

AD-A175 408

EIGENFUNCTION METHODS IN MAGNETOSPHERIC
RADIAL-DIFFUSION THEORY(U) AEROSPACE CORP EL SEGUNDO CA
SPACE SCIENCES LAB M SCHULZ 30 SEP 86
TR-0086(6940-05)-1 SD-TR-86-86

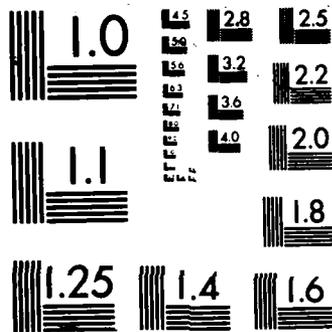
1/1

UNCLASSIFIED

F/G 4/1

NL





COPY RESOLUTION TEST CHART

Eigenfunction Methods in Magnetospheric Radial-Diffusion Theory

M. SCHULZ
Space Sciences Laboratory
Laboratory Operations
The Aerospace Corporation
El Segundo, CA 90245

30 September 1986

DTIC
ELECTE
DEC 31 1986
S D

AD-A175 408

APPROVED FOR PUBLIC RELEASE;
DISTRIBUTION UNLIMITED

FILE COPY

Prepared for
SPACE DIVISION
AIR FORCE SYSTEMS COMMAND
Los Angeles Air Force Station
P.O. Box 92960, Worldway Postal Center
Los Angeles, CA 90009-2960

This report was submitted by The Aerospace Corporation, El Segundo, CA 90245, under Contract No. F04701-85-C-0086 with the Space Division, P.O. Box 92960, Worldway Postal Center, Los Angeles, CA 90009-2960. It was reviewed and approved for The Aerospace Corporation by H. R. Rugge, Director, Space Sciences Laboratory.

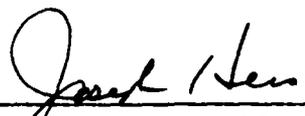
Capt Douglas R. Case/YCM was the project officer for the Mission-Oriented Investigation and Experimentation (MOIE) Program.

This report has been reviewed by the Public Affairs Office (PAS) and is releasable to the National Technical Information Service (NTIS). At NTIS, it will be available to the general public, including foreign nationals.

This technical report has been reviewed and is approved for publication. Publication of this report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.



DOUGLAS R. CASE, Capt, USAF
MOIE Project Officer
SD/YCM



JOSEPH HESS, GM-15
Director, AFSTC West Coast Office
AFSTC/WCO OL-AB

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER SD-TR-86-86	2. GOVT ACCESSION NO. <i>AD-A175-408</i>	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) EIGENFUNCTION METHODS IN MAGNETOSPHERIC RADIAL-DIFFUSION THEORY		5. TYPE OF REPORT & PERIOD COVERED
		6. PERFORMING ORG. REPORT NUMBER TR-0086(6940-05)-1
7. AUTHOR(s) Michael Schulz		8. CONTRACT OR GRANT NUMBER(s) F04701-85-C-0086
9. PERFORMING ORGANIZATION NAME AND ADDRESS The Aerospace Corporation El Segundo, Calif. 90245		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Space Division Los Angeles Air Force Station Los Angeles, Calif. 90009-2960		12. REPORT DATE 30 September 1986
		13. NUMBER OF PAGES 15
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Bessel Functions Radiation Belts Transport Theory		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Complete sets of orthonormal basis functions constructed according to a generalization of the quantum-mechanical WKB approximation can be used to generate a nearly diagonal matrix representation of the radial-transport operator for ring-current ions in the presence of radial diffusion and charge exchange. The resulting eigenfunctions (constructed by weighting the basis functions in proportion to the respective components of the eigenvectors of the matrix representation) and eigenvalues provide a spatial and temporal description of the evolving phase-space density during and		

DD FORM 1473
(FACSIMILE)

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

19. KEY WORDS (Continued)

20. ABSTRACT (Continued)

following a magnetospheric disturbance (e.g., a magnetic storm). A linear superposition of the basis functions can also be used to eliminate any discrepancy between the steady-state solution of the transport equation and the appropriate WKB approximation of this steady-state solution.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

