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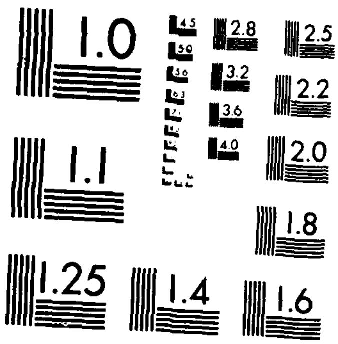

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VARIATIONAL PRINCIPLES FOR DYNAMICS OF LINEAR ELASTIC FLUID-SATURATED SOILS

Ranbir S. Sandhu and S. J. Hong
Department of Civil Engineering

For the Period
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Variational Principles for dynamics of the fluid-saturated porous media are derived assuming that solid is linear elastic and deformation is small. Starting with basic mathematical concepts related to the inverse problem of calculus of variation and following the methodology proposed by Sandhu for coupled problems, general variational principles for the problem are developed. Complementary as well as direct formulation are discussed with reference to finite element approximation. Discontinuities in the field variables, the approximation space and the excitation are allowed for. Extensions of the variational principles to relax smoothness requirements on certain field variables are introduced along with some specializations.

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VARIATIONAL PRINCIPLES FOR DYNAMICS OF LINEAR ELASTIC FLUID-SATURATED SOILS

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FOREWORD

The investigation reported herein is part of the research project at The Ohio State University, Columbus, Ohio supported by the Air Force Office of Scientific Research Grant 83-00-55. Lt. Col. Lawrence D. Hokanson is the Program Manager. The present report documents part of the work done from February 1, 1984 to January 31, 1985. At The Ohio State University, the project is supervised by Dr. Ranbir S. Sandhu, Professor, Department of Civil Engineering.

ABSTRACT

Variational Principles for dynamics of the fluid-saturated porous media are derived assuming that ~~solid~~ (soil) is linear elastic and deformation is small. Starting with basic mathematical concepts related to the inverse problem of calculus of variation and following the methodology proposed by Sandhu for coupled problems, general variatonal principles for the problem are developed. Complementary as well as direct formulation are discussed with reference to finite element approximation. Discontinuities in the field variables, the approximation space and the excitation are allowed for. Extensions of the variational principles to relax smoothness requirements on certain field variables are introduced along with some specializations.

TABLE OF CONTENTS

SECTION I

INTRODUCTION

Direct methods of obtaining approximations to solutions of boundary value problems often rely on variational formulations. Finite element procedures for numerical solution of various engineering problems are also based on variational principles. For linear operators many investigators studied the subject in an inner product space. Tonti [1] noted that the self-adjointness of an operator depends on the given bilinear map. A given operator may be non-self adjoint with respect to one bilinear mapping but self-adjoint with respect to another. Following Tonti, Magri [2] showed that for every linear operator there is an infinity of bilinear mappings with respect to which it is self-adjoint. Guntin [3,4] used convolution product as a bilinear mapping for initial-boundary value problems. Sandhu and Pister [5,6] extended the application of this bilinear mapping to linear coupled initial-boundary value problems using a generalization of Mikhlin's basic theorem [7].

Mikhlin [7] assumed homogeneous boundary conditions in stating the variational principle so that a self-adjoint operator was symmetric [10]. The conventional procedure to treat nonhomogeneous boundary conditions has been to find the particular solution for the nonhomogeneous boundary conditions and use change of the field variable to homogenize the boundary conditions. This approach, though theoretically elegant, is cumbersome in implementation of direct methods. Gurtin [3,4] introduced nonhomogeneous boundary terms explicitly into the governing function. Nickell and Sackman [8] and Sandhu and Pister [6] followed Gurtin's approach. In the context of application of finite element method, Prager [9] included, in the variational formulation, jump discontinuities which may exist across interelement boundaries. This development assumes special significance in case continuity of approximating functions cannot be ensured up to the desired degree. Sandhu and Salaam [10] examined the general case of linear operators with nonhomogeneous boundary conditions and internal jump discontinuities based on extension of Mikhlin's theorem. By introducing the concept of boundary operators consistent with the field operators, a systematic procedure to obtain variational principles for linear coupled problem was developed. In line with this approach, Sandhu [11] presented a comprehensive treatment on the variational principles for soil consolidation using convolution product as the bilinear mapping.

Ghaboussi and Wilson [12] derived a variational principle for the dynamic analysis of saturated porous elastic soil. Biot's equations of motion [14] were restated in integral form through Laplace transformation followed by rearrangement of terms and inversion. The general procedure followed Sandhu and Pister [13,14]. Herrera and Bielak [15] pointed out that the equivalent variational formulation could be written without transforming the field equations and, instead, using Tonti's approach [16]. Ghaboussi's [12] treatment of the boundary conditions was incomplete. The motivation for the present work stems from the need to write the boundary conditions in a consistent fashion and to develop a systematic procedure for variational formulation governing dynamics of fluid-saturated linear porous media. Both direct and complementary formulation of field equation are studied following Sandhu's approach. In chapter II, some mathematical concepts and definitions basic to the development of variational principles are introduced. Biot's field equations for dynamics of fluid-saturated linear elastic porous media are given in Chapter III. Integral form of the equations is the same as used by Ghaboussi and Wilson [12]. In Chapter IV, general variational principles for the problem are developed following Sandhu and Salaam [10] and Sandhu [11]. The governing functional for the operator equations explicitly includes the initial conditions, the nonhomogeneous boundary conditions as well as any internal jump discontinuities. As an alternative procedure, complementary formulation of the problem is presented. Extended variational formulations based on self-adjointness of the operator matrix are introduced along with several specializations. These should be useful as the starting points for alternative approachs to finite element formulation.

SECTION II

MATHEMATICAL PRELIMINARIES

2.1 BOUNDARY VALUE PROBLEM

Consider a boundary value problem

$$A(u) = f \quad \text{on } R \tag{1}$$

$$C(u) = g \quad \text{on } \partial R \tag{2}$$

where R is an open connected region in an Euclidean space. ∂R is the boundary of R and $\bar{R}$ its closure. We suppose that the field operator A and the boundary operator C are bounded and defined such that

$$A : W_R \longrightarrow V_R$$

$$C : W_{\partial R} \longrightarrow V_{\partial R}$$

V_R, $V_{\partial R}$ are linear vector spaces defined on the regions indicated by the subscripts and W_R, $W_{\partial R}$ are dense subsets in V_R, $V_{\partial R}$, respectively. Throughout, A and C are assumed to be linear so that

$$A(\alpha u + \beta v) = \alpha A(u) + \beta A(v) \quad \forall\, u,v \in W_R \tag{3}$$

and $$C(\alpha u + \beta v) = \alpha C(u) + \beta C(v) \quad \forall\, u,v \in W_{\partial R} \tag{4}$$

where α, β are arbitrary scalars.

