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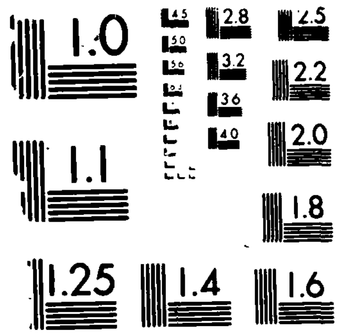

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THE INITIAL STAGE OF WAKE DEVELOPMENT IN LINEARIZED INCOMPRESSIBLE AND COMPRESSIBLE FLOWS

Karl G. Guderley

University of Dayton Research Institute
Dayton, Ohio

March 1987

Interim Report for the Period May 1986 to August 1986

FLIGHT DYNAMICS LABORATORY
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NOTICE

This technical report has been reviewed and is approved for publication.

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The report investigates the initial stage of the wake formation in linearized incompressible or compressible subsonic flows after a change of the upwash at the wing in the form of a step function in time. By a superposition of solutions of this kind all other changes of the upwash at the wing can be treated. The potential due to the above perturbation can be expressed by a circulation-free flow that satisfies the boundary condition imposed by the sudden change of upwash and the potential due to the wake vortices which leaves the boundary conditions unchanged. The latter is expressed by a superposition of the potentials due to individual vortices emanating from the trailing edge at different times and traveling downstream with the free stream velocity. The intensity of these vortices is determined by the Kutta condition, namely that at the trailing edge and at all times the upwash in the wake obtained by the combination of circulation-free flow and wake vortices be finite. In the incompressible case the analysis can be carried out in all details. In the compressible case the flow fields needed can be described by similarity solutions, but of a rather complex character. The feature essential for the present

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19) problem, namely the upwash singularity at the trailing edge can, however, be gleaned by a discussion in general terms so that one can determine the vortex distribution within the wake except for one constant factor.

FOREWORD

This report was written under Contract F33615-86-C-3200 entitled "A Study of Linearized Integral Equations for Steady and Oscillatory Supersonic Flow," to the University of Dayton for the Aeroelastic Group, Analysis and Optimization Branch, Structures and Dynamics Division (AFWAL/FIBRC), Air Force Wright Aeronautical Laboratories, Wright Patterson Air Force Base, Ohio. The work was conducted under Program Element No. 61102F, Project No. 2304, Task N1, and Work Unit 22.

The work was performed during the period of May 1986 through June 1986. Dr. Karl G. Guderley of the University of Dayton Research Institute was Principal Investigator. Dr. Charles L. Keller, AFWAL/FIBRC, (513) 255-7384, was Program Manager.

The author would like to express his appreciation for the excellent typing of Ms. Carolyn Gran.

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SECTION I

INTRODUCTION

During the development of a method for unsteady linearized subsonic flows the question arose, how the vorticity within the wake is distributed during the first stage of the wake formation. This problem is treated here for both compressible and incompressible flows.

The incompressible flow has been explored without the restriction to the initial stage by Herbert Wagner (Reference 1). The present study uses Wagner's concepts. With the restriction to the initial stage one obtains rather specific results.

The nature of the problem changes if one takes compressibility into account. In incompressible flow the velocity of sound is infinite; the flow field adjusts itself immediately to changing boundary conditions and a changing vortex distribution, even if the changes occur within short times. In compressible flows perturbations travel during a finite time only over a finite distance. In the beginning stage the wake formation has an effect on only a small part of the flow field in the vicinity of the trailing edge, but within this part the unsteady effects cannot be neglected. Nevertheless, the basic results are the same in the two cases.

In both cases one obtains, conceptually, the actual flow field by the superposition of two types of solutions of the partial differential equation. One of these solutions satisfies the upwash conditions at the wing, but does not allow for wake vortices. The second solution is obtained by a superposition of flow fields each with zero upwash at the wing, they are generated by one vortex shed from the leading edge at a certain time and traveling from then on downstream with the flow velocity. In an incompressible unsteady flow the potential is, of course, time dependent, but the time dependence does not appear in the potential equation (given by the Laplace equation). Here these flow fields can be represented by explicit formulae. In the compressible case, the potential equation can be brought into similarity form, so that instead of three independent variables (two space variables and time) one deals only with two independent variables. But the resulting partial differential equation is rather complicated. Fortunately, only a very limited amount of information is needed to find the vortex distribution within the wake. A general discussion is sufficient; only one constant (dependent upon the Mach number) remains undetermined. In the incompressible case these data are expressed in explicit formulae. The availability of such formulae allows one to give specific information about the resulting flow field. This facet is derived in an Appendix B.

SECTION II

INCOMPRESSIBLE FLOWS, BASIC CONCEPTS

Considered is a two-dimensional nonsteady incompressible flow. Let x and y be a system of Cartesian coordinates in which the x-axis has the free stream direction, and let t be the time. A thin profile is subject to small time dependent motions or deformations in a flow with the velocity U. The wake vortices move with the local velocity. The perturbation of the original parallel flow due to the wing are considered as small. The vortices, therefore, move with the free stream velocity U. The airfoil is replaced by a slit extending along part of the x-axis. An upwash will be imposed along this slit that depends upon x; with respect to time it is given by a step function. With results for such an upwash, it is then possible to generate the response to an upwash which has the same x dependence and arbitrary time dependence. Without loss of generality, one can assume that the step occurs at time zero.

The perturbation potential depends on x,y, and t. But the potential equation

$$\phi_{xx} + \phi_{yy} = 0$$

does not contain the time explicity. We do not allow the airfoil thickness to change with time. The upwash is, therefore, the same on the upper and lower sides of the airfoil. It follows, that the upwash ϕ_y is symmetric with respect to the x-axis. The potential, its derivatives with respect to x and t, and the pressure are then antisymmetric. There will be, of course, a jump of the potential between the upper and lower sides of the wing and of the wake. At the wake the pressure is continuous, and therefore, because of its antisymmetry, zero.

Because of the assumption of small perturbations, the perturbation pressure is given by

$$\Delta p = -\rho(U\phi_x + \phi_t) \tag{1}$$

Since at the wake the pressure is zero, it follows that the potential at the upper side of the wake has the form

$$\phi(t, x, 0^+) = f(t - (x - x_{tr})/U) \tag{2}$$

where x_{tr} is the x coordinate of the trailing edge. (0^+ means that one approaches y = 0 from above.) For the lower side one has the same expression with the opposite sign.

The perturbation field is decomposed into two parts. A steady part is given by the circulation-free flow determined by the upwash condition (after the step has occurred) at the wing. In such a flow one finds adjacent to the trailing edge an infinite pressure at points of the wing and an infinite upwash at points of the wake. Superimposed to this steady flow is one caused by the time dependent vortex distribution within the wake (as it exists at the current time). At the wing zero upwash is prescribed. Outside the wake and the wing this flow field satisfies the Laplace equation. This second field gives singularities at the trailing edge of the same kind as the steady field. The potential at the upper side of the wake must have the form of Eq. (2). The function f must be chosen so that the trailing edge singularities cancel those of the circulation-free flow.

The potential of the circulation-free flow can be developed with respect to the distance from the trailing edge. Let the origin of the x,y-system lie at the trailing edge and let

$$z = x + iy$$

Then the lowest order terms of the development of the perturbation potential are given by

$$\phi = \mathrm{Im}\Omega$$

with

$$\Omega = a_{1/2}\, z^{1/2} + a_1 z + a_{3/2}\, z^{3/2} + \ldots$$

Since for this flow the boundary conditions are independent of time, the coefficients "a" are independent of time. No constant term occurs because at the location of the wake there is no potential jump in the circulation-free flow. Then

$$\phi_x = \mathrm{Im}(d\Omega/dz)$$

$$\phi_y = \mathrm{Re}(d\Omega/dz)$$

Specifically within the wake, i.e., for $z = x > 0$

$$\phi_x = 0$$

$$\phi_y = (1/2)a_{1/2}\, x^{-1/2} + a_1 + (3/2)a_{3/2}\, x^{1/2}$$

and at the upper side of the wing $z = x < 0$ $(z = -|x|)$

$$\phi_x = -(1/2)a_{1/2}|x|^{-1/2} + (3/2)a_{3/2}|x|^{1/2}$$

$$\phi_y = a_1 + 2a_2 x + \ldots$$

Specific examples can be readily obtained, for instance, from the formulae derived in Appendix E of Reference 2.