CONTENTS

INTRODUCTION..... 3
TRANSFORMATIONS..... 4
STEADY STATE..... 6
EIGENFUNCTIONS AND EIGENVALUES..... 8
APPLICATIONS..... 12
REFERENCES..... 13



Accession For	
NTIS CRA&I	<input checked="" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By	
Distribution/	
Availability Codes	
Dist	Avail and/or Special
A-1	

INTRODUCTION

If energy degradation (Coulomb loss) and pitch-angle diffusion can be neglected, then the radial transport of magnetospheric ring-current ions is governed by an equation of the form

$$\begin{aligned} \partial \bar{f} / \partial t &= L^2 (\partial / \partial L) [(D_{LL} / L^2) (\partial \bar{f} / \partial L)] - (\bar{f} / \tau_q) \\ &= (\partial / \partial \Phi) [D_{\Phi\Phi} (\partial \bar{f} / \partial \Phi)] - (\bar{f} / \tau_q), \end{aligned} \quad (1)$$

where \bar{f} is the phase-space density at fixed M and J (first two adiabatic invariants), D_{LL} is the diffusion coefficient for transport in L (dimensionless shell parameter), Φ is the third adiabatic invariant (inversely proportional to L), and τ_q is the ionic lifetime against charge exchange. Jentsch [1984] has described a method for obtaining approximate steady-state solutions of this equation when D_{LL} is exactly proportional to a fixed power (β) of L , i.e., when $D_{\Phi\Phi}$ is exactly proportional to a fixed power ($4-\beta$) of Φ . The purpose of the present work is to describe an alternative method for obtaining time-dependent as well as steady-state solutions of (1) while permitting the dependence of D_{LL} upon L (or equivalently, of $D_{\Phi\Phi}$ upon Φ) to deviate somewhat from a strict power law.

The alternative method is highly advantageous because even the simplest dynamical models for magnetospheric radial diffusion lead to diffusion coefficients D_{LL} that deviate in fact from strict power laws except in certain limits. For example, the standard model [e.g., Cornwall, 1972; Schulz, 1983] for charged-particle diffusion in a dipolar magnetic field leads to a diffusion coefficient of the form

$$\begin{aligned} D_{LL} \approx & \frac{1 \times 10^{-10} L^{10} \text{ day}^{-1}}{(M/y^2 \gamma Z M_0)^2 [2D(y)/T(y)]^2 + 10^{-6} L^4} \\ & + 7 \times 10^{-9} [Q(y)/180D(y)]^2 L^{10} \text{ day}^{-1}, \end{aligned} \quad (2)$$

where y is the sine of the equatorial pitch angle α_0 , γ is the ratio of relativistic mass m to rest mass m_0 , Z is the integer that specifies charge state, and $M_0 \equiv 1 \text{ GeV/gauss}$. The auxiliary functions $Q(y)$, $D(y)$, and $T(y)$ in (2) are well approximated [Schulz and Lanzerotti, 1974, pp. 20, 21, 44; Davidson, 1976] by the algebraic expressions

$$Q(y) = -27.12667 - 45.39913y^4 + 5.88256y^8, \quad (3a)$$

$$D(y) = 0.4600577 + 0.1066154y^{3/4} - 0.1997662y, \quad (3b)$$

and

$$T(y) = 1.3801730 - 0.6396925y^{3/4}. \quad (3c)$$

A further complication is that y and γ in (2) typically vary with L at fixed M and J . The variation of y is given approximately [Chen and Stern, 1975] by

$$y^{-2} = 1 + 1.38048X - 0.030425X^{4/3} + 0.10066X^{5/3} + [X/2T(0)]^2, \quad (4)$$

where $X \equiv (La/8m_0\mu M)^{1/2}J$, a is the radius of the earth, and μ is the earth's magnetic moment. The variation of γ with L is given by

$$\gamma^2 = 1 + (2\mu M/L^3 a^3 y^2 m_0 c^2), \quad (5)$$

where c is the speed of light. The limiting cases $X = 0$ ($J = 0$) and $X = \infty$ ($M = 0$) correspond to $y = 1$ and $y = 0$, respectively, but $X = \infty$ implies $M/y^2 = J^2 La/32\mu m_0 [T(0)]^2$ upon evaluation of the indeterminate form.

