AD-A175 034 REPLACEMENT WITH NON-CONSTANT OPERATING COST(U) NORTH CAROLINA UNIV AT CHARLOTTE DEPT OF MATHEMATICS R F ANDERSON 1986 AFOSR-TR-86-2144 AFOSR-80-0245 1/1

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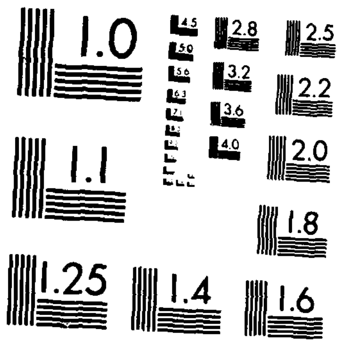

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REPLACEMENT WITH NON-CONSTANT OPERATING COST

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SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION Unclassified	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S)	5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TR- 86-2144
6a. NAME OF PERFORMING ORGANIZATION University of North Carolina 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION AFOSR/NM
6c. ADDRESS (City, State and ZIP Code) University of North Carolina at Charlotte Charlotte, North Carolina 28223	7b. ADDRESS (City, State and ZIP Code) Bldg. 410 Bolling AFB, DC 20332-6448
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR 8b. OFFICE SYMBOL (If applicable) NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER AFOSR-80-0245
8c. ADDRESS (City, State and ZIP Code) Bolling Air Force Base Washington, DC 20332-6448	10. SOURCE OF FUNDING NOS. PROGRAM ELEMENT NO.: 61102F PROJECT NO.: 2304 TASK NO.: A5 WORK UNIT NO.:
11. TITLE (Include Security Classification) "Replacement with Non-Constant Operating Cost"	

12. PERSONAL AUTHOR(S)
R.F. Anderson

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT
reprint	FROM ______ TO ______		56 (fifty-six)

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB. GR.	

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

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20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐	21. ABSTRACT SECURITY CLASSIFICATION Unclassified
22a. NAME OF RESPONSIBLE INDIVIDUAL Capt Thomas	22b. TELEPHONE NUMBER (Include Area Code) 767-5025 22c. OFFICE SYMBOL NM

DD FORM 1473, 83 APR — EDITION OF 1 JAN 73 IS OBSOLETE.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

REPLACEMENT WITH NON-CONSTANT OPERATING COST

R. F. Anderson†*

ABSTRACT

The long run average cost problem is considered in the case of a non-decreasing Markov wear process with failure determined by a random threshold. The method of analysis is to first consider the discounted problem and then let the discount factor go to zero.

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* This work was partially supported by AFOSR 80-0245

§1. Introduction.

Assume that one has a machine whose failure is described by a wear process $x_t : t \geq 0$ which is a positi e non-decreasing Markov process and a random threshold $Y \geq 0$ independent of $x_t : t \geq 0$ with failure occuring at time $\sigma = \inf\{t : x_t \geq Y\}$. At any time one can replace the machine by a new one with the same mode of operation. There is an operating cost $f(x)$ per unit of time and a replacement cost $g(x)$ if replacement is done before failure and replacement cost c_0 if replacement is done at failure. Note that replacement is always to replace the machine by a new one. Models of this type have been considered in the reliability literature by Abdel-Hameed [1], [2], Drosen [5], and Taylor [10].

The stochastic control problem of minimizing the cost is generally called the Optimal Replacement Problem and has been considered by the above authors in the case that f is constant. The interest has been in the long run average cost problem. In this work we will consider the case of general $f(x)$. We first view the problem as a discounted cost problem, and as in Robin [8], we obtain the long run average cost by letting the discount factor go to zero. The main difference with the work of Robin is that the invariant measure is not obtianed exponentially fast by the Markov transition probabilities of the replacement process as time goes to infinity. (See §4).

In Section 2 we state the Long Run Average Cost Optimal Replacement Problem and deal with a preliminary discounted optimal stopping problem. Section 3 formulates and solves the discounted optimal replacement problem. The replacement process is introduced in Section 4, the invariant measure is found, and ergodic results are derived for the linear problem. Section 5 proves the main technical result and Section 6 contains the main result that the solution to the discounted problem suitably modified converges to the solution of the long run average problem.

§2. Notation, Statement of the Problem, and a Preliminary Stopping Problem.

Let $\Omega = D(R^+, R^+)$ be the space of right continuous functions with left limits. Here $R^+ = [0, \infty)$. Let $x_t(\omega) = \omega(t)$ for $\omega \varepsilon \Omega$, $F_t^0 = \sigma(x_s : 0 \leqslant s \leqslant t)$, $F^0 = F_\infty^0$, and F_t, F the universally completed σ-fields F_t^0 and F^0 respectively. Let $(\Omega, F_t, x_t : t \geqslant 0, P_x)$ be a homogeneous, non-decreasing, non-negative Markov process with associated semi-group $T_t : t \geqslant 0$ defined on $C_b(R^+)$, the space of bounded real valued continuous functions defined on R^+ with norm taken to be supremum norm. We assume

(2.1) $T_t : t \geqslant 0$ is Feller, that is, for $f \varepsilon C_b(R^+)$
$T_t f \varepsilon C_b(R^+)$ and $T_t f \to f$ in supremum norm as $t \to 0$.

Let A denote the infinitesimal generator of $T_t : t \geqslant 0$ and D_A its domain in $C_b(R^+)$. Assume also

(2.2) $x_t : t \geqslant 0$ is quasi-left continuous, that is if $\tau_n : n \geqslant 1$ is a sequence of stopping times with $\tau_n \uparrow \tau$, then $x_{\tau_n} \to x_\tau$ a.s. P_x on the set $(\tau < \infty)$. See Dynkin [6], Vol. 1 pp. 103.

Let Y be a positive random variable independent of $x_t : t \geqslant 0$ with a continuous distribution function $G(y)$. Let $\overline{G}(y) = 1 - G(y)$. Define

$$\sigma = \text{int}\{t : x_t \geqslant Y\}$$

and let $H(t) = P_0(\sigma \leqslant t) = P_0(x_t \geqslant Y)$. Assume

(2.3) $E_0[\sigma] < \infty$ and $H(0) = 0$.

Let

(2.4) $f, g \varepsilon C_b(R^+)$ $f, g \geqslant 0$ and $c_o > 0$ a constant.

We say $\tau : \Omega \to [0, \infty]$ is a stopping time with respect to $x_t : t \geqslant 0$ if $(\tau \leqslant t) \varepsilon F_t$ for all t.

Let

$$\lambda = \inf_{\tau} \frac{E_0\left[\int_0^{\sigma\wedge\tau} f(x_s)ds + I_{(\tau<\sigma)}g(x_\tau) + I_{(\tau\geq\sigma)}c_0\right]}{E_0[\sigma\wedge\tau]}$$

The Long Run Average Optimal Replacement Problem is to find $\hat{\tau}$ so that

$$\lambda = \frac{E_0\left[\int_0^{\sigma\wedge\hat{\tau}} f(x_s)ds + I_{(\hat{\tau}<\sigma)}g(x_{\hat{\tau}}) + I_{(\hat{\tau}\geq\sigma)}c_0\right]}{E_0[\sigma\wedge\hat{\tau}]}.$$

Our first step is to establish a result for a discounted optimal stopping problem. For the following see Robin [9] or Bensoussan [3]:

<u>Lemma 2.1</u>. For $b\geq 0$ fixed and $\alpha>0$, let $V_b \in C_b(R^+)$ be the maximal solution of

$$U \in C_b(R^+) \qquad U(x) \leq b + g(x)\,\overline{G}(x) + c_0 G(x)$$

$$U \leq e^{-\alpha t}T_t U + \int_0^t e^{-\alpha s}T_s(f\overline{G} + \alpha(b+c_o)G)ds$$

then

$$V_b(x) = \inf_{\tau} J_x^b(\tau)$$

where

$$J_x^b(\tau) = E_x\left[\int_0^{\tau} e^{-\alpha s}(f(x_s)\overline{G}(x_s) + \alpha(b+c_0)G(x_s))ds + e^{-\alpha\tau}(b + g(x_\tau)\overline{G}(x_\tau) + c_o G(x_\tau))\right]$$

Moreover if

$$\hat{\tau}_b = \inf\{t : V_b(x_t) = b + g(x_t)\overline{G}(x_t) + c_0 G(x_t)\}.$$

then

$$V_b(x) = J_x^b(\hat{\tau}_b).$$

Lemma 2.2. (i) For $0 \leq b_0 < b_1$, $V_{b_0}(x) \leq V_{b_1}(x)$

(ii) $|V_{b_0}(x) - V_{b_1}(x)| \leq 2|b_0 - b_1|$

Proof. (i) If $0 \leq b_0 < b_1$, then $J_x^{b_0}(\tau) \leq J_x^{b_1}(\tau)$ for any stopping time τ.

Hence $V_{b_0}(x) \leq V_{b_1}(x)$.

(ii) For any τ,

$$J_x^{b_0}(\tau) - J_x^{b_1}(\tau) = E_x\left[\int_0^\tau \alpha e^{-\alpha s}(b_0 - b_1)G(x_s)ds + e^{-\alpha\tau}(b_0 - b_1)\bar{G}(x_\tau)\right]$$

Since G is a distribution function,

$$|J_x^{b_0}(\tau) - J_x^{b_1}(\tau)| \leq 2|b_0 - b_1| \text{ and therefore}$$

$$|V_{b_0}(x) - V_{b_1}(x)| \leq 2|b_0 - b_1|.$$

Lemma 2.3. If $H(0) = 0$, there exists a $b_0 > 0$ so that

$$V_{b_0}(0) < b_0.$$

For such a b_0, define inductively

$$b_k = V_{b_{k-1}}(0) \qquad k \geq 1$$

Then $b_k \downarrow \bar{b}$ where $\bar{b} \geq 0$. Moreover $V_{b_k}(x) \downarrow V_{\bar{b}}(x)$ in $C_b(R^+)$ and $V_{\bar{b}}(0) = \bar{b}$.

Proof. For any $b \geq 0$ take $\tau \equiv 0$ to obtain

$$V_b(0) \leq E_0\left[\int_0^\infty e^{-\alpha s}(f(x_s)\bar{G}(x_s) + \alpha(b + c_o)G(x_s))ds\right]$$

Note that

$$H(t) = P_0(\sigma \leq t) = P_0(x_t \geq Y) = E_0[G(x_t)]$$

and so

$$\hat{H}(\alpha) = E_0\left[e^{-\alpha\sigma}\right] = E_0\left[\int_0^\infty \alpha e^{-\alpha s}G(x_s)ds\right]$$

Since $\hat{H}$ is not point mass at 0, $H(\alpha) < 1$.

Let

$$z_0 = E_0\left[\int_0^\infty e^{-\alpha s}(f(x_s)\bar{G}(x_s) + \alpha\, c_0\, G(x_s))ds\right]$$

and we have

$$V_b(0) \leqslant z_0 + b\,\hat{H}(\alpha).$$

Select b_0 large enough so that

$$z_0 + b_0\hat{H}(\alpha) < b_0$$

Next for such a b_0, we have

$$b_1 = V_{b_0}(0) < b_0$$

and by Lemma 2.2, it follows inductively that

$$b_k = V_{b_{k-1}}(0) \leqslant V_{b_{k-1}}(0) = b_{k-1}$$

Define $\bar{b} = \lim b_k$, and let $V_{\bar{b}}(x)$ be given by Lemma 2.1.

From Lemma 2.2,

$$|V_{b_k}(x) - V_{\bar{b}}(x)| \leqslant 2|b_k - \bar{b}|$$

and so $V_{b_k} \downarrow V_{\bar{b}}$ in $C_b(R^+)$ and moreover

$$\bar{b} = \lim_k b_k = \lim_k V_{b_{k-1}}(0) = V_{\bar{b}}(0).$$

We thus have established the following.

<u>Theorem 2.4</u>. Under the assumption (2.1) - (2.4), let $V^\alpha(x) = V_{\bar{b}}(x)$, $\bar{b}$ as in Lemma 2.3. Then

$$V^\alpha(x) = \inf_\tau J_x^\alpha(\tau)$$

where

$$J_x^\alpha(\tau) = E_x\left[\int_0^\tau e^{-\alpha s}(f(x_s)\bar{G}(x_s) + \alpha(V^\alpha(0) + c_0)G(x_s))ds + e^{-\alpha\tau}(V^\alpha(0) + g(x_\tau)\bar{G}(x_\tau) + c_0 G(x_\tau))\right]$$

Moreover if

$$\hat{\tau} = \inf\{t : V^\alpha(x_t) = V^\alpha(0) + g(x_t)\bar{G}(x_t) + c_0\, G(x_t)\}$$

then

$$V^\alpha(x) = J_x^\alpha(\hat{\tau}).$$

Remark 2.1. It will be shown in Section 3 that $V^\alpha(x)$ is the maximal solution of a quasi-variational inequality.

Corollary 2.5. Under the assumption $g(0) > 0$ and these of Theorem 2.4, $\hat{\tau} > 0$ a.s. P_0

Proof. If $g(0) > 0$ then taking $\tau \equiv 0$

$$V^\alpha(0) < V^\alpha(0) + g(0)$$

and hence there is a $\delta > 0$ so that if $0 \leqslant x < \delta$ then

$$V^\alpha(x) + \varepsilon < V^\alpha(0) + g(x)\bar{G}(x) + c_0 G(x)$$

for $\varepsilon > 0$ sufficiently small. Hence

$$\hat{\tau} > \tau_{\delta/2} = \inf\{t : x(t) \geqslant \delta/2\}$$

and since $x_t : t \geqslant 0$ is right continuous, $\tau_{\delta/2} > 0$ a.s. P_0.

§3. The Discounted Optimal Replacement Problem.

