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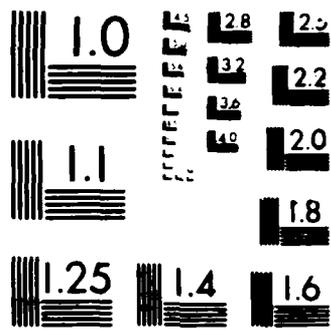
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TECHNICAL REPORTS AND PREPRINTS

Number 68: December 1986

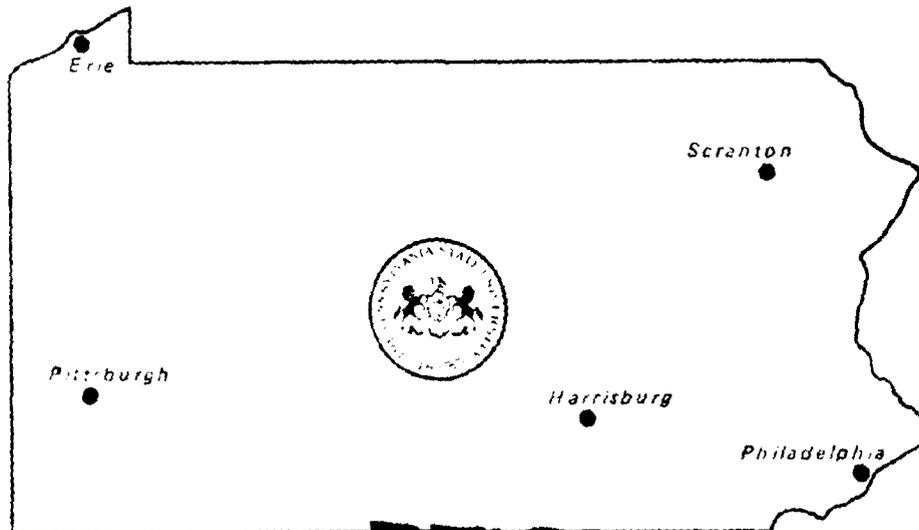
INVARIANT TESTS IN BIVARIATE MODELS  
AND THE  $L_1$  CRITERION

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\*This author's work was partially supported by ONR Contract N00014-80-C0741.

Abstract

Oja (1983) introduced an affine invariant multivariate median. Brown and Hettmansperger (1987) developed affine invariant rank methods in the bivariate location model based on Oja's measure of scatter. Analogs of the Wilcoxon signed rank and the Mann-Whitney-Wilcoxon rank sum tests were presented. In this paper, we discuss scale, rotation and affine invariance. ~~We then discuss tests in the bivariate linear model for an overall regression effect.~~ *In discussion* In particular we present tests in the one- and two-way layouts that are analogs of the Kruskal-Wallis and Friedman tests.

Key words: Affine invariance, AOV, Kruskal-Wallis, Friedman, linear models, spatial ranks.

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## 1. Introduction and Summary

In this paper we present affine invariant rank-like tests and estimates in the bivariate linear model. We describe a general test for an overall regression effect and then specialize the test to the one- and two-way layouts. The tests are affine invariant analogs of the Kruskal-Wallis (1952) test and the Friedman (1937) test. These tests are based on a notion of an affine invariant rank vector that can be developed from Oja's (1983) work on an affine invariant median. The analogs of the Wilcoxon signed rank test and the Mann-Whitney-Wilcoxon rank sum test are developed by the authors (1987).

Along the way, we describe three possible extensions of the  $L_1$  notion to the bivariate setting. The three types of bivariate medians that result are: the scale invariant vector of component medians, the rotation invariant spatial median, and the affine invariant Oja generalized median. These form the basis for the extension to three types of rank vectors in the linear model. The main emphasis will be on the affine invariant version which is illustrated on a data set in Section 3.

We indicate in Section 2 that the Oja generalized median has better efficiency than the other two medians for an underlying normal distribution. Although we do not have asymptotic efficiency results on the affine invariant tests in the linear model, we illustrate their robustness to outliers in the example. In addition, affine invariance may be an important consideration in a particular application and the tests presented here provide natural alternatives to the traditional tests based on means.