TRANSFORMATIONS

The factors $[2D(y)/y^2 T(y)]^2$ and $[Q(y)/180D(y)]^2$ in (2) vary approximately as powers of L [Schulz and Lanzerotti, 1974, pp. 91, 93]. The exponents of L are given by

$$2L \frac{\partial}{\partial L} \left[\ln \frac{D(y)}{y^2 T(y)} \right]_{M,J} = \frac{Y(y)}{24D(y)} \left\{ 10 - \left[1 + \frac{yT'(y)}{T(y)} \right] \frac{Y(y)}{T(y)} \right\} = \begin{cases} 0, & y = 1 \\ 2, & y = 0 \end{cases} \quad (6a)$$

and

$$2L \frac{\partial}{\partial L} \left[\ln \frac{Q(y)}{D(y)} \right]_{M,J} = \left[\frac{yD'(y)}{2D(y)} - \frac{yQ'(y)}{2Q(y)} \right] \frac{Y(y)}{T(y)} = \begin{cases} 0, & y = 1 \\ 0, & y = 0 \end{cases} \quad (6b)$$

respectively, where [Schulz and Lanzerotti, 1974, pp. 20, 21]

$$\left(\frac{\partial \ln y}{\partial \ln L} \right)_{M,J} = -\frac{Y(y)}{4T(y)} = \begin{cases} 0, & y = 1 \\ -1/2, & y = 0 \end{cases} \quad (7a)$$

and

$$Y(y) = 2y \int_y^1 (y')^{-2} T(y') dy' = 6[T(y) - 2D(y)] = \begin{cases} 0, & y = 1 \\ 2T(0), & y = 0 \end{cases} \quad (7b)$$

The exponents of L implied by (6) remain approximately (but only approximately) independent of L for fixed (but nonvanishing) M and J . They remain strictly independent of L only for $y = 1$ and for $y = 0$, and even in these cases the form of D_{LL} specified by (2) is not strictly proportional to a fixed power of L , e.g., proportional to L^β with β a function of M and J only.

The transformation $z = (\beta - 3) \ln L$ proposed by Jentsch [1984] brings (1) into an equation of the form

$$\partial w / \partial t = (\beta - 3)^2 L^{-2} D_{LL} [(\partial^2 / \partial z^2) - (1/4)] w - (w / \tau_q), \quad (8)$$

where $w = L^{(\beta-3)/2} \bar{f}$, if $D_{LL} \propto L^\beta$ for fixed β . The transformation

$$\zeta = \int_L^\infty (L')^2 D_{LL}^{-1} dL' = L^2 (d\Phi/dL) \int_0^\Phi D_{\Phi\Phi}^{-1} d\Phi \quad (9)$$

introduced in the present work brings (1) into an equation of the form

$$\partial \bar{f} / \partial t = (L^4 / D_{LL}) (\partial^2 \bar{f} / \partial \zeta^2) - (\bar{f} / \tau_q) \quad (10)$$

without recourse to the assumption that $D_{LL} \propto L^\beta$ for fixed β . The factor $L^2(d\Phi/dL)$ in (9) is a constant, since $\Phi \propto 1/L$, and the factor L^4/D_{LL} in (10) is proportional to $D_{\Phi\Phi}^{-1}$ for the same reason. The fixed limit of integration $L' = \infty$ ($\Phi' = 0$) in (9) lies outside the domain of validity of (1) and thus requires the integrand of (9) to be evaluated by analytical extrapolation for $L' > L_1$. However, it follows from (9) that $\zeta = (\beta-3)^{-1} L^3 D_{LL}^{-1} \propto L^{3-\beta}$ if $D_{LL} \propto L^\beta$ for fixed $\beta > 3$, and no other fixed limit of integration in (9) would lead to such a simple result for ζ . The form of (10) suggests a time-dependent solution

$$\bar{f}(L,t) = \bar{f}_\infty(L) + \sum_{n=0}^{\infty} a_n(t) g_n(L) \quad (11)$$

in which $\bar{f}_\infty(L)$ is the steady-state solution of (10) and the $g_n(L)$ are the eigenfunctions of the operator $\Lambda \equiv - (L^4/D_{LL})(\partial^2/\partial\zeta^2) + (1/\tau_q)$, corresponding (respectively) to the eigenvalues λ_n . The expansion coefficients $a_n(t)$ are thus given by $a_n(t) = a_n(0)\exp(-\lambda_n t)$ if the transport coefficients and boundary conditions are time-independent for $t > 0$.

