u})$$

So we have

$$(\bar{X}\bar{\lambda}, -\bar{Y}\bar{\lambda}) = (\hat{x}, \ -\hat{y}) + (\hat{v} + \bar{v}, \hat{u} + \bar{u}) \in (\hat{x}, -\hat{y}) + (V^*, U^*) \qquad (1)$$

and

$$(\hat{v} + \bar{v}, \hat{u} + \bar{u}) \neq 0 \qquad (2)$$

(In fact, if $(v + \bar{v}, \hat{u} + \bar{u}) = 0$, we would have $(\bar{v}, \bar{u}) = (\hat{v}, -\hat{u}) \in (V^*, U^*)$ Since $(\hat{v}, \hat{u}) \neq 0$, without loss of generality, let $\hat{v} \neq 0$. Then we have $\bar{v} = -\hat{v} \in V^*$. This yields a contradiction to $V^* \cap (-V^*) = \{0\}$).

Let

$$v^* = \hat{v} + \bar{u} \in V^*, \qquad u^* = \hat{u} + \bar{u} \in U^*.$$

By (1) and (2), we have

$$(\bar{X}\bar{\lambda}, -\bar{Y}\bar{\lambda}) = (\hat{x}, -\hat{y}) + (v^*, u^*), \quad (v^*, u^*) \neq 0$$

so

$$\bar{X}\bar{\lambda} = \hat{x} + v^* = x_{j_0} - s^{0-} + v^*,$$

$$-\bar{Y}\bar{\lambda} = -\hat{y} + u^* = -y_{j_0} - s^{0+} + u^*.$$

Then we get

$$\begin{cases} \bar{X}\bar{\lambda} + (s^{0-} - v^*) = x_{j_0}, \\ -\bar{Y}\bar{\lambda} + (s^{0+} - u^*) = -y_{j_0}, \\ \bar{\lambda} \in -K^*, \quad s^{0-} - v^* \in -V^*, \quad s^{0+} - u^* \in -U^*. \end{cases}$$

Further, since $\tau \in \text{int } V$, $v^* \in V^*$, $\hat{\tau} \in \text{int } U$, $u^* \in U^*$, we have

$$\tau^T v^* \leq 0, \ \hat{\tau}^T u^* \leq 0.$$

We know that $(v^*, u^*) \neq 0$, so

$$\tau^T v^* + \hat{\tau}^T u^* < 0.$$

Thus

$$\tau^T(s^{o-} - v^*) + \hat{\tau}^T(s^{o+} - u^*)$$
$$= (\tau^T s^{o-} + \hat{\tau}^T s^{o+}) - (\tau^T v^* + \hat{\tau}^T u^*)$$
$$> \tau^T s^{o-} + \hat{\tau}^T s^{o+}.$$

This contradicts the fact that $(\lambda^o, s^{o-}, s^{o+})$ is an optimal solution of (P_{j_o}). Thus $(\hat{x}, \hat{y})$ is a nondominated solution of (VP) associated with $V^* \times U^*$.

Q.E.D.

Corollary 1. Let

$$(x_{n+1}, y_{n+1}) = (\hat{x}, \hat{y})$$

where $(\hat{x}, \hat{y})$ is the projection of DMU_{j_0}. Then DMU_{n+1} is DEA efficient.

Proof: By Theorem 1 and Theorem 2, DEA efficiency and nondominated solution of (VP) are equivalent properties.

Q.E.D.

Theorem 6 Suppose

(i) For arbitrary $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T \in -K^*$, we have

$$\lambda_j V^* \subset V^*, \quad \lambda_j U^* \subset U^*, \quad j = 1, 2, \ldots, n.$$

where

$$\lambda_j V^* = \{\lambda_j v^* : v^* \in V^*\}, \quad \lambda_j U^* = \{\lambda_j u^* : u^* \in U^*\}.$$

(ii) For arbitrary $\lambda^i = (\lambda_1{}^i, \lambda_2{}^i, \ldots, \lambda_n{}^i)^T \in -K^*, \quad i = 0, 1, \ldots, n,$

we have

$$(\lambda^1, \lambda^2, \ldots, \lambda^n)\lambda^o = \left(\sum_{k=1}^{n} \lambda_1{}^k \lambda_k{}^o, \sum_{k=1}^{n} \lambda_2{}^k \lambda_k{}^o, \ldots, \sum_{k=1}^{n} \lambda_n{}^k \lambda_k{}^o\right) \in -K^*$$

Then there exists at least one DMU_{j_0} $(1 \le j_0 \le n)$ which is DEA efficient.