Notice that the term in ϕ_x and ϕ_y with the factor $|x|^{-1/2}$ occurs with the same coefficient. The terms in ϕ_y at the wing do not contain fractional powers. (They are determined by the boundary conditions for the upwash.) The wake development is solely determined by the coefficient $a_{1/2}$.

Formulae for a single vortex in the wake in the presence of a finite wing are derived in Appendix A. These are the formulae on which the work of Wagner is based. In the present discussion where we restrict ourselves from the outset to small times and consequently to small distances from the trailing edge one obtains simplified formulae, because then the wing chord is very large in comparison to the distances under consideration. Therefore, we assume for the evaluation of the wake potential that the wing extends along the x-axis from $-\infty$ to zero. The formulae so obtained could alternatively be obtained by developing the complete expression of the appendix under the assumption that the distances from the trailing edge are small. To obtain the potential for a single vortex in the presence of such an infinite slit we first consider in an x_1 y_1 plane (Figure 1)

$$\phi(z_1) = \mathrm{Im}\,\log \frac{z_1 - a}{z_1 + a} \qquad a > 0 \text{ real}$$

with

$$z_1 = x_1 + iy_1$$

This is the flow field with two vortices of opposite sign at the points $x_1 = a$ and $x_1 = -a$. The velocity component normal to the y_1-axis is zero. The y_1-axis is mapped into a slit from $-\infty$ to zero along the x-axis by setting

$$z_1 = z^{1/2}$$

$$a = \xi^{1/2} \qquad \xi > 0 \text{ real}$$

then one obtains

$$\phi = \mathrm{Im}\,\log \frac{z^{1/2} - \xi^{1/2}}{z^{1/2} + \xi^{1/2}} = \mathrm{Im}[\log(z - \xi) - 2\log(z^{1/2} + \xi^{1/2})] \quad (3)$$

This expression has one logarithmic singularity at $z = \xi$. Other singularities occur at the trailing edge because of the power $z^{1/2}$. One obtains

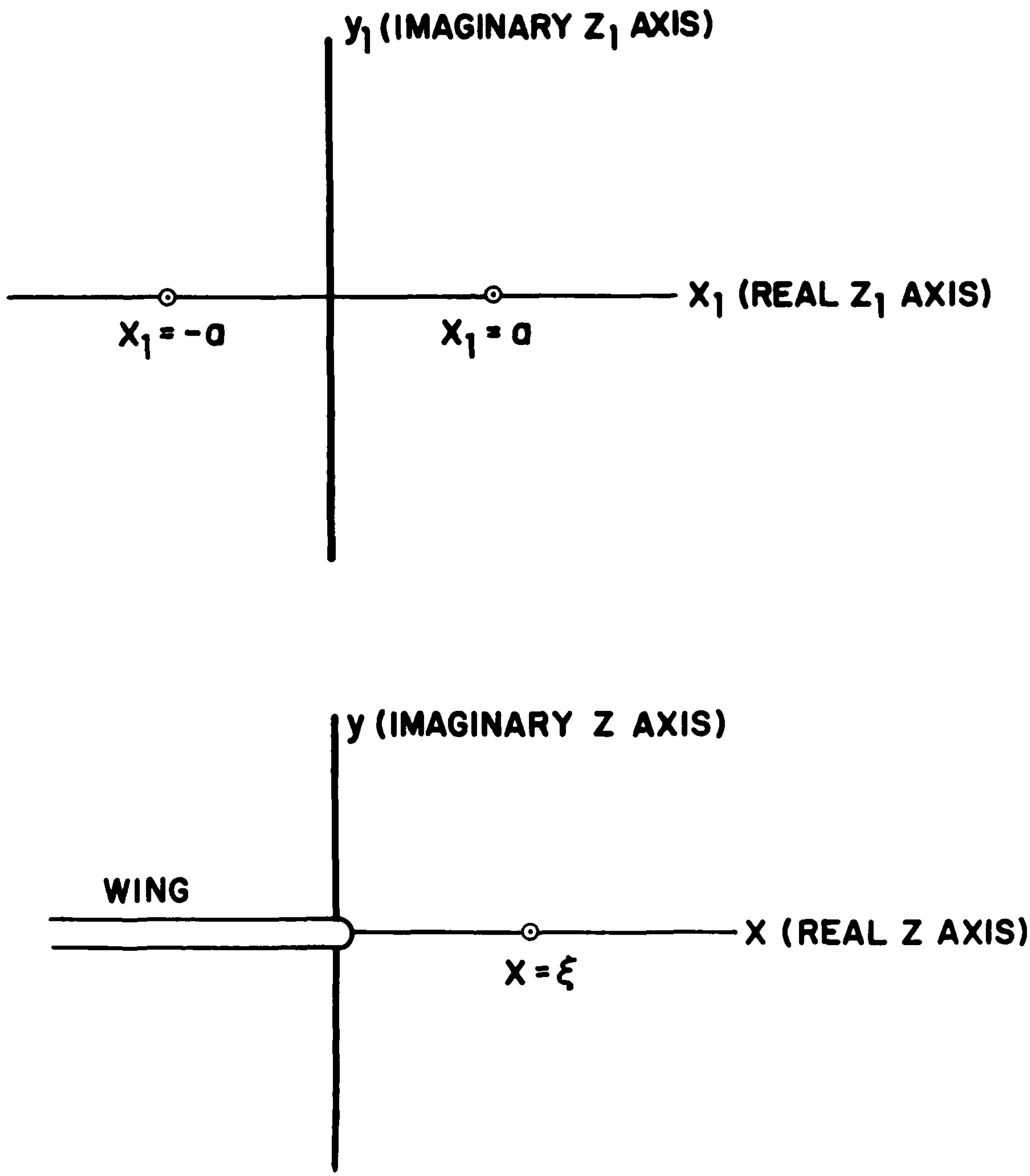

Figure 1. Conformal mapping of the right half of the x_1, y_1 plane with logarithmic singularities at $x_1 = a$ and $x_1 - a$ into an x,y plane with cut along the negative x-axis and a logarithmic singularity at $x = \xi = z^{1/2}$.

$$\phi = 0 \qquad \text{for } z = x > \xi$$

$$\phi = \pi \qquad \text{for } 0 < z = x + i0 < \xi$$

$$\phi = \pi - 2 \operatorname{Im} \log(\xi^{1/2} + i|x|^{1/2}) \qquad \text{for } z = x + i0 < 0$$

Moreover,

$$\phi_x = \operatorname{Im}(z^{-1/2} \frac{\xi^{1/2}}{z - \xi})$$

$$\phi_y = \operatorname{Re}(z^{-1/2} \frac{\xi^{1/2}}{z - \xi})$$

Then for $z = x > 0$ (wake)

$$\phi_x = 0$$

$$\phi_y = x^{-1/2} \frac{\xi^{1/2}}{x - \xi}$$

for $z = x + i0^+ < 0$ (wing)

$$\phi_x = -|x|^{-1/2} \frac{\xi^{1/2}}{x - \xi}$$

$$\phi_y = 0$$

These expressions have the same singularities at the trailing edge as the circulation-free flow.