STEADY STATE

The steady-state solution $\bar{f}_\infty(L)$ thus satisfies, according to (10), the equation

$$(d^2\bar{f}_\infty/d\zeta^2) - (D_{LL}/L^4\tau_q)\bar{f} = 0, \quad (12)$$

subject to the inner boundary condition that $\bar{f}_\infty(L_0) = 0$ at the top of the atmosphere and the outer boundary condition that $\bar{f}_\infty(L_1)$ correspond to the phase-space density at the inner edge of the plasma sheet. The exact solution of (12) is expressible [cf. Schiff, 1955, p. 187; Walt, 1970, p. 414]

in terms of modified Bessel functions of fractional order if $D_{LL}/L^4 \tau_q$ is exactly proportional to a fixed power (p) of ζ . This fact suggests a modified WKB approximation of the form

$$\bar{f}_\infty(L) \approx \hat{f}_\infty(L) \equiv \frac{\hat{\theta}^{1/2} L (\tau_q / D_{LL})^{1/4}}{\hat{\theta}_1^{1/2} L_1 (\tau_q / D_{LL})_1^{1/4}} \times \frac{I_\nu(\hat{\theta}) K_\nu(\hat{\theta}_0) - K_\nu(\hat{\theta}) I_\nu(\hat{\theta}_0)}{I_\nu(\hat{\theta}_1) K_\nu(\hat{\theta}_0) - K_\nu(\hat{\theta}_1) I_\nu(\hat{\theta}_0)} \bar{f}_\infty(L_1) \quad (13)$$

for the steady-state solution of (1) and (10), where

$$\begin{aligned} \hat{\theta} &\equiv \int_0^\zeta (D_{LL})^{1/2} [(L')^4 \tau_q(L')]^{-1/2} d\zeta = \int_L^\infty [\tau_q(L') D_{LL}']^{-1/2} dL' \\ &= \int_0^\Phi [\tau_q(\Phi') D_{\Phi\Phi}']^{-1/2} d\Phi'. \quad (14) \end{aligned}$$

The name "modified WKB approximation" is suggested by the appearance of modified (rather than ordinary) Bessel functions in (13) as a consequence of the negative (minus) sign in (12). The optimal order ν of the modified Bessel functions $I_\nu(\hat{\theta})$ and $K_\nu(\hat{\theta})$ in (13) is given by $\nu = 1/(\bar{p}+2)$, where \bar{p} is a representative value of

$$\begin{aligned} \hat{p} &\equiv (d \ln D_{LL} / d \ln \zeta) - 4(d \ln L / d \ln \zeta) - (d \ln \tau_q / d \ln \zeta) \\ &= (d \ln D_{\Phi\Phi} / d \ln \zeta) - (d \ln \tau_q / d \ln \zeta) \quad (15) \end{aligned}$$

within the interval $L_0 \leq L \leq L_1$. For $D_{LL} \propto L^\beta$ and $\tau_q \propto L^\gamma$ exactly (i.e., with fixed β and γ) one obtains $\hat{\theta} = 2(\beta+\gamma-2)^{-1} L^{-1/2} D_{LL}^{-1/2} \propto L^{(2-\beta-\gamma)/2}$, $\hat{p} = (\beta-\gamma-4)/(3-\beta)$, and $\nu = (\beta-3)/(\beta+\gamma-2)$, in which case $\bar{f}_\infty(L)$ is given exactly by (13) if $\beta > 3$ and $\beta + \gamma > 2$. The subscripts 0 and 1 in (13) denote evaluation at $L = L_0$ and $L = L_1$, respectively. The limit $\tau_q(L) \rightarrow \infty$ in (12) yields $\bar{f}_\infty(L) = [(\zeta_0 - \zeta)/(\zeta_0 - \zeta_1)] \bar{f}_\infty(L_1)$ exactly. Expansion of the modified Bessel functions in (13) for small argument yields this same result for $\bar{f}_\infty(L)$ if β and γ are fixed (i.e., independent of L).