Let $x_t^k : t \geq 0$ and Y^k $k \geq 1$ be independent copies of $x_t : t \geq 0$ and Y. Define

$$\sigma_k = \inf\{t : x_t^k \geq Y^k\}$$

Let f, g and c_o satisfy (2.4). Suppose $\tau_k : k \geq 1$ are stopping times respectively with respect to $F_t^k : t \geq 0$ the universal completion of $\sigma(x_s^k : s \leq t)$ and assume for all n $\sum_{k \geq n} \tau_k = \infty$ a.s. P_o. We use the notation

$\vec{\tau} = (\tau_1, \tau_2, \ldots)$ and refer to $\vec{\tau}$ as replacement times. Define

$$\tilde{J}_0^\alpha(\vec{\tau}) = E_0\Bigg[\int_0^{\sigma_1 \wedge \tau_1} e^{-\alpha s} f(x_s^1)\,ds + e^{-\alpha\,\sigma_1 \wedge \tau_1} \big(g(x_{\tau_1}^1) I_{(\tau_1 < \sigma_1)} + c_o I_{(\tau_1 \geq \sigma_1)}\big)$$

$$+ \sum_{n=2}^{\infty} \prod_{\ell=1}^{n-1} e^{-\alpha\sigma \wedge \tau_\ell} \Bigg(\int_0^{\sigma_n \wedge \tau_n} e^{-\alpha s} f(x_s^n)\,ds$$

$$+ e^{-\alpha\sigma_n \wedge \tau_n} \big(g(x_{\tau_n}^n) I_{(\tau_n < \sigma_n)} + c_o\ I_{(\tau_n \geq \sigma_n)}\big)\Bigg)\Bigg].$$

Here it is assumed that $P_0(x_0^k = 0) = 1$ for all k. Let

$$V_o^\alpha = \inf_{\vec{\tau}} \tilde{J}_o^\alpha(\vec{\tau})$$

We seek $\hat{\vec{\tau}}$ so that

$$V_o^\alpha = \tilde{J}_o^\alpha(\hat{\vec{\tau}})$$

This we refer to as the Discounted Optimal Replacement Problem.

Define

$$J_o(\tau_1) = E_o\Bigg[\int_0^{\sigma_1 \wedge \tau_1} e^{-\alpha s} f(x_s^1)\,ds + e^{-\alpha\sigma_1 \wedge \tau_1} \big(V_o^\alpha + g(x_{\tau_1}^1) I_{(\tau_1 < \sigma_1)}$$

$$+ c_o I_{(\tau_1 \geq \sigma_1)}\big)\Bigg].$$

Lemma 3.1. $V_o^\alpha = \inf_{\tau_1} J_o^\alpha(\tau_1)$

Proof. Let $\tau = (\tau_1, \tau_2, \ldots)$ be any replacement sequence. By independence letting $\hat{\vec{\tau}} = (\tau_2, \tau_3, \ldots)$

$$\tilde{J}_o^\alpha(\tau) = E_o\left[\int_0^{\sigma_1 \wedge \tau_1} e^{-\alpha s} f(x_s^1)\,ds + e^{-\alpha\sigma_1 \wedge \tau_1}(\tilde{J}_o^\alpha(\hat{\vec{\tau}}) + g(x_{\tau_1}^1) I_{(\tau_1 < \sigma_1)} + c_o I_{(\tau_1 \geqslant \sigma_1)})\right] \geqslant J_o^\alpha(\tau_1)$$

Therefore

$$V_o^\alpha \geqslant \inf_{\tau_1} J_o^\alpha(\tau_1) \equiv d$$

For $\varepsilon > 0$ choose τ_1 and $\vec{\tau}$ so that

$$J_o^\alpha(\tau_1) \leqslant d + \varepsilon \quad \text{and} \quad \tilde{J}_o^\alpha(\vec{\tau}) \leqslant V_o^\alpha + \varepsilon$$

Let $\vec{\tau}_o = (\tau_1, \tau)$ then

$$V_o^\alpha \leqslant \tilde{J}_o^\alpha(\vec{\tau}_o) \leqslant J_o^\alpha(\tau_1) + \varepsilon \leqslant d + 2\varepsilon.$$

Since ε was arbitrary, we have equality.

In what follow $x_t : t \geqslant 0$ and Y wil be generic copies of the wear process and the random threshold and $\sigma = \inf\{t : x_t \geqslant Y\}$. Note (2.3) insures $\sigma < \infty$ a.s. P_o.

Lemma 3.2. for any stopping time τ with respect to $F_t : t \geqslant 0$ on the set $(\tau < \infty)$

$$P_o(\sigma \leqslant \tau \mid F_\tau) = P_o(x_\tau \geqslant Y \mid F_\tau) = G(x_\tau)$$

and $$P_o(\sigma > \tau \mid F_\tau) = P_o(x_\tau < Y \mid F) = \bar{G}(x_\tau).$$

On the set $(\tau = \infty)$,

$$P_o(\sigma \leqslant \tau \mid F_\tau) = 1 \quad \text{and} \quad P_o(\sigma > \tau \mid F_\tau) = 0.$$

Proof. First note that by the independence of $x_t : t \geq 0$ and Y, and the assumption that $x_t : t \geq 0$ is pathwise non-decreasing, for any fixed t.

$$P_o(\sigma \leq t | F_t) = P_o(x_t \geq Y | F_t) = G(x_t)$$

For any τ an $F_t : t \geq 0$ stopping time, define

$$\tau_n = \begin{cases} k/n & \text{on } (k-1/n < \tau \leq k/n) = A_{k/n} \\ \infty & \text{on } (\tau = \infty) \end{cases}$$

Since $A_{k/n} \varepsilon F_{k/n}$, on the set $(\tau_n < \infty)$

$$P_o(\sigma \leq \tau_n | F_{\tau_n}) = \sum_1^\infty P_o(\sigma \leq k/n | F_{k/n}) I_{A_{k/n}}$$

$$= \sum_1^\infty G(x_{k/n}) I_{A_{k/n}} = G(x_{\tau_n})$$

Because $\tau \leq \tau_n$, $F_\tau \leq F_{\tau_n}$ and so on the set $(\tau < \infty)$

$$P_o(\sigma \leq \tau | F_\tau) = P_o(\bigcap_1^\infty (\sigma \leq \tau_n) | F_\tau)$$

$$= \lim_n E_o[P_o(\sigma \leq \tau_n | F_{\tau_n}) | F_\tau]$$

$$= \lim_n E_o[G(x_{\tau_n}) | F_\tau] = G(x_\tau)$$

The last follows since $\tau_n \downarrow \tau$ and $x_t : t \geq 0$ is right continuous.

On the set $(\tau = \infty)$ there is nothing to show.

Lemma 3.3. Let f and g be bounded and continuous and $\alpha > 0$. Then for any stopping time τ with respect to $F_t : t \geq 0$.

(i) $$E_o\left[\int_0^{\sigma \wedge \tau} e^{-\alpha s} f(x_s) ds\right] = E_o\left[\int_0^{\tau} e^{-\alpha s} f(x_s) \bar{G}(x_s) ds\right]$$

(ii) $$E_o\left[e^{-\alpha\sigma\wedge\tau} g(x_\tau) I_{(\tau<\sigma)}\right] = E_o\left[e^{-\alpha\tau} g(x_\tau)\bar{G}(x_\tau)\right]$$

(iii) $$E_o\left[e^{-\alpha\sigma\wedge\tau} I_{(\tau\geqslant\sigma)}\right] = E_o\left[e^{-\alpha\tau}G(x_\tau) + \int_0^\tau \alpha e^{-\alpha s}G(x_s)ds\right]$$

Proof. (i) By Lemma 3.2

$$P_o(x_s < Y|F_s) = \bar{G}(x_s)$$

and so on the set $(s<\tau)$

$$P_o(x_s < Y|F_\tau) = \bar{G}(x_s)$$

Hence

$$E_o\left[\int_0^{\tau\wedge\sigma} e^{-\alpha s} f(x_s)ds\right] = E_o\left[\int_0^\tau e^{-\alpha s} f(x_s) I_{(x_s<Y)}ds\right]$$

$$= E_o\left[\int_0^\tau e^{-\alpha s} f(x_s)\bar{G}(x_s)ds\right]$$

(ii) This follows directly from Lemma 3.2.

(iii) Note that

$$e^{-\alpha\sigma\wedge\tau} I_{(\tau\geqslant\sigma)} = e^{-\alpha\tau} I_{(x_\tau\geqslant Y)} + \int_0^\tau \alpha e^{-\alpha s} I_{(x_s\geqslant Y)}ds\ I_{(x_\tau\geqslant Y)}.$$

Since $x_t : t\geqslant 0$ is pathwise non-decreasing

$$\int_0^\tau \alpha e^{-\alpha s}\ I_{(x_s\geqslant Y)}ds\ I_{(x_\tau\geqslant Y)} = \int_0^\tau \alpha e^{-\alpha s} I_{(x_s\geqslant Y)}ds$$

The rest is Lemma 3.2.

Lemma 3.4. $$V_o^\alpha = \inf_\tau E_o\left[\int_0^\tau e^{-\alpha s}(f(x_s)\bar{G}(x_s) + \alpha(V_o^\alpha + c_o)G(x_s))ds + e^{-\alpha\tau}(V_o^\alpha + g(x_\tau)G(x_\tau) + c_oG(x_\tau))\right].$$

Proof. This is immediate from Lemma 3.1 and Lemma 3.3.

Theorem 3.5. Suppose $g(o) > 0$ and (2.1)-(2.4) are satisfied. Let $V^\alpha(x)$ be the maximal solution of $U(x) \leq U(0)$

(3.1) $$U \in C_b(R^+) \quad U(x) \leq U(0) + g(x)\bar{G}(x) + c_0 G(x) \quad \text{and}$$

$$U(x) \leq e^{-\alpha t} T_t U(x) + \int_0^t e^{-\alpha s} T_s [f\bar{G} + \alpha(U(0) + c_0)G](x)ds$$

then

(3.2) $$V^\alpha(x) = \inf_\tau J_x^\alpha(\tau) \quad \text{and}$$

with $$\hat{\tau} = \inf\{t : V^\alpha(x_t) = V^\alpha(0) + g(x_t)\bar{G}(x_t) + c_0 G(x_t)\}$$

(3.3) $$V^\alpha(x) = J_x^\alpha(\hat{\tau})$$

Moreover

(3.4) $$V^\alpha(0) = \inf_\tau \tilde{J}_o^\alpha(\tau)$$

and if $\hat{\tau} = (\hat{\tau}_1, \hat{\tau}_2, \ldots)$ where $\hat{\tau}_k$ is defined the same as $\hat{\tau}$ above except x_t^k replaces x_t, then

(3.5) $$V^\alpha(0) = J_o^\alpha(\tau).$$

<u>Proof</u>. Let $V^\alpha(x)$ be as in Theorem 2.4. By Lemma 2.1, $V^\alpha(x)$ solves (3.1). Moreover (3.2) and (3.3) are consequences of Theorem 2.4. Also Lemmas 3.1 and 3.4 prove (3.4). What is left is to show (3.5) and $V^\alpha(x)$ is the maximal solution of (3.1).

Let $U(x)$ be any solution of (3.1) and τ be any stopping time. By the Markov property,

$$e^{-\alpha t} U(x_t) + \int_0^t e^{-\alpha s} f(x_s)\bar{G}(x_s) + \alpha(U(0) + c_o G(x_s))ds$$

is a submartingale. Hence

$$U(x) \leq E_x \left[e^{-\alpha t \wedge \tau} U(x_{t \wedge \tau}) + \int_0^{t \wedge \tau} e^{-\alpha s}(f(x_s)\bar{G}(x_s) + \alpha(U(0) + c_o)G(x_s))ds \right]$$

By (2.2) letting $t \to \infty$,

$$(3.6) \qquad U(x) \leq E_x\left[e^{-\alpha\tau}U(x\ ) + \int_0^{\tau} e^{-\alpha s}(f(x_s)\bar{G}(x_s) + \alpha(U(0) + c_o)G(x_s))ds\right]$$

$$\leq E_x\left[\int_0^{\tau} e^{-\alpha s}f(x_s)\bar{G}(x_s) + \alpha(U(0) + c_o)G(x_s)ds + e^{-\alpha\tau}(U(0) + g(x_\tau)\bar{G}(x_\tau) + c_oG(x_\tau))\right]$$

If we can show $U(0) \leq V^{\alpha}(0)$ then

$$U(x) \leq J_{\alpha}^{x}(\tau) \quad \text{and so} \quad U(x) \leq V^{\alpha}(x)$$

By (3.6) and Lemma 3.3

$$(3.7) \qquad U(0) \leq E_o\left[\int_0^{\sigma\wedge\tau} e^{-\alpha s}f(x_s)ds + e^{-\alpha\sigma\wedge\tau}(U(0) + g(x_\tau)I_{(\tau<\sigma)} + c_oI_{(\tau\geq\sigma)})\right]$$

If $\vec{\tau} = (\tau_1, \tau_2, \ldots)$ is any replacement sequence, then using (3.7) inductively, after n steps we have

$$(3.8) \qquad U(0) \leq E_o\left[\int_0^{\sigma_1\wedge\tau_1} e^{-\alpha s}f(x_s^1)ds + e^{-\alpha\sigma\wedge\tau_1}(g(x_{\tau_1}^1)I_{(\tau_1<\sigma_1)} + c_oI_{(\tau_1\geq\sigma_1)}) + \sum_{j=2}^{n}\prod_{\ell=1}^{j-1} e^{-\alpha\sigma_\ell\wedge\tau_\ell}\int_0^{\sigma_j\wedge\tau_j} e^{-\alpha s}f(x_s^j)ds + e^{-\alpha\sigma_j\wedge\tau_j}(g(x_{\tau_s}^j)I_{(\tau_j<\sigma_j)} + c_0 I_{(\tau_j\geq\sigma_j)} + \prod_{\ell=1}^{n} e^{-\alpha\sigma_\ell\wedge\tau_\ell}U(o)\right]$$

By (2.3) since the σ_ℓ's are independent, and $\sum_{j\geq n}\tau_j = \infty$ for all n a.s. P_0,

we obtain by letting $n \to \infty$

$$U(0) \leq \vec{J}_o^{\alpha}(\vec{\tau}), \text{ i.e. } U(0) \leq V_o^{\alpha} = V^{\alpha}(0).$$

To obtain (3.5), repeat the above argument using $V^{\alpha}(0)$ and $\hat{\vec{\tau}}$. At each step we have equality and so (3.8) is an equality. Corollary 2.5 assures that $\sum \hat{\tau}_j = \infty$ a.s. P_o, and so $V^{\alpha}(0) = J_o^{\alpha}(\tau)$.