Proof: By Theorem 1 and Theorem 2, it is only necessary to show that there exists some $(x_{jo}, y_{jo}) \in S$ such that it is a nondominated solution of (VP) associated with $V^* \times U^*$.

Suppose for an arbitrary j $(j = 1, \ldots, n)$, (x_j, y_j) is not a nondominated solution of (VP) associated with $V^* \times U^*$, then there exist $(\bar{x}_j, \bar{y}_j) \in T$ and $\bar{\lambda}^j \in -K^*$ such that

$$(\bar{x}_j, \bar{y}_j) \in (\bar{X}\bar{\lambda}^j, \ \bar{Y}\bar{\lambda}^j) + (-V^*, U^*) \tag{3}$$

and

$$(x_j, -\bar{y}_j) \in (x_j, -y_j) + (V^*, U^*), \ (\bar{x}_j, \bar{y}_j) \neq (x_j, y_j), \quad j = 1, 2, \ldots, n \tag{4}$$

By (3), there exist $\bar{v}^j \in V^*$, $\bar{u}^j \in U^*$ such that

$$(\bar{x}_j, \bar{y}_j) = (\bar{X}\bar{\lambda}^j, \bar{Y}\bar{\lambda}^j) + (-\bar{v}^j, \bar{u}^j) \tag{3'}$$

By (4), there exist $v \in V^*$, $u \in U^*$ such that

$$(x_j, y_j) = (x_j, y_j) + (v^j, \ -u^j), \ (v^j, u^j) \neq 0 \tag{4'}$$

By Theorem 5, there exists $\lambda^o \in -K^*$, $\lambda^o \neq 0$ such that

$$(\hat{x}, \hat{y}) = (\bar{X}\lambda^o, \ \bar{Y}\lambda^o) \tag{5}$$

is a nondominated solution of (VP).

Multiplying (4') by $\lambda_j{}^o$ and summing over j, we get

$$\begin{pmatrix} \sum_{j=1}^{n} \bar{x}_j\lambda_j{}^o \\ \\ \sum_{j=1}^{n} \bar{y}_j\lambda_j{}^o \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} x_j\lambda_j{}^o \\ \\ \sum_{j=1}^{n} y_j\lambda_j{}^o \end{pmatrix} + \begin{pmatrix} \sum_{j=1}^{n} v^j\lambda_j{}^o \\ \\ -\sum_{j=1}^{n} u^j\lambda_j{}^o \end{pmatrix}$$

namely,

$$\begin{pmatrix} \sum_{j=1}^{n} \bar{x}_j\lambda_j{}^o \\ \\ -\sum_{j=1}^{n} \bar{y}_j\lambda_j{}^o \end{pmatrix} = \begin{pmatrix} \bar{X}\lambda^o \\ \\ -\bar{Y}\lambda^o \end{pmatrix} + \begin{pmatrix} \sum_{j=1}^{n} v^j\lambda_j{}^o \\ \\ \sum_{j=1}^{n} u^j\lambda_j{}^o \end{pmatrix} \tag{6}$$

By (6), (5) and assumption (i), we have

$$\begin{pmatrix} \sum_{j=1}^{n} \bar{x}_j \lambda_j{}^{o} \\ \\ -\sum_{j=1}^{n} \bar{y}_j \lambda_j{}^{o} \end{pmatrix} = \begin{pmatrix} \hat{x} \\ \\ -\hat{y} \end{pmatrix} + \begin{pmatrix} \sum_{j=1}^{n} v^j \lambda_j{}^{o} \\ \\ \sum_{j=1}^{n} u^j \lambda_j{}^{o} \end{pmatrix} \in \begin{pmatrix} \hat{x} \\ \\ -\hat{y} \end{pmatrix} + \begin{pmatrix} V^* \\ \\ U^* \end{pmatrix} \tag{7}$$