The above development of a modified WKB approximation for $\bar{f}_\infty(L)$ is somewhat reminiscent of the Green-Liouville solutions described by Jentsch [1984]. His Green-Liouville solutions involved hyperbolic-sine (sinh) functions, which are in fact proportional to modified Bessel functions of order $\nu = 1/2$. It seems that (13) is the appropriate generalization of the procedure described by Jentsch [1984] to situations in which $\bar{p} \neq 0$.

EIGENFUNCTIONS AND EIGENVALUES

The development of time-dependent solutions of (1) is facilitated by the eigenfunction expansion shown in (11), where

$$\begin{aligned} L^2(d/dL)[(D_{LL}/L^2)(dg_n/dL)] - \tau_q^{-1}g_n(L) + \lambda_n g_n(L) \\ = (d/d\Phi)[D_{\Phi\Phi}(dg_n/d\Phi)] + \tau_q^{-1}g_n(L) + \lambda_n g_n(L) = 0. \end{aligned} \quad (16)$$

The eigenfunctions $g_n(L)$ are required to vanish both at $L = L_0$ and at $L = L_1$. Eigenfunctions corresponding to distinct eigenvalues λ_n and λ_m are necessarily orthogonal in the sense that

$$\int_{L_0}^{L_1} L^{-2} g_n(L) g_m(L) dL = \delta_{nm} \equiv \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases} \quad (17)$$

Given a complete set $\{\bar{g}_n(L)\}$ of orthonormal basis functions satisfying (17) and the boundary conditions $\bar{g}_n(L_0) = \bar{g}_n(L_1) = 0$, the required eigenfunctions $g_n(L)$ and eigenvalues λ_n can be obtained by diagonalizing the matrix representation

$$\Lambda_{nm} = \int_{L_0}^{L_1} L^{-2} \bar{g}_n(L) D_{LL} \bar{g}_m(L) dL + \int_{L_0}^{L_1} L^{-2} \bar{g}_n(L) \tau_q^{-1} \bar{g}_m(L) dL \quad (18)$$

of the transport operator $\Lambda \equiv -L^2(\partial/\partial L)[(D_{LL}/L^2)(\partial/\partial L)] + (1/\tau_q)$. An optimal set $\{\bar{g}_n(L)\}$ of basis functions would be one that can be constructed by means of a fairly simple prescription, but one that makes the off-diagonal elements

of Λ_{nm} especially small in absolute value.

It is evident from (18) that all the eigenvalues of Λ_{nm} are positive. This situation corresponds, of course, to temporal decay of the expansion coefficients $a_n(t)$ in (11). Moreover, if the eigenvalues λ_n are ordered (as usual) so that $0 < \lambda_0 < \lambda_1 < \lambda_2 < \dots$, then it follows from (18) that (with increasing n) radial diffusion becomes increasingly important (compared to charge exchange) for the determination of λ_n . The presence of derivatives of the $\bar{g}_n(L)$ in the first term (but not in the second term) on the right-hand side of (18) assures this. Except for the term $r_q^{-1} \bar{g}_n(L)$, which has no counterpart in their paper, the eigenvalue equation specified by (16) is identical in form to the one for which Schulz and Boucher [1984] successfully constructed an optimal set of orthonormal basis functions by means of a variant of the WKB approximation. A further variant of that construction is required here, since the boundary conditions of the present radial-diffusion problem differ from the boundary conditions appropriate to the pitch-angle diffusion problem treated by Schulz and Boucher [1984].

The analogous construction appropriate to the radial-diffusion problem yields orthonormal basis functions of the form

$$\begin{aligned} \bar{g}_n(L) = & 2L(\theta_n/\theta_n^0)^{1/2} D_{LL}^{-1/4} \left[\int_{L_0}^{\infty} D_{LL}^{-1/2} dL \right]^{-1/2} C_{\nu}^*(\theta_n) \\ & \times \{ [C_{\nu \pm 1}(\theta_n^0)]^2 - [\alpha C_{\nu \pm 1}(\alpha \theta_n^0)]^2 \}^{-1/2}, \end{aligned} \quad (19)$$

where

$$C_{\nu}^*(\theta_n) \equiv J_{\nu}(\theta_n) Y_{\nu}(\alpha \theta_n^0) - J_{\nu}(\alpha \theta_n^0) Y_{\nu}(\theta_n) \quad (20a)$$

and

$$C_{\nu \pm 1}(\theta_n) \equiv J_{\nu \pm 1}(\theta_n) Y_{\nu}(\alpha \theta_n^0) - J_{\nu}(\alpha \theta_n^0) Y_{\nu \pm 1}(\theta_n). \quad (20b)$$