§4. The Replacement Process.

Let $x_t^k : t \geq 0$ and Y^k be as in Section 3 and as before $\sigma_k = \inf\{t : x_t^k \geq Y^k\}$. Assume that $P_0(x_0^k = 0$ for all $k) = 1$ and define $\sigma_0 \equiv 0$. Let

$$(4.1) \qquad z_t = x_t^k \text{ on the set } \sum_{\ell=0}^{k-1} \sigma_\ell \leq t < \sum_{\ell=0}^{k} \sigma_\ell .$$

We refer to $z_t : t \geq 0$ as the replacement process. Let $H(t)$ be as in (2.3). For $B \subset R^+$, Borel, by Lemma 3.2 and the independence of $x_t^k : t \geq 0$ and Y^k,

$$P_0(z_t \in B) = \sum_{k=1}^{\infty} P_0\left(\sum_{\ell=0}^{k-1} \sigma_\ell \leq t,\ x^k_{t - \sum_{\ell=0}^{k-1} \sigma_\ell} \in B,\ x^k_{t - \sum_{\ell=0}^{k-1} \sigma_\ell} < Y^k\right)$$

$$= \sum_{k=1}^{\infty} \int_0^t E_0[I_B(x_{t-u})\bar{G}(x_{t-u})]\, H^{(k-1)}(du)$$

$$= \int_0^t \int_B p_{t-u}(0,dz)\bar{G}(z)R(du)$$

where $H^{(k)}(t)$ is the k-fold convolution of $H(t)$, $R(t) = \sum_0^\infty H^{(k)}(t)$, and $p_t(0,dz) = P_0(x_t \in dz)$. It is standard from renewal theory (see Feller [7]), that $R(t) < \infty$.

Suppose $G(x) < 1$. Considerations of x for which $G(x) = 1$ are not necessary because $z_t : t \geq 0$ never reaches x. Define Y_x by defining for $y > x$

$$P(Y_x \leq y) = P(Y \leq y \mid Y > x) = \frac{G(y) - G(x)}{\bar{G}(x)} = G_x(y)$$

We use the notation $P_{x,0}$ to stand for the condition $P_{x,0}(x_0^1 = x,\ x_0^k = 0,\ k \geq 2) = 1$. Take Y_x^1 to have distribution $G_x(y)$ and maintain the usual independence.

Define

$$\sigma_x = \inf\{t : x_t^1 \geq Y_x^1\}.$$

Let $z_t : t \geq 0$ be as in (4.1) except that $\sigma_1 = \sigma_x$.

Define $H_x(t) = P_x(\sigma_x \leq t)$. By a slight modification of Lemma 3.2,

$$(4.2) \qquad P_x(z_t \in B) = P_{x,0}(x_t^1 \in B,\ x_t^1 \leq Y_x^1)$$

$$+ \sum_{k=2}^{\infty} P_{x,0}\left(\sum_{\ell=0}^{k-1} \sigma_\ell \leq t,\ X^k_{t - \sum_{\ell=0}^{k-1}\sigma_\ell} \in B,\ X^k_{t - \sum_{\ell=0}^{k-1}\sigma_\ell} < Y^k\right)$$

$$= E_x[I_B(x_t)\overline{G}_x(x_t)] = \sum_{k=2}^{\infty}\int_0^t E_0[I_B(x_{t-n})\overline{G}(x_{t-u})]H_x * H^{(k-2)}(du)$$

$$= \int_B p_t(x,dy)\frac{\overline{G}(y)}{\overline{G}(x)} + \int_0^t \int_B p_{t-u}(0,dy)\overline{G}(y)H_x * R(du).$$

where $p_t(x,dy) = P_x(x_t \in dy)$.

To establish the Markov property, first note that

$$(4.3) \qquad P_x(x_s \in A, x_s < Y_x \leq x_{s+t}) = E_x[I_A(x_s)(G_x(x_{s+t}) - G_x(x_s))]$$

$$= E_x[I_A(x_s)\frac{\overline{G}(x_s)}{\overline{G}(x)} G_{x_s}(x_{s+t})]$$

$$= \int_A p_s(x,dy)\frac{\overline{G}(y)}{\overline{G}(x)} H_y(t).$$

Let

$$P_o(z_t \in dy) = q_t(0,dy) = \int_0^t p_{t-u}(0,dy)\overline{G}(y)R(du)$$

and

$$P_x(Z_t \in dy) = p_t(x,dy)\frac{\overline{G}(y)}{\overline{G}(x)} + \int_0^t p_{t-u}(0,dy)\overline{G}(y)H_x * R(du)$$

The Markov property for $z_t : t \geq 0$ will follow if we establish

$$P_x(z_s \in A, z_{s+t} \in B) = \int_B \int_A q_s(x,dy) q_t(y,dz).$$

Now

(4.4) $$P_x(x_s^1 \in A, x_s^1 < Y_x^1, x_{s+t}^1 \in B, {}^1_{s+t} < Y_x^1)$$

$$= \int_A P_s(x,dy) \int_B P_t(y,dz) \frac{\bar{G}(t)}{\bar{G}(y)}$$

$$= \int_A P_s(x,dy) \frac{\bar{G}(y)}{\bar{G}(x)} \int_B P_t(y,dz) \frac{\bar{G}(z)}{\bar{G}(x)}$$

By (4.3) we have

(4.5) $$\sum_{j=2}^{\infty} P_{x,0}\Big(x_s^1 \in A, x_s^1 < Y_x^1, \sum_{\ell=0}^{j-1} \sigma_\ell \leq s+t, x^j_{s+t-\sum_{\ell=0}^{j-1}\sigma_\ell} \in B, x^j_{s+t-\sum_{\ell=0}^{j-1}\sigma_\ell} < Y^j\Big)$$

$$= \sum_{j=2}^{\infty} \int_A P_s(x,dy) \frac{\bar{G}(y)}{\bar{G}(x)} \int_0^t \int_B P_{t-u}(0,dz)\bar{G}(z) H_y * H^{(j-2)}(du)$$

$$= \int_A P_s(x,dy) \frac{\bar{G}(y)}{\bar{G}(x)} \int_0^t \int_B P_{t-u}(0,dz)\bar{G}(z) H_y * R(du)$$

Also

(4.6) $$\sum_{k=2}^{\infty} P_{x,0}\Big(\sum_{\ell=0}^{k-1} \sigma_\ell \leq s, x^k_{s-\sum_{\ell=0}^{k-1}\sigma_\ell} \in A, x^k_{s-\sum_{\ell=0}^{k-1}\sigma_\ell} < Y^k, \sum_{\ell=0}^{k-1}\sigma_\ell \leq s+t,$$

$$x^k_{s+t-\sum_{\ell=0}^{k-1}\sigma_\ell} \in B, x^k_{s+t-\sum_{\ell=0}^{k-1}\sigma_\ell} < Y^k\Big)$$

$$= \sum_{k=2}^{\infty} \int_0^s \int_A P_{s-u}(0,dy)\bar{G}(y) \int_B P_t(y,dz) \frac{\bar{G}(y)}{\bar{G}(z)} H_x * H^{(k-2)}(du)$$

$$= \int_0^s \int_A P_{s-u}(0,dy)\bar{G}(y) \int_B P_t(y,dz) \frac{\bar{G}(z)}{\bar{G}(y)} H_x * R(du).$$

Again by (4.2)

$$(4.7)\qquad \sum_{k=2}^{\infty}\ \sum_{j=k+1}^{\infty} P_{x,0}\Bigg(\sum_{\ell=0}^{k-1}\sigma_\ell \le s,\ x^k_{s-\sum_{\ell=0}^{k-1}\sigma_\ell} \in A,\ x^k_{s-\sum_{\ell=0}^{k-1}\sigma_\ell} < Y_x,$$

$$\sum_{\ell=0}^{j-1}\sigma_\ell \le s+t,\ x^j_{s+t-\sum_{\ell=0}^{j-1}\sigma_\ell} \in B,\ x^j_{s+t-\sum_{\ell=0}^{j-1}\sigma_\ell} < Y^j\Bigg)$$

$$= \sum_{k=2}^{\infty}\ \sum_{j=k+1}^{\infty} \int_0^s \int_A p_{s-u}(0,dy)\bar{G}(y) \int_0^t \int_B p_{t-u}(0,dz)\bar{G}(z) H_y * H^{(j-k-1)}(du)\ H_x * H^{(k-2)}(du)$$

$$= \int_0^s \int_A p_{s-u}(0,dy)\bar{G}(y) \int_0^t \int_B p_{t-u}(0,dt)\bar{G}(z) H_y * R(du) H_x * R(du)$$

Now by combining (4.4)-(4.7),

$$P_x(z_s \in A, z_{t+s} \in B) = \sum_{j=1}^{\infty}\ \sum_{j=k}^{\infty} P_{x,0}\Bigg(\sum_{\ell=0}^{k-1}\sigma_\ell \le s,\ x^k_{s-\sum_{\ell=0}^{k-1}\sigma_\ell} \in A,$$

$$x^k_{s-\sum_{\ell=0}^{k-1}\sigma_\ell} < Y^k,\ \sum_{\ell=0}^{k-1}\sigma_\ell \le s+t,\ x^j_{t+s-\sum_{\ell=0}^{j-1}\sigma_\ell} \in B,\ x^j_{s+t-\sum_{\ell=0}^{j-1}\sigma_\ell} < Y^j\Bigg)$$

$$= \int_A p_s(x,dy)\frac{\bar{G}(y)}{\bar{G}(x)} \int_B p_t(y,dz)\frac{\bar{G}(z)}{\bar{G}(x)} + \int_0^t \int_B p_{t-u}(0,dz)\bar{G}(z) H_y * R(du)$$

$$= \int_0^s \int_A p_{s-u}(0,dy)\bar{G}(y) \int_B p_t(y,dz) + \int_0^t \int_B p_{t-u}(0,dz)\bar{G}(z) H_y * R(du)\ H_x * R(du)$$

$$= \int_A \int_B q_s(x,dy) q_t(y,dz)$$

Next let

$$S_t f(x) = E_x[f(z_t)] = \frac{1}{\bar{G}(x)} T_t(\bar{G}f)(x) + \int_0^t T_{t-s}(\bar{G}f)(0) H_x * R(ds)$$

For $f \varepsilon C_b(R^+)$, since $G(x)$ is assumed continuous and $T_t : t \geqslant 0$ is Feller

$$\frac{1}{\overline{G}(x)} T_t(\overline{G}f)(x) \varepsilon C_b(R^+)$$

Moreover

$H_x(t) = E_x[G_x(X_t)]$ is continuous in x and therefore the family of distributors $\{H_x(t) : x \varepsilon R^+\}$ is continuous in distribution in x. Since $T_{t-s}(\overline{G}f)(0)$ is continuous in s and $H_x{*}R(t)$ is continuous in t,

$\int_0^t T_{t-s}(Gf)(0)H_x{*}R(ds)$ is continuous in x and so $S_t f \varepsilon C_b(R^+)$.

To show that $S_t : t \geqslant 0$ is strongly continuous we assume

(4.8) $\quad G \varepsilon D_A$ and $\dfrac{AG(x)}{\overline{G}(x)}$ is continuous and bounded.

Note that

(4.9) $$\int_0^t T_{t-s}(\overline{G})(0)H_x{*}R(ds) = E_x[G_x(x_t)].$$

so

$$|S_t f(x) - f(x)| \leqslant |T_t f(x) - f(x)| + 2\|f\| T_t G_x(x)$$

From (4.8) by Dynkins formula

$$0 \leqslant T_t G_x(x) \leqslant \frac{1}{\overline{G}(x)}(T_t G(x) - G(x)) = \frac{1}{\overline{G}(x)} \int_0^t T_s AG(x)$$

$$\leqslant t \left\| \frac{AG}{\overline{G}} \right\|$$

and therefore

$$\|S_t f - f\| \leqslant \|T_t f - f\| + 2\|f\| \left\|\frac{AG}{\overline{G}}\right\| t$$

and we have strong continuity.