## 2. Bivariate Medians and Invariance

We first consider the development of rank based methods in the univariate linear model from the  $L_1$  norm operating on a single sample. For a given set of numbers  $y_1, \dots, y_n$  define their center by  $\hat{\theta}$  such that

$$D(\theta) = \sum_{i=1}^n |y_i - \theta| \quad (1)$$

is a minimum. It is well known that  $\hat{\theta}$  is the median of the set  $\{y_1, \dots, y_n\}$ , denoted  $\hat{\theta} = \text{med } y_n$ . The derivative with respect to  $\theta$  exists almost everywhere and is given by

$$Q(\theta) = \sum_{i=1}^n \text{sgn}(\theta - y_i). \quad (2)$$

The estimate  $\hat{\theta} = \text{med } y_i$  can be considered the solution of  $Q(\theta) = 0$  where "=" is interpreted as a sign change for  $n$  odd.

Next, Let  $r_i(\beta) = y_i - x_i' \beta$ ,  $i = 1, \dots, n$  represent residuals in a linear model, where  $x_i' = (x_{i1}, \dots, x_{ip})$  and  $\beta' = (\beta_1, \dots, \beta_p)$ . Then an estimate of  $\beta$  is determined by minimizing

$$\begin{aligned} D^*(\beta) &= \sum_{i=1}^n D(r_i(\beta)) & (3) \\ &= \frac{1}{2} \sum_{i < j} |r_i(\beta) - r_j(\beta)| \\ &= \sum_{i=1}^n \left\{ \text{Rank}(r_i(\beta)) - \frac{N+1}{2} \right\} r_i(\beta) \end{aligned}$$

$$= \frac{1}{2} \sum_{i=1}^n Q(r_i(\beta)) r_i(\beta).$$

The second equality relates the estimate of  $\beta$  to the  $L_1$  norm and the third equality, achieved after a bit of algebra, relates the estimate of  $\beta$  to the ranks of the residuals. Jaeckel (1972) proposed the rank version of  $D^*(\beta)$  as a criterion for estimating  $\beta$ . McKean and Hettmansperger (1976) proposed using the reduction in  $D^*(\beta)$  when passing from a reduced to full model as a criterion for testing hypotheses in the linear model. The fourth equality results from  $Q(r_i(\beta)) = \sum_j \text{sgn}(r_i(\beta) - r_j(\beta)) = 2(\text{Rank}(r_i(\beta)) - (N+1)/2)$ . We consider  $Q(r_i(\beta))$  to be a centered rank or quantile of  $r_i(\beta)$  in the sample. Note that the dispersion measure  $D^*(\beta)$  is a linear function of the residuals with weights determined by the quantiles.

Our goal is to carry an  $L_1$  notion over to the bivariate setting. We then will have a bivariate sample median with efficiency and robustness properties related to the univariate median. Then we extend, via the ideas in (3), to the linear model. The gradient  $Q(\theta)$  acts as a bivariate quantile and the result is rank-like tests and estimates in the bivariate linear model. The bivariate case is more complex than the univariate case and we will consider three different  $L_1$  notions.

Let  $y_i^T = (y_{i1}, y_{i2})$ ,  $i = 1, \dots, n$  be a set of bivariate observations, and define  $\dot{y}_i^T = (-y_{i2}, y_{i1})$ ,  $i = 1, \dots, n$ . We now present three bivariate dispersion measures each of which entails different invariance properties for the resulting estimates and tests. Let  $\theta^T = (\theta_1, \theta_2)$ , then

$$D_1(\underline{\theta}) = \sum_{i=1}^n (|y_{i1} - \theta_1| + |y_{i2} - \theta_2|) \quad (4)$$

$$D_2(\underline{\theta}) = \sum_{i=1}^n \{(y_{i1} - \theta_1)^2 + (y_{i2} - \theta_2)^2\}^{1/2} \quad (5)$$

$$D_3(\underline{\theta}) = \sum_{i < j} \left| \det \begin{bmatrix} y_{i1} & y_{j1} & \theta_1 \\ y_{i2} & y_{j2} & \theta_2 \\ 1 & 1 & 1 \end{bmatrix} \right| \quad (6)$$