By (3′), we have

$$\begin{pmatrix} \sum_{j=1}^{n} \bar{x}_j \lambda_j{}^{o} \\ \\ \sum_{j=1}^{n} \bar{y}_j \lambda_j{}^{o} \end{pmatrix} = \begin{pmatrix} \sum_{K=1}^{n} \left(\sum_{j=1}^{n} x_j \bar{\lambda}_j{}^{K} - \bar{v}^{K} \right) \lambda_K{}^{o} \\ \\ \sum_{K=1}^{n} \left(\sum_{j=1}^{n} y_j \bar{\lambda}_j{}^{K} + \bar{u}^{K} \right) \lambda_K{}^{o} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^{n} \left(\sum_{K=1}^{n} \bar{\lambda}_j{}^{K} \lambda_K{}^{o} \right) x_j \\ \\ \sum_{j=1}^{n} \left(\sum_{K=1}^{n} \bar{\lambda}_j{}^{K} \lambda_K{}^{o} \right) y_j \end{pmatrix} + \begin{pmatrix} -\sum_{K=1}^{n} v^{K} \lambda_K{}^{o} \\ \\ \sum_{K=1}^{n} u^{K} \lambda_K{}^{o} \end{pmatrix}$$

By assumption (ii), we have

$$\left(\sum_{K=1}^{n} \lambda_1{}^{K} \lambda_K{}^{o}, \ \sum_{K=1}^{n} \bar{\lambda}_2{}^{K} \lambda_K{}^{o}, \ldots, \ \sum_{K=1}^{n} \bar{\lambda}_n{}^{K} \lambda_K{}^{o} \right)^{T} \in -K^*$$

By assumption (i), we have

$$\sum_{K=1}^{n} \bar{v}^{K} \lambda_K{}^{o} \in V^*, \quad \sum_{K=1}^{n} \bar{u}^{K} \lambda_K{}^{o} \in U^*$$

so we get

$$\begin{pmatrix} \sum_{j=1}^{n} \bar{x}_j \lambda_j^0 \\ \\ \sum_{j=1}^{n} \bar{y}_j \lambda_j^0 \end{pmatrix} \in T \tag{8}$$

Since $\lambda^0 \neq 0$, then

$$\begin{pmatrix} \sum_{j=1}^{n} v^j \lambda_j^0 \\ \\ \sum_{j=1}^{n} u^j \lambda_j^0 \end{pmatrix} \neq 0 \tag{9}$$

In fact, if

$$\begin{pmatrix} \sum_{j=1}^{n} v^j \lambda_j^0 \\ \\ \sum_{j=1}^{n} u^j \lambda_j^0 \end{pmatrix} = 0 \tag{10}$$

by $(v^j, u^j) \neq 0$, $j = 1, \dots, n$, and $\lambda^0 \neq 0$, without loss of generality, let $\lambda_{j'} \neq 0$ and $v^{j'} \neq 0$. Then by (10), we have

$$\sum_{j \neq j'} v^j \lambda_j^0 = -v^{j'} \lambda_{j'}^0 \neq 0$$

By assumption (i), we get

$$v^{j'} \lambda_{j'}^0 \in V^* \cap (-V^*),$$

a contradiction.

By (7), (8) and (9), we get a contradiction to $(\hat{x}, \hat{y})$ is a nondominated solution of (VP) associated with $V^* \times U^*$.

Q.E.D.

Appendix

Consider the following pair of dual programming problems

$$(P)\begin{cases} \min & c^T x \\ \text{s.t.} & Ax - b \in K \end{cases}$$

and

$$(D)\begin{cases} \max & y^T b \\ \text{s.t.} & y^T A - c^T = 0 \\ & y \in -K^* \end{cases}$$

where A is an $m \times n$ matrix, $b \in E^m$, $c \in E^n$, $K \subset E^m$ is a closed convex cone and $\operatorname{Int} K \neq 0$ (let $K^o = \operatorname{Int} K$).