SECTION III

THE FORM OF THE WAKE POTENTIAL IN INCOMPRESSIBLE FLOW

We express the potential due to the wake by a linear combination of expressions (3). Let $f(\xi)$ be some function of the umbral variable ξ. This potential is then given by

$$\phi_{wake}(t,x,y) = \int_0^{Ut} f(\xi) \,\mathrm{Im}\, \log \frac{z^{1/2} - \xi^{1/2}}{z^{1/2} + \xi^{1/2}} \, d\xi$$

$$= \int_0^{Ut} f(\xi) \,\mathrm{Im}\, \log (z - \xi) d\xi - 2 \int_0^{Ut} f(\xi) \,\mathrm{Im}\, \log(z^{1/2} + \xi^{1/2}) d\xi \qquad (4)$$

In the limits of integration we have taken into account that the wake extends from 0 to Ut. The second term in the last equation is analytic. The relation between the function f and the jump of the potential at a point x_o between the upper and lower side of the wake can be determined in the following manner. The circulation integral $\oint \mathrm{grad}\, \phi \cdot d\vec{s}$ for a single vortex $\mathrm{Im}\, \log(z - \xi)$ is 2π. Consider now a path which starts and ends at the same point x_o of the wake and which encloses the wake downstream from the point x_o (the wake ends at a finite distance namely $x = Ut$). (See Figure 2.) The second term on the right in Eq. (4) is regular at and in the region within this path and therefore gives no contribution. One obtains

$$\oint \mathrm{grad}\, \phi\ (t,x,y) . d\vec{s} = \phi(t,x_o,0^+) - \phi(t,x_o,0^-) = 2\pi \int_{x_o}^{Ut} f(\xi) d\xi$$

But ϕ is antisymmetric. Hence,

$$\phi(t,x_o,0^+) = \pi \int_{x_o}^{Ut} f(\xi) d\xi$$

$$f(x_o) = -\frac{1}{\pi} \frac{\partial \phi}{\partial x}(t,x_o,0^+) \qquad (5)$$

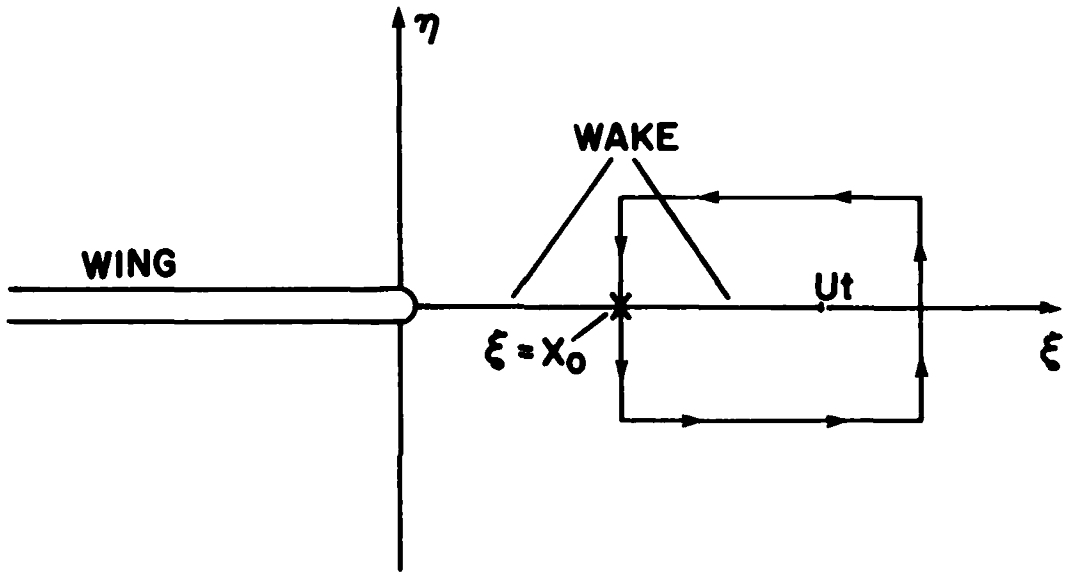

Figure 2. Wing and wake at time t. Path of integration for the determination of the potential difference between the upper and the lower sides of the wake at a point x_o. The path starts and ends at x_o. One proceeds around the vortices enclosed between $\xi = x_o$ and $\xi = Ut$ in the counterclockwise sense.

The form of the potential within the wake has been found in Eq. (2). The derivative $\partial\phi/\partial x$ has, of course, the same form. The function $f(\xi)$ therefore has the form $g(t - (\xi/U))$, and one obtains

$$\phi_{wake}(t,x,y) = \int_0^{Ut} g(t - (\xi/U)) \operatorname{Im} \log \frac{z^{1/2} - \xi^{1/2}}{z^{1/2} + \xi^{1/2}} \, d\xi \qquad (6)$$

The singularity in ϕ_y which arises in this expression at points of the wake adjacent to the trailing edge must cancel the corresponding singularity in the circulation-free flow around the wing. This is the condition which determines the form of the function g. We form

$$\frac{\partial\phi_{wake}}{\partial y} = \int_0^{Ut} g(t - (\xi/U)) \operatorname{Re} z^{-1/2} \frac{\xi^{1/2}}{z - \xi} \, d\xi$$

For $z = x > 0$ one obtains

$$\frac{\partial\phi_{wake}}{\partial y} = \frac{1}{x^{1/2}} \int_0^{Ut} g(t - (\xi/U)) \frac{\xi^{1/2}}{x - \xi} \, d\xi$$

In order for this expression to cancel the corresponding singularity in the circulation-free flow one must have

$$I = \lim_{x \to 0} \int_0^{Ut} g(t - (\xi/U)) \frac{\xi^{1/2}}{x - \xi} \, d\xi = \text{const}$$

where the right-hand side does not depend upon the time. Let

$$\xi = q\, Ut$$

$$x = \tilde{x}\, Ut$$

Then one obtains

$$\lim_{\tilde{x} \to 0} U^{1/2} \int_0^1 t^{1/2} \, g(t(1 - q)) \frac{q^{1/2}}{\tilde{x} - q} \, dq$$

and one postulates

$$\lim_{\tilde{x}\to 0} \int_0^1 t^{1/2} g(t(1-q)) \frac{q^{1/2}}{\tilde{x}-q}\, dq = \text{const}$$

This gives the requirement that $t^{1/2}\, g(t(1-q))$ be solely a function of q.

$$g(t(1-q)) = \tilde{f}(q) t^{-1/2}$$

The argument of g is t(1 - q). It follows that $\tilde{f}(q)$ has the form $\text{const}(1-q)^{-1/2}$. Therefore,

$$g(t(1-q)) = t^{-1/2}(1-q)^{-1/2}\ \text{const}$$

or, after one returns to the original coordinate with a different choice of the constant,

$$g = \text{const}(Ut - \xi)^{-1/2} \tag{7}$$

Thus,

$$\phi_{wake}(t,x,y) = \text{const} \int_0^{Ut} (Ut - \xi)^{-1/2}\ \text{Im} \log \frac{z^{1/2} - \xi^{1/2}}{z^{1/2} + \xi^{1/2}}\, d\xi \tag{8}$$

Eq. (8) is the crucial result for the incompressible flow.

SECTION IV

THE WAKE IN LINEARIZED COMPRESSIBLE FLOW

As in the incompressible flow, one suddenly imposes at time zero the boundary condition of constant upwash at the wing. If the wing extended along the entire x-axis, one would obtain a compression wave and an expansion wave respectively on the upper and lower sides which at a (positive) time t generate fields of positive and negative perturbation pressure and constant upwash out to a distance a t. (See Figure 3.)

Perturbations float downstream with the free stream velocity U and expand with the velocity of sound a. If the plate ends at $x = 0$, the above flow field will terminate at a circle around the point $x = Ut$ with radius a t. We consider times which are sufficiently small, so that perturbations coming from the leading edge do not affect the region in the vicinity of the trailing edge. (See Figure 4.)

Assume first that no vortices are shed from the trailing edge. Within the circle described above one then obtains a flow field caused by the pressure difference between the upper and lower side. This flow field will have similarity form with respect to time, that is the velocities will depend only upon x/t and y/t. At the portion of the plate within the circle one will have the desired upwash. At the portion 1, 2, 3 of the circle (which moves with time) the solution within the circle must match the solution for the infinite plate, at the remaining portion of the circle it must match the undisturbed flow. One expects that at points of the x-axis downstream of the trailing edge and adjacent to it this flow has a singularity in the upwash of the same character as in an incompressible flow. This will be discussed in some detail.