The respective gradient vectors are given by

$$Q_1^T(\underline{\theta}) = (\sum \text{sgn}(\theta_1 - y_{i1}), \sum \text{sgn}(\theta_2 - y_{i2})) \quad (7)$$

$$Q_2(\underline{\theta}) = \sum \{(y_{i1} - \theta_1)^2 + (y_{i2} - \theta_2)^2\}^{-1/2} (\theta - y_i) \quad (8)$$

$$Q_3(\underline{\theta}) = \sum_{i < j} \text{sgn}(\det \begin{bmatrix} y_{i1} & y_{j1} & \theta_1 \\ y_{i2} & y_{j2} & \theta_2 \\ 1 & 1 & 1 \end{bmatrix}) (\dot{y}_j - \dot{y}_i) \quad (9)$$

The dispersion  $D_1(\underline{\theta})$  in (4) with  $Q_1(\underline{\theta})$  in (7) defines componentwise  $L_1$  methods. The estimate  $\hat{\underline{\theta}}^T = (\text{med } y_{i1}, \text{med } y_{i2})$ , is scale invariant but not rotation invariant; see Bickel (1964). The dispersion  $D_2(\underline{\theta})$  in (5) with  $Q_2(\underline{\theta})$  in (8) defines the spatial median which is rotation invariant but not scale or affine invariant; see Brown (1983). Finally, the dispersion  $D_3(\underline{\theta})$  in (6) with  $Q_3(\underline{\theta})$  in (9) is the sum of areas of triangles formed by taking  $\underline{\theta}$  and pairs of observations as the vertices. Minimizing the sum of areas results in the generalized median  $\hat{\underline{\theta}}$  proposed by Oja (1983). The Oja generalized median is affine invariant.

Both  $D_1(\underline{\theta})$  and  $D_2(\underline{\theta})$  reduce to the  $L_1$  norm in one dimension. This is

not the case, however, with  $D_3(\theta)$ . It seems that  $D_3(\theta)$  is more intrinsically bivariate than the others. The three bivariate medians: componentwise medians, spatial median, and Oja generalized median, have somewhat different efficiency properties, but all are related to the efficiency of the univariate median. The univariate median has efficiency .637 relative to the mean when the underlying distribution is normal.

Following Bickel (1964, p.1083), we define the efficiency of the bivariate median relative to the bivariate mean to be  $[g\text{-var}(\text{mean})/g\text{-var}(\text{median})]^{1/2}$ , where  $g\text{-variance}$  is  $\sigma_1^2\sigma_2^2(1-\rho^2)$  and the parameters are taken from the asymptotic covariance matrix. Bickel's Theorem 5.1 provides a formula for the efficiency of the componentwise medians when the underlying distribution is bivariate normal. The efficiency, independent of the variances, declines as a function of the correlation coefficient  $\rho$  from .637 when  $\rho = 0$ . Some values of  $(\rho, \text{efficiency})$  are  $(0, .637)$ ,  $(.2, .629)$ ,  $(.4, .605)$ ,  $(.6, .558)$ ,  $(.8, .473)$  and  $(.9, .396)$ . The spatial median, on the other hand, has efficiency that does depend upon the variances but not on  $\rho$ . Using Table 1 of Brown (1983) and Bickel's definition of efficiency, it is easy to see that the spatial median is generally more efficient relative to the mean for an underlying bivariate normal distribution than the vector of componentwise medians. The efficiency depends upon the ratio of standard deviations of the two components. If  $\lambda$  is that ratio, then some values of  $(\lambda, \text{efficiency})$  are:  $(1, .785)$ ,  $(.8, .783)$ ,  $(.6, .773)$ ,  $(.4, .747)$ ,  $(.2, .678)$ ,  $(.05, .593)$  and  $(.01, .321)$ . The efficiency deteriorates as the contour ellipses of the bivariate normal distribution become very narrow. Oja and Niinimaa (1985) show that the affine invariant Oja generalized median has efficiency relative to the mean equal to .785, independent of the variances

and correlation. It is strictly better than the spatial median unless the bivariate normal distribution is circular in which case the efficiency is the same. Hence, the Oja generalized median has superior efficiency properties as well as enjoying affine invariance.

### 3. Tests in the Linear Model

The next step is to extend these  $L_1$  notions for a single sample to different types of quantiles in the linear model. Using (7) - (9) we arrive at the vector of component ranks, the spatial ranks and the affine invariant ranks, respectively. These ranks then provide the basis for tests.