To compute the infinitesimal generator, observe that

$$\tilde{A}f(x) = \lim_{t \to 0} \frac{S_t f(x) - f(x)}{t}$$

$$= \lim_{t \to 0} \frac{1}{\overline{G}(x)} \frac{T_t(\overline{G}f)(x) - \overline{G}(x)f(x)}{t} + \lim_{t \to 0} \frac{1}{t} \int_0^t p_{t-u}(0,dy)\overline{G}(y)f(y)H_x{*}R(du)$$

Pointwise then

$$\lim_{t\to 0} \frac{1}{\overline{G}(x)} \frac{T_t(\overline{G}f)(x) - \overline{G}(x)f(x)}{t} = \frac{A(Gf)(x)}{\overline{G}(x)}$$

and

$$\lim_{t\to 0} \frac{1}{t}\int_0^t P_{t-u}(0,dy)\overline{G}(y)f(y)H_x{*}R(du) = \lim_{t\to 0} \frac{1}{t}\int_0^t P_{t-u}(0,dy)\overline{G}(y)(f(y)-f(0))H_x{*}R(du)$$

$$(4.10) \qquad + f(0) \lim_{t\to 0} \frac{1}{t}\int_0^t P_{t-u}(0,dy)\overline{G}(y)H_x{*}R(du).$$

Clearly the first term on the right of (4.10) goes to 0 and by (4.9)

$$\lim_{t\to 0} \frac{1}{t}\int_0^t P_{t-u}(0,dy)\overline{G}(y)H_x{*}R(du) = \lim_{t\to 0} \frac{1}{\overline{G}(x)} \frac{T_tG(x) - G(x)}{t}$$

$$= \frac{AG(x)}{\overline{G}(x)}.$$

Therefore pointwise

$$(4.11) \qquad \tilde{A}f(x) = \frac{A(Gf)(x)}{\overline{G}(x)} + f(0)\frac{AG(x)}{\overline{G}(x)}$$

With respect to supremum norm on $C_b(R^+)$ we establish the following: Under the condition (4.8), $f \in D_{\tilde{A}}$ if and only if $\overline{G}f \in D_A$ and $\frac{A\overline{G}f}{\overline{G}}$ is bounded and continuous. We first note that

$$(4.12) \qquad \left| \frac{1}{t}\int_0^t P_{t-u}(0,dy)\overline{G}(y)f(y)H_x{*}R(du) - f(0)\frac{AG(x)}{\overline{G}(x)} \right|$$

$$\leq \left| \frac{1}{t}\int_0^t T_{t-u}(\overline{G}(f - f(0))(0)H_x{*}R(du) \right| + \frac{1}{t}|f(0)| \left| \int_0^t T_{t-u}\overline{G}(0)H_x{*}R(du) - \frac{AG(x)}{\overline{G}(x)} \right|$$

From (4.8), (4.9) and Dynkin's formula, for t small enough

$$\left\| \frac{1}{t}\int_0^t T_{t-u}(\overline{G}(f - f(0))(0)H_x{*}R(du) \right\| \leq \sup_{0\leq s\leq t} \|T_s\overline{G}(f - f(0))\| \left\| \frac{1}{t}H_x{*}R(t) \right\|$$

$$\leq \sup_{0\leq s\leq t} \|T_s(\overline{G}(f - f(0))\| \left\| \frac{1}{t} E_x \frac{[G(x_t) - G(x)]}{\overline{G}(x)} \right\|$$

$$\leq \sup_{0\leq s\leq t} \|T_s(\overline{G}(f - f(0))\| \; C \left\| \frac{AG}{\overline{G}} \right\| \to 0.$$

Moreover by (4.8), (4.9) and Dynkin's formula again,

$$\left| \frac{1}{t}\int_0^t T_{t-u}(\overline{G})(0)H_x{*}R(du) - \frac{AG(x)}{\overline{G}(x)} \right|$$

$$= \left| \frac{T_t G(x) - G(x)}{\overline{G}(x)t} - \frac{AG(x)}{\overline{G}(x)} \right|$$

$$\leqslant \frac{1}{t}\int_0^t \left\| T_s\left(\frac{AG}{\overline{G}}\right) - \frac{AG}{\overline{G}} \right\| ds + \frac{1}{t}\left\| \int_0^t T_s\left(\frac{AG}{\overline{G}}\left(\frac{\overline{G}}{\overline{G}(x)} - 1\right)\right) ds \right\|$$

Now the first term goes to 0 and for the second,

$$(4.13) \qquad \frac{1}{t}\left\| \int_0^t T_s\left(\frac{AG}{\overline{G}}\left(\frac{\overline{G}}{\overline{G}(x)} - 1\right)\right) ds \right\| \leqslant \left\| \frac{AG}{\overline{G}} \right\| \frac{1}{t}\int_0^t \left\| \frac{T_s G - G}{\overline{G}} \right\| ds$$

$$\leqslant \left\| \frac{AG}{\overline{G}} \right\|^2 \frac{1}{t}\int_0^t s\, ds$$

and so we have uniform convergence for the left hand side of (4.12).

Next if $f \in D_{\tilde{A}}$, then by (4.11) and the fact that we have uniform convergence in (4.12)

$$\left\| \frac{1}{t}(T_t(\overline{G}f) - \overline{G}f) - A\overline{G}f \right\| \leqslant \left\| \frac{T_t(\overline{G}f) - \overline{G}f}{t\,\overline{G}} - \frac{A\overline{G}f}{\overline{G}} \right\|$$

$$\leqslant \left\| \frac{S_t f - f}{t} - \tilde{A}f \right\| + \left\| \int_0^t T_{t-u}(\overline{G}f)(0)\; H_x{*}R(du) - f(0)\; \frac{AG(x)}{\overline{G}(x)} \right\|$$

and hence

$$\lim_{t \to 0} \left\| \frac{1}{t}(T_t(\overline{G}f) - \overline{G}f) - A\overline{G}f \right\| = 0$$

That $\frac{A\overline{G}f}{\overline{G}}$ is bounded follows from (4.11).

Conversely if $\overline{G}f \in D_A$ and $\frac{A\overline{G}f}{\overline{G}}$ is bounded to show

$$\left\| \frac{S_t f - f}{t} - \tilde{A}f \right\| \to 0 \quad \text{we must show}$$

$$\left\| \frac{T_t(\overline{G}f) - \overline{G}f}{\overline{G}} - \frac{A(Gf)}{\overline{G}} \right\| \to 0.$$

This is so since

$$\left\| \frac{T_t(\overline{G}f) - \overline{G}f}{\overline{G}} - \frac{A(\overline{G}f)}{\overline{G}} \right\| \leq \frac{1}{t} \int_0^t \left\| T_s\left(\frac{A(\overline{G}f)}{\overline{G}}\right) - \frac{A(\overline{G}f)}{\overline{G}} \right\| ds$$

$$+ \left\| \frac{A(\overline{G}f)}{\overline{G}} \right\| \frac{1}{t} \int_0^t \left\| \frac{T_s(G) - G}{\overline{G}} \right\| ds \to 0.$$ The latter follows as in (4.13).

To summerize, we have established:

Theorem 4.1. Assume the conditions (2.1) and (4.8). Let

$$z_t = x_{t - \sum_{\ell=0}^{k-1} \sigma_\ell} \quad \text{on the set} \quad \sum_{\ell=0}^{k-1} \sigma_\ell \leq t < \sum_{\ell=0}^{k} \sigma_\ell .$$

Then $z_t : t \geq 0$ is a strong Markov process and has transition probabilities given by

$$q_t(x,dy) = p_t(x,dy) \frac{\overline{G}(y)}{\overline{G}(x)} + \int_0^t p_{t-u}(0,dy)\overline{G}(y)H_x * R(du)$$

Moreover the associated semigroup $S_t : t \geq 0$ acting on $C_b(R^+)$ is strongly continuous and Feller

Further letting $\hat{A}f$ denote the infinitesimal generator then $f \in D_{\hat{A}}$ if and only if $\overline{G}f \in D_A$ and $\frac{\hat{A}(\overline{G}f)}{\overline{G}}$ is bounded. Lastly

$$\hat{A}f(x) = \frac{A(\overline{G}f)(x)}{\overline{G}(x)} + f(0) \frac{AG(x)}{\overline{G}(x)} .$$

Theorem 4.2. Suppose (2.1), (4.13) and

(4.14) $\quad E_x[\sigma_x] < \infty$ for all x.

Then $z_t : t \geq 0$ has a unique invariant probability measure Π given by

$$\Pi(A) = \frac{E_0\left[\int_0^\sigma I_A(x_t)dt\right]}{E_0[\sigma]} = \frac{E_0\left[\int_0^\infty I_A(x_t)\overline{G}(x_t)dt\right]}{E_0\left[\int_0^\infty \overline{G}(x_t)dt\right]} .$$

Proof. First note that by Lemma 3.3, the two representations of Π are equivalent. Recall the notation $\sigma_1 = \sigma_x = \inf\{t: x_t^1 \geq Y_x^1\}$ and $\sigma_k = \inf\{t: x_t^k \geq Y^k\}$. Since $\sigma_k: k \geq 1$ are independent as well as $\int_0^{\sigma_k} f(x_t^k)dt : k \geq 1$ and for $k \geq 2$, are respectively identically distributed with means

$$E[\sigma_k] = E_0\left[\int_0^\infty \overline{G}(x_t)dt\right] \; k \geq 2$$

and

$$E\left[\int_0^{\sigma_k} f(x_t^k)dt\right] = E_0\left[\int_0^\infty f(x_t)\overline{G}(x_t)dt\right] \; k \geq 2$$

we have by the Strong Law of Large Number for $f \in C_b(R^+)$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(z_s)ds = \lim_{n\to\infty} \frac{\frac{1}{n}\sum_{k=1}^n \int_0^{\sigma_k} f(x_s^k)ds}{\frac{1}{n}\sum_{k=1}^n \sigma_k}$$

$$= \frac{E_0\left[\int_0^\infty f(x_t)\overline{G}(x_t)dt\right]}{E_0\left[\int_0^\infty \overline{G}(x_t)dt\right]} \quad \text{a.s. } P_z \text{ for all } z.$$

Hence Π is an invariance measure.

If $\tilde{\Pi}$ is any other invariant measure, then for $f \in C_b(R^+)$,

$$\int f(z)\tilde{\Pi}(dt) = \int E_z[f(z_s)]\tilde{\Pi}(dt)$$

Therefore by Lebesque Dominated Convergence

$$\int f(z)\tilde{\Pi}(dt) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \int E_z[f(z_s)]\tilde{\Pi}(dz)ds$$

$$= \int E_z\left[\lim_{t\to\infty} \frac{1}{t}\int_0^t f(z_s)ds\right]\tilde{\Pi}(dz) = \int f(z)\Pi(dz)$$

Hence $\tilde{\Pi} = \Pi$.

Remark 4.1. In the work of Robin [8] it is required that

$$P_x(z_t \in \Gamma) - \Pi(\Gamma) \leq Be^{-\gamma t} \qquad \gamma > 0$$

and as a consequence

$$S_t g(x) - \Pi(g) \leq B_1 e^{-\gamma_1 t} \qquad \gamma > 0$$

where

$$\Pi(g) = \int g(x)\,\Pi(dx).$$

For our case we have only that

$$\lim_{t \to 0} \frac{1}{t} \int_0^t S_u g(x)du = \Pi(g).$$

Since $z_t : t \geq 0$ is periodic, we will not even have pointwise convergence.

Let $f \in C_b(R^+)$ and $u_\alpha(x)$ be the unique solution of $\tilde{A}u_\alpha - \alpha u_\alpha = -f(x)$.

It is standard that

$$(4.15) \qquad u_\alpha(x) = \int_0^\infty e^{-\alpha t} q_t(x,dy)f(y).$$

Define

$$\bar{f} = \Pi(f) = \frac{E_o\left[\int_0^\infty \bar{G}(x_s)f(x_s)ds\right]}{E_o\left[\int_0^\infty \bar{G}(x_s)ds\right]}.$$

Theorem 4.3. Under the assumption (2.1), (4.8) and the assumption

$$(4.16) \qquad E_x[\sigma_x] = \frac{E_x\left[\int_0^\infty \bar{G}(x_s)ds\right]}{\bar{G}(x)} \leq C \quad \text{independent of } x,$$

(4.17) $\quad P_x(X_t < \infty) = 1$ for all t and x, and

$$\lim_{t\to\infty} x_t = \infty \quad \text{a.s. } P_x \text{ for all } x.$$

(i) $\lim_{\alpha\to 0} \alpha u_\alpha(x) = \bar{f}$.

(ii) Let $v_\alpha(x) = u_\alpha(x) - u_\alpha(0)$. Then

$$\lim_{\alpha\to 0} v_\alpha(x) = \bar{v}(x) \quad \text{uniformly on compact sets where}$$

$$\bar{v}(x) = \frac{E_x\left[\int_0^\infty (f(x_s) - \bar{f})\bar{G}(x_s)ds\right]}{\bar{G}(x)}$$

and $\bar{v}$ is the unique solution of

$$-\tilde{A}v = f - \bar{f} \quad \text{with} \quad \bar{v}(0) = 0.$$

Proof. From (4.2) and (4.15)

$$u_\alpha(x) = \frac{1}{\bar{G}(x)} E_x\left[\int_0^\infty e^{-\alpha t} f(x_t)\bar{G}(x_t)dt\right]$$

$$+ \int_0^\infty e^{-\alpha t} \int_0^t E_o\left[f(x_{t-s})\bar{G}(x_{t-s})\right] H_x * R(ds)dt.$$

By (4.16)

$$\left|\frac{1}{\bar{G}(x)} E_x\left[\int_0^\infty e^{-\alpha t} f(x_t)\bar{G}(x_t)dt\right]\right| \leq f \; \frac{E_x\left[\int_0^\infty \bar{G}(x_t)dt\right]}{\bar{G}(x)}$$

and so

$$\lim_{\alpha\to 0} \alpha \left|\frac{1}{\bar{G}(x)} E_x\left[\int_0^\infty e^{-\alpha t} f(x_t)\bar{G}(x_t)dt\right]\right| \to 0$$

Next by the convolution property of Laplace transforms,

$$\int_0^\infty e^{-\alpha t} \int_0^t E_o\left[f(x_{t-s})\overline{G}(x_{t-s})\right] H_x * R(ds)dt$$

$$= \int_0^\infty e^{-\alpha t} E_o[\overline{G}(x_t)f(x_t)]dt \int_0^\infty e^{-\alpha t} H_x(dt) \int_0^\infty e^{-\alpha t} R(dt)$$

Let

$$\hat{H}_x(\alpha) = \int_0^\infty e^{-\alpha t} H_x(dt) \quad \text{and note that} \quad \hat{H}_x(\alpha) \longrightarrow 1 \quad \text{as} \quad \alpha \to 0.$$

Since

$$\int_0^\infty e^{-\alpha t} R(dt) = \frac{1}{1 - \hat{H}(\alpha)} \quad \text{where} \quad \hat{H}(\alpha) = \int_0^\infty e^{-\alpha t} H(dt)$$

we have

$$\lim_{\alpha \to 0} \frac{1 - \hat{H}(\alpha)}{\alpha} = E_o[\sigma] = E_o\left[\int_0^\infty \overline{G}(x_t)dt\right] \quad \text{showing (i).}$$

For (ii), note that

$$u_\alpha(0) = E_o\left[\int_0^\infty e^{-\alpha t}\overline{G}(x_t)f(x_t)dt\right] \frac{1}{1 - \hat{H}(\alpha)} .$$

Therefore

$$(4.18) \qquad u_\alpha(x) - u_\alpha(0) = \frac{1}{\overline{G}(x)} E_x\left[\int_0^\infty e^{-\alpha t}\, \overline{G}(x_t)f(x_t)dt\right]$$

$$+ E_o\left[\int_0^\infty e^{-\alpha t}\, \overline{G}(x_t)f(x_t)dt\right] \frac{\hat{H}_x(\alpha)-1}{1 - \hat{H}(\alpha)} .$$

It is enough to establish the uniform convergence on compact set for $f \geqslant 0$, the general case easily follows.