The argument θ_n of the ordinary Bessel functions $J_{\nu}(\theta_n)$ and $Y_{\nu}(\theta_n)$ in (19)

and (20) is given by

$$\theta_n \equiv \theta_n^0 \int_L^\infty D_{LL}^{-1/2} dL \div \int_{L_0}^\infty D_{LL}^{-1/2} dL, \quad (21)$$

where θ_n^0 is the n^{th} positive root ($n=0,1,2,\dots$) of $C_\nu^*(\theta_n^0) = 0$, i.e., where

$$C_\nu^*(\theta_n^0) = J_\nu(\theta_n^0)Y_\nu(\alpha\theta_n^0) - J_\nu(\alpha\theta_n^0)Y_\nu(\theta_n^0) = 0, \quad (22)$$

and where

$$\alpha \equiv \int_{L_1}^\infty D_{LL}^{-1/2} dL \div \int_{L_0}^\infty D_{LL}^{-1/2} dL < 1. \quad (23)$$

Since $C_\nu^*(\theta_n^0) = C_\nu^*(\alpha\theta_n^0) = 0$ for each value of n , it thus follows from (19)-(23) that $\bar{g}_n(L_0) = \bar{g}_n(L_1) = 0$, as is required. The normalization and mutual orthogonality [in the sense of (17)] of the basis functions $\bar{g}_n(L)$ specified by (19) can be verified by using θ_n as the variable of integration and invoking certain indefinite integrals evaluated by Watson [1944, pp. 148-149]. The optimal order ν of the ordinary Bessel functions in (20a) is given by $\nu = 1/(\bar{p}+2)$, where \bar{p} is a representative value of

$$p \equiv (d \ln D_{LL} / d \ln \zeta) - 4(d \ln L / d \ln \zeta) = d \ln D_{\Phi\Phi} / d \ln \zeta \quad (24)$$

within the interval $L_0 \leq L \leq L_1$ ($\theta_n^0 \geq \theta_n \geq \alpha\theta_n^0$). For $D_{LL} \propto L^\beta$ exactly (i.e., with fixed $\beta > 3$) one obtains $\theta_n = (L_0/L)^{(\beta-2)/2} \theta_n^0$, $\alpha = (L_0/L_1)^{(\beta-2)/2}$, $p = (4-\beta)/(\beta-3)$, and $\nu = (\beta-3)/(\beta-2)$. The basis functions specified by (19) should lead to a nearly diagonal matrix representation Λ_{nm} , as defined by (18), of the transport operator $\Lambda \equiv -(\partial/\partial\Phi)[D_{\Phi\Phi}(\partial/\partial\Phi)] + (1/\tau_q)$. In other words, the diagonal element Λ_{nn} (at least for $n \geq 4$) should greatly exceed the absolute value of each off-diagonal element Λ_{nm} ($= \Lambda_{mn}$) in the same row or column of the matrix. This major benefit of the WKB construction of basis functions enables the eigenvalues and eigenvectors of Λ_{nm} (and therefore the eigenvalues and eigenfunctions of the transport operator Λ) to be evaluated

by means of a rapidly convergent perturbation theory. The formal results [Schulz and Boucher, 1984] are

$$\lambda_n \approx \Lambda_{nn} - \sum_{k \neq n} \frac{\Lambda_{nk} \Lambda_{kn}}{\Lambda_{kk} - \Lambda_{nn}} \quad (25a)$$

and

$$g_n(L) = U_{nn} \bar{g}_n(L) + \sum_{k \neq n} U_{kn} \bar{g}_k(L), \quad (25b)$$

where

$$\frac{U_{kn}}{U_{nn}} = \frac{1}{\Lambda_{nn} - \Lambda_{kk}} \left[\Lambda_{kn} + \sum_{j \neq k, n} \frac{\Lambda_{kj} \Lambda_{jn}}{\Lambda_{nn} - \Lambda_{jj}} \right] \quad (25c)$$

for $k \neq n$ and

$$U_{nn} = [1 + \sum_{k \neq n} (U_{kn}/U_{nn})^2]^{-1/2} \quad (25d)$$

to assure the unitarity of the transformation from the $\{\bar{g}_n(L)\}$ to the $\{g_n(L)\}$, i.e., to assure that the $g_n(L)$ are likewise normalized in accordance with (17).