2.2 BILINEAR MAPPING

A bilinear mapping $B : W \times V \longrightarrow S$, where W, V, S are linear vector spaces, for given $w \in W$, $v \in V$, is defined as a function to assign an element in S corresponding to an ordered pair (w,v). B is said to be bilinear if

$$B(\alpha w_1 + \beta w_2, v) = \alpha B(w_1, v) + \beta B(w_2, v) \quad (5)$$

$$B(w, \alpha v_1 + \beta v_2) = \alpha B(w, v_1) + \beta B(w, v_2) \quad (6)$$

where α, β are scalars. We shall use the notation

$$B_R(w,v) = \langle w,v \rangle_R \quad (7)$$

B is said to be nondegenerate if

$$\langle w,v \rangle_R = 0 \quad \forall\, w \in W \text{ if and only if } v = 0 \quad (8)$$

For W = V, B is symmetric if

$$\langle w,v \rangle_R = \langle v,w \rangle_R \quad (9)$$

2.3 SELF-ADJOINT OPERATOR

Let $A: V \longrightarrow W$ be an operator on the linear vector space V defined on spatial region R. Operator A^* is said to be adjoint of A with respect to a bilinear mapping $< , >_R : W \times W \longrightarrow S$ if

$$<w,Av>_R = <v,A^*w>_R + D_{\partial R}(v,w) \tag{10}$$

for all $w \in W$ and $v \in V$. Here, $D_{\partial R}(v,w)$ represents quantities associated with the boundary ∂R of R. If $A = A^*$, then A is said to be self-adjoint. In particular, a self-adjoint operator A on V is symmetric with respect to the bilinear mapping if V = W and

$$<w,Av>_R = <v,Aw>_R \tag{11}$$

2.4 GATEAUX DIFFERENTIAL OF A FUNCTION

The Gateaux differential of a continuous function $F : V \longrightarrow S$ is defined as

$$\Delta_v F(u) = \lim_{\lambda \to 0} \frac{1}{\lambda} [F(u+\lambda v) - F(u)] \tag{12}$$

provided the limit exists. v is referred to as the 'path' and λ is a scalar. We note that for $u,v \in V$, $u+\lambda v \in V$. Equation (12) can be equivalently written as

$$\Delta_v F(u) = \frac{d}{d\lambda} F(u+\lambda v)\bigg|_{\lambda=0} \tag{13}$$

2.5 BASIC VARIATIONAL PRINCIPLE

For the boundary value problem given by (1) with homogeneous boundary condition, Mikhlin [7] showed that for self-adjoint, positive definite operator A, the unique solution u_0 minimizes the functional

$$\Omega(u) = \langle Au,u\rangle_R - 2\langle u,f\rangle_R \tag{14}$$

where $\langle\ ,\ \rangle_R$ denotes inner product over the separable space of square integrable functions. Conversely, u_0 which minimizes the functional (14) is the solution of the problem (1).

Taking Gateaux differential of (14),

$$\Delta_v\Omega(u) = \lim_{\lambda\to 0} \frac{1}{\lambda}[\langle A(u+\lambda v),u+\lambda v\rangle - 2\langle u+\lambda v,f\rangle - \langle Au,u\rangle + 2\langle u,f\rangle]$$

$$= \langle Au,v\rangle + \langle Av,u\rangle - 2\langle v,f\rangle$$

$$= 2\langle v,Au-f\rangle = 0 \tag{15}$$

In writing (15) we have only assumed linearity and self-adjointness of A with respect to the bilinear mapping and the symmetry of the bilinear mapping. The Gateaux differential evidently vanishes at the solution u_0 such that $Au_0 - f = 0$. For the vanishing of the Gateaux differential at $u = u_0$ to imply $Au_0-f = 0$, the bilinear mapping has to be nondegenerate. To prove the minimization property, the bilinear mapping has to be into the real line and the operator must be positive. However, in general, it is only necessary to use vanishing of the Gateaux differential as equivalent to (1) being satisfied.

SECTION III

FIELD EQUATIONS FOR DYNAMICS OF FLUID-SATURATED POROUS SOILS

3.1 PRELIMINARIES

In this section, field equations for dynamics of fluid-saturated linear elastic porous media are stated following Biot [13,14]. The domain of definition of all functions is the cartesian product $R \times [0,\infty)$, where R is the closure of the spatial region R and $[0,\infty)$ is the positive time interval. The soil skeleton is assumed to be linear elastic. Integral form of the field equations is obtained by Laplace transformation of the Biot's equatins of motion followed by inversion. Throughout, standard indicial notation is used. The Latin indices take on range of values 1,2,3 and summation on repeated indices is implied. A superposed dot indicates time derivative.

3.2 DIFFERENTIAL FORM OF FIELD EQUATIONS

(a) Dynamic Equilibrium

Biot's equations of motion for the binary mixture of fluid and solid are [14];

$$\tau_{ij,j} + \rho b_i = \rho \ddot{u}_i + \rho_2 \ddot{w}_i \quad (16)$$

$$\pi_{,i} + \rho_2 b_i = \rho_2 \ddot{u}_i + (\rho_2/f)\ddot{w}_i + (1/k)\dot{w}_i \quad (17)$$

where τ_{ij}, b_i, u_i, w_i are, respectively, components of the total stress tensor, the body force vector per unit volume of the mixture, the solid displacement vector and the relative displacement of the fluid with respect to the solid. π is the pore pressure. ρ is the mixture density and ρ_2 is the mass of fluid per unit volume of the mixture.

(b) Kinematics

For small displacement, the strain-displacement relations are;

$$e_{ij} = u_{(i,j)} = 1/2\ (u_{i,j} + u_{j,i}) \tag{18}$$

$$\xi = w_{i,i} \tag{19}$$

where e_{ij} are components of the symmetric strain tensor of solid and ξ is the rate of volume change of the fluid per unit volume of the solid.

(c) Constitutive Equations

For linear elastic fluid-saturated soil, Biot [14] proposed the constitutive equations

$$\tau_{ij} = E_{ijkl}\ e_{kl} + \alpha M\ \delta_{ij}\ (\alpha\,\delta_{kl}\ e_{kl} + \xi) \tag{20}$$

$$\pi = M\ (\alpha\ \delta_{ij}\ e_{ij} + \xi) \tag{21}$$

The inverse relationships are;

$$e_{ij} = C_{ijkl}\ (\ \delta_{kl} - \alpha\pi\delta_{kl}) \tag{22}$$

$$\xi = \pi(1/M + \alpha^2\ C_{ijkl}\ \delta_{kl}\ \delta_{ij}) - \alpha\ C_{ijkl}\ \delta_{kl}\ \tau_{ij} \tag{23}$$

Here, E_{ijkl} and C_{ijkl} are, respectively, components of the elasticity and compliance tensors of the elastic solid, α is the compressibility of the solid and M that of the fluid.

(d) Boundary Conditions

$$
\begin{aligned}
u_i(\underline{x},t) &= \hat{u}_i(\underline{x},t) && \text{on } S_1 \times [0,\infty) \\
w_i(\underline{x},t) &= \hat{w}_i(\underline{x},t) && \text{on } S_2 \times [0,\infty) \\
\tau_{ij}(\underline{x},t)n_j &= T_i(\underline{x},t) = \hat{T}_i(\underline{x},t) && \text{on } S_3 \times [0,\infty) \\
\pi(\underline{x},t)n_i &= \hat{\pi}_i(\underline{x},t) && \text{on } S_4 \times [0,\infty)
\end{aligned}
\tag{24}
$$

The first two in (24) are the displacement boundary conditions and the last two the traction boundary conditions. n_i are the components of the unit outward normal to ∂R and $\hat{T}_i$ are components of the prescribed traction in i-th direction. Each of $\{S_1,S_3\}$ and $\{S_2,S_4\}$ consists of disjoint complementary subsets of ∂R.