Now

$$\frac{1}{\overline{G}(x)} E_x\left[\int_0^\infty e^{-\alpha t}\, \overline{G}(x_t)f(x_t)dt\right] \uparrow \frac{1}{\overline{G}(x)} E_x\left[\int_0^\infty \overline{G}(x_t)f(x_t)dt\right]$$

which is bounded by (4.16). Next

$$\lim_{\alpha\to 0} E_0\left[\int_0^\infty e^{-\alpha t}\,\bar G(x_t)f(x_t)dt\right]\frac{\alpha}{1-\hat H(\alpha)} = \bar f\ .$$

For the other part of the second term in (4.18),

$$\frac{\hat H_x(\alpha)-1}{\alpha} = \frac{E_x\left[\int_0^\infty \alpha e^{-\alpha t}(G(x_t)-G(x))dt - \bar G(x)\right]}{\alpha\bar G(x)}$$

$$= -\frac{E_x\left[\int_0^\infty e^{-\alpha t}\,\bar G(x_t)dt\right]}{\bar G(x)} + -\frac{E_x\left[\int_0^\infty \bar G(x_t)dt\right]}{\bar G(x)}\ .$$

Therefore, uniformly on compact sets

$$\lim_{\alpha\to 0} u_\alpha(x) - u_\alpha(0) = \frac{E_x\left[\int_0^\infty \bar G(x_t)(f(x_t)-\bar f)dt\right]}{\bar G(x)}.$$

To show that

$$-\bar A\bar v(x) = f(x) - \bar f$$

first note that $\bar v(0) = 0$ and so by Theorem 4.1 it is enough to show

$$-A(\bar G\ \bar v)(x) = \bar G(x)(f(x)-\bar f).$$

Note that by the Markov property

$$\bar G(x)\bar v(x) = E_x\left[\int_0^\infty \bar G(x_s)(f(x_s)-\bar f)ds\right]$$

$$= E_x\left[\int_0^t \bar G(x_s)(f(x_s)-\bar f)ds + \int_0^\infty \bar G(x_{t+s})(f(x_{t+s})-\bar f)ds\right]$$

$$= E_x\left[\int_0^t \bar G(x_s)(f(x_s)-\bar f)ds + \bar G(x_t)\bar v(x_t)\right]$$

And so

$$\left\| \frac{T_t \bar{G}\bar{v} - \bar{G}\bar{v}}{t} + \bar{G}(f - \bar{f}) \right\| \le \frac{1}{t} \int_0^t \| T_s \bar{G}(f - \bar{f}) - \bar{G}(f - \bar{f}) \| ds \to 0.$$

To show uniqueness, suppose $\bar{v}_1$ and $\bar{v}_2$ are two solutions of $-\breve{A} v = f - \bar{f}$, $v(0) = 0$. Let $w = \bar{v}_1 - \bar{v}_2$. Then $\breve{A} w = 0$ and since $w(0) = 0$ we must have $A\bar{G}w = 0$. By Dynkin's formula

$$\bar{G}(x) w(x) = E_x[\bar{G}(x_t) w(x_t)] \quad \text{for all } t$$

and by (4.18) it follows that

$$\lim_{t \to \infty} E_x[\bar{G}(x_t) w(x_t)] = 0 \text{ i.e. } w(x) = 0 \quad \text{for all } x.$$

It is well known that a necessary condition that $\breve{A} v = -f$ have a solution is that $\Pi(f) = 0$. See Robin [8].

Theorem 4.4. Under the assumption (2.1), (4.8), (4.16) and (4.17) a necessary and sufficient condition that

(4.19) $\breve{A} v = - f$

have a solution is that $\Pi(f) = 0$.

Moreover if $\Pi(f) = 0$, then any two solution of (4.19) differ by a constant.

Proof. We need only prove the sufficiency. Suppose $\breve{A} v = - f$. Then by Theorem 4.1

$$Av = -\bar{G}f - v(0) AG.$$

Therefore by Dynkin's formula

$$v(0) = E_0\left[\bar{G}(x_t) v(x_t) + \int_0^t \bar{G}(x_s) f(x_s) ds + v(0) G(x_t)\right]$$

Letting $t \to \infty$ (4.17)

$$v(0) = E_o\left[\int_0^\infty \overline{G}(x_s)f(x_s)ds\right] + v(0),$$

that is $\Pi(f) = 0$.

Next suppose v_1 and v_2 are two solutions of

$$\tilde{A}v = -f.$$

Then by Theorem 4.1

$$A\,\overline{G}(v_1 - v_2) = -(v_1 - v_2)[0]\ AG$$

and so by Dynkin's formula again

$$\overline{G}(x)(v_1(x) - v_2(x)) = E_x[\overline{G}(x_t)(v_1(x_t) - v_2(x_t))]$$
$$+ (v_1 - v_2)(0)\ E_x\left[\int_0^t AG(x_s)ds\right]$$

Thus by (4.17)

$$\overline{G}(x)(v_1(x) - v_2(x)) = \lim_{t\to\infty} E_x[G(x_t) - G(x)](v_1 - v_2)(0)$$

or

$$v_1(x) - v_2(x) = v_1(0) - v_2(0).$$

§5. Asymptotics of a Stopping Problem.

Assume that

(5.1) $f,\psi \in C_b(R^+)$ with $f,\psi \geq 0$

and consider the stopping problem

(5.2) $$u(x) = \inf_{\tau} J_x(\tau)$$

where

$$J_x(\tau) = E_x\left[\int_0^{\tau} f(x_s)ds + \psi(x_\tau)I_{(\tau<\infty)}\right]$$

Define for $\alpha > 0$

$$J_x^{\alpha}(\tau) = E_x\left[\int_0^{\tau} f(x_s)ds + e^{-\alpha\tau}\psi(x_\tau)\right]$$

and

(5.3) $$u_\alpha(x) = \inf_{\tau} J_x^{\alpha}(\tau).$$

See Bensoussan [3] or Robin [9] for the following:

Theorem 5.1. Under the assumption (2.1), (2.2) and (5.1)

(i) u_α is the maximal element of the set of functions $h \in C_b(R^+)$

$$h \leq e^{-\alpha t}T_t h + \int_0^t e^{-\alpha s}T_s f\,ds \qquad h \leq \psi$$

(ii) $\hat{\tau}_\alpha = \inf\{t : u_\alpha(x_t) = \psi(x_t)\}$ is optimal, is

$$u_\alpha(x) = J_x^{\alpha}(\hat{\tau}_\alpha).$$

(iii) For $\varepsilon > 0$, define

$$J_x^{\alpha,\varepsilon}(V) = E_x\left[\int_0^{\infty} e^{-\alpha t}\exp -\frac{1}{\varepsilon}\int_0^t V_s ds\ (f(x_t) + \frac{1}{\varepsilon}V_t\,\psi(x_t))dt\right]$$

where $0 \leq V_t \leq 1$ and $V_t \in F_t : t \geq 0$. Let

$$w_{\alpha,\varepsilon}(x) = \inf_{V} J_x^{\alpha,\varepsilon}(V)$$

Then $w_{\alpha,\varepsilon}$ is the unique solution in $C_b(R^+)$ of

$$w_{\alpha,\varepsilon} = \int_0^\infty e^{-\alpha t} T_t [f - \frac{1}{\varepsilon}(w_{\alpha,\varepsilon} - \psi)^+] dt$$

and moreover if

$$\hat{V}_t = \begin{cases} 1 & w_{\alpha,\varepsilon}(x_t) \geqslant \psi(x_t) \\ 0 & w_{\alpha,\varepsilon}(x_t) < \psi(x_t) \end{cases}$$

then

$$w_{\alpha,\varepsilon}(x) = J_x^{\alpha,\varepsilon}(\hat{V}).$$

(iv) $w_{\alpha,\varepsilon} \downarrow v_\alpha$ as $\varepsilon \to 0$ uniformly on compact sets.

Suppose we have the additional condition

(5.4) $f(x) \geqslant \beta > 0.$

Remark 5.1. For the consideration of (5.1), under (5.3), we can restrict our attention to stopping times τ so that

$$E_x(\tau) \leqslant \frac{\|\psi\|}{\beta}$$

since by taking $\tau \equiv 0$, we obtain $u(x) \leqslant \|\psi\|$ and for any stopping time τ, $\beta\, E_x[\tau] \leqslant J_x^\alpha(\tau)$.

For the following see Robin [8], Theorem 3.1 and Remark 3.3:

Theorem 5.2. Under (2.1), (2.2), (5.1) and (5.4)

(i) u is the maximal element of the set of funcetions

$$h \in C_b(R^+) \quad h \leqslant T_t h + \int_0^t T_s f ds \quad h \leqslant \psi$$

(ii) $\hat{\tau} = \inf\{t : u(x_t) = \psi(x_t)\}$ is optimal, i.e.

$$u(x) = J_x(\hat{\tau})$$

(iii) $u_\alpha \uparrow u$ uniformly on compact sets as $\alpha \to 0$

(iv) Let $\varepsilon > 0$ and

$$w_\varepsilon(x) = \inf_V J_x^\varepsilon(V)$$

where

$$J_x^\varepsilon(V) = E_x\left[\int_0^\infty \exp\left(-\frac{1}{\varepsilon}\int_0^t V_s ds\right)(f(x_t) + \frac{1}{\varepsilon}\ V_t\psi(x_t)dt\right]$$

and $0 \leqslant V_s \leqslant 1$, $V_t \in F_t : t \geqslant 0$. Then w_ε is the unique solution in $C_b(R^+)$ of

$$w_\varepsilon = T_t w_\varepsilon + \int_0^t T_s(f - \frac{1}{\varepsilon}(w_\varepsilon - \psi)^+)ds$$

and if

$$\hat{V}_t = \begin{cases} 1 & w_\varepsilon(x_t) \geqslant \psi(x_t) \\ 0 & w_\varepsilon(x_t) < \psi(x_t) \end{cases}$$

then

$$w_\varepsilon(x) = J_x^\varepsilon(\hat{V}).$$

(v) $w_{\alpha,\varepsilon} \uparrow w_\varepsilon$ uniformly on compact sets.

(vi) $w_\varepsilon \downarrow u$ uniformly on compact sets.

Define for $\alpha > 0$

$$J_x^\alpha(\tau) = E_x\left[\int_0^\tau \bar{G}(x_t)f(x_t)dt + e^{-\alpha\tau}\bar{G}(x_\tau)\psi(x_\tau)\right]$$

where f and ψ satisfy (5.1) and (5.4) and $G(x)$ is a continuous distribution function as before. Let

$$J_x(\tau) = E_x\left[\int_0^\tau \bar{G}(x_t)f(x_t)dt + \bar{G}(x_\tau)\psi(x_\tau)\right].$$

Under the assumption (4.17), it is consistant to define

$$\bar{G}(x_\tau)\psi(x_\tau) = 0 \quad \text{on the set} \quad (\tau = \infty).$$

Hence under (4.16), $J_x(\tau) < \infty$ for any stopping time τ and moreover by quasi-left continuity, see (2.2)

(5.5) $$\lim_{t\to\infty} J_x(\tau \wedge t) = J_x(\tau) \quad \text{and} \quad \lim_{t\to\infty} J_x^\alpha(\tau\wedge t) = J_x^\alpha(\tau).$$

Let

$$u(x) = \inf_{\tau} J_x(\tau) \quad \text{and} \quad u_\alpha(x) = \inf J_x^\alpha(\tau).$$

We wish to establish the following generalization of Theorem 5.2:

<u>Theorem 5.3</u>. Under (2.1), (2.2), (4.16), (4.17), (5.1), and (5.4)

(i) u is the maximal element of the set of functions

$$h \in C_b(R^+), \quad h \leq T_t h + \int_0^t T_s(\overline{G}f)ds \qquad h \leq \overline{G}\psi.$$

(ii) $\hat{\tau} = \inf\{t: u(x_t) = \overline{G}(x_t)\ \psi(x_t)\}$ is optimal, i.e.

$$u(x) = J_x(\hat{\tau})$$

(iii) $u_\alpha \uparrow u$ uniformly on compact sets.

The proof will follow after a series of lemmas. Let

$$J_x^n(\tau) = E_x\left[\int_0^\tau (\overline{G}(x_t) V \tfrac{1}{n}\ \ f(x_t)dt + \overline{G}(x_\tau)\psi(x_\tau)\right]$$

and

$$u_n(x) = \inf_{\tau} J_x^n(\tau).$$

<u>Lemma 5.4</u>. $u_n \in C_b(R^+)$ and $u_n(x) \downarrow u(x)$.

<u>Proof</u>. Note that by (5.4) and Remark 5.1

$$u_n(x) = \inf\{J_x(\tau):\ \ E_x[\tau] < \infty\}.$$

Since for any τ with $E_x[\tau] < \infty$

$$J_x^n(\tau) \downarrow J_x(\tau)$$

it follows that

$$u_n(x) \downarrow \breve{u}(x) \qquad \breve{u}(x) \geq u(x).$$

Now Theorem 5.4 applies to $u_n(x)$ and so for any stopping time τ with $E_x[\tau] < \infty$

$$u_n(x) \leq J_x^n(\tau)$$

Letting $n \to \infty$

$$\tilde{u}(x) \leq J_x(\tau) \quad \text{and so} \quad \tilde{u}(x) \leq u(x),$$

i.e. $\tilde{u}(x) = u(x)$.

Let

$$J_x^{\varepsilon,n}(V) = E_x\left[\int_0^\infty e^{-1/\varepsilon\int_0^t V_s ds} (\bar{G}(x_t)V\tfrac{1}{n})\, f(x_t) + \tfrac{1}{\varepsilon}V_t\bar{G}(x_t)\psi(x_t)dt\right]$$

$$J_x^{\varepsilon,\alpha}(V) = E_x\left[\int_0^\infty e^{-\alpha t}e^{-1/\varepsilon\int_0^t V_s ds} \bar{G}(x_t)\, f(x_t) + \tfrac{1}{\varepsilon}V_t\bar{G}(x_t)\psi(x_t)dt\right]$$

and

$$J_x^{\varepsilon}(V) = E_x\left[\int_0^\infty e^{-1/\varepsilon\int_0^t V_s ds} \bar{G}(x_t)f(x_t) + \tfrac{1}{\varepsilon}V_t\bar{G}(x_t)\psi(x_t)dt\right]$$

where

$V_t \in F_t : t \geq 0$ with $0 \leq V_t \leq 1$. Define

$$u_{\varepsilon,n}(x) = \inf_V J_x^{\varepsilon,n}(V), \quad u_{\varepsilon,\alpha}(x) = \inf_V J_x^{\varepsilon,\alpha}(V), \quad \text{and} \quad u_\varepsilon(x) = \inf_V J_x^\varepsilon(V).$$

<u>Lemma 5.5</u>.