A further use of the orthonormal basis functions $\bar{g}_n(L)$ specified by (19) is to eliminate altogether the presumably small discrepancy between the exact steady-state solution $\bar{f}_\infty(L)$ of (1) and the modified WKB-approximate steady-state solution $\hat{f}_\infty(L)$ given by (13). This can be done by formally expanding the discrepancy as a general linear superposition of the $\bar{g}_m(L)$ and inserting the formal expansion, viz.,

$$\bar{f}_\infty(L) = \hat{f}_\infty(L) - \sum_m A_m \bar{g}_m(L), \quad (26)$$

into (1) for $\partial \bar{f} / \partial t = 0$. The result (after the usual straightforward steps) is a set of coupled linear equations given by

$$\sum_m \Lambda_{nm} A_m = \int_{L_0}^{L_1} L^{-2} \bar{g}_n(L) D_{LL} \left(\frac{d\hat{f}_\infty}{dL} \right) dL + \int_{L_0}^{L_1} L^{-2} \bar{g}_n(L) \tau_q^{-1}(L) \hat{f}_\infty(L) dL \quad (27)$$

for the expansion coefficients A_m that should be inserted in (26). The solution of (27) is numerically well-determined, since the matrix Λ_{nm} given by (18) is supposed to be nearly diagonal when the basis functions $\bar{g}_n(L)$ are constructed in accordance with (19)-(24).

APPLICATIONS

This report was previously prepared as a short paper. This prevented the inclusion of numerical results illustrating the usefulness of analytical methods described above for solving problems in radial-diffusion theory. However, the numerical results of Schulz and Boucher [1984], showing (for example) that the off-diagonal elements of Λ_{nm} are consistently smaller (by one to several orders of magnitude) in absolute value than the corresponding diagonal elements when analogous methods are applied to a prototypical pitch-angle diffusion problem, suggest that the present approach will be found highly advantageous when applied numerically to radial-diffusion problems as well. Moreover, the ease with which charge exchange can be incorporated, as in (19), into the radial-diffusion problem suggests that charge exchange could similarly be handled together with pitch-angle diffusion in problems that require this, e.g., in studies of the evolving pitch-angle distribution of ring-current ions after charge exchange has made the equatorial distribution anisotropic enough to generate electromagnetic ion-cyclotron waves [cf. Cornwall, 1977]. For this latter application the basis functions could be constructed according to the prescription of Schulz and Boucher [1984]. A long-range goal is to treat the simultaneous occurrence of radial diffusion and pitch-angle diffusion, in which case the L-dependent eigenvalues of the pitch-angle diffusion operator will presumably enter the mathematical description of radial transport [cf. Walt, 1970] in somewhat the same way that the charge-exchange rate $\tau_q^{-1}(L)$ enters (1). However, the bimodal (radial/

pitch-angle) diffusion problem is complicated by the absence of a kinematical quantity that both modes of diffusion simultaneously conserve, and the solution is presumably much more elusive for it than for the isolated radial-diffusion problem treated here.

REFERENCES

- Chen, A. J., and D. P. Stern, Adiabatic Hamiltonian of charged-particle motion in a dipole field, J. Geophys. Res., 80, 690-693, 1975.
- Cornwall, J. M., Radial diffusion of ionized helium and protons: A probe for magnetospheric dynamics, J. Geophys. Res., 77, 1756-1770, 1972.
- Cornwall, J. M., On the role of charge exchange in generating unstable waves in the ring current, J. Geophys. Res., 82, 1188-1196, 1977.
- Davidson, G. T., An improved empirical description of the bounce motion of trapped particles, J. Geophys. Res., 81, 4029-4030, 1976.
- Jentsch, V., The radial distribution of radiation-belt protons: Approximate solution of the steady-state transport equation at arbitrary pitch angle, J. Geophys. Res., 89, 1527-1539, 1984.
- Schiff, L. I., Quantum Mechanics, p. 187, McGraw-Hill, New York, 1955.