(e) Initial Conditions

The initial conditions for the problem are

$$
\begin{aligned}
u(\underline{x},0) &= u_0(\underline{x}) \\
\dot{u}(\underline{x},0) &= \dot{u}_0(\underline{x}) \\
w(\underline{x},0) &= w_0(\underline{x})
\end{aligned}
\tag{25}
$$

$$\dot{w}(\underline{x},0) = \dot{w}_0(\underline{x})$$

The equations (16) through (25) completely define the initial-boundary value problem of small motion of fluid-saturated porous media.

3.3 INTEGRAL FORM OF THE FIELD EQUATIONS

For development of variational principles, we need to rewrite the field equations in the form of convolution product so that the time derivatives are avoided. This can be done through applying Laplace transform and taking inverse after appropriate rearrangement. Following the procedure originally suggested by Gurtin [3,4], Ghaboussi and Wilson [12] presented the following results for the Equations (16) through (25).

(a) Dynamic Equilibrium

Laplace transformation of (16) and (17) followed by inversion gives;

$$t * \tau_{ij,j} + F_i - \rho u_i - \rho_2 w_i = 0 \tag{26}$$

$$t * \pi_{,i} - 1 * (1/k) w_i + G_i - \rho_2 u_i - (\rho_2/f) w_i = 0 \tag{27}$$

where

$$F_i = t * \rho b_i + \rho[t\dot{u}_i(0) + u_i(0)] + \rho_2 [t\dot{w}_i(0) + w_i(0)] \tag{28}$$

$$G_i = t * \rho_2 b_i + \rho_2 [t\dot{u}_i(0) + u_i(0)] + (\rho_2/f) [t\dot{w}_i(0) + w_i(0)] + (1/k)\, t w_i(0) \tag{29}$$

Here, symbol * denotes the convolution product defined as

$$f * g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau \tag{30}$$

which satisfies the commutative law, the associative law and the distributive law.

(b) Kinematics

Equations (18) and (19) need to be restated in the form [11,12]

$$t * e_{ij} = (1/2)\, t * (u_{i,j} + u_{j,i}) \tag{31}$$

$$t * \xi = t * w_{i,i} \tag{32}$$

(c) Constitutive Equations

Equations (20) to (23) must be restated so that the constitutive relations show the dependence of quantities appearing in the equilibrium equations upon corresponding kinematic quantities in them.

$$t * \tau_{ij} = t * E_{ijkl}\, e_{kl} + t * \alpha M \delta_{ij} (\alpha \delta_{kl} e_{kl} + \xi) \tag{33}$$

$$t * \pi = t * M (\alpha \delta_{ij} e_{ij} + \xi) \tag{34}$$

$$t * e_{ij} = t * C_{ijkl} (\tau_{kl} - \alpha \pi \delta_{kl}) \tag{35}$$

$$t * \xi = t * \pi (1/M + \alpha^2 C_{ijkl}\, \delta_{kl}\, \delta_{ij}) - t * \alpha C_{ijkl}\, \delta_{kl}\, \tau_{ij} \tag{36}$$

SECTION IV

VARIATIONAL PRINCIPLES

4.1 PRELIMINARIES

To transform the coupled initial-boundary value problem of wave equations for the fluid-saturated porous media into an equivalent variational problem, the set of field variables are regarded as an n-tuple in the admissible space whose elements are defined in $R \times [0, \infty)$. A solution of the mixed problem is, then, an admissible state of the field variables which satisfies the field equation, the initial conditions and the boundary conditions to the problem. The linear vector space W consisting of all admissible states is referred to as the product space, i.e.

$$W = W_1 \times W_2 \times \ldots\ldots\ldots\ldots\ldots \times W_n \tag{37}$$

where w_i is an subspace whose elements represent the admissible state for an specific field variable u_i.

Consider operator equations of the coupled boundary value problem

$$A(u) = f \quad \text{on } R \tag{38}$$

$$C(u) = g \quad \text{on } \partial R \tag{39}$$

in which u is the vector of the field variables and A is an operator matrix which is self-adjoint in the sense of an appropriate bilinear mapping. Each element of the operator matrix A may be viewed as a transformation

$$A_{ij} : M_{ij} \longrightarrow P_i \tag{40}$$

where M_{ij}, P_i are, respectively, the domain and the range of A_{ij} which are both the linear vector spaces on R. The explicit form of the field equations is

$$\sum_j A_{ij} u_j = f_i \tag{41}$$

$$\sum_j C_{ij} u_j = g_i, \quad i = 1,2, \ldots\ldots, n \tag{42}$$

in which n is the number of independent field variables.

Consider a bilinear mapping

$$< , >_R : V_i \times V_i \longrightarrow S, \; i = 1,2,\ldots\ldots\ldots n \tag{43}$$

The matrix of operators is self-adjoint with respect to the bilinear mapping if [17,18]

$$\sum_j^n <u_j, A_{ji} u_i>_R = <u_i, \sum_j^n A_{ij} u_j>_R + D_{\partial R}(u_i, u_j) \; , \; i=1,2,\ldots n \tag{44}$$

where $D_{\partial R}(u_j,u_i)$ is a quantity associated with ∂R. As a generalization of Mikhlin's theorem, the governing function for the operator equation (38) and (39) is defined as

$$\Omega = \sum_i \sum_j \left(\langle u_i, A_{ij} u_j - 2f_i \rangle_R + \langle u_i, C_{ij} u_j - 2g_i \rangle_{\partial R} \right) \tag{45}$$

For the present problem, we use the bilinear mapping introduced by Gurtin viz.

$$\langle f, g \rangle_R = \int_R f * g \, dR \tag{46}$$

4.2 CONSISTENT BOUNDARY CONDITIONS AND INTERNAL DISCONTINUITY

Sandhu [10] pointed out that appropriate boundary terms should be included *in the governing function even if they are homogeneous.* This is important for certain approximation procedures, e.g. the finite element method, where the functions of limited smoothness are used. The boundary operators must be in a form consistent [10] with the field operator. Consider the boundary value problem of multi-variables given by (41) and (42). Referring to (44), Sandhu [17] defined consistency of boundary operators with the field operators to be the property;

$$D_{\partial R}(u_i, u_j) = \langle v_i, \sum_j^n C_{ij} u_j \rangle_{\partial R} - \sum_j^n \langle u_j, C_{ji} v_i \rangle_{\partial R} \ , \ i=1,2,\ldots,n \tag{47}$$

In seeking approximation to the exact solution by the finite element method, the function space with limited smoothness over the entire domain is sometimes used. In order to properly handle this limited smoothness problem in the variational formulation, Sandhu [11] introduced internal discontinuity conditions in the form;

$$(Cu)' = g \qquad \text{on } \partial R_i \tag{48}$$

where a prime denotes the internal jump discontinuity along element boundary ∂R_i embedded in the region R. Sandhu and Salaam [10] and Sandhu [11] showed that this condition can be included explicitly in the governing function.