(i) $u_{\varepsilon,n} \in C_b(R^+)$ and $u_{\varepsilon,n}(x) \downarrow u_\varepsilon(x)$

(ii) $u_{\varepsilon,\alpha}(x) \in C_b(R^+)$ and $u_{\varepsilon,\alpha}(x) \uparrow u_\varepsilon(x)$

(iii) $u_\varepsilon(x)$ is the unique non-negative solution in $C_b(R^+)$ of

$$u_\varepsilon = T_t u_\varepsilon + \int_0^t T_s(\bar{G}f - 1/\varepsilon(u_\varepsilon - \bar{G}\psi)^+)ds$$

and if

$$\hat{V}_t = \begin{cases} 1 & u_\varepsilon(x_t) \geq \bar{G}(x_t)\psi(x_t) \\ 0 & u_\varepsilon(x_t) < \bar{G}(x_t)\psi(x_t) \end{cases}$$

then $u_\varepsilon(x) = J_x^\varepsilon(\hat{V})$.

Proof. We first claim

$$u_\varepsilon(x) = \inf\{J_x^\varepsilon(V) : \int_0^\infty V_t\,dt = \infty \quad \text{a.s. } P_x\}$$

Suppose $P_x\left(\int_0^\infty V_t\,dt < \infty\right) > 0$ and let $\delta > 0$. It is enough to show that there is a $\widetilde{V}_t : t \geq 0$ so that

$$\int_0^\infty \widetilde{V}_t\,dt = \infty \quad \text{a.s. } P_x \text{ and}$$

$$J_x^\varepsilon(\widetilde{V}) \leq J_x^\varepsilon(V) + \delta$$

By (4.16)

$$J_x(V) \leq E_x\left[\int_0^\infty \overline{G}(x_t)\,dt\right] \; (\|f\| + 1/\varepsilon\|\psi\|) < \infty$$

there is a T_o so if $T \geq T_o$.

$$(5.6) \qquad E_x\left[\int_T^\infty e^{-1/\varepsilon\int_0^t V_s ds}(\overline{G}(x_t)f(x_t) + 1/\varepsilon\, V_t\, \overline{G}(x_t)\psi(x_t)dt\right] < \delta/2$$

Not that (4.17) implies that

$$\lim_{t\to\infty} E_x[\overline{G}(x_t)] = 0$$

and so there is a T_1 so that if $T \geq T_1$

$$E_x[\overline{G}(x_T)] < \delta/2\|\psi\|.$$

For $T \geq T_o \vee T_1$, define

$$V_t = \begin{cases} V_t & t \leq T \\ 1 & t > T \end{cases}$$

Now

$$J_x^\varepsilon(\widetilde{V}) = E_x\left[\int_0^T e^{-1/\varepsilon\int_0^t \widetilde{V}_s ds}\,\overline{G}(x_t)f(x_t) + 1/\varepsilon\widetilde{V}_t\overline{G}(x_t)\psi(x_t)dt\right]$$

$$+ E_x\left[\int_T^\infty e^{-1/\varepsilon\int_0^t \widetilde{V}_s ds}\,\overline{G}(x_t)f(x_t) + 1/\varepsilon\widetilde{V}_t\overline{G}(x_t)\psi(x_t)dt\right]$$

Moreover

$$E_x\left[\int_T^\infty e^{-1/\varepsilon\int_0^t \widehat{V}_s ds}\,\overline{G}(x_t)f(x_t)dt\right] \leqslant E_x\left[\int_T^\infty e^{-1/\varepsilon\int_0^t V_s ds}\,\overline{G}(x_t)f(x_t)dt\right]$$

and

$$E_x\left[\int_T^\infty e^{-1/\varepsilon\int_0^t \widehat{V}_s ds}\,1/\varepsilon\ \widehat{V}_t\overline{G}(x_t)\psi(x_t)dt\right] \leqslant \|\psi\|\ E_x\left[e^{-1/\varepsilon\int_0^T V_s ds}\,\overline{G}(x_t)\ < \delta/2\right]$$

Also by (5.6)

$$E_x\left[\int_T^\infty e^{-1/\varepsilon\int_0^t V_s ds}\ 1/\varepsilon V_t\overline{G}(x_t)\psi(x_t)dt\right] < \delta/2$$

and thus

$$J_x^\varepsilon(\widehat{V}) \leqslant J_x^\varepsilon(V) + \delta.$$

To show (i), note that if $P_x\left(\int_0^\infty V_t dt < \infty\right) > 0$ then (5.1) and (5.4) imply

$$J_x^{\varepsilon,n}(V) \geqslant \beta/n\ \ E_x\left[\int_0^\infty e^{-1/\varepsilon\int_0^t V_s d_s}\,dt\right] = \infty$$

Also for $V_t \equiv 1$

$$J_x^{\varepsilon,n}(V) \leqslant \varepsilon/n\ \|f\| + \|\psi\| < \infty$$

and hence

$$u_{\varepsilon,n}(x) = \inf\left\{J_x^{\varepsilon,n}(V):\ \int_0^\infty V_t dt = \infty \quad \text{a.s. } P_x\right\}$$

Now note that if $P_x\left(\int_0^\infty V_t dt = \infty\right) = 1$ then

$$(5.7)\qquad J_x^{\varepsilon,n}(V) \downarrow J_x(V)$$

and so $u_{\varepsilon,n}(x) \downarrow \widetilde{u}_\varepsilon(x)$ where $\widetilde{u}_\varepsilon(x) \geqslant u_\varepsilon(x)$.

Since

$$u_{\varepsilon,n}(x) \leqslant J_x^{\varepsilon,n}(V)$$

and for $V_t : t \geq 0$ with $P_x\left(\int_0^\infty V_t dt = \infty\right) = 1$ by (5.7)

we have $\tilde{u}_\varepsilon(x) = u_\varepsilon(x)$. That $u_{\varepsilon,n} \in C_b(R^+)$ follows since Theorem 5.2 (iv) applies.

For (ii) first note that $J_x^\varepsilon(V) < \infty$ for all $V_t : t \geq 0$ and

$$J_x^{\varepsilon,\alpha}(V) \uparrow J_x^\varepsilon(V) \qquad \text{as} \quad \alpha \to 0. \quad \text{Hence}$$

$$u_{\varepsilon,\alpha}(x) \uparrow \tilde{u}_\varepsilon(x) \qquad \tilde{u}_\varepsilon(x) \leq u_\varepsilon(x).$$

By taking $V_t \equiv 1$ we see that

$$\tilde{u}_\varepsilon(x) \leq u_\varepsilon(x) \leq \bar{G}(x) E_x\left[\int_0^\infty e^{-1/\varepsilon t}(f(x_t) + 1/\varepsilon\ \psi(x_t))dt\right]$$

$$\leq \bar{G}(x)\ (\varepsilon\|f\| + \|\psi\|)$$

and so

$$(5.8) \qquad \lim_{x\to\infty} \tilde{u}^\varepsilon(x) = 0.$$

By Theorem 5.1

$$u_{\varepsilon,\alpha}(x) = e^{-\alpha t} T_t u_{\varepsilon,\alpha}(x) + \int_0^t e^{-\alpha s} T_s(\bar{G}f - 1/\varepsilon(u_{\varepsilon,\alpha} - \bar{G}\psi)^+)(x)ds$$

and so

$$\tilde{u}_\varepsilon(x) = T_t \tilde{u}_\varepsilon(x) + \int_0^t T_s(\bar{G}f - 1/\varepsilon(\tilde{u}_{\varepsilon,\alpha} - \bar{G}\psi)^+)(x)ds$$

For any $V_t : t \geq 0$ integrating by parts yields

$$e^{-1/\varepsilon\int_0^t V_s ds}\tilde{u}_\varepsilon(x_t) + \int_0^t e^{-1/\varepsilon\int_0^s V_u du}\bar{G}(x_s)f(x_s) + 1/\varepsilon V_s \tilde{u}_\varepsilon(x_s)$$

$$- 1/\varepsilon(\tilde{u}_\varepsilon - \bar{G}\psi)^+(x_s)ds$$

is a martingale. Taking

$$\hat{V}_t = \begin{cases} 0 & \breve{u}_\varepsilon(x_t) < \overline{G}(x_t)\psi(x_t) \\ 1 & \breve{u}_\varepsilon(x_t) \geqslant \overline{G}(x_t)\psi(x_t) \end{cases}$$

yields

$$\breve{u}_\varepsilon(x) = E_x\left[e^{-1/\varepsilon\int_0^t \hat{V}_s ds}\,\breve{u}_\varepsilon(x_t) + \int_0^t e^{-1/\varepsilon\int_0^s \hat{V}_u du}\,\overline{G}(x_s)f(x_s) + 1/\varepsilon\hat{V}_s\overline{G}(x_s)\psi(x_s)ds\right]$$

and by (4.17) and (5.8) it follows by letting $t \to \infty$ that

$$\breve{u}_\varepsilon(x) = J_x(\hat{V}) \quad \text{and so} \quad \breve{u}_\varepsilon(x) = u(x).$$

What is left to prove in (iii) is to show the uniqueness. If $w_\varepsilon \in C_b(R^+)$ is any non-negative solution of

$$w_\varepsilon = T_t w_\varepsilon + \int_0^t T_s(\overline{G}f - 1/\varepsilon(w_\varepsilon - \overline{G}\psi)^+)ds$$

then as above for any $V_t : t \geqslant 0$

$$w_\varepsilon(x) = E_x\left[e^{-1/\varepsilon\int_0^t V_s ds}\,w_\varepsilon(x_t) + \int_0^t e^{-1/\varepsilon\int_0^s V_u du}\,\overline{G}(x_s)f(x_s) + 1/\varepsilon V_s w_\varepsilon(x_s) - 1/\varepsilon(w_\varepsilon - \overline{G}\psi)^+(x_s)ds\right] \tag{5.9}$$

Taking $V_t \equiv 1$ yields

$$0 \leqslant w_\varepsilon(x) \leqslant E_x\left[e^{-t/\varepsilon}\,w_\varepsilon(x_t) + \int_0^t e^{-s/\varepsilon}\,\overline{G}(x_s)f(x_s) + 1/\varepsilon\overline{G}(x_s)\psi(x_s)ds\right]$$

and letting $t \to \infty$ we obtain

$$0 \leqslant w_\varepsilon(x) \leqslant E_x\left[\int_0^\infty e^{-t/\varepsilon}\,\overline{G}(x_t)f(x_t) + 1/\varepsilon\overline{G}(x_t)\psi(x_t)dt\right]$$

$$\leqslant \overline{G}(x)(\varepsilon\|f\| + \|\psi\|).$$

Thus $\lim_{x \to \infty} w_\varepsilon(x) = 0$.

From (5.9) it follows for any V

$$w_\varepsilon(x) \leq E_x\left[e^{-1/\varepsilon\int_0^t V_s ds} w_\varepsilon(x_t) + \int_0^t e^{-1/\varepsilon\int_0^s V_u du}\overline{G}(x_s)f(x_s) + 1/\varepsilon V_s \overline{G}(x_s)\psi(x_s)ds\right]$$

and letting $t \to \infty$ yields by (4.17)

$$w_\varepsilon(x) \leq J_x^\varepsilon(V).$$

Lastly letting

$$\hat{V}_t = \begin{cases} 0 & w_\varepsilon(x_t) < \overline{G}(x_t)\psi(x_t) \\ 1 & w_\varepsilon(x_t) \geq \overline{G}(x_t)\psi(x_t) \end{cases}$$

it follows that

$$w_\varepsilon(x) = J_x(\hat{V}) \quad \text{and we have uniqueness.}$$

<u>Remark 5.2</u>. Note that Lemmas 5.4 and 5.5 are valid for the case when $\overline{G}(x)\psi(x)$ is replaced by

$$(5.10) \qquad \psi_\beta(x) = E_x\left[\int_0^\infty e^{-\beta s}\overline{G}(x_s)\psi(x_s)ds\right]$$

since

$$\psi_\beta(x) \leq \overline{G}(x)\|\psi\|.$$

<u>Lemma 5.6</u>. $u_\varepsilon(x) \downarrow u(x)$ as $\varepsilon \to 0$.

<u>Proof</u>. An argument due to Menaldi (see [3] or [4]) shows that for $\alpha > 0$

$$u_{\varepsilon,\alpha} \downarrow \quad \text{as} \quad \varepsilon \to 0.$$

Hence Lemma 5.5 yields $u_\varepsilon \downarrow$ as $\varepsilon \to 0$.

Define

$$\tau_\varepsilon = \inf\{t : u_\varepsilon(x_t) \geqslant \psi(x_s)\bar{G}(x_t)\}.$$

Lemma 5.5 implies that

$$u_\varepsilon(x) = E_x\left[u_\varepsilon(x_{t\wedge\tau_\varepsilon}) + \int_0^{t\wedge\tau_\varepsilon} \bar{G}(x_s)f(x_s)ds\right].$$

Since on $\tau_\varepsilon = \infty$

$u_\varepsilon(x_t) < \bar{G}(x_t)\psi(x_t)$, we have

$$\lim_{t\to\infty} u_\varepsilon(x_{t\wedge\tau_\varepsilon}) = 0.$$

Because of quasi-left continuity cn $(\tau < \infty)$

$$\lim_{t\to\infty} u_\varepsilon(x_{t\wedge\tau_\varepsilon}) = u_\varepsilon(x_{\tau_\varepsilon}) \geqslant \bar{G}(x_{\tau_\varepsilon})\psi(x_{\tau_\varepsilon})$$

and we have

$$u_\varepsilon(x) \geqslant J_x(\tau_\varepsilon) \geqslant u(x).$$

Let $\psi_\beta(x)$ be as in (5.10). Define $w_\varepsilon(x)$ and $w(x)$ by substituting ψ_β in place of $\bar{G}\psi$ in the definitions of u_ε and u. The above proof again yields $w_\varepsilon \geqslant w$.