Let (see [13], [14] and [15])

$$R = \{x:\ Ax - b \in K\}$$

$$I(K^o, \bar{z}) = \{z - \alpha\bar{z}:\ z \in K^o,\ \alpha \geq 0\},\quad \bar{z} \in K$$

$$T(R, \bar{x}) = \{z:\ \exists\, x^K \in R \text{ and } \alpha_K > 0, \text{ such that } \lim_{K \to \infty} \alpha_K(x^K - \bar{x}) = z\}$$

$$L(\bar{x}) = \{\, z:\ Az \in \overline{I(K^o, A\bar{x} - b)}\,\}$$

$$L^o(\bar{x}) = \operatorname{Int} L(\bar{x})$$

$$D(\bar{x}) = \{-A^T y:\ y \in -K^*,\ y^T(A\bar{x} - b) = 0\}$$

where $\bar{x} \in R$.

It is easy to establish the following lemma:

Lemma 1.

(i) $I(K^o, \bar{z})$ is an open convex cone.

(ii) $L(x)$ is a closed convex cone.

(iii) $D(\bar{x})$ is a convex cone.

Lemma 2. $I^*(K^o, \bar{z}) = \{y:\ y \in K^*,\ y^T\bar{z} = 0\}$.

Proof: Let $y \in I^*(K^o, \bar{z})$, then for arbitrary $z \in K^o$ and $\alpha \geq 0$ we have

$$y^T(z - \alpha\bar{z}) \leq 0 \qquad (*)$$

Let $\alpha = 0$, we get

$$y^T z \le 0, \quad \forall z \in K^0.$$

namely, $y \in (K^0)^* = K^*$.

Since $\bar{z} \in K$, we have $y^T\bar{z} \le 0$. By (*), we get $y^T\bar{z} \ge 0$, so $y^T\bar{z} = 0$.

Therefore

$$I^*(K^0, \bar{z}) \subset \{y: \ y \in K^*, \ y^T\bar{z} = 0\}.$$

Let $y \in \{y: \ y \in K^*, \ y^T\bar{z} = 0\}$. Then for arbitrary $z \in K^0$, $\alpha \ge 0$, we have

$$\begin{aligned} & y^T(z - \alpha\bar{z}) \\ = \ & y^T z - \alpha y^T\bar{z} \\ = \ & y^T z \\ \le \ & 0, \end{aligned}$$

so

$$y \in I^*(K^0, \ \bar{z}).$$

Therefore

$$\{y: \ y \in K^*, y^T\bar{z} = 0\} \subset I^*(K^0, \bar{z})$$

Q.E.D.

<u>Lemma 3.</u>

(i) $L(\bar{x}) = D^*(\bar{x})$.

(ii) If $D(x)$ is closed, then $L^*(\bar{x}) = D(\bar{x})$.

<u>Proof</u>:

(i) Let $z \in D^*(\bar{x})$, then for an arbitrary

$$y \in I^*(K^0, A\bar{x} - b) = \{y: \ y \in K^*, \ y^T(A\bar{x} - b) = 0\},$$

we have $-A^T(-y) \in D(\bar{x})$, hence

$$(Az)^T y = z^T(-A^T(-y)) \le 0.$$

Therefore

$$Az \in (I^*(K^0, A\bar{x} - b))^* = \overline{I(K^0, A\bar{x} - b)}.$$

Namely,

$$D^*(\bar{x}) \subset L(\bar{x}).$$

Now, let $z \in L(\bar{x})$, i.e.

$$Az \in \overline{I(K^o, A\bar{x} - b)}.$$

Then for arbitrary y satsifying

$$y \in -K^*, \quad y^T(A\bar{x} - b) = 0$$

we have

$$z^T(-A^T y) = (Az)^T(-y) \leq 0$$

(Since $I^*(K^o, A\bar{x} - b) = \{y: \ y \in K^*, \ y^T(A\bar{x} - b) = 0\}$, so $-y \in I^*(K^o, A\bar{x} - b)$.) Since $-A^T y \subset D(\bar{x})$, we get $z \in D^*(\bar{x})$, namely

$$L(\bar{x}) \subset D^*(\bar{x}).$$

(ii) Since $D(x)$ is a closed convex cone, from (i) we have

$$L^*(\bar{x}) = D^{**}(\bar{x}) = D(\bar{x}).$$

Q.E.D.