4.3 VARIATIONAL PRINCIPLES FOR DYNAMICS OF FLUID-SATURATED SOILS

4.3.1 Field Equations

Equations (26) through (36) in self-adjoint matrix form are;

$$A(u) = f \qquad \text{on } R \times [0,\infty) \tag{49}$$

Here,

$$A = \begin{bmatrix} \rho & \rho_2 & 0 & -L & 0 & 0 \\ \rho_2 & \rho_2/f + 1^*(1/k) & -t^*\frac{\partial}{\partial m} & 0 & 0 & 0 \\ 0 & t^*\frac{\partial}{\partial m} & 0 & 0 & 0 & -t^* \\ L & 0 & 0 & 0 & -t^* & 0 \\ 0 & 0 & 0 & -t^* & P & t^*\alpha M\delta_{ij} \\ 0 & 0 & -t^* & 0 & t^*\alpha M\delta_{kl} & t^*M \end{bmatrix} \tag{50}$$

where

$$L = (1/2)\ t^* \left(\delta_{lm} \frac{\partial}{\partial k} + \delta_{km} \frac{\partial}{\partial l}\right) \tag{51}$$

$$P = t^* \left(E_{ijkl} + \alpha^2 M\, \delta_{ij}\, \delta_{kl}\right) \tag{52}$$

$$u = \begin{Bmatrix} u_m \\ w_m \\ \pi \\ \tau_{ij} \\ e_{kl} \\ \xi \end{Bmatrix} \quad \text{and} \quad f = \begin{Bmatrix} F_m \\ G_m \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{53}$$

Elements of A satisfy self-adjointness in the sense of Equation (44). The operators on the diagonal are symmetric and the off-diagonal operators constitute adjoint pairs with respect to the bilinear mapping (46). Consistent boundary conditions for the Equations (49) are

$$\begin{aligned} -t^* u_i n_j &= -t^* \hat{u}_i n_j && \text{on } S_1 \times [0,\infty) \\ -t^* w_i n_i &= -t^* \hat{w}_i n_i && \text{on } S_2 \times [0,\infty) \\ t^* \pi n_i &= t^* \hat{\pi} n_i && \text{on } S_3 \times [0,\infty) \\ t^* \tau_{ij} n_j &= t^* \hat{T}_i && \text{on } S_4 \times [0,\infty) \end{aligned} \tag{54}$$

Consistent form of the internal jump discontinuities is

$$-t^* (u_i n_j)' = -t^* (g_1)_i n_j \quad \text{on } S_{1i} \times [0,\infty)$$

$$-t^* (w_i n_i)' = -t^* g_2 \quad \text{on } S_{2i} \times [0,\infty)$$

$$t^* (\pi n_i)' = t^* g_3 n_i \quad \text{on } S_{3i} \times [0,\infty) \qquad (55)$$

$$t^* (\tau_{ij} n_j)' = t^* g_4 n_i \quad \text{on } S_{4i} \times [0,\infty)$$

Here, surfaces S_{1i}, S_{2i}, S_{3i} and S_{4i} are embedded in the interior of R. Operators in the self-adjoint operator matrix equation (49) have the following relationships;

$$\langle t^* u_{i,j}, \tau_{ij} \rangle_R = - \langle t^* u_i, \tau_{ij,j} \rangle_R$$

$$+ \langle t^* u_i n_j, \tau_{ij} \rangle_{S_1} + \langle t^* u_i, \tau_{ij} n_j \rangle_{S_4}$$

$$+ \langle t^* (u_i n_j)', \tau_{ij} \rangle_{S_{1i}} + \langle t^* u_i, (\tau_{ij} n_j)' \rangle_{S_{4i}} \qquad (56)$$

$$\langle t^* w_i, \pi_{,i} \rangle_R = - \langle t^* w_{i,i}, \pi \rangle_R$$

$$+ \langle t^* w_i n_i, \pi \rangle_{S_2} + \langle t^* w_i, \pi n_i \rangle_{S_3}$$

$$+ \langle t^* (w_i n_i)', \pi \rangle_{S_{2i}} + \langle t^* w_i, (\pi n_i)' \rangle_{S_{3i}} \qquad (57)$$

In writing (56) and (57) we assume that $\langle \, , \, \rangle_R$ can be evaluated as the sum of quantities evaluated over subregions of R such that all the surfaces S_{1i}, S_{2i}, S_{3i}, S_{4i} are contained in the union of the boundaries of these subregions.

4.3.2 A General Variational Principle

For the operator equations (49), we define the governing function, following (45) as;

$$
\begin{aligned}
\Omega(u) = {} & \langle \rho u_i, u_i \rangle_R + 2\langle \rho_2 w_i, u_i \rangle_R - \langle t^* \tau_{ij,j}, u_i \rangle_R + \langle (\frac{\rho_2}{f} + 1^* \frac{1}{k}) w_i, w_i \rangle_R \\
& - \langle t^* \pi_{,i}, w_i \rangle_R + \langle t^* w_{i,i}, \pi \rangle_R - 2\langle t^* \xi, \pi \rangle_R + \langle t^* u_{i,j}, \tau_{ij} \rangle_R \\
& - 2\langle t^* e_{ij}, \tau_{ij} \rangle_R + \langle t^* (E_{ijkl} + \alpha^2 M \delta_{ij} \delta_{kl}) e_{kl}, e_{ij} \rangle_R \\
& + 2\langle t^* \alpha M \delta_{ij} e_{ij}, \xi \rangle_R + \langle t^* M \xi, \xi \rangle_R \\
& - 2\langle u_i, F_i \rangle_R - 2\langle w_i, G_i \rangle_R \\
& - \langle \tau_{ij}, t^*(u_i - 2u_i) n_j \rangle_{S_1} - \langle \pi, t^*(w_i - 2w_i) n_i \rangle_{S_2} \\
& + \langle w_i, t^*(\pi - 2\pi) n_i \rangle_{S_3} + \langle u_i, t^*(\tau_{ij} n_j - 2T_i) \rangle_{S_4} \\
& - \langle \tau_{ij}, t^*((u_i n_j)' - 2(g_1)_i n_j) \rangle_{S_{1i}} - \langle \pi, t^*((w_i n_i) - 2g_2) \rangle_{S_{2i}} \\
& + \langle w_i, t^*((\pi n_i)' - 2g_3 n_i) \rangle_{S_{3i}} + \langle u_i, t^*((\tau_{ij} n_j)' - 2g_4 n_i) \rangle_{S_{4i}} \qquad (58)
\end{aligned}
$$

The Gateaux differential of this function along $v = \{\bar{u}_i, \bar{w}_i, \bar{\pi}, \bar{\tau}_{ij}, \bar{e}_{ij}, \bar{\xi}\}$ is;

$$
\begin{aligned}
\Delta_v \Omega(u) = {} & \langle \bar{u}_i, \rho u_i + \rho_2 w_i - t^* \tau_{ij,j} - 2F_i \rangle_R \\
& + \langle u_i, \rho \bar{u}_i + \rho_2 \bar{w}_i - t^* \bar{\tau}_{ij,j} \rangle_R \\
& + \langle \bar{w}_i, \rho_2 u_i + (\frac{\rho_2}{f} + 1^* \frac{1}{k}) w_i - t^* \pi_{,i} - 2G_i \rangle_R \\
& + \langle w_i, \rho_2 \bar{u}_i + (\frac{\rho_2}{f} + 1^* \frac{1}{k}) \bar{w}_i - t^* \bar{\pi}_{,i} \rangle_R
\end{aligned}
$$

$$+ \langle \bar{\pi}, t^* w_{i,i} - t^* \xi \rangle_R + \langle \pi, t^* \bar{w}_{i,i} - t^* \bar{\xi} \rangle_R$$

$$+ \langle \bar{\tau}_{ij}, t^* u_{i,j} - t^* e_{ij} \rangle_R + \langle \tau_{ij}, t^* \bar{u}_{i,j} - t^* \bar{e}_{ij} \rangle_R$$

$$+ \langle \bar{e}_{ij}, -t^* \tau_{ij} + t^*(E_{ijkl} + \alpha^2 M \delta_{ij} \delta_{kl}) e_{kl} + \alpha M \delta_{ij} \xi \rangle_R$$