Suppose $E_x(\tau) < \infty$ and define

$$V_t^\tau = \begin{cases} 0 & t < \tau \\ 1 & t \geqslant \tau \end{cases}$$

Then for w_ε and w

$$J_x^\varepsilon(V^\tau) - J_x^\varepsilon(\tau) = E_x\left[\int_\tau^\infty e^{-1/\varepsilon(s-\tau)}(\bar{G}(x_s)f(x_s) + 1/\varepsilon\,\psi_\beta(x_s))ds - \psi_\beta(x_\tau)\right]$$

By the Markov property since $\psi_\beta \varepsilon D_A$, it follows that

$$\psi_\beta(x_{\tau+t})e^{-t/\varepsilon} - \int_0^t e^{-s/\varepsilon} (A\psi_\beta - 1/\varepsilon\,\psi_\beta)(x_{\tau+t})ds$$

is a martingale with respect to $G_s = F_{s+\tau}$. Hence

$$-E_x[\psi_\beta(x_\tau)] = E_x\left[\int_0^\infty e^{-s/\varepsilon} (A\psi_\beta - 1/\varepsilon\psi_\beta)(x_{\tau+s})ds\right]$$

$$= E_x\left[\int_0^\infty e^{-s-\tau/\varepsilon} (A\psi_\beta - 1/\varepsilon\psi_\beta)(x_s)ds\right]$$

Hence

$$J_x^\varepsilon(V^\tau) - J_x^\varepsilon(\tau) = E_x\left[\int_\tau^\infty e^{-s-\tau/\varepsilon} \overline{G}(x_s)f(x_s) - A\psi_\beta(x_s)ds\right]$$

$$\leqslant \varepsilon\|f - A\psi_\beta\|$$

For the same reasons as in Lemma 5.4

$$w(x) = \inf\{J_x(\tau) : E_x[\tau] < \infty\}$$

and so

$$w \leqslant w_\varepsilon \leqslant w + \varepsilon\|f - A\psi_\beta\|$$

showing $w_\varepsilon \downarrow w$.

Lastly since

$$\|w - u\| \leqslant \|\psi_\beta - \overline{G}\psi\|$$

and

$\|w_\varepsilon - u_\varepsilon\| \leqslant \|\psi_\beta - \overline{G}\psi\|$. It is a standard fact from semi-group theory that

$\lim_{\alpha\to\infty} \|\psi_\beta - \overline{G}\psi\| = 0$ and we have $u_\varepsilon \downarrow u$ as $\varepsilon \to 0$.

<u>Lemma 5.7</u>. $u_\alpha \downarrow u$.

<u>Proof</u>. Let w_α be given by

$$w_\alpha(x) = \inf_\tau J_x^\alpha(\tau)$$

where

$$J_x^\alpha(\tau) = E_x\left[\int_0^\tau e^{-\alpha t}\,\overline{G}(x_t)f(x_t)dt + e^{-\alpha\tau}\psi_\beta(x_\tau)\right]$$

and $w_{\varepsilon,\alpha}(x)$ be defined as $u_{\varepsilon,\alpha}$ but with ψ_β replacing $\overline{G}\psi$. Then it follows as in Lemma 5.6 that $\|w_{\varepsilon,\alpha} - w_\alpha\| \leq \varepsilon\|f - A\psi_\beta\|$.

Hence letting w_ε and w be as in Lemma 5.6,

$$|w_\alpha(x) - w(x)| \leq \|w_\alpha - w_{\alpha,\varepsilon}\| + |w_{\alpha,\varepsilon}(x) - w_\varepsilon(x)| + \|w_\varepsilon - w\|$$

$$\leq 2\,\|\psi_\beta - \overline{G}\psi\| + |w_\alpha(x) - w(x)|$$

Letting $\alpha \to 0$ and then $\beta \to \infty$ shows $u_\alpha(x) \to u(x)$. That $u_\alpha(x) \uparrow u(x)$ follows since $J_x^\alpha(\tau) \uparrow J_x(\tau)$.

<u>Proof of Theorem 5.3</u>. That u is in the set of solution in (i) follows since Lemmas 5.4 and 5.7 show $u \in C_b(R^+)$ and letting $\alpha \to 0$ in Theorem 5.1 (i) yields that

$$u \leq T_t u + \int_0^t T_s(\overline{G}f)ds \qquad u \leq \overline{G}\psi.$$

It is standard that if $h(x)$ is any other solution then

$h(x) \leq J_x(\tau)$ for any stopping time τ. Hence u is the maximal solution.

What remains to prove is that

$$\hat{\tau} = \inf\{t : u(x_t) = \overline{G}(x_t)\psi(x_t)\} \quad \text{is optimal.}$$

Let $B = \{x : u(x) = \psi(x)\overline{G}(x)\}$. If $x \in B$, $\hat{\tau} \equiv 0$ and there is nothing to prove. If $x \in B$, find δ so that

$$u(x) + \delta < \psi(x)\overline{G}(x).$$

Define

$$\tau_R = \inf\{t : |x_t - x| \geq R\}$$

and

$$\tau_\delta = \inf\{t : u(x_t) \geq \overline{G}(x_t)\psi(x_t) - \delta\}$$

Since $u_\varepsilon \to u$ uniformly on compact sets choose ε_δ so for $\varepsilon < \varepsilon_\delta$

$$\sup_{|x-y| \leq R} |u_\varepsilon(y) - u(y)| < \delta/2.$$

For $s \in [0, \tau_\delta \wedge \tau_R)$

$$u_\varepsilon(x_s) \leq u(x_s) + \delta/2 \leq \overline{G}(x_s)\psi(x_s) - \delta/2$$

and so Lemma 5.5 says

$$u_\varepsilon(x) = E_x\left[u_\varepsilon(x_{\tau_\delta \wedge \tau_R}) + \int_0^{\tau_\delta \wedge \tau_R} \overline{G}(x_s)f(x_s)ds\right]$$

Letting $\varepsilon \to 0$

$$u(x) = E_x\left[u(x_{\tau_\delta \wedge \tau_R}) + \int_0^{\tau_\delta \wedge \tau_R} \overline{G}(x_s)f(x_s)ds.\right]$$

Again since $u = 0$ at ∞ and we have quasi-left continuity, we can let $R \to \infty$ yielding

$$(5.11) \qquad u(x) = E_x\left[u(x_{\tau_\delta}) + \int_0^{\tau_\delta} \overline{G}(x_s)f(x_s)ds\right].$$

Lastly note that $\tau_\delta \uparrow \hat{\tau}$. This is so since $\tau_\delta \uparrow \breve{\tau}$ as $\delta \to 0$ and by quasi-left continuity

$$x_{\tau_\delta} \to x_{\breve{\tau}} \quad \text{on } (\breve{\tau} < \infty)$$

Thus $u(x_{\breve{\tau}}) \geq \overline{G}(x_{\breve{\tau}})\psi(x_{\breve{\tau}})$ on $(\breve{\tau} < \infty)$ and since $\breve{\tau} \leq \hat{\tau}$, $\breve{\tau} = \hat{\tau}$ on $(\breve{\tau} < \infty)$. Therefore $\breve{\tau} = \hat{\tau}$ a.s. P_x. Therefore (5.11) becomes

$$u(x) = E_x\left[u(x_{\hat{\tau}}) + \int_0^{\hat{\tau}} \overline{G}(x_s)f(x_s)ds\right] = J_x(\hat{\tau}).$$

§6. The Long Run Average Cost Problem.

Let $V^{\alpha}(x)$ be as in Theorem 3.5. Recall that

$$(6.1)\qquad V^{\alpha}(x) = \inf_{\tau} E_x\Big[\int_0^{\tau} e^{-\alpha s}\overline{G}(x_s)f(x_s) + \alpha(V^{\alpha}(0)+c_0)G(x_s)ds$$
$$+ e^{-\alpha\tau}(V^{\alpha}(0)+g(x_{\tau})\overline{G}(x_{\tau})+c_0G(x_{\tau}))\Big]$$

Assume (4.8) and let

$$r(x) = \frac{AG(x)}{\overline{G}(x)} .$$

Since $x_t : t \geqslant 0$ and $G(x)$ are both non-decreasing, $r(x) \geqslant 0$. Define

$$(6.2)\qquad u_{\alpha}(x) = \inf_{\tau} E_x\Big[\int_0^{\tau} e^{-\alpha s}\overline{G}(x_s)(f(x_s)+c_0r(x_s)-\alpha V^{\alpha}(0))ds$$
$$+ e^{-\alpha\tau}\overline{G}(x_{\tau})g(x_{\tau})\Big]$$

By Dynkin's formula

$$G(x) = E_x\Big[e^{-\alpha\tau}G(x_{\tau}) - \int_0^{\tau} e^{-\alpha s}(AG(x_s)-\alpha G(x_s))ds\Big]$$

and

$$(1-e^{-\alpha\tau})V^{\alpha}(0) = \int_0^{\tau} \alpha\, e^{-\alpha s}ds\; V^{\alpha}(0)$$

Thus

$$(6.3)\qquad u_{\alpha}(x) = V^{\alpha}(x) + V^{\alpha}(0) - c_0G(x)$$

and if $\hat{\tau}_{\alpha}$ is optimal for V^{α} as given by Theorem 3.5, then it is also optimal for $u_{\alpha}(x)$ in (6.2).

Define

$$(6.4)\qquad \overline{V} = \frac{E_0\Big[\int_0^{\infty} \overline{G}(x_s)(f(x_s)+c_0r(x_s))ds\Big]}{E_0\Big[\int_0^{\infty}\overline{G}(x_s)ds\Big]}$$

Note that

$$0 = E_0\left[G(x_t) - \int_0^t AG(x_s)ds\right]$$

which yields by (4.17) by letting $t \to \infty$

$$1 = E_0\left[\int_0^\infty AG(x_s)ds\right] = E_0\left[\int_0^\infty \overline{G}(x_s)r(x_s)ds\right].$$

Hence

$$\overline{V} = \frac{E_0\left[\int_0^\infty \overline{G}(x_s)f(x_s)ds\right] + c_0}{E_0\left[\int_0^\infty \overline{G}(x_s)ds\right]} \qquad (6.5)$$

Lemma 6.1. $0 \leq \overline{\lim}\ \alpha V^\alpha(0) \leq \overline{V}$.

Proof. Let

$$\hat{H}(\alpha) = E_0\left[\int_0^\infty \alpha\, e^{-\alpha s} G(x_s)ds\right] = E_0\left[e^{-\alpha\sigma}\right]$$ where σ is given by (2.3). From (6.1), we obtain that

$$0 \leq V^\alpha(0) \leq E_0\left[\int_0^\infty e^{-\alpha s}\overline{G}(x_s)f(x_s)ds\right] + (V^\alpha(0) + c_0)\hat{H}(\alpha)$$

and so

$$0 \leq V^\alpha(0) \leq \frac{E_0\left[\int_0^\infty e^{-\alpha s}\overline{G}(x_s)f(x_s)ds\right] + c_0\,\hat{H}(\alpha)}{\frac{1-\hat{H}(\alpha)}{\alpha}}$$

Since $\lim_{\alpha \to 0} \frac{1 - H(\alpha)}{\alpha} = E_0(\sigma) = E_0\left[\int_0^\infty \overline{G}(x_s)ds\right]$,

$$0 \leq \overline{\lim}\ \alpha\ V^\alpha(0) \leq \overline{V}.$$

Let

$$v_\alpha(x) = \inf_\tau E_x \left[\int_0^\tau e^{-\alpha s}\overline{G}(x_s)(f(x_s) - V^\alpha(0)) + \alpha c_0 \, G(x_s)ds \right.$$

$$\left. + e^{-\alpha\tau}(g(x_\tau)\overline{G}(x_\tau) + c_0 \, G(x_\tau)) \right]$$

and

$$\overline{v}_0(x) = \frac{E_x\left[\int_0^\infty \overline{G}(x_s)(f(x_s) + c_0 r(x_s) - \overline{V})ds\right]}{\overline{G}(x)} .$$

Suppose $\rho = \sup \overline{v}_0(x)$

(6.6) $\overline{V} + \rho r(x) - \lambda \geqslant \beta > 0$

Theorem 6.1. Under (2.1) - (4.2), (4.8), (4.16), (4.17) and (6.6)

(i) $v_\alpha \to v$ uniformly on compact sets where

$$v(x) = \inf_\tau E_x \left[\int_0^\tau \overline{G}(x_s)(f(x_s) - \lambda)ds + g(x_\tau)\overline{G}(x_\tau) + c_0 G(x_\tau) \right]$$

(ii) $\hat{\tau} = \inf\{t : v(x_t) = \overline{G}(x_t)g(x_t) + c_0 G(x_t)\}$ is optimal,

(iii) $v(x)$ is the maximal element of the set of solutions h of $h \in C_b(R^+)$

$$h \leqslant T_t h + \int_0^t T_s((f-\lambda)\overline{G})ds \qquad h \leqslant \overline{G}g + c_0 G.$$

Moreover $v(0) = 0$.