Lemma 4. $T(R, \bar{x}) \subset L(\bar{x})$.

Proof: For an arbitrary $z \in T(R, \bar{x})$, there exist $x^K \in R$ and $\alpha_K > 0$ such that

$$\lim_{K \to \infty} \alpha_K(x^K - \bar{x}) = z.$$

From $Ax^K - b \in K$ and $K^o \neq 0$ we know that there exists $\{y^{K,\ell}\} \subset K^o$ such that

$$\lim_{\ell \to \infty} y^{K,\ell} = Ax^K - b.$$

Because $y^{K,\ell} \in K^o$ and $\alpha_K > 0$ we have

$$\alpha_K(y^{K,\ell} - (A\bar{x}^K - b)) \in I(K^o, A\bar{x} - b).$$

Let $\ell \to \infty$, we get

$$\alpha_K(Ax^K - b) - \alpha_K(A\bar{x} - b) \in \overline{I(K^o, A\bar{x} - b)}.$$

But

$$A\alpha_K(x^K - \bar{x}) = \alpha_K(Ax^K - b) - \alpha_K(A\bar{x} - b).$$

Thus

$$A\alpha_K(x^K - \bar{x}) \in \overline{I(K^o, A\bar{x} - b)}.$$

Let $K \to \infty$, we have

$$Az \in \overline{I(K^o, A\bar{x} - b)},$$

namely

$$T(R, \bar{x}) \subset L(\bar{x})$$

Q.E.D

<u>Lemma 5.</u> $L^o(\bar{x}) \subset T(R, \bar{x})$.

<u>Proof:</u> Since $K^o \neq 0$, it is easy to show that

$$L^o(x) = \{z: \ Az \in I(K^o, A\bar{x} - b)\}.$$

For an arbitrary $z \in L^o(x)$, there exist $u \in K^o$, $\alpha \geq 0$ such that

$$Az = u - \alpha(Ax - b).$$

<u>Case (1),</u> $\alpha = 0$. For an arbitrary $\beta \geq 0$, we have

$$A(\bar{x} + \beta z) - b$$

$$= (A\bar{x} - b) + \beta Az$$

$$= (A\bar{x} - b) + \beta u \in K \quad (\text{because } \bar{x} \in R \text{ and } u \in K^o).$$

Take $\{\beta_K\}$ satisfying

$$\beta_1 > \beta_2 > \ldots > 0, \quad \lim_{K \to \infty} \beta_K = 0.$$

Let

$$x^K = \bar{x} + \beta_K z, \quad \alpha_K = 1 \quad \beta_K,$$

we have $x^K \in R$, $\lim_{K \to \infty} x^K = \bar{x}$, $\alpha_K > 0$ and

$$z = \alpha_K(x^K - \bar{x}).$$

Therefore

$$z \in T(R, \bar{x}).$$

Case (ii), $\alpha > 0$. For an arbitrary $\beta \in [0, 1/\alpha]$ we have

$$A(\bar{x} + \beta z) - b$$

$$= A\bar{x} - b + \beta Az$$

$$= (A\bar{x} - b) + \beta(u - \alpha(A\bar{x} - b))$$

$$= (1 - \alpha\beta)(A\bar{x} - b) + \beta u \in K \quad (\text{because } \bar{x} \in R,\ u \in K^0).$$

Take $\{\beta_K\}$ satisfying $1/\alpha \geq \beta_1 > \beta_2 > \ldots > 0$, $\lim_{K\to\infty} \beta_K = 0$.

Let

$$x^K = \bar{x} + \beta_K z, \quad \alpha_K = 1\ \beta_K$$

We have $x^K \subset R$, $\alpha_K > 0$, $\lim_{K\to\infty} x^K = \bar{x}$ and $z = \alpha_K(x^K - x)$

Therefore

$$z \subset T(R, x).$$

Q.E.D

Theorem 1. (Weak Duality Theorem) Let x be a feasible solution of (P), y be a feasible solution of (D). Then

$$c^T x \geq y^T b.$$

Proof. Since $Ax - b \in K$, there exists $u \subset K$ such that $Ax = b + u$, hence

$$c^T x = y^T Ax$$

$$= y^T(b+u)$$

$$\geq y^T b$$

Q.E.D.