Superimposed to this field is another one due to the vortices shed from the trailing edge. A single vortex which leaves the trailing edge at time $\bar{t}$ moves downstream with the velocity U. It generates a field which also has similarity character; it depends solely upon $x/(t - \bar{t})$ and $y/(t - \bar{t})$. The intensity of the vortices

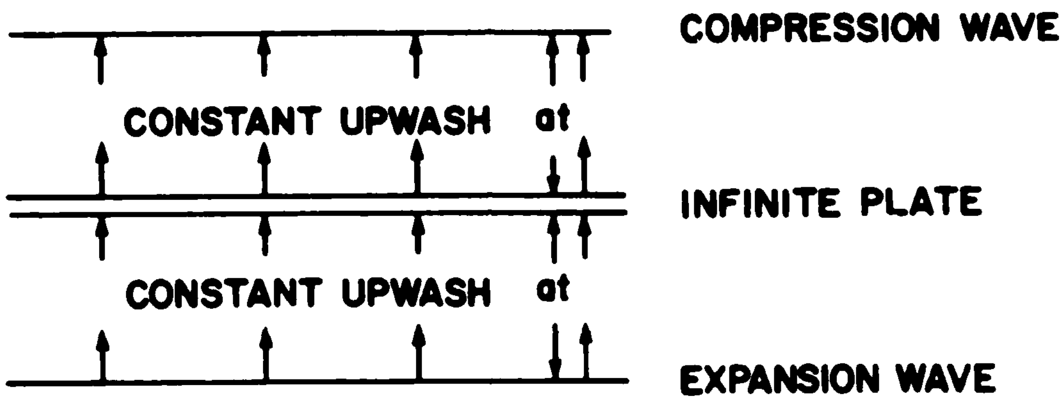

Figure 3. Perturbation generated in a parallel flow with velocity U by an infinite plate (double line) suddenly set in motion in the y-direction with a constant velocity. One obtains a constant upwash for $|y| < at$ and no upwash at $|y| > at$. At $y = at$ and $y = -at$ one has, respectively, a compression and a rarefaction wave.

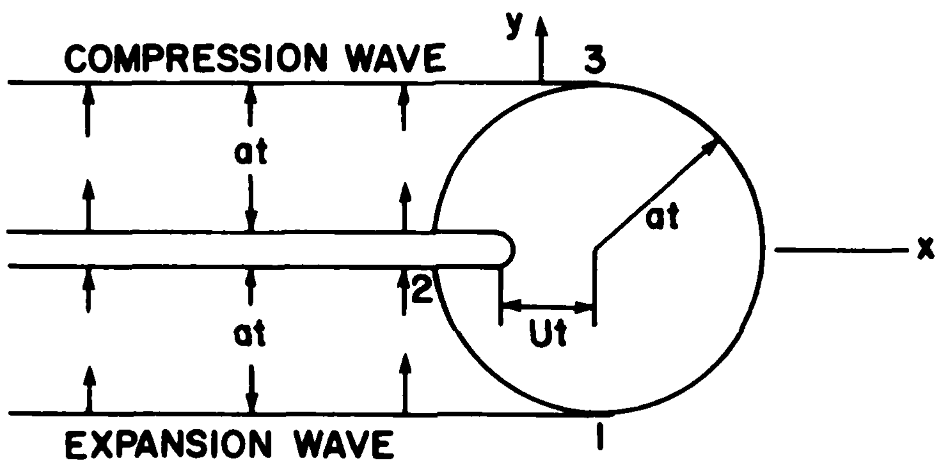

Figure 4. Perturbation generated in a parallel flow with velocity U by a half-infinite plate (double line) suddenly set in motion in the y-direction with constant velocity. Along the negative x-axis outside of the circle one had the same field as for the infinite plate; within the circle there is a complicated transition field.

shed in this manner will depend upon $\bar{t}$. The function which gives the intensity must be determined in such a manner that the singularity generated in the circulation-free flow is cancelled at all times by the flow field generated by the vortices shed form the trailing edge. In the following we shall develop these ideas in detail.

After a linearization for the vicinity of a parallel flow the perturbation potential in the two-dimensional case satisfies the equation.

$$(a^2 - U^2)\phi_{xx} + a^2\phi_{yy} - 2U\phi_{xt} - \phi_{tt} = 0 \qquad (9)$$

The potential of a vortex moving downstream with the velocity U (in other words, with zero velocity with respect to the surrounding fluid) is given by

$$\phi = \text{Im}\Omega$$

$$\Omega = \log(x + iy - Ut) \qquad (10)$$

The real part in Eq. (10) would give a moving source. This expression should satisfy Eq. (9). Let

$$z = x + iy$$

One has, indeed,

$$\Omega_{xx} = -(z - Ut)^{-2}$$

$$\Omega_{yy} = (z - Ut)^{-2}$$

$$\Omega_{xt} = U(z - Ut)^{-2}$$

$$\Omega_{tt} = -U^2(z - Ut)^2$$

It is readily seen that Eq. (9) is satisfied.

Similarity solutions are obtained by introducing independent variables

$$\xi = x/t \quad , \quad \eta = y/t \quad , \quad \tau = t \tag{11}$$

and

$$\phi(x,y,t) = \tau^{\alpha}\psi(\xi,\eta,\tau) \tag{12}$$

If ψ actually depends upon the third variable τ, this is merely a coordinate transformation. A similarity solution arises if the function ψ is independent of τ. To describe the circulation-free flow, we must set $\alpha = 1$. It gives for fixed ξ and η velocities which are independent of time (t or τ). This is in accordance with the boundary condition of constant upwash. In the velocity field due to a single vortex moving with the velocity U one must set $\alpha = 0$ for then the circulation around this vortex (the jump of ϕ) is constant if one travels around the vortex along some path returning to the same point.

One obtains from Eq. (12)

$$\phi_{xx} = \tau^{\alpha-2}\psi_{\xi\xi}$$

$$\phi_{yy} = \tau^{\alpha-2}\psi_{\eta\eta}$$

$$\phi_{xt} = \tau^{\alpha-2}\{(\alpha-1)\psi_{\xi} - \xi\psi_{\xi\eta} + \tau\psi_{\xi\tau}\}$$

$$\phi_{tt} = \tau^{\alpha-2}\{\alpha(\alpha-1)\psi + 2(\alpha-1)(-\xi\psi_{\xi} - \eta\psi_{\eta}) + 2\alpha\tau\psi_{\tau}$$
$$+ \xi^2\psi_{\xi\xi} + 2\xi\eta\psi_{\xi\eta} + \eta^2\psi_{\eta\eta} - 2\xi\tau\psi_{\xi\tau} - 2\eta\tau\psi_{\eta\tau} + \tau^2\psi_{\tau\tau}\}$$

Then one obtains from Eq. (9) for $\alpha = 0$

$$\begin{aligned} &\psi_{\xi\xi}(a^2 - U^2 + 2\xi U - \xi^2) + \psi_{\eta\eta}(a^2 - \eta^2) + \psi_{\xi\eta}(2U\eta - 2\xi\eta) \\ &+ \psi_{\xi}(2U - 2\xi) + \psi_{\eta}(-2\eta) + \tau\psi_{\xi\tau}(-2U + 2\xi) + \tau\psi_{\eta\tau}(2\eta) - \tau^2\psi_{\tau\tau} = 0 \end{aligned} \tag{13}$$

Futhermore, for $\alpha = 1$ and ψ independent of τ

$$\psi_{\xi\xi}(a^2 - U^2 + 2\xi U - \xi^2) + \psi_{\eta\eta}(a^2 - \eta^2) + \psi_{\xi\eta}(2U\eta - 2\xi\eta) = 0 \tag{14}$$

The last two equations have the same principal parts (coefficients of the highest derivatives).