We first introduce the notation of the bivariate linear model. Let  $Y$  be an  $n \times 2$  observation matrix in which the  $n$  rows are independent random vectors such that

$$E Y = E \begin{bmatrix} Y_1^T \\ \vdots \\ Y_n^T \end{bmatrix} = Z \beta, \quad (10)$$

where  $Z$  is an  $n \times (p+1)$  matrix of known regression constants and  $\beta$  is a  $(p+1) \times 1$  matrix of unknown parameters. Let  $r_1(\beta) = Y_1 - \beta^T z_1$  denote the  $i$ th  $2 \times 1$  residual vector where  $z_1^T$  is the  $i$ th row of  $Z$ .

In each of the three cases (4) - (6), it can be shown that, for  $i = 1, 2, 3$ ,

$$D_1^{\#}(\beta) = \sum_{j=1}^n D_{1j}(\beta) \quad (11)$$

$$\sum_{j=1}^n \mathbf{w}_j^T \mathbf{C}_j \mathbf{B} \mathbf{C}_j^T \mathbf{B}$$

It is obvious from (1) for  $\mathbf{B} = \mathbf{I}$  (Brown and Hettmansperger, 1977) that (2) holds for  $\mathbf{B} = \mathbf{I}$ . Straightforward computations show that also (2) holds for  $\mathbf{B} = \mathbf{I}$ . The vector  $\mathbf{q}_j \in \mathbb{R}^k$  is called the *aligned centered quantile of entered rank*. This is partly expressed by the property  $\mathbf{C}_j \mathbf{q}_j \in \mathbb{R}^k$  in each case  $j = 1, \dots, n$  and is analogous to the deviation of entered ranks from  $\mu_j$  and  $\sigma_j$  as used after (1). The second form in (2) shows that our dispersion criterion is similar to that proposed by Jaeckel (1972) in the univariate linear model and is related to the third equality in (3). For  $\mathbf{C}_j = \mathbf{D}_j \mathbf{A}_j^T$   $\mathbf{B} = \mathbf{A}_j \mathbf{A}_j^T$  is a rotation invariant and affine invariant, respectively, and will provide similar invariance in the estimates and tests based on them.

We now refine the notation of the linear model. To separate the  $k$ -dimensional vector from the regression portion, henceforth we start to write  $\mathbf{y}_j$  as  $\mathbf{y}_j = \mathbf{u}_j \mathbf{0}_k^T + \mathbf{q}_j$ .

Let

$$\mathbf{B} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{\Delta}_1 & \mathbf{\Delta}_2 \\ \vdots & \vdots \\ \mathbf{\Delta}_{1-} & \mathbf{\Delta}_{2-} \end{pmatrix} = \begin{pmatrix} \mathbf{B} \\ \mathbf{\Delta} \end{pmatrix}$$

and let  $\mathbf{C}_j^T = (\mathbf{y}_{j1}^T, \dots, \mathbf{y}_{jp}^T)$  with  $\mathbf{y}_{j\alpha}$  an  $n$ -vector with  $\alpha$ th row  $(\mathbf{y}_{j\alpha})_i = \mathbf{y}_{ij}$ , then



$$\begin{aligned} \underline{L} &= \left( \left( -\frac{\partial}{\partial \underline{\Delta}} D(\underline{\Delta}) \Big|_{\underline{\Delta}=\underline{0}} \right) \right) \\ &= \underline{X}^T \underline{Q} \end{aligned} \quad (15)$$

where

$$\underline{Q} = \begin{bmatrix} q_1^{(1)} & q_1^{(2)} \\ \vdots & \vdots \\ q_n^{(1)} & q_n^{(2)} \end{bmatrix}.$$

To test  $H_0: \underline{\Delta} = \underline{0}$  we must assess the size of  $\underline{L}$ . The statistic we will use is the Hotelling-Lawley trace statistic given by

$$\begin{aligned} S &= (n-1) \text{Trace} \left[ \underline{L}^T (\underline{X}_C^T \underline{X}_C)^{-1} \underline{L} (\underline{Q}^T \underline{Q})^{-1} \right] \\ &= (n-1) \text{Trace} \left[ \underline{L} (\underline{Q}^T \underline{Q})^{-1} \underline{L}^T (\underline{X}_C^T \underline{X}_C)^{-1} \right] \end{aligned} \quad (16)$$

where  $\underline{X}_C$  is the mean centered design matrix and we assume that  $\underline{Q}^T \underline{Q}$  and  $\underline{X}_C^T \underline{X}_C$  are nonsingular. The trace form of  $S$  is not immediately intuitive and we will provide two motivations for  $S$ .