Lemma 6. Let $\bar{x} \in R$ be an optimal solution of (P). Then

$$-c \in T^*(R, \bar{x}).$$

Proof. It is only necessary to show

$$c^T z \geq 0, \quad \text{for } \forall z \in T(R, \bar{x}).$$

Now for an arbitrary $z \in T(R, \bar{x})$, there exist $\{x^K\} \subset R$, $\alpha_K > 0$ and $\lim_{K\to\infty} x^K = \bar{x}$

such that

$$\lim_{K\to\infty} \alpha_K(x^K - \bar{x}) = z.$$

Since $\bar{x}$ is an optimal solution of (P), we have

$$c^T \alpha_K(x^K - \bar{x}) = \alpha_K(c^T x^K - c^T\bar{x}) \geq 0.$$

Let $k\to\infty$, we have

$$c^T z \geq 0.$$

Q.E.D.

Lemma 7. Let $\bar{x} \in R$ be an optimal solution of (P) and let $D(\bar{x})$ be a closed set. Then

$$-c \in D(\bar{x}).$$

Proof. From Lemma 3, Lemma 4 and Lemma 5 we get

$$L^o(\bar{x}) \subset T(R, \bar{x}) \subset L(\bar{x}) = D^*(\bar{x}),$$

hence

$$L^*(\bar{x}) = (L^o(\bar{x}))^* \supset T^*(R, \bar{x}) \supset L^*(\bar{x}) = D^{**}(\bar{x}) = D(\bar{x}).$$

Thus

$$L^*(\bar{x}) = T^*(R, \bar{x}) = D(\bar{x}).$$

From Lemma 6, we get

$$-c \subset D(\bar{x}).$$

Q.E.D

Theorem 2. (Dual Theorem) Let $\bar{x} \in R$ be an optimal solution of (P) and let $D(\bar{x})$ be a closed set. Then (D) has an optimal solution $\bar{y}$, and $c^T\bar{x} = \bar{y}^T b$.

Proof: By Lemma 6, we have

$$-c \in D(\bar{x}).$$

Namely, there exists $\bar{y} \in E^m$ such that

$$\bar{y} \in -K^*,$$

$$\bar{y}^T(A\bar{x} - b) = 0,$$

$$-c = -A^T\bar{y}.$$

Therefore

$$\begin{cases} A\bar{x} - b \in K, \\ \bar{y}^T A - c^T = 0, \quad \bar{y} \in -K^* \end{cases}$$

and

$$c^T\bar{x} = \bar{y}^T A\bar{x} = \bar{y}^T b.$$

By Theorem 1, $\bar{y}$ is an optimal solution of (D), and

$$c^T\bar{x} = \bar{y}^T b.$$

Q.E.D.

Note: Take $K = E_+^m$ (namely, (P) and (D) are linear programming problems). Let

$$I = \{i: a_i\bar{x} = b_i, \quad 1 \le i \le m\},$$

then

$$D(x) = \left\{ \sum_{i \in I} y_i a_i^T : \; y_i \ge 0, \; i \in I \right\},$$

where

$$A = (a_1, a_2, \ldots, a_m), \qquad b = (b_1, b_2, \ldots, b_m)$$

It is easy to show that $D(\bar{x})$ is a closed set.

Let us consider the following pair of dual programs:

$$(\bar{P}) \begin{cases} \min & (w^T x_{jo} - \mu^T y_{jo}) \\ \text{s.t.} & w^T\bar{X} - \mu^T Y \in K \\ & w - \tau \in V \\ & \mu - \hat{\tau} \in U \end{cases}$$

and

$$(\bar{D}) \begin{cases} \max & (\tau^T s^- + \hat{\tau}^T s^+) \\ \text{s.t.} & \bar{X}\lambda - x_{jo} + s^- = 0 \\ & -\bar{Y} + y_{jo} + s^+ = 0 \\ & \lambda \in -K^*, \; s^- \in -V^*, \; s^+ \in -U^*. \end{cases}$$