The behavior of particle solutions at the trailing edge can be studied by a development with respect to ξ and η. As mentioned above, ϕ is antisymmetric. The form of the lowest terms is obtained by setting $\xi = 0$ and $\eta = 0$ in the coefficients of Eqs. (13) and (14). Suppressing the time dependence in ψ one obtains

$$\psi_{\xi\xi}(a^2 - U^2) + \psi_{\eta\eta}a^2 = 0 \quad \text{for } \alpha = 1 \tag{15}$$

$$\psi_{\xi\xi}(a^2 - U^2) + \psi_{\eta\eta}a^2 + 2U\psi_{\xi} = 0 \quad \text{for } \alpha = 0 \tag{16}$$

The term $U\psi_{\xi}$ in the second equation already gives a contribution of higher order in ψ. In essence one deals with the Laplace equation. Setting

$$\zeta = (\xi + i(1 - M^2)^{1/2}\eta) \quad , \quad M = U/a$$

one obtains for the lowest order terms in the development in Eq. (15)

$$\psi = a_{1/2}\, \mathrm{Im}\zeta^{1/2} + a_1 \mathrm{Im}\zeta$$

The upwash in the case $\alpha = 1$ is then given by

$$\frac{\partial\phi}{\partial y} = \frac{\partial\psi}{\partial\eta} = (1 - M^2)^{1/2}[(a_{1/2}/2)\mathrm{Re}(\zeta^{-1/2}) + a_1]$$

At the wing ($\xi < 0$), $\mathrm{Re}(\zeta^{-1/2}) = 0$. The coefficient a_1 is given by the boundary condition for ϕ_y prescribed at the plate. So far the coefficient $a_{1/2}$ is unknown. Its determination requires the solution (in terms of ξ and η) of the boundary value problem for the circulation-free flow.

The flow due to a single vortex moving in the xy-system with a velocity U is required to satisfy the boundary condition $\phi_y = 0$ at the wing. Its behavior in the vicinity of the trailing edge is expected to be given by

$$\phi = b_{1/2}\ \mathrm{Im}(\zeta^{1/2})$$

(The coefficient corresponding to a_1 is zero.)

To obtain an expression which gives the moving vortex singularity we rewrite Eq. (10)

$$\phi = \mathrm{Im}[\log(\frac{x}{\tau} + \frac{iy}{\tau} - U) + \log \tau] = \mathrm{Im}[\log(\xi + i\eta - U) + \log \tau] \tag{17}$$

Except for the additive term log τ this expression has similarity form. The entire expression satisfies Eq. (13), because the original expression satisfies Eq. (9). The real and imaginary parts of the expression (17) substituted into Eq. (13) will give, respectively, a real and an imaginary expression. But Im log τ = 0. Therefore the imaginary part satisfies Eq. (13), even if one omits the terms that contain time derivatives. In the real part the τ-dependent part gives $-\tau^2 \frac{d^2 \log\tau}{d\tau^2} = 1$. It may come as a surprise that the expression for the moving vortex (Eq. (17)) fits the similarity hypothesis, while that for the moving source fails to do so. This can, of course, be shown by direct substitution into Eq. (13). For this purpose it is practical to introduce

$$\xi + i\eta = \zeta$$

Details are omitted. Accordingly, the expression describing a moving vortex is given by

$$\phi = \text{Im}\log(\xi + i\eta - U) \tag{18}$$

This expression fails to satisfy the boundary conditions, zero upwash at the wing, and zero perturbation at the circle around the point $\xi = 0$, $\eta = 0$ with radius "a." Therefore, a particular solution is superimposed which corrects for this failure. It depends on the parameter $U/a = M$. At the trailing edge it will have a singularity

$$b_{1/2}(M)\,\text{Im}(\xi + (1 - M^2)^{1/2}\, i\eta)^{1/2}$$

The value of $b_{1/2}(M)$ is the only information needed to determine the potential in the wake. Without detailed computation the value of this constant is not available.

We return to the coordinates (x,y,t). The expression with zero upwash at the wing for a vortex generated at the trailing edge at time $\bar{t}$, has a logarithmic singularity at $\xi = U$, $\eta = 0$ given by

$$\phi = \text{Im}\log\left(\frac{x}{t - \bar{t}} + \frac{iy}{t - \bar{t}} - U\right)$$

At the origin, this particular solution gives a term

$$b_{1/2}(M)\,\text{Im}\left(\frac{x + (1 - M^2)^{1/2}\, iy}{t - \bar{t}}\right)^{1/2}$$

Let the intensity of the vortex shed at time $\bar{t}$ be $g(\bar{t})$. The time dependent singularity at the trailing edge is then given by

$$b(M)^{1/2} \int_0^t g(\bar{t}) \operatorname{Im}\left(\frac{x + i(1 - M^2)^{1/2} y}{t - \bar{t}}\right)^{1/2} d\bar{t}$$

$$= b(M)^{1/2} \operatorname{Im}(x + i(1 - M^2)^{1/2} y)^{1/2} \int_0^t \frac{g(\bar{t})}{(t - \bar{t})^{1/2}} d\bar{t}$$

Now the postulate is imposed that this expression cancel the singularity in the circulation-free flow which was given by

$$a_{1/2}(M) \operatorname{Im}(x + i(1 - M^2)^{1/2} y)$$

This expression does not depend upon time. The integral

$$\int_0^t \frac{g(\bar{t})}{(t - \bar{t})^{1/2}} d\bar{t}$$

must therefore be a constant, independent upon time. Let

$$1 - \frac{\bar{t}}{t} = v$$

The integral then transforms into

$$\int_0^1 v^{-1/2} [t^{1/2} g(t(1 - v)] dv$$

In order for $t^{1/2} g(t(1 - v))$ to be independent of t, one must have

$$g = \text{const } t^{-1/2} (1 - v)^{-1/2}$$

or after substitution of v

$$g = \text{const } \bar{t}^{\,-1/2}$$

This gives the time history of the vortex shedding. Because of the antisymmetry of the potential with respect to the x-axis, the circulation-free flow and the potential added to the expression (18) (to satisfy the boundary conditions) give zero potential for $x > 0$ (for $x < 0$ at the wing there will, of course, be a potential jump). The potential as one approaches the wake from above or below is, therefore, solely given by the superposition of expressions, Eq. (18).

$$\phi(t,x,+0) = \int_0^t g(\bar{t}) \operatorname{Im} \log\left(\frac{x + iy}{t - \bar{t}} - U\right) d\bar{t}$$

A clearer picture for a fixed time t is obtained by introducing

$$U(t - \bar{t}) = \tilde{x}$$

Then one obtains, with a different constant,

$$\phi(t,x,+0) = \text{const} \int_0^{Ut} \frac{1}{(Ut - \tilde{x})^{1/2}} \operatorname{Im}(\log(x + iy) - \tilde{x}) d\tilde{x}$$

which shows the vortices which appear at the station $\tilde{x}$. The vortex distribution extends from zero to Ut, the intensity is given by $\text{const}(Ut - \tilde{x})^{-1/2}$. According to the discussion given in conjunction with the incompressible case (Eq. (5)), one then finds

$$\phi(t,x,+0) = 2\pi \text{ const}(Ut - x)^{1/2}$$

The coefficient of the $x^{-1/2}$ singularity for ϕ_x at $x < 0$ and for ϕ_y at $x > 0$ in the expression due to the shed vortices is independent of time. The constant in all the expressions is chosen so that they cancel the corresponding term in the circulation-free flow.

Notice the $(Ut - x)^{-1/2}$ singularity that arise at the downstream end of the wake. It is caused by the fact that the wake ends rather abruptly. The potential goes down to zero as

$(Ut - x)^{1/2}$. A sudden jump of the potential would correspond to a single vortex, and then ϕ_y would behave as $(Ut - x)^{-1}$. Under present conditions the transition is somewhat smoother, and the power for ϕ_y is -1/2 rather than -1.