$$+ \langle e_{ij}, -t^* \bar{\tau}_{ij} + t^*(E_{ijkl} + \alpha^2 M \delta_{ij} \delta_{kl}) \bar{e}_{kl} + \alpha M \delta_{ij} \bar{\xi} \rangle_R$$

$$+ \langle \bar{\xi}, -t^* \pi + t^* \alpha M \delta_{kl} e_{kl} + t^* M \xi \rangle_R$$

$$+ \langle \xi, -t^* \bar{\pi} + t^* \alpha M \delta_{kl} \bar{e}_{kl} + t^* M \bar{\xi} \rangle_R$$

$$- \langle \bar{\tau}_{ij}, t^*(u_i n_j - 2\hat{u}_i n_j) \rangle_{S_1} - \langle \tau_{ij}, t^* \bar{u}_i n_j \rangle_{S_1}$$

$$- \langle \bar{\pi}, t^*(w_i n_i - 2\hat{w}_i n_i) \rangle_{S_2} - \langle \pi, t^* \bar{w}_i n_i \rangle_{S_2}$$

$$+ \langle \bar{w}_i, t^*(\pi n_i - 2\hat{\pi} n_i) \rangle_{S_3} + \langle w_i, t^* \bar{\pi} n_i \rangle_{S_3}$$

$$\langle \bar{u}_i, t^*(\tau_{ij} n_j - 2\hat{T}_i) \rangle_{S_4} + \langle u_i, t^* \tau_{ij} n_j \rangle_{S_4}$$

$$- \langle \bar{\tau}_{ij}, t^*((u_i n_j)' - 2(g_1)_i n_j) \rangle_{S_{1i}} - \langle \tau_{ij}, t^*(\bar{u}_i n_j)' \rangle_{S_{1i}}$$

$$- \langle \bar{\pi}, t^*((w_i n_i)' - 2g_2 \rangle_{S_{2i}} - \langle \pi, t^*(\bar{w}_i n_i)' \rangle_{S_{2i}}$$

$$+ \langle \bar{w}_i, t^*((\pi n_i)' - 2g_3 n_i) \rangle_{S_{3i}} + \langle w_i, t^*(\bar{\pi} n_i)' \rangle_{S_{3i}}$$

$$+ \langle \bar{u}_i, t^*((\tau_{ij} n_j)' - 2(g_4) n_i) \rangle_{S_{4i}} + \langle u_i, t^*(\bar{\tau}_{ij} n_j)' \rangle_{S_{4i}} \quad (59)$$

Using equations (56) and (57), the Gateaux differential can be rewritten as;

$$\Delta_v \Omega(u) = 2\langle \bar{u}_i, \rho u_i + \rho_2 w_i - t^* \tau_{ij,j} - F_i \rangle_R$$

$$+ 2\langle \bar{w}_i, \rho u_i + (\frac{\rho_2}{f} + 1 * \frac{1}{k}) w_i - t * \pi_{,i} - G_i \rangle_R$$

$$+ 2\langle \bar{\pi}, t * w_{i,i} - t * \xi \rangle_R + 2\langle \mathcal{T}_{ij}, t * u_{i,j} - t * e_{ij} \rangle_R$$

$$+ 2\langle \bar{e}_{ij}, -t * \mathcal{T}_{ij} + t * (E_{ijkl} + \alpha^2 M \delta_{ij} \delta_{kl}) e_{kl} + \alpha M \delta_{ij} \xi \rangle_R$$

$$+ 2\langle \bar{\xi}, -t * \pi + t * \alpha M \delta_{kl} e_{kl} + t * M \xi \rangle_R$$

$$- 2\langle \bar{\mathcal{T}}_{ij}, t * (u_i n_j - u_i n_j) \rangle_{S_1}$$

$$- 2\langle \bar{\pi}, t * (w_i n_i - w_i n_i) \rangle_{S_2}$$

$$+ 2\langle \bar{w}_i, t * (\pi n_j - \pi n_i) \rangle_{S_3}$$

$$+ 2\langle \bar{u}_i, t * (\mathcal{T}_{ij} n_j - T_i) \rangle_{S_4}$$

$$- 2\langle \bar{\mathcal{T}}_{ij}, t * ((u_i n_j)' - (g_1)_i n_j) \rangle_{S_{1i}}$$

$$- 2\langle \bar{\pi}, t * ((w_i n_i)' - g_2 \rangle_{S_{2i}}$$

$$+ 2\langle \bar{w}_i, t * ((\pi n_i)' - g_3 n_i) \rangle_{S_{3i}}$$

$$+ 2\langle \bar{u}_i, t * ((\mathcal{T}_{ij} n_j)' - (g_4) n_i) \rangle_{S_{4i}} \qquad (60)$$

The Gateaux differential vanishes if and only if all the field equations along with the boundary conditions (54) and the jump conditions (55) are satisfied because of linearity and nondegeneracy of bilinear mapping (46). Hence, vanishing of $\Delta_v \Omega(v)$ for all $v \in W$ implies satisfaction of (49), (54) and (55).

4.3.3 Extended Variational Principles

Equations (56) and (57) relate pairs of operators in the operator matrix (49). These relations may be used to eliminate either of the elements in each pair of the function $\Omega(u)$ in (58). Eight alternative forms can be obtained by using either or both relations. Elimination of an operator A_{ij} from the function implies that state of variable u_j needs not be in the domain M_{ij} of A_{ij}. Where A_{ij} are differential operators, this results in relaxing the requirement of smoothness in u_j thereby extending the space of admissible states. In the context of finite element method, it is clear that the extension of the admissible space provides greater freedom in selection of approximation function. In the following, the possible extensions are explicitly stated.

Using (56) to eliminate $\tau_{ij,j}$ from (58),

$$
\begin{aligned}
\Omega_1 = {} & <\rho u_i, u_i>_R + 2<\rho_2 w_i, u_i>_R + <(\frac{\rho_2}{f} + 1*\frac{1}{k}) w_i, w_i>_R \\
& - <t*\pi_{,i}, w_i>_R + <t*w_{i,i}, \pi>_R - 2<t*\xi, \pi>_R + 2<t*u_{i,j}, \tau_{ij}>_R \\
& - 2<t*e_{ij}, \tau_{ij}>_R + <t*(E_{ijkl} + \alpha^2 M\,\delta_{ij}\,\delta_{kl})\,e_{kl}, e_{ij}>_R \\
& + 2<t*\alpha M\,\delta_{ij} e_{ij}, \xi>_R + <t*M\,\xi, \xi>_R \\
& - 2<u_i, F_i>_R - 2<w_i, G_i>_R \\
& - 2<\tau_{ij}, t*(u_i - \hat{u}_i) n_j>_{S_1} - <\pi, t*(w_i - 2\hat{w}_i) n_i>_{S_2} \\
& + <w_i, t*(\pi - 2\hat{\pi}) n_i>_{S_3} - 2<u_i, t*\hat{T}_i>_{S_4} \\
& - 2<\tau_{ij}, t*((u_i n_j)' - (g_1)_i n_j)>_{S_{1i}} - <\pi, t*((w_i n_i)' - 2g_2)>_{S_{2i}}
\end{aligned}
$$

$$+ \langle w_i, t^*((\pi n_i)' - 2g_3 n_i)\rangle_{S_{3i}} - 2\langle u_i, t^* g_4 n_i\rangle_{S_{4i}} \tag{61}$$