Let $(\lambda^o, s^{o-}, s^{o+})$ be a feasible solution of $(\bar{D})$ and

$$\bar{D}(\lambda^o, s^{o-}, s^{o+}) = \left\{ \begin{pmatrix} \bar{X}^T w - \bar{Y}^T\mu + y_1 \\ w + y_2 \\ \mu + y_3 \end{pmatrix} : \begin{array}{l} y_1 \in K,\ y_2 \in V,\ y_3 \in U \\ y_1{}^T\lambda^o = y_2{}^T s^{o-} = y_3{}^T s^{o+} = 0 \end{array} \right\}$$

Assumption (A): $\bar{D}(\lambda^o, s^{o-}, s^{o+})$ is a closed set.

Theorem 3 Let $(\lambda^o, s^{o-}, s^{o+})$ be an optimal solution of $(\bar{D})$ and let Assumption (A) hold. Then $(\bar{P})$ has an optimal solution (w^o, μ^o), and

$$w^{oT}x_{jo} - \mu^{oT}y_{jo} = \tau^T s^{o-} + \hat{\tau}^T s^{o+}.$$

Proof Since the dual of $(\bar{D})$ is $(\bar{P})$, and Assumption (A) holds. By Theorem 2, we can get the results.

Q.E.D

Now let us consider the following pair of dual programs:

$$(\hat{P}) \begin{cases} \min & (w^T x_{jo} - \mu^T y_{jo}) \\ \text{s.t.} & w^T\bar{X} - \mu^T\bar{Y} \in K \\ & w - v \in V \\ & \mu - u \in U \\ & \tau^T v + \hat{\tau}^T u = 1 \\ & v \in V,\quad u \in U \end{cases}$$

and

$$(\hat{D}) \begin{cases} \max & z \\ \text{s.t.} & \bar{X}\lambda - x_{jo} + s^- = 0 \\ & -\bar{Y}\lambda + y_{jo} + s^+ = 0 \\ & z\tau - s^- \in V^* \\ & z\hat{\tau} - s^+ \in U^* \\ & \lambda \in -K^*,\quad s^- \in -V^*,\quad s^+ \in -U^* \end{cases}$$

Let $(\lambda^o, s^{o-}, s^{o+}, z^o)$ be a feasible solution of $(\hat{D})$ and

$$\hat{D}(\lambda^o, s^{o-}, s^{o+}, z^o) = \left\{ \begin{pmatrix} \bar{X}^T w - \bar{Y}^T \mu + y_1 \\ w - v + y_2 \\ \mu - u + y_3 \\ \tau^T v + \hat{\tau}^T u \end{pmatrix} : \begin{array}{l} v \in -V, \; u \in -U \\ y_1 \in K, \; y_2 \in V, \; y_3 \in U \\ v^T(z^o \tau - s^{o-}) = 0 \\ u^T(s^o \hat{\tau} - s^{o+}) = 0 \\ y_1{}^T \lambda^o = y_2{}^T s^{o-} = y_3{}^T s^{o+} = 0 \end{array} \right\}$$

<u>Assumption (B)</u>: $\hat{D}(\lambda^o, s^{o-}, s^{o+}, z^o)$ is a closed set.

<u>Theorem 4</u>. Let $(\lambda^o, s^{o-}, s^{o+}, z^o)$ be an optimal solution of $(\hat{D})$, and let Assumption (B) hold. Then $(\hat{P})$ has an optimal solution (w^o, μ^o, v^o, u^o) and

$$w^{oT} x_{jo} - \mu^{oT} y_{jo} = z^o.$$

<u>Proof</u> It is similar to the proof of Theorem 3.

Q.E.D.

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A new "cone-ratio" Data Envelopment Analysis model which substantially generalizes the CCR model and the Charnes-Cooper Thrall approach characterizing its efficiency classes is herein developed and studied. It allows for infinitely many DMU's and arbitrary closed convex cones for the virtual multipliers as well as the cone of positivity of the vectors involved. Generalizations of linear programming and polar cone dualizations are the analytical vehicles employed.

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