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APPENDIX A

THE FLOW FIELD OF A POTENTIAL VORTEX IN THE VICINITY OF A PLATE IN TWO-DIMENSIONAL INCOMPRESSIBLE FLOW

The flow field is obtain by a sequence of conformal mappings. Figure A.1.

$$z_1 = x_1 i + iy_1$$

We begin (as in the main text) with

$$\phi(z_1) = \text{Im} \log \frac{z_1 - a}{z_1 + a} \quad a > 0, \text{ real}$$

The transformation

$$z_2 = \frac{1 + z_1}{1 - z_1} \quad , \quad z_1 = \frac{z_2 - 1}{z_2 + 1}$$

maps the right half of the z_1-plane into the outside of the unit circle in the z_2-plane. (The left half is mapped into its inside.)

One obtains

$$\phi_z(z_2) = \text{Im} \log \frac{z_2 - b}{z_2 - b^{-1}}$$

with

$$b = \frac{1 + a}{1 - a}$$

The transformation

$$z = (1/2)(z_2 + z_2^{-1}) \quad , \quad z_2 = z + \sqrt{z^2 - 1}$$

maps the outside of the unit circle in the z_2-plane into the whole z-plane with a slit extending from -1 +1. One obtains

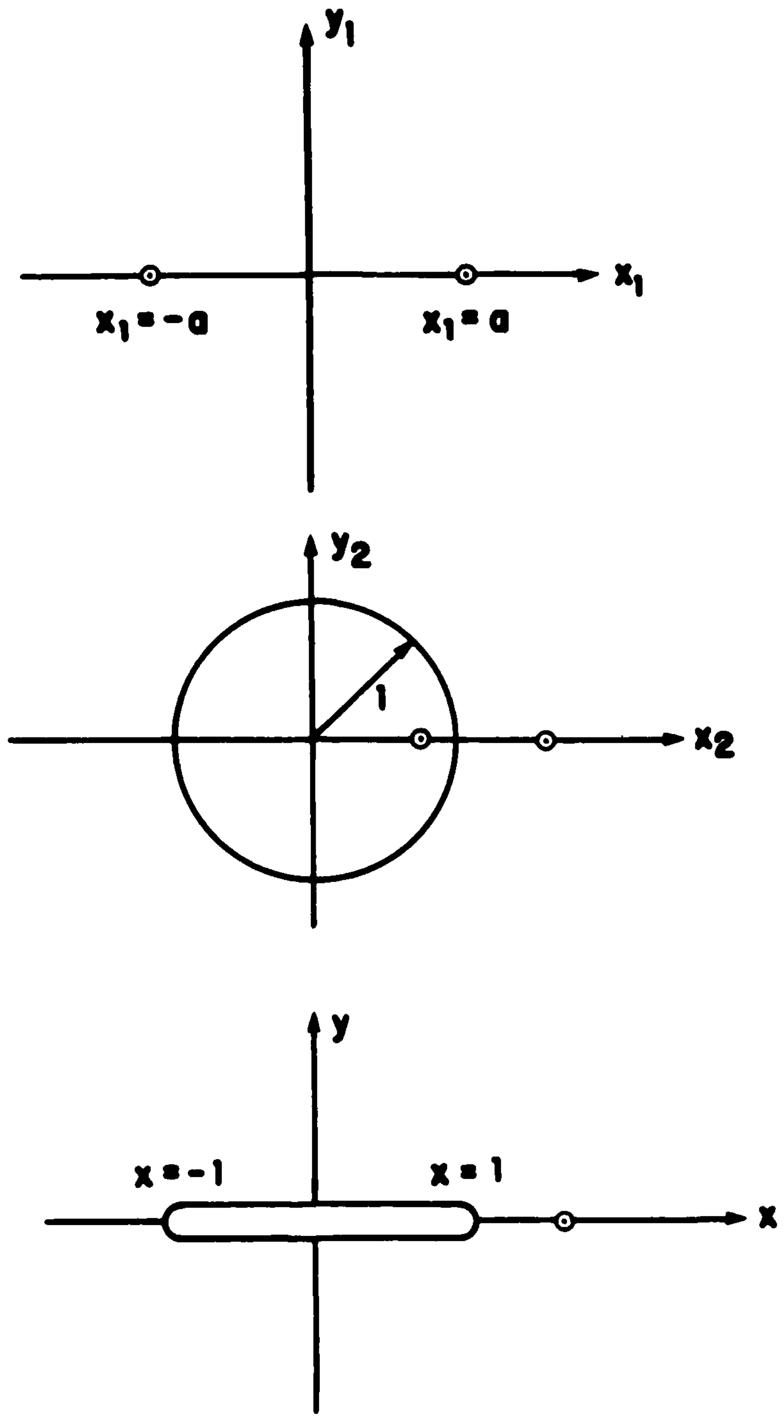

Figure A.1. Sequence of conformal mappings from the right half of an x_1, y_1 plane with a logarithmic singularity to a slit in the x,y plane extending from x = -1 to x = 1 again with a logarithmic singularity.

$$\phi = \text{Im} \log \frac{(z + \sqrt{z^2 + 1}) - (\xi + \sqrt{\xi^2 + 1})}{(z + \sqrt{z^2 + 1}) - (\xi - \sqrt{\xi^2 - 1})} \qquad (A.1)$$

For the constant b in the above formula we now introduce

$$b = \xi + \sqrt{\xi^2 - 1}$$

then

$$b^{-1} = \xi - \sqrt{\xi^2 - 1}$$

Eq. (A.1) must be applied if the extension of the wake is not small in comparison to the span of the wing. It is the expression underlying the work of Wagner.

APPENDIX B

DETAILED EVALUATIONS FOR THE INCOMPRESSIBLE CASE

The constant occurring in Eq. (8) (for incompressible flow) is found from the requirement that the expression ϕ_{wake} cancel the corresponding singularity in the circulation-free flow around the profile. This requires the evaluation of the integral occurring in this equation. Beyond this we are interested in the properties of the flow field due to ϕ_{wake}, in particular along the x-axis.

There may be a question whether Eq. (6) (which is more general than Eq. (8)) gives zero pressure within the wake. Therefore, we evaluate ϕ_x and ϕ_t directly from this equation. The result to be expected is found from Eq. (5)

$$\phi_x = -\pi g(t - (x/U)) \qquad 0 < x < Ut \tag{B.1}$$

Then, because the wake perturbation pressure is zero, from Eq. (1)

$$\phi_t = \pi U(g(t - (x/U)) \qquad 0 < x < Ut \tag{B.2}$$

We restrict ourselves to an evaluation of ϕ_t. To avoid in Eq. (6) a differentiation of g with respect to t, we introduce, in essence, instead of ξ the argument of g as a new variable.

$$\begin{aligned} Ut - \xi &= v \\ -d\xi &= dv \end{aligned} \tag{B.3}$$

The the limit $\xi = 0$ gives $v = Ut$,
the limit $\xi = Ut$ gives $v = 0$.

One obtains

$$\phi_{wake} = \int_0^{Ut} g(v/U) \operatorname{Im} \log \frac{z^{1/2} - (Ut - v)^{1/2}}{z^{1/2} + (Ut - v)^{1/2}} \, dv$$

The differentiation of the upper limit with respect to t gives

$$Ug(t)\mathrm{Im}\ \log\frac{z^{1/2}}{z^{1/2}}$$

The imaginary part of log 1 is not one-valued. Consider a point $z = x > Ut$. This is a point outside of the incipient wake and because the antisymmetry of the potential with respect to the x-axis $\phi = 0$. Therefore, log 1 must be taken equal to zero at these points and, by continuation, everywhere else. We are, therefore, left with differentiations under the integral sign. One obtains

$$\frac{\partial\phi_{wake}}{\partial t} = \int_0^{Ut} g(v/U)\frac{U}{(Ut - v)^{1/2}}\ \mathrm{Im}\ \frac{-z^{1/2}}{z - (Ut - v)}\ dv$$

$$\frac{\partial\phi_{wake}}{\partial x} = \int_0^{Ut} g(v/U)(Ut - v)^{1/2}\ \mathrm{Im}\ \frac{z^{-1/2}}{z - (Ut - v)}\ dv$$