In (61) the stress components need not be differentiable. Using (56) to eliminate, alternatively, $u_{i,j}$, (58) gives

$$\Omega_2 = \langle \rho u_i, u_i\rangle_R + 2\langle \rho_2 w_i, u_i\rangle_R - 2\langle t^* \tau_{ij,j}, u_i\rangle_R + \langle(\frac{\rho_2}{f} + 1^* \frac{1}{k}) w_i, w_i\rangle_R$$
$$- \langle t^* \pi_{,i}, w_i\rangle_R + \langle t^* w_{i,i}, \pi\rangle_R - 2\langle t^* \xi, \pi\rangle_R$$
$$- 2\langle t^* e_{ij}, \tau_{ij}\rangle_R + \langle t^*(E_{ijkl} + \alpha^2 M \delta_{ij} \delta_{kl}) e_{kl}, e_{ij}\rangle_R$$
$$+ 2\langle t^* \alpha M \delta_{ij} e_{ij}, \xi\rangle_R + \langle t^* M \xi, \xi\rangle_R$$
$$- 2\langle u_i, F_i\rangle_R - 2\langle w_i, G_i\rangle_R$$
$$+ 2\langle \tau_{ij}, t^* \hat{u}_i n_j\rangle_{S_1} - \langle \pi, t^*(w_i - 2\hat{w}_i) n_i\rangle_{S_2}$$
$$+ \langle w_i, t^*(\pi - 2\hat{\pi}) n_i\rangle_{S_3} + 2\langle u_i, t^*(\tau_{ij} n_j - \hat{T}_i)\rangle_{S_4}$$
$$+ 2\langle \tau_{ij}, t^*(g_1)_i n_j\rangle_{S_{1i}} - \langle \pi, t^*((w_i n_i)' - 2g_2)\rangle_{S_{2i}}$$
$$+ \langle w_i, t^*((\pi n_i)' - 2g_3 n_i)\rangle_{S_{3i}} + 2\langle u_i, t^*((\tau_{ij} n_j)' - g_4 n_i)\rangle_{S_{4i}} \tag{62}$$

in which the displacement u_i need not be differentiable. Elimination of $w_{i,i}$ by (57) from (58) gives

$$\Omega_3 = \langle \rho u_i, u_i\rangle_R + 2\langle \rho_2 w_i, u_i\rangle_R - \langle t^* \tau_{ij,j}, u_i\rangle_R + \langle(\frac{\rho_2}{f} + 1^* \frac{1}{k}) w_i, w_i\rangle_R$$
$$- 2\langle t^* \pi_{,i}, w_i\rangle_R - 2\langle t^* \xi, \pi\rangle_R + \langle t^* u_{i,j}, \tau_{ij}\rangle_R$$
$$- 2\langle t^* e_{ij}, \tau_{ij}\rangle_R + \langle t^*(E_{ijkl} + \alpha^2 M \delta_{ij} \delta_{kl}) e_{kl}, e_{ij}\rangle_R$$

$$+ 2\langle t^*\alpha M\, \delta_{ij}e_{ij}, \xi\rangle_R + \langle t^*M\, \xi, \xi\rangle_R$$

$$- 2\langle u_i, F_i\rangle_R - 2\langle w_i, G_i\rangle_R$$

$$- \langle \tau_{ij}, t^*(u_i - 2\hat{u}_i)n_j\rangle_{S_1} + 2\langle \pi, t^*\hat{w}_i n_i\rangle_{S_2}$$

$$+ 2\langle w_i,\ t^*(\pi - \hat{\pi})n_i\rangle_{S_3} + \langle u_i,\ t^*(\tau_{ij}n_j - 2\hat{T}_i)\rangle_{S_4}$$

$$- \langle \tau_{ij}, t^*((u_in_j)' - 2(g_1)_in_j)\rangle_{S_{1i}} + 2\langle \pi, t^*g_2\rangle_{S_{2i}}$$

$$+ 2\langle w_i,\ t^*((\pi n_i)' - g_3 n_i)\rangle_{S_{3i}} + \langle u_i,\ t^*((\tau_{ij}n_j)' - 2g_4 n_i)\rangle_{S_{4i}} \qquad (63)$$

Here, w_i need not be differentiable. In the same way $\pi_{,i}$ can be dropped out by using (57), yielding

$$\Omega_4 = \langle \rho u_i, u_i\rangle_R + 2\langle \rho_2 w_i, u_i\rangle_R - \langle t^*\tau_{ij,j}, u_i\rangle_R + \langle (\frac{\rho_2}{f} + 1^*\frac{1}{k})w_i, w_i\rangle_R$$

$$+ 2\langle t^*w_{i,i}, \pi\rangle_R - 2\langle t^*\xi, \pi\rangle_R + \langle t^*u_{i,j}, \tau_{ij}\rangle_R$$

$$- 2\langle t^*e_{ij}, \tau_{ij}\rangle_R + \langle t^*(E_{ijkl} + \alpha^2 M\, \delta_{ij}\, \delta_{kl})\, e_{kl},\ e_{ij}\rangle_R$$

$$+ 2\langle t^*\alpha M\, \delta_{ij}e_{ij}, \xi\rangle_R + \langle t^*M\, \xi, \xi\rangle_R$$

$$- 2\langle u_i, F_i\rangle_R - 2\langle w_i, G_i\rangle_R$$

$$- \langle \tau_{ij}, t^*(u_i - 2\hat{u}_i)n_j\rangle_{S_1} - 2\langle \pi, t^*(w_i - \hat{w}_i)n_i\rangle_{S_2}$$

$$- 2\langle w_i,\ t^*\hat{\pi}n_i\rangle_{S_3} + \langle u_i,\ t^*(\tau_{ij}n_j - 2\hat{T}_i)\rangle_{S_4}$$

$$- \langle \tau_{ij}, t^*((u_in_j)' - 2(g_1)_in_j)\rangle_{S_{1i}} - 2\langle \pi, t^*((w_in_i)' - g_2)\rangle_{S_{2i}}$$

$$-2\langle w_i,\ t^*g_3 n_i\rangle_{S_{3i}} + \langle u_i,\ t^*((\tau_{ij}n_j)' - 2g_4 n_i)\rangle_{S_{4i}} \qquad (64)$$

In (64) the fluid pressure is not required to be differentiable. As can be seen from (61) to (64), both the differential operators in an adjoint pair cannot be removed at the same time. Use of (56) and (57), however, eliminates the differentiability of two field variables from (58). Eliminating $\tau_{ij,j}$ and $\pi_{,i}$ from (58), we have

$$\begin{aligned}
\Omega_5 = {} & \langle \rho u_i, u_i \rangle_R + 2\langle \rho_2 w_i, u_i \rangle_R + \langle (\frac{\rho_2}{f} + 1*\frac{1}{k}) w_i, w_i \rangle_R \\
& + 2\langle t*w_{i,i}, \pi \rangle_R - 2\langle t*\xi, \pi \rangle_R + 2\langle t*u_{i,j}, \tau_{ij} \rangle_R \\
& - 2\langle t*e_{ij}, \tau_{ij} \rangle_R + \langle t*(E_{ijkl} + \alpha^2 M\, \delta_{ij}\, \delta_{kl})\, e_{kl}, e_{ij} \rangle_R \\
& + 2\langle t*\alpha M\, \delta_{ij} e_{ij}, \xi \rangle_R + \langle t*M\, \xi, \xi \rangle_R \\
& - 2\langle u_i, F_i \rangle_R - 2\langle w_i, G_i \rangle_R \\
& - 2\langle \tau_{ij}, t*(u_i - \hat{u}_i) n_j \rangle_{S_1} - 2\langle \pi, t*(w_i - \hat{w}_i) n_i \rangle_{S_2} \\
& - 2\langle w_i,\ t*\hat{\pi} n_i \rangle_{S_3} - 2\langle u_i,\ t*\hat{T}_i) \rangle_{S_4} \\
& - 2\langle \tau_{ij}, t*((u_i n_j)' - (g_1)_i n_j) \rangle_{S_{1i}} - 2\langle \pi, t*((w_i n_i)' - g_2) \rangle_{S_{2i}} \\
& - 2\langle w_i,\ t*g_3 n_i \rangle_{S_{3i}} - 2\langle u_i,\ t*g_4 n_i) \rangle_{S_{4i}}
\end{aligned} \tag{65}$$