- Schulz, M., Principles of magnetospheric ion composition, in Energetic Ion Composition in the Earth's Magnetosphere, edited by R. G. Johnson, pp. 1-21, Terra Sci. Publ. Co., Tokyo, 1983.
- Schulz, M., and D. J. Boucher, Jr., Orthogonal basis functions for pitch-angle diffusion theory, in Physics of Space Plasmas (1982-4), edited by J. Belcher, H. Bridge, T. Chang, B. Coppi, and J. R. Jasperse, pp. 159-168, Scientific Publ., Inc., Cambridge, Mass., 1984.
- Schulz, M., and L. J. Lanzerotti, Particle Diffusion in the Radiation Belts, pp. 19-21, 44, 89-93, Springer, Heidelberg, 1974.
- Walt, M., Radial diffusion of trapped particles, in Particles and Fields in the Magnetosphere, edited by B. M. McCormac, pp. 410-415, Reidel, Dordrecht, 1970.
- Watson, G. N., A Treatise on the Theory of Bessel Functions, pp. 148-149, Cambridge Univ. Press, Cambridge, England, 1944.

LABORATORY OPERATIONS

The Aerospace Corporation functions as an "architect-engineer" for national security projects, specializing in advanced military space systems. Providing research support, the corporation's Laboratory Operations conducts experimental and theoretical investigations that focus on the application of scientific and technical advances to such systems. Vital to the success of these investigations is the technical staff's wide-ranging expertise and its ability to stay current with new developments. This expertise is enhanced by a research program aimed at dealing with the many problems associated with rapidly evolving space systems. Contributing their capabilities to the research effort are these individual laboratories:

Aerophysics Laboratory: Launch vehicle and reentry fluid mechanics, heat transfer and flight dynamics; chemical and electric propulsion, propellant chemistry, chemical dynamics, environmental chemistry, trace detection; spacecraft structural mechanics, contamination, thermal and structural control; high temperature thermomechanics, gas kinetics and radiation; cw and pulsed chemical and excimer laser development including chemical kinetics, spectroscopy, optical resonators, beam control, atmospheric propagation, laser effects and countermeasures.

Chemistry and Physics Laboratory: Atmospheric chemical reactions, atmospheric optics, light scattering, state-specific chemical reactions and radiative signatures of missile plumes, sensor out-of-field-of-view rejection, applied laser spectroscopy, laser chemistry, laser optoelectronics, solar cell physics, battery electrochemistry, space vacuum and radiation effects on materials, lubrication and surface phenomena, thermionic emission, photo-sensitive materials and detectors, atomic frequency standards, and environmental chemistry.

Computer Science Laboratory: Program verification, program translation, performance-sensitive system design, distributed architectures for spaceborne computers, fault-tolerant computer systems, artificial intelligence, microelectronics applications, communication protocols, and computer security.

Electronics Research Laboratory: Microelectronics, solid-state device physics, compound semiconductors, radiation hardening; electro-optics, quantum electronics, solid-state lasers, optical propagation and communications; microwave semiconductor devices, microwave/millimeter wave measurements, diagnostics and radiometry, microwave/millimeter wave thermionic devices; atomic time and frequency standards; antennas, rf systems, electromagnetic propagation phenomena, space communication systems.

Materials Sciences Laboratory: Development of new materials: metals, alloys, ceramics, polymers and their composites, and new forms of carbon; non-destructive evaluation, component failure analysis and reliability; fracture mechanics and stress corrosion; analysis and evaluation of materials at cryogenic and elevated temperatures as well as in space and enemy-induced environments.

Space Sciences Laboratory: Magnetospheric, auroral and cosmic ray physics, wave-particle interactions, magnetospheric plasma waves; atmospheric and ionospheric physics, density and composition of the upper atmosphere, remote sensing using atmospheric radiation; solar physics, infrared astronomy, infrared signature analysis; effects of solar activity, magnetic storms and nuclear explosions on the earth's atmosphere, ionosphere and magnetosphere; effects of electromagnetic and particulate radiations on space systems; space instrumentation.

...

END

2-87

DTIC