For $z = x > 0$, but different from $Ut - v$, the imaginary parts are obviously zero. A contribution to ϕ_t, therefore, comes only from the immediate vicinity of the point $v = Ut - x$. There one obtains as dominant terms

$$\frac{\partial\phi_{wake}}{\partial t} = g\left(\frac{(Ut - x)}{U}\right)U \int_{v = Ut - x - a}^{Ut - x + b} \mathrm{Im}\ \frac{-dv}{v - (Ut - z)}$$

$$= -g(t - \frac{x}{U})U\ \mathrm{Im}\ \log(v - (Ut - z))\Big|_{v = Ut - x - a}^{Ut - x + b}$$

$$= -(g(t - \frac{x}{U})U\ \mathrm{Im}\ \log\frac{z - x + b}{z - x - a}$$

$a > 0$ small, $b > 0$ small, a and b real. Assume that z approaches the point x from within the upper half plane.

$$z = x + i\varepsilon$$

Then one has

$$\lim_{\varepsilon \to 0} \log \frac{z - x + b}{z - x - a} = \lim_{\varepsilon \to 0} \log \frac{b + i\varepsilon}{-a + i\varepsilon} = -\pi i$$

Hence in accordance with Eq. (5)

$$\frac{\partial \phi_{wake}}{\partial t}(t,x,+0) = U\pi g(t - \frac{x}{U})$$

The procedure for $\frac{\partial \phi_{wake}}{\partial x}$ is in essence the same.

For the remaining evaluations the specific form of g (Eq. (8)) is substituted. Introducing v as above (Eq. (B.3)) one obtains

$$\phi_{wake}(t,x,y) = \text{const} \int_0^{Ut} v^{-1/2} \text{ Im } \log \frac{z^{1/2} - (Ut - v)^{1/2}}{z^{1/2} + (Ut - v)^{1/2}} dv$$

Then

$$\frac{\partial \phi_{wake}}{\partial t} = - \text{const } U \text{ Im} \int_0^{Ut} v^{-1/2} (Ut - v)^{-1/2} \frac{z^{1/2}}{v - (Ut - z)} dv$$

$$-\frac{\partial \phi_{wake}}{\partial x} = \text{const Im} \int_0^{Ut} v^{-1/2} (Ut - v)^{1/2} \frac{1}{z^{1/2}(v - (Ut - z))} dv$$

$$-\frac{\partial \phi_{wake}}{\partial y} = \text{const Re} \int_0^{Ut} v^{-1/2} (Ut - v)^{1/2} \frac{1}{z^{1/2}(v - (Ut - z))} dv$$

We make the lengths dimensionless with Ut

$$v = Ut \cdot w$$

$$z = Ut \cdot \tilde{z}$$

Then

$$\frac{\partial \phi_{wake}}{\partial t} = -\text{ const } U^{1/2} t^{-1/2} \operatorname{Im} I_1$$

$$\frac{\partial \phi_{wake}}{\partial x} = \text{const}(Ut)^{-1/2} \operatorname{Im} I_2$$

$$\frac{\partial \phi_{wake}}{\partial y} = \text{const}(Ut)^{-1/2} \operatorname{Re} I_2 \qquad (B.4)$$

with

$$I_1 = \int_0^1 w^{-1/2}(1 - w)^{-1/2} \frac{\tilde{z}^{1/2}}{w - (1 - \tilde{z})} dw \qquad (B.5)$$

$$I_2 = \int_0^1 w^{-1/2}(1 - w)^{1/2} \frac{1}{\tilde{z}^{1/2}(w - (1 - \tilde{z}))} dw \qquad (B.6)$$

We set $w = q^2$. Then

$$I_1 = 2\tilde{z}^{1/2}\int_0^1 (1 - q^2)^{-1/2}[q^2 - (1 - \tilde{z})]^{-1} dq$$

$$I_2 = 2\tilde{z}^{-1/2}\int_0^1 (1 - q^2)^{1/2}[q^2 - (1 - \tilde{z})]^{-1} dq \qquad (B.7)$$

I_2 is rewritten

$$I_2 = -2\tilde{z}^{-1/2}\int_0^1 (1 - q^2)^{-1/2} dq + 2\tilde{z}^{1/2}\int_0^1 (1 - q^2)^{-1/2}[q^2 - (1 - \tilde{z})]^{-1} dq$$

$$I_2 = -\pi\tilde{z}^{-1/2} + I_1 \qquad (B.8)$$

I_1 can be evaluated in terms of elementary functions. The basic formula, which, of course, can be verified by differentiation, is found in Reference 2 (Eqs. 236, 3c, and 3d), $x < a$ is real.

$$\int \frac{dx}{(x - \alpha)\sqrt{a^2 - x^2}} = \tag{B.9a}$$

$$\frac{-1}{\sqrt{a^2 - \alpha^2}}\left\{\log(-\alpha x + a^2) + \sqrt{(a^2 - \alpha^2)(a^2 - x^2)} - \log(x - \alpha)\right\}$$

$$\int \frac{dx}{(x - \alpha)\sqrt{a^2 - x^2}} = \tag{B.9b}$$

$$\frac{1}{\sqrt{\alpha^2 - a^2}} \text{ arc sin } \frac{a^2 - \alpha x}{a(x - \alpha)} \quad ; \quad |\alpha| > a > 0, \ \alpha \text{ real}$$

For real α the first version is practical for $|\alpha| < a$; it can be used also for complex α. The expressions (B.7) and (B.8) can also be evaluated by the calculus of residues. The necessary transformations are shown in Appendix C, but in principle the calculus of residue requires the same transformations as the systematic derivation of Eqs. (B.9). In the present case a = 1 and x = q.

$$\int \frac{dq}{(q - \alpha)\sqrt{1 - q^2}} = \tag{B.10a}$$

$$-\frac{1}{\sqrt{1 - \alpha^2}}\left\{\log(1 - \alpha q + \sqrt{(1 - \alpha^2)(1 - q^2)}) - \log(q - \alpha)\right\}$$

$$\int \frac{dq}{(q - \alpha)} \sqrt{1 - a^2} = \tag{B.10b}$$

$$\frac{1}{\sqrt{\alpha^2 - 1}} \text{ arc sin } \frac{1 - \alpha q}{|q - \alpha|} \quad ; \quad |\alpha| > 1, \ \alpha \text{ real}$$

For $\tilde{z} = \tilde{x} > 1$ we set

$$\tilde{z} - 1 = \tilde{x} - 1 = b^2 \tag{B.11}$$

Then from Eq. (B.7)

$$I_1 = 2\tilde{z}^{-1/2}\int_0^1 (1 - q^2)^{-1/2}(q^2 + b^2)^{-1}dq$$

$$= \frac{\tilde{z}^{-1/2}}{ia}\int_0^1 (1 - q^2)^{-1/2}\left(\frac{1}{q - ib} - \frac{1}{q + ib}\right) dq$$

Here Eq. (B.10a) is applied. We set

$$\alpha = \pm\, ib$$

$$1 - \alpha^2 = 1 + b^2 = \tilde{x}$$

One obtains

$$I_1 =$$

$$- \frac{\tilde{x}^{1/2}}{ib}(1 + b^2)^{-1/2}\Big\{\log(1 - ibq + \sqrt{1 + b^2)(1 - q^2)}\Big|_0^1 - \log(q - ib)\Big|_0^1$$

$$- \log(1 + ibq + \sqrt{(1 + b^2)(1 - q^2)}\Big|_0^1 + \log(q + ib)\Big|_0^1\Big\}$$

After substitution of the limits, the terms left within the braces are $\log(-ib) - \log(ib) = -i\pi$. Substituting b (Eq. (B.11)) one obtains

$$I_1 = \pi(\tilde{x} - 1)^{-1/2} \qquad \tilde{x} > 1 \tag{B.12}$$

For $0 < \tilde{z} = \tilde{x} < 1$ one must delay setting $\tilde{z} = \tilde{x}$ because of the singularity which arises in the integrand at $q = 1 - x$. We write

$$1 - \tilde{z} = \alpha^2$$

Then

$$I_1 = 2\tilde{z}^{1/2} \int_0^1 (1 - q^2)^{-1/2}(q^2 - \alpha^2)^{-1} dq$$

$$= \tilde{z}^{1/2}\alpha^{-1}\int_0^1 (1 - q^2)^{-1/2}[(q - \alpha)^{-1} - (q + \alpha)^{-1}]dq$$