In (65), the total stress field and the fluid pressure need not be differentiable. Elimination of $u_{i,j}$ and $\pi_{,i}$ from (58) gives

$$\begin{aligned}
\Omega_6 = {} & \langle \rho u_i, u_i \rangle_R + 2\langle \rho_2 w_i, u_i \rangle_R - \langle t*\tau_{ij,j}, u_i \rangle_R + \langle (\frac{\rho_2}{f} + 1*\frac{1}{k}) w_i, w_i \rangle_R \\
& + \langle t*w_{i,i}, \pi \rangle_R - 2\langle t*\xi, \pi \rangle_R
\end{aligned}$$

$$- 2\langle t^{*}e_{ij}, \tau_{ij}\rangle_R + \langle t^{*}(E_{ijkl} + \alpha^2 M\, \delta_{ij}\, \delta_{kl})\, e_{kl},\, e_{ij}\rangle_R$$

$$+ 2\langle t^{*}\alpha M\, \delta_{ij}e_{ij}, \xi\rangle_R + \langle t^{*}M\, \xi, \xi\rangle_R$$

$$- 2\langle u_i, F_i\rangle_R - 2\langle w_i, G_i\rangle_R$$

$$+ 2\langle \tau_{ij}, t^{*}\hat{u}_i n_j\rangle_{S_1} - 2\langle \pi, t^{*}(w_i - \hat{w}_i)n_i\rangle_{S_2}$$

$$- 2\langle w_i,\ t^{*}\hat{\pi} n_i\rangle_{S_3} + 2\langle u_i,\ t^{*}(\tau_{ij}n_j - \hat{T}_i)\rangle_{S_4}$$

$$+ 2\langle \tau_{ij}, t^{*}(g_1)_i n_j\rangle_{S_{1i}} - 2\langle \pi, t^{*}((w_i n_i)' - g_2)\rangle_{S_{2i}}$$

$$- 2\langle w_i,\ t^{*}g_3 n_i\rangle_{S_{3i}} + 2\langle u_i,\ t^{*}((\tau_{ij}n_j)' - g_4 n_i)\rangle_{S_{4i}} \qquad (66)$$

where u_i and π need not be differentiable. Eliminating $\tau_{ij,j}$ and $w_{i,i}$ from (58),

$$\Omega_7 = \langle \rho u_i, u_i\rangle_R + 2\langle \rho_2 w_i, u_i\rangle_R + \langle (\frac{\rho_2}{f} + 1^{*}\frac{1}{k}) w_i, w_i\rangle_R$$

$$- 2\langle t^{*}\ \pi_{,i}, w_i\rangle_R - 2\langle t^{*}\xi, \pi\rangle_R + 2\langle t^{*}u_{i,j}, \tau_{ij}\rangle_R$$

$$- 2\langle t^{*}e_{ij}, \tau_{ij}\rangle_R + \langle t^{*}(E_{ijkl} + \alpha^2 M\, \delta_{ij}\, \delta_{kl})\, e_{kl},\, e_{ij}\rangle_R$$

$$+ 2\langle t^{*}\alpha M\, \delta_{ij}e_{ij}, \xi\rangle_R + \langle t^{*}M\, \xi, \xi\rangle_R$$

$$- 2\langle u_i, F_i\rangle_R - 2\langle w_i, G_i\rangle_R$$

$$- 2\langle \tau_{ij}, t^{*}(u_i - \hat{u}_i)n_j\rangle_{S_1} + 2\langle \pi, t^{*}\hat{w}_i n_i\rangle_{S_2}$$

$$+ 2\langle w_i,\ t^{*}(\pi - \hat{\pi})n_i\rangle_{S_3} - 2\langle u_i,\ t^{*}\hat{T}_i\rangle_{S_4}$$

$$- 2\langle \tau_{ij}, t^{*}((u_i n_j)' - (g_1)_i n_j)\rangle_{S_{1i}} + 2\langle \pi, t^{*}g_2\rangle_{S_{2i}}$$

$$+ 2\langle w_i,\ t^*((\pi n_i)'-g_3 n_i)\rangle_{S_{3i}} - 2\langle u_i,\ t^* g_4 n_i)\rangle_{S_{4i}} \quad (67)$$

which does not require the total stress and the relative displacement of fluid to be differentiable. Using (56) and (57) to eliminate $u_{i,j}$ and $w_{i,i}$, (58) is

$$\begin{aligned}
\Omega_8 = {} & \langle \rho u_i, u_i\rangle_R + 2\langle \rho_2 w_i, u_i\rangle_R - \langle t^* \tau_{ij,j}, u_i\rangle_R + \langle (\frac{\rho_2}{f} + 1^* \frac{1}{k}) w_i, w_i\rangle_R \\
& - \langle t^* \pi_{,i}, w_i\rangle_R - 2\langle t^* \xi, \pi\rangle_R \\
& - 2\langle t^* e_{ij}, \tau_{ij}\rangle_R + \langle t^*(E_{ijkl} + \alpha^2 M\, \delta_{ij}\, \delta_{kl})\, e_{kl},\ e_{ij}\rangle_R \\
& + 2\langle t^* \alpha M\, \delta_{ij} e_{ij}, \xi\rangle_R + \langle t^* M\, \xi, \xi\rangle_R \\
& - 2\langle u_i, F_i\rangle_R - 2\langle w_i, G_i\rangle_R \\
& + 2\langle \ _{ij}, t^* \hat{u}_i n_j\rangle_{S_1} + 2\langle \pi, t^* \hat{w}_i n_i\rangle_{S_2} \\
& + 2\langle w_i,\ t^*(\pi - \hat{\pi}) n_i\rangle_{S_3} \cdot 2\langle u_i,\ t^*(\tau_{ij} n_j - \hat{T}_i)\rangle_{S_4} \\
& + 2\langle \tau_{ij}, t^*(g_1)_i n_j\rangle_{S_{1i}} + 2\langle \pi, t^* g_2\rangle_{S_{2i}} \\
& + 2\langle w_i,\ t^*((\pi n_i)' - g_3 n_i\rangle_{S_{3i}} + 2\langle u_i,\ t^*((\tau_{ij} n_j)' - g_4 n_i)\rangle_{S_{4i}} \quad (68)
\end{aligned}$$

Here, the solid displacement and the relative displacement of fluid need not be differentiable.

4.3.3 Specializations

If the admissible state is constrained to satisfy some field equations and/or the boundary conditions, certain specialized forms of the variational principle are realized. This procedure is used to reduce the number of free variables in the governing function. Also, certain assumptions in the spatial or temporal variation of some of the variables lead to approximate theories. In the context of direct methods of approximation the constraints assumed in the specialization must be satisfied by admissible states. If it is difficult to satisfy the constraints, such specialization of the variational formulation will not be useful in practice. Some specializations of the extended variational stated in the previous section are presented below.