In the first motivation, we roll out the  $p \times 2$  matrix  $\underline{L}$  by columns into

$$\underline{L}^T = (L_1^{(1)} \dots L_p^{(1)} L_1^{(2)} \dots L_p^{(2)}). \quad (17)$$

where the first half of the vector consists in the statistics relevant to the first component and similarly for the second half.

Recall that the direct or Kronecker product of two matrices is defined

where



$$S = \frac{(n-1)}{1-r_{12}^2} \left( \frac{1}{q_{11}} \underline{L}^{(1)T} (\underline{X}_C^T \underline{X}_C)^{-1} \underline{L}^{(1)} + 2 \frac{q_{12}}{q_{11}q_{22}} \underline{L}^{(1)T} (\underline{X}_C^T \underline{X}_C)^{-1} \underline{L}^{(2)} + \frac{1}{q_{22}} \underline{L}^{(2)T} (\underline{X}_C^T \underline{X}_C)^{-1} \underline{L}^{(2)} \right). \quad (19)$$

This shows that the multivariate test statistic is composed of the corresponding componentwise test statistics along with a cross component statistic. The combining weights depend on the variances and covariances of the quantiles.

For the second motivation, we roll out  $\underline{L}$  by rows into

$$\underline{L}_{row}^T = (L_1^{(1)} L_1^{(2)} L_2^{(1)} L_2^{(2)} \dots L_p^{(1)} L_p^{(2)}).$$

Under  $H_0: \underline{\Delta} = \underline{0}$ , the permutation covariance matrix of  $\underline{L}_{row}$  is

$$\text{Cov}(\underline{L}_{row}) = \frac{1}{n-1} (\underline{X}_C^T \underline{X}_C)^{-1} (\underline{Q}^T \underline{Q}).$$

Then the second form of  $S$  in (16) reduces to

$$S = (n-1) \underline{L}_{row}^T [(\underline{X}_C^T \underline{X}_C)^{-1} (\underline{Q}^T \underline{Q})^{-1}] \underline{L}_{row} \quad (20)$$

another natural representation of  $S$  as a quadratic form for assessing the size of  $\underline{L}$ .

Let  $x_c^{ij}$  denote the  $(i,j)$  element of  $(\underline{X}_C^T \underline{X}_C)^{-1}$  then  $S$  in (20) can be written

$$S = (n-1) \sum_{i=1}^p \sum_{j=1}^p x_c^{ij} \underline{L}_{(i)}^T (\underline{Q}^T \underline{Q})^{-1} \underline{L}_{(j)} \quad (21)$$

where  $\underline{L}_{(i)} = (L_i^{(1)} L_i^{(2)})$ . This provides a nice interpretation because the square of the univariate statistic is replaced by a quadratic form in the components of the bivariate statistic.

We finally note that to test  $H_0: \underline{\Delta} = \underline{0}$  vs  $H_A: \underline{\Delta} \neq \underline{0}$ , that is, test for an overall regression effect, we reject  $H_0$  if

$$S > \chi_{\alpha}^2(2p) \tag{22}$$

where  $\chi_{\alpha}^2(2p)$  is a chi-square critical value. This test has approximate significance level  $\alpha$ . The limiting chi-square distribution follows from standard permutation arguments; see, for example, Puri and Sen (1985).

We next specialize the statistic  $S$  to the one- and two-way layouts. In the one-way layout, we suppose that we have  $k$  samples of size  $n_1, \dots, n_k$ , from continuous, bivariate distributions with location vectors  $\theta_1, \dots, \theta_k$ , and we wish to test  $H_0: \theta_1 = \dots = \theta_k$ . The design matrix  $X$  consists of columns of zeros and ones. In the regression setting one column can be dropped to insure full rank. If we drop the first column, then the matrix  $L$ , rolled out by columns is  $L_{col}^T = (L_2^{(1)} \dots L_k^{(1)} L_2^{(2)} \dots L_k^{(2)})$  and, for example,  $L_2^{(1)}$  is the sum of the first components of the combined sample quantiles corresponding to the second sample. The matrix  $X_c$  along with  $(X_c^T X_c)^{-1}$  is given by Hettmansperger (1984, p.258). The element  $X_c^{1j} = n_1^{-1} + n_j^{-1}$  if  $j = 1$  and  $X_c^{1j} = n_j^{-1}$  if  $j \neq 1$ . We can now describe the two versions of  $S$ , (19) and (21), for the one-way layout. From (19) we find