With Eq. (B.10a) one obtains

I_1 =

$$- \tilde{z}^{1/2}\alpha^{-1}(1 - \alpha^2)^{1/2}\{\log(1 - \alpha q + \sqrt{(1 - \alpha^2)(1 - q)}\,\Big|_{q=0}^{1} - \log(q - \alpha)\Big|_0^1$$

$$- \log(1 + \alpha q + \sqrt{(1 - \alpha^2)(1 - q)^2}\,\Big|_0^1 + \log(q + \alpha)\Big|_{q=0}^{1}\ \}$$

The singularity of the integrand expresses itself by the term

$\log(q - \alpha)\Big|_0^1$. We let $\tilde{z}$ approach the point $\tilde{x}$ in the upper half plan.

$$\tilde{z} = \tilde{x} + i\varepsilon \qquad \varepsilon > 0 \text{ small}$$

$$\alpha(\tilde{z}) = \sqrt{1 - \tilde{z}} = \sqrt{1 - \tilde{x}}\ \frac{i}{2}\ \frac{\varepsilon}{\sqrt{1 - \tilde{x}}} + \ldots$$

is then a point of the lower half plane, the variable of integration q is real and ranges from zero to one. See Figure B.1. A cut from $q = \alpha$ to $q = -i_\infty$ needed to make $\log(q - \alpha)$ one valued, does not intersect the path of integration. In the limit $\varepsilon \to 0$ the modulus of $q - \alpha$ is 0 for $q = 1$ and $+\pi$ for $q = 0$. Figure B.1 therefore

$$\lim_{\varepsilon \to 0} \log(q - \alpha)\Big|_0^1 = \log(1 - \alpha(\tilde{x})) - \log\alpha(\tilde{x})) = -i\pi$$

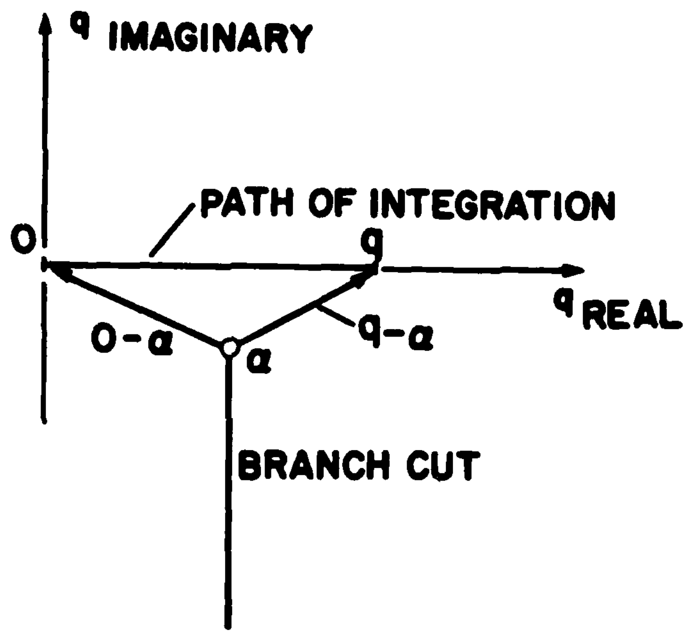

Figure B.1. Determination of the argument of $(q - \alpha)$ and $(0 - \alpha)$ in the complex q plane. A cut from the point α to negative imaginary infinity does not intersect the path of the integration from $q = 0$ to $q = 1$. As one moves in an $\tilde{x}$, $\tilde{y}$ plane, $(\tilde{y} > 0)$ to $\tilde{y} = 0$, α moves in the lower half plane toward the real q axis.

In the remaining terms one can replace immediately α by $\alpha(\tilde{x}) = \sqrt{1 - \tilde{x}}$. One obtains

$$I_1 = -i\pi(1 - \tilde{x})^{-1/2}$$

For $\tilde{z} = \tilde{x} < 0$ one can immediately replace z by $-|x|$. Then

$$z^{-1/2} = -i|x|^{-1/2}$$

$$z^{1/2} = i|x|^{1/2}$$

We set

$$1 - \tilde{z} = 1 + |\tilde{x}| = \alpha^2$$

Then

$$I_1 = \frac{i|\tilde{x}|^{1/2}}{\alpha} \int_0^1 (1 - q^2)^{-1/2}((q - \alpha)^{-1} - (q + \alpha)^{-1})$$

Here $|\alpha| > 1$ and the second version of Eq. (B.10) is applied

$$I = \frac{i|\tilde{x}|^{1/2}}{\alpha} (\alpha^2 - 1)^{-1/2} \left\{ \text{arc sin} \frac{1 - \alpha q}{\alpha - q}\Bigg|_0^1 - \text{arc sin} \frac{1 + \alpha q}{\alpha + q}\Bigg|_0^1 \right\}$$

Substituting the limits one obtains the same expression as for $0 < \tilde{x} < 1$

$$I_1 = -i\pi(1 - \tilde{x})^{-1/2} \qquad \tilde{x} < 1$$

We repeat Eq. (B.12)

$$I_1 = \pi(\tilde{x} - 1)^{-1/2} \qquad \tilde{x} > 1$$

Returning to Eqs. (B.4), using Eq. (B.8), and substituting $\tilde{x} = \frac{x}{Ut}$ one finds

$$\phi_t = 0 \qquad x > Ut$$

$$\phi_t = +\text{const } U\pi(Ut - x)^{-1/2} \qquad x < Ut$$

$$\phi_x = 0 \qquad x > Ut$$

$$\phi_x = -\text{const } \pi(Ut - x)^{-1/2} \qquad 0 < x < Ut$$

$$\phi_x = \text{const } \pi((x)^{-1/2} - (Ut - x)^{-1/2}) \qquad x < 0$$

$$\phi_y = \pi \text{ const}(-x^{-1/2} + (Ut - x)^{-1/2}) \qquad x > Ut$$

$$\phi_y = -\pi \text{ const}(x^{-1/2}) \qquad 0 < x < Ut$$

$$\phi_y = 0 \qquad x < 0$$

APPENDIX C

TRANSFORMATIONS OF THE EXPRESSION I_1, EQ. (B.7)

$$I_1 = 2 \int_0^1 \tilde{z}^{1/2} \frac{dq}{(1 - q^2)^{1/2}(q^2 - (1 - \tilde{z}))}$$

Since the integrand is an even function of q, we can write

$$I_1 = \tilde{z}^{1/2} \int_{-1}^{+1} \frac{dq}{(1 - q^2)^{1/2}(q^2 - (1 - \tilde{z})}$$

The transformation

$$p^2 = \frac{1 + q}{1 - q}$$

$$q = \frac{p^2 - 1}{p^2 + 1}$$

generates a rational integrand and maps the points $q = -1$ and $q = +1$ into $p = 0$ and $p = \infty$, respectively. One obtains

$$I_1 = \tilde{z}^{1/2} \int_0^{\infty} \frac{(p^2 + 1)2dp}{[(\tilde{z} - 1)(p^2 + 1)^2 + (p^2 - 1)^2]}$$

Since the integrand is an even function of p, one can write

$$I_1 = \tilde{z}^{1/2} \int_{-\infty}^{+\infty} \frac{(p^2 + 1)dp}{[(z - 1)(p^2 + 1)^2 + (p^2 - 1)^2]}$$

In this form the integral is suitable for the evaluation by the calculus of residues. The denominator can be rewritten

$$(p^2 - 1)^2 - (1 - \tilde{z})(p^2 + 1)^2$$

$$= \tilde{z}\, p^2 \left[- \frac{(1 + \sqrt{1 - \tilde{z}})^2}{\tilde{z}}\right]\left[p^2 - \frac{(1 - \sqrt{1 - \tilde{z}})^2}{\tilde{z}}\right]$$

From this expression the location of the poles is readily found.

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