(iv) $$\lambda = \frac{E_0\left[\int_0^{\hat{\tau}} \overline{G}(x_s)f(x_s)ds + \overline{G}(x_\tau)g(x_\tau) + c_0 G(x_\tau)\right]}{E_0\left[\int_0^{\hat{\tau}} \overline{G}(x_s)ds\right]}$$

and

$$\lambda = \inf_{\tau} \frac{E_0\left[\int_0^{\tau} \overline{G}(x_s)f(x_s)ds + \overline{G}(x_\tau)g(x_\tau) + c_0 G(x_\tau)\right]}{E_0\left[\int_0^{\tau} \overline{G}(x_s)ds\right]}$$

(v) $\lim_{\alpha \to 0} \alpha V^{\alpha}(0) = \lambda$

(vi) $$\lambda = \frac{E_0\left[\int_0^{\hat{\tau} \wedge \sigma} f(x_s)ds + g(x_\tau)I(\hat{\tau} < \sigma) + c_0\ I(\hat{\tau} \geq \sigma)\right]}{E_0[\hat{\tau} \wedge \sigma]}$$

and

$$\lambda = \inf_{\tau} \frac{E_0\left[\int_0^{\tau \wedge \sigma} f(x_s)ds + g(x_\tau)I(\tau < \sigma) + c_0 I(\tau \geq \sigma)\right]}{E_0[\tau \wedge \sigma].}$$

<u>Proof</u>. By Theorem 3.5 and (6.3)

$$v_\alpha(x) = V^{\alpha}(x) - V^{\alpha}(0) = u_\alpha(x) + c_0 G(x)$$

and

$$v(x) = u(x) + c_0 G(x)$$

where

$$u(x) = \inf_{\tau} E_x\left[\int_0^{\tau} \overline{G}(x_s)(f(x_s) + c_0 r(x) - \lambda)ds + \overline{G}(x_\tau)g(x_\tau)\right]$$

Let

$$\overline{v}_0(x) = \frac{E_x\left[\int_0^{\infty} \overline{G}(x_s)(f(x_s) + c_0 r(x_s) - \overline{V}))ds\right]}{\overline{G}(x)}$$

and $\overline{v}(x) = \overline{v}_0(x) - \rho$. Note that $\overline{v}(x) \leq 0$ and $\overline{v}(0) = -\rho \leq 0$. Define

$$w_\alpha(x) = u_\alpha(x) - \overline{G}(x)\overline{v}(x) \quad \text{and}$$

$$w(x) = u(x) - \overline{G}(x)\overline{v}(x).$$

By Theorems 4.1 and 4.4, $\bar{v}(x)$ is the unique solution of $-\hat{A}\bar{v} = f(x) + c_0 r(x) - \bar{V}$ with $\bar{v}(0) = -\rho$ and

$$-A\bar{v} = \frac{A(\bar{G}\,\bar{v})}{\bar{G}} + \bar{v}(0)\frac{AG}{\bar{G}} .$$

So

$$-A(\bar{G}\,\bar{v})(x) = \bar{G}(x)(f(x) + c_0 r(x) - \bar{V}) + v(0)AG(x)$$

Therefore

$$e^{-\alpha t}\bar{G}(x_t)\bar{v}(x_t) + \int_0^t e^{-\alpha s}\bar{G}(x_s)(f(x_s) + c_0 r(x_s) - \bar{V}) + \bar{v}(0)AG(x_s) - \alpha\bar{G}(x_s)\bar{v}(x_s)ds$$

and

$$\bar{G}(x_t)\bar{v}(x_t) + \int_0^t \bar{G}(x_s)(f(x_s) + c_0 r(x_s) - \bar{V}) + \bar{v}(0)\ AG(x_s)ds$$

are both martingales. Hence

$$w_\alpha(x) = \inf_\tau E_x\Big[\int_0^\tau e^{-\alpha\tau}\ \bar{G}(x_s)(\bar{V} + \rho\ r(x_s) - \alpha V^\alpha(0))ds + e^{-\alpha\tau}\bar{G}(x)(g(x_\tau) - \bar{v}(x_\tau))\Big]$$

By (4.17), $\lim_{t\to\infty} \bar{G}(x_t) = 0$ and by (2.2), $x_t : t \geqslant 0$ is quasi-left continuous and so for all stopping times τ,

$$E_x\Big[\int_0^\tau \bar{G}(x_s)(f(x_s) + c_0 r(x_s) - \lambda)ds + \bar{G}(x_\tau)g(x_\tau) \quad - \bar{G}(x)\bar{v}(x)\Big]$$

$$= E_x\Big[\int_0^\tau \bar{G}(x_s)(\bar{V} + \rho\ r(x_s) - \lambda) + \bar{G}(x_\tau)(g(x_\tau) - \bar{v}(x_\tau))\Big]$$

and so

$$w(x) = \inf_\tau E_x\Big[\int_0^\tau \bar{G}(x_s)(\bar{V} + \rho r(x_s) - \lambda)ds + \bar{G}(x_\tau)(g(x_\tau) - \bar{v}(x_\tau))\Big]$$

Define

$$\tilde{w}_\alpha(x) = \inf_\tau E_x\Big[\int_0^\tau e^{-\alpha s}\bar{G}(x_s)(\bar{V} + \rho r(x_s) - \lambda)ds + e^{-\alpha\tau}\ \bar{G}(x_\tau)(g(x_\tau) - \bar{v}(x_\tau))\Big]$$

Let $\alpha_n \to 0$ so that by Lemma 6.1,

$$\lim_n \alpha_n V^{\alpha_n}(0) = \lambda$$

By Theorem 5.3

$$\tilde{w}_{\alpha_n}(x) \to w(x) \quad \text{uniformly on compact sets.}$$

Now

$$|w_{\alpha_n}(x) - \tilde{w}_{\alpha_n}(x)| \leq (|\lambda - \alpha_n V^{\alpha_n}(0)| + \alpha_n \|\bar{v}\|) E_x\left[\int_0^\infty \bar{G}(x_s)ds\right]$$

Since (4.16) say

$$E_x\left[\int_0^\infty \bar{G}(x_s)ds\right] \leq C \quad \text{independent of } x$$

$$\lim_{n\to\infty} \|w_{\alpha_n} - \tilde{w}_{\alpha_n}\| = 0$$

Also

$$|v_{\alpha_n}(x) - v(x)| \leq |w_{\alpha_n}(x) - w(x)|$$

$$\leq \|\tilde{w}_{\alpha_n} - w_\alpha\| + |\tilde{w}_\alpha(x) - w(x)|$$

and so (i) follows for the sequence $\{\alpha_n\}^\infty$.

From Theorem 5.3

$$\hat{\tau} = \inf\{t : w(x_t) = \bar{G}(x_t)(g(x_t) - \bar{v}(x_t))\}$$

is optimal. It follows that

$$\hat{\tau} = \inf\{t : v(x_t) = \bar{G}(x_t)g(x_t) + c_o G(x_t)\}$$

is optimal for v. All we need notice is that

$$v(x) = w(x) + \bar{G}(x)\bar{v}(x) + c_o G(x).$$

Since $v_\alpha(0) = 0$, it follows that $v(0) = 0$ and

$$\lambda = \inf_{\tau} \frac{E_x\left[\int_0^\tau \bar{G}(x_s)f(x_s)ds + \bar{G}(x_\tau)g(x_\tau) + c_0G(x_\tau)\right]}{E_0\left[\int_0^\tau \bar{G}(x_s)ds\right]}$$

Thus if for any other subsequence $\alpha_n \to 0$ with $\alpha_n V^{\alpha_n}(0) \to \bar{\lambda}$, then repeating the above argument we see that $\bar{\lambda} = \lambda$. Hence (i) and (v) follow. Now (ii) and (iv) are also done and (v) follows from Lemmas 3.2 and 3.3.

For (iii), we have from Theorem 5.4 that w is the maximal solution of

$$\ell \in C_b(R^+) \qquad \ell \leq T_t\ell + \int_0^t T_s(\bar{G}(\bar{V} + \rho r - \lambda))ds \qquad \ell \leq \bar{G}(g - \bar{v})$$

Since

$$\bar{G}\ \bar{v} = T_t\bar{G}\ \bar{v} - \int_0^t T_s(\bar{G}(f + c_0 r - \bar{V}) + \rho A G)ds$$

and

$$G = T_tG - \int_0^t T_s(AG)ds$$

$$= T_tG - \int_0^t T_s(\bar{G}r)ds$$

we see that

$$v(x) = w(x) + \bar{G}(x)\bar{v}(x) + c_0G(x)$$

$$\leq T_tw + \int_0^t T_s(\bar{G}(\bar{V} + \rho r - \lambda))ds$$

$$+ T_t\bar{G}\ \bar{v} - \int_0^t T_s(\bar{G}(f + c_0 r - \bar{V} - \rho r))ds$$

$$+ c_0T_tG - \int_0^t T_s(\bar{G}\ c_0\ r)ds$$

$$= T_tv + \int_0^t T_s(\bar{G}(f - \lambda))ds$$

and

$$v \leq \overline{G}g + c_0 G.$$

Now if h is any solution of (iv), then

$$h(x_t) + \int_0^t \overline{G}(x_s)(f(x_s) - \lambda)ds$$ is a submartingale and so for any

stopping time τ,

$$h(x) \leq E_x \left[h(x_{t\wedge\tau}) + \int_0^{t\wedge\tau} \overline{G}(x_s)(f(x_s) - \lambda)ds \right]$$

$$\leq E_x \left[\int_0^{t\wedge\tau} \overline{G}(x_s)(f(x_s) - \lambda)ds + \overline{G}(x_{t\wedge\tau})\, g(x_{t\wedge\tau}) + c_o G(x_{t\wedge\tau}) \right]$$

By (4.17) and (2.2) letting $t \to \infty$

$$h(x) \leq E_x \left[\int_0^{\tau} \overline{G}(x_s)(f(x_s) - \lambda)ds + \overline{G}(x_\tau)g(x_\tau) + c_o G(x_\tau) \right].$$

Since $\hat{\tau}$ is optimal

$$h(x) \leq v(x).$$

<u>Remark 6.1.</u> Recall that

$$\hat{\tau} = \inf\{t : v(x_t) = \overline{G}(x_t)g(x_t) + c_o G(x_t)\}.$$

Under the assumption $g(0) > 0$ since $v(0) = 0$ it follows that

$$P_0(\hat{\tau} > 0) = 1.$$

<u>Remark 6.2.</u> Since $\overline{v}_0(0) = 0$ it follows from Theorems 4.1 and 4.4 that for any stopping time τ,

$$0 = E_0 \left[\overline{G}(x_t)\overline{v}_0(x\ ) + \int_0^{\tau} \overline{G}(x_s)(f(x_s) + c_o r(x_s) - \overline{V})ds \right]$$

Theorem 6.2. Suppose for $U \subset R^+$ open and not containing a neighborhood of 0 that if $\tau = \inf\{t : x(t) \in U\}$, then $P_0(\tau < \infty) > 0$. Assume further that $G(x) < 1$ for all x. Under the conditions (2.1) - (2.4), (4.8) (4.16), (4.17) and $g(0) > 0$, if $\lambda < \bar{V}$ then $\{g(x) < \bar{v}_0(x)\} \neq \phi$. If $r(x) > \beta > 0$ and $\{g(x) < \bar{v}_0(x)\} \neq \phi$ then $\lambda < \bar{V}$.

Proof. Suppose $\lambda < \bar{V}$ and let $\hat{\tau}$ be optimal. Not that

(6.7) $$P_0(\hat{\tau} < \infty) > 0.$$

Now

(6.8) $$0 = E_0\left[\int_0^{\hat{\tau}} \bar{G}(x_s)(f(x_s) + c_0 r(x_s) - \lambda)ds + \bar{G}(x_{\hat{\tau}})g(x_{\hat{\tau}})\right]$$

because for any stopping time τ,

$$E_0[G(x_\tau) = E_0\left[\int_0^{\tau} \bar{G}(x_s)r(x_s)ds\right].$$

So by Remark 6.2 and (6.8)

$$0 = E_0\left[\int_0^{\hat{\tau}} \bar{G}(x_s)(\bar{v} - \lambda)ds + \bar{G}(x_{\hat{\tau}})(g(x_{\hat{\tau}}) - \bar{v}_0(x_{\hat{\tau}}))\right]$$

and by Remark 6.1 and (6.7)

$$P_0(0 < \hat{\tau} < \infty) > 0.$$

Hence

$$\{g(x) < \bar{v}_0(x)\} \neq \phi.$$

Suppose $r(x) > \beta > 0$ and $\{g(x) < \bar{v}_0(x)\} \neq \phi$. Since both g and $\bar{v}_0(x)$ are continuous, for $\delta > 0$ but small enough

$$\phi \neq \{g(x) + \delta < \bar{v}_0(x)\}$$ and does not contain a neighborhood of the origin. Define

$\tau = \inf\{t : g(x_t) + \delta < \overline{v}_0(x_t)\}$ and note then that $P_0(0 < \tau < \infty) > 0$. Also since $g(x) \geq 0$ and $\{\overline{v}_0(x) > g(x)\} \neq \phi$ then $\rho = \sup \overline{v}_0(x) > 0$ and Theorem 6.1 applies. Arguing as above

$$0 \leq E_0\left[\int_0^\tau \overline{G}(x_s)(\overline{V} - \lambda)ds + G(x_\tau)(g(x_\tau) - v_0(x_\tau))\right]$$

and so

$$0 < \delta \; E_0[\overline{G}(x_\tau)] \leq E_0\left[\int_0^\tau \overline{G}(x_s)(\overline{V} - \lambda)ds\right], \quad \text{that is } \lambda < \overline{V}.$$

Theorem 6.3.

(i) Under the conditions (2.1)-(2.4), (4.8), (4.16), (4.17) and $g(x) > 0$, if $\lambda = \overline{V}$ and $r(x) \geq \beta > 0$ then the do nothing policy is optimal.

(ii) Under the conditions (2.1)-(2.4), (4.8), (4.17) and if $\rho = \sup \overline{v}_0(x) = 0$ and $\lambda = \overline{V}$ then the do nothing policy is optimal.

Proof. For (i), note that Theorem 6.1 applies and since

$$\lambda = \overline{V} = \frac{E_0\left[\int_0^\infty \overline{G}(x_s)f(x_s)ds\right] + c_o}{E_0\left[\int_0^\infty \overline{G}(x_s)ds\right]}$$

the do nothing policy is optimal.

For (ii) if $\rho = \sup\overline{v}_0(x) = 0$ we have for any stopping time τ

$$E_0\left[\int_0^\tau \overline{G}(x_s)(f(x_s) - \lambda)ds + \overline{G}(x_\tau)g(x_\tau) + c_o G(x_\tau)\right]$$

$$= E_0\left[\int_0^\tau \overline{G}(x_s)(\overline{V} - \lambda)ds + \overline{G}(x_\tau)(g(x_\tau) - \overline{v}_0(x_\tau))\right] \geq 0$$

and since

$$E_0\left[\int_0^{\infty}\overline{G}(x_s)(f(x_s)-\lambda)ds\right]+c_o=0$$

again the do nothing policy is optimal.

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