$$S = \frac{H^{(1)}}{D^{(1)}} = \frac{1}{D^{(1)}} H^{(1)} = \frac{2q_{12}}{4 + 3q_{22}} H^{(1)} + \frac{1}{4q_{22}} H^{(2)} \tag{23}$$

where  $H^{(1)} = \sum_{j=1}^k n_j^{-1} (1 - \delta_{1j})$ ,  $H^{(2)} = \sum_{j=1}^k n_j^{-1} \delta_{1j}$  and  $D^{(1)} = \sum_{j=1}^k n_j^{-1} (1 - \delta_{1j})^2$ . Note

that  $\frac{2q_{12}}{4 + 3q_{22}} H^{(1)}$  is a Hotelling-Wallis (1951) type statistic computed on the

first component of the quantiles, similarly for  $(n-1)q_{22}^{-1} H^{(2)}$ , while  $H^{(12)}$  is a cross component statistic.

From (21) we find

$$S = (n-1) \sum_{i=1}^k n_i^{-1} (L_i^{(1)} L_i^{(2)}) (Q^T Q)^{-1} (L_i^{(1)} L_i^{(2)})^T. \quad (24)$$

This corresponds to a Kruskal-Wallis calculation using the  $k$  quadratic forms.

If  $q_j^{(i)} = R_j^{(i)} - (n+1)/2$ ,  $i = 1, 2$ ,  $j = 1, \dots, n$  are the centered componentwise ranks in the combined sample, then  $S$  is the test discussed by Puri and Sen (1971, p.186) and is a direct generalization of the univariate Kruskal-Wallis test. If  $q_j$  is taken to be a bivariate spatial quantile, then a rotation invariant test results. Our primary interest is in describing the affine invariant version and to that purpose we now present an example.

Example:

In Table 1 we present data on tail length and wing length for 4 subspecies of birds, and we wish to carry out a test to see if the location vectors differ among the subspecies.

- Table 1 about here -

Since the data is arrayed in a one-way layout, we will use the affine invariant test based on  $S$  in (22). We find  $S = 36.5$ , and if  $\alpha = .01$ , the approximate critical value is  $\chi_{.01}^2(6) = 16.8$ . Hence, the null hypothesis of no difference among the 4 population location vectors is easily rejected.

In Figure 1a, we present plots of the data and the corresponding quantiles. To illustrate the effects of an outlier, we changed the first component of the first observation in Table 1 from 207 to 2.7. The resulting value of  $S$  is 33.0, still highly significant. In Figure 1b, we present the

new plots of data and quantiles.

- Figure 1 about here -

Generally, the statistic  $S$  will be less effected by outlying data than the normal theory based Hotelling-Lawley test. The authors (1987) provide a two-sample example in which the introduction of an outlier has little effect on the quantile test but switches the traditional test from significance to nonsignificance.

We now turn to the two-way layout. We consider a randomized block design and describe an analog to the Friedman (1937) test. Suppose we have  $k$  treatments and  $n$  blocks. In the univariate two-way layout the observations are replaced by their ranks within blocks. Then these ranks are combined into a quadratic form based on the  $k$  treatment rank sums. In the bivariate two-way layout we replace the observation vectors by the bivariate quantiles computed within blocks.

Let  $q_{ij}^T = (q_{ij}^{(1)}, q_{ij}^{(2)})$  denote the quantile in the  $(i, j)$  cell, corresponding to block  $i$  and treatment  $j$ . Then  $L_j^T = (L_j^{(1)}, L_j^{(2)}) = (\sum_{i=1}^n q_{ij}^{(1)}, \sum_{i=1}^n q_{ij}^{(2)})$  is the sum of quantiles for the  $j$ th treatment,  $j = 1, \dots, k$ .

It remains to determine the permutation covariance matrix of the quantiles. This is done by estimating within each block separately then averaging over blocks. We define

$$\frac{1}{n(k-1)} Q^T Q = \frac{1}{n(k-1)} \sum_{i=1}^n \sum_{j=1}^k q_{i,j} q_{i,j}^T. \quad (25)$$

The natural analog of Friedman's statistic is

$$\begin{aligned}
 S &= (k-1) \sum_{i=1}^n (L_i^{(1)} L_i^{(2)}) (Q^T Q)^{-1} (L_i^{(1)} L_i^{(2)})^T & (26) \\
 &= \frac{k-1}{1-r_{12}^2} \left( \frac{1}{q_{11}} F^{(1)} - \frac{2q_{12}}{q_{11}q_{22}} F^{(12)} + \frac{1}{q_{22}} F^{(2)} \right)
 \end{aligned}$$

where  $F^{(j)} = \sum_{i=1}^k (L_i^{(j)})^2$ ,  $j = 1, 2$ ,  $F^{(12)} = \sum_{i=1}^k L_i^{(1)} L_i^{(2)}$ , and  $q_{ij}$  is the  $(i, j)$  element of  $Q^T Q$  in (24), and  $r_{12} = q_{12}/(q_{11}q_{22})^{1/2}$ .

If  $q_{ij}^{(s)}$ ,  $s = 1, 2$ , are the componentwise centered ranks in cell  $(i, j)$ , then we get the scale invariant test described by Puri and Sen (1971, p.279). If we use spatial quantiles, we get a rotation invariant test and if we use the affine invariant quantiles, we get an affine invariant test. Standard permutation theory shows that an approximate size  $\alpha$  test rejects the null hypothesis of no treatment effect if  $S > \chi_{\alpha}^2(k-1)$ .

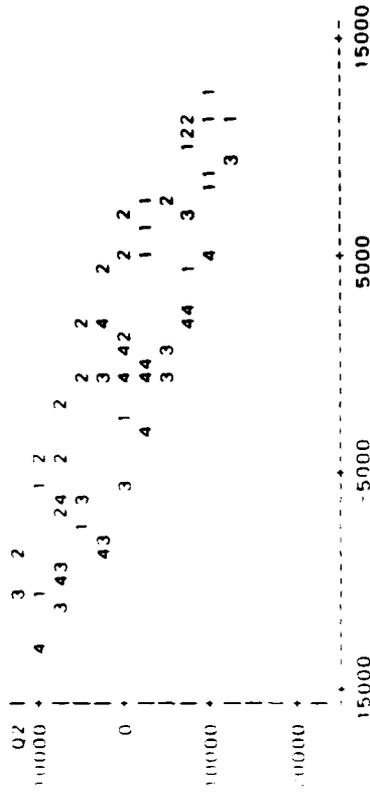
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Subspecies							
1		2		3		4	
T	W	T	W	T	W	T	W
207	294	193	282	176	262	186	275
200	287	196	290	185	286	167	276
198	275	210	306	187	270	191	280
192	259	201	283	177	264	183	271
188	270	199	279	165	256	181	269
182	261	205	294	182	257	177	257
197	279	194	294	182	281	180	277
195	282	208	300	169	253	196	285
178	273	195	282	163	265	190	274
185	264	186	281	162	258	191	285
200	277	206	299	185	265	169	255
184	269	211	309	193	283	179	265
198	285	212	303	178	274	163	251
179	258	202	290	171	271	174	262
202	296	184	279	180	274	174	259

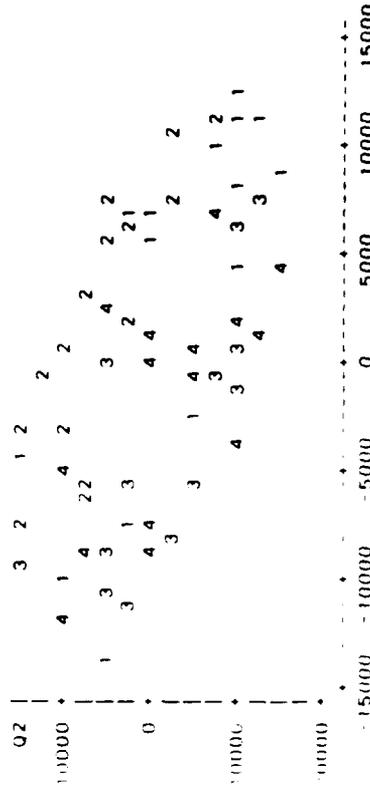
Table 1. Tail and wing length data.

QUANTILE PLOT: NUMBER INDICATES SAMPLE



NOTE: 8 OBS HIDDEN

QUANTILE PLOT: NUMBER INDICATES SAMPLE



NOTE: 2 OBS HIDDEN



END

2-87.

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