For the functional Ω_5 in Equation (65), in which the soil stress and the fluid pressure need not be differentiable, specialization to satisfy $(49)_3$ and $(49)_4$, i.e. satisfying identically the kinematic relationships gives

$$\begin{aligned}\Omega_9 = & <\rho u_i, u_i>_R + 2<\rho_2 w_i, u_i>_R + <(\frac{\rho_2}{f} + 1*\frac{k}{k})w_i, w_i>_R \\ & + <t*(E_{ijkl} + \alpha^2 M \delta_{ij}\delta_{kl})e_{kl}, e_{ij}>_R \\ & + 2<t*\alpha M \delta_{ij}e_{ij}, \xi>_R + <t*M\xi, \xi>_R \\ & - 2<u_i, F_i>_R - 2<w_i, G_i>_R \\ & - 2<\tau_{ij}, t*(u_i - \hat{u}_i)n_j>_{S_1} - 2<\pi, t*(w_i - \hat{w}_i)n_i>_{S_2} \\ & - 2<w_i, t*\hat{\pi}n_i>_{S_3} - 2<u_i, t*\hat{T}_i)>_{S_4}\end{aligned}$$

$$- 2\langle \tau_{ij}, t^*((u_i n_j)' - (g_1)_i n_j) \rangle_{S_{1i}} - 2\langle \pi, t^*((w_i n_i)' - g_2) \rangle_{S_{2i}}$$

$$- 2\langle w_i, t^* g_3 n_i \rangle_{S_{3i}} - 2\langle u_i, t^* g_4 n_i \rangle_{S_{4i}} \tag{69}$$

If the field variables over the domain are continuous, the jump discontinuity terms drop out giving the specialization;

$$\Omega_{10} = \langle \rho u_i, u_i \rangle_R + 2\langle \rho_2 w_i, u_i \rangle_R + \langle (\frac{\rho_2}{f} + 1^* \frac{1}{k}) w_i, w_i \rangle_R$$

$$+ \langle t^*(E_{ijkl} + \alpha^2 M \delta_{ij} \delta_{kl}) e_{kl}, e_{ij} \rangle_R$$

$$+ 2\langle t^* \alpha M \delta_{ij} e_{ij}, \xi \rangle_R + \langle t^* M \xi, \xi \rangle_R$$

$$- 2\langle u_i, F_i \rangle_R - 2\langle w_i, G_i \rangle_R$$

$$- 2\langle \tau_{ij}, t^*(u_i - \hat{u}_i) n_j \rangle_{S_1} - 2\langle \pi, t^*(w_i - \hat{w}_i) n_i \rangle_{S_2}$$

$$- 2\langle w_i, t^* \hat{\pi} n_i \rangle_{S_3} - 2\langle u_i, t^* \hat{T}_i) \rangle_{S_4} \tag{70}$$

Further specialization of (70) to the case where displacement boundary conditions are identically satisfied yields the function governing the two field formulation proposed by Ghaboussi and Wilson [12] except that in the present formulation the boundary terms are consistent.

Alternatively, specializing Equation (67) to satisfy the $(49)_4$,

$$\Omega_{11} = \langle \rho u_i, u_i \rangle_R + 2\langle \rho_2 w_i, u_i \rangle_R + \langle (\frac{\rho_2}{f} + 1^* \frac{1}{k}) w_i, w_i \rangle_R$$

$$- 2\langle t^* \pi_{,i}, w_i \rangle_R - 2\langle t^* \xi, \pi \rangle_R$$

$$+ \langle t^*(E_{ijkl} + \alpha^2 M \delta_{ij} \delta_{kl}) e_{kl}, e_{ij} \rangle_R$$

$$+ 2\langle t^* \alpha M \delta_{ij} e_{ij}, \xi \rangle_R + \langle t^* M \xi, \xi \rangle_R$$

$$- 2\langle u_i, F_i \rangle_R - 2\langle w_i, G_i \rangle_R$$

$$- 2\langle \tau_{ij}, t^*(u_i - \hat{u}_i) n_j \rangle_{S_1} + 2\langle \pi, t^* \hat{w}_i n_i \rangle_{S_2}$$

$$+ 2\langle w_i, t^*(\pi - \hat{\pi}) n_i \rangle_{S_3} - 2\langle u_i, t^* \hat{T}_i \rangle_{S_4}$$

$$- 2\langle \tau_{ij}, t^*((u_i n_j)' - (g_1)_i n_j) \rangle_{S_{1i}} + 2\langle \pi, t^* g_2 \rangle_{S_{2i}}$$

$$+ 2\langle w_i, t^*((\pi n_i)' - g_3 n_i) \rangle_{S_{3i}} - 2\langle u_i, t^* g_4 n_i \rangle_{S_{4i}} \qquad (71)$$

Furthermore, assuming that the internal discontinuities and the boundary conditions on S_1 and S_3 are identically satisfied and eliminating ξ by using $(49)_6$, Ω_{11} gives

$$\Omega_{12} = \langle \rho u_i, u_i \rangle_R + 2\langle \rho_2 w_i, u_i \rangle_R + \langle (\frac{\rho_2}{f} + 1^* \frac{1}{k}) w_i, w_i \rangle_R$$

$$- 2\langle t^* \pi_{,i}, w_i \rangle_R + \langle t^* E_{ijkl} e_{kl}, e_{ij} \rangle_R + 2\langle t^* \alpha \delta_{ij} e_{ij}, \pi \rangle_R$$

$$- \langle t^* \pi / M, \pi \rangle_R - 2\langle u_i, F_i \rangle_R - 2\langle w_i, G_i \rangle_R$$

$$+ 2\langle \pi, t^* \hat{w}_i n_i \rangle_{S_2} - 2\langle u_i, t^* \hat{T}_i \rangle_{S_4} \qquad (72)$$

Ω_{12}, which is three field formulation, has important meaning in the context of finite element analysis, since thismixed formulation of u-w-π can produce the continuity of the pore pressure which is the physically important quantity in the analysis of dynamic response of the fluid-saturated solid. Similar three field formulation can be obtained by specializing Ω_1 to satisfy $(49)_3$, $(49)_4$ and $(49)_6$. Clearly, a large number of other specializations are possible even if they are not listed here and left to interested readers.

4.4 COMPLEMENTARY FORMULATION

4.4.1 Complementary Form of Field Equations

An alternative procedure to set up the variational principles governing the problem is to write the operator equations in complementary form instead of the direct form of equation (49). In writing the equations, it is assumed that kinematics of the solid and the fluid, i.e. (30) and (31) are satisfied. The complementary form of field equations is

$$\begin{bmatrix} \rho & \rho_2 & 0 & -L \\ \rho_2 & \frac{\rho_2}{f} + 1 * \frac{1}{k} & -t * \frac{\partial}{\partial m} & 0 \\ 0 & t * \frac{\partial}{\partial m} & P & t * \alpha \, C_{ijkl} \delta_{kl} \\ L & 0 & t * \alpha \, C_{ijkl} \delta_{ij} & -t * C_{ijkl} \end{bmatrix} \begin{Bmatrix} u_m \\ w_m \\ \pi \\ \tau_{ij} \end{Bmatrix} = \begin{Bmatrix} F_i \\ G_i \\ 0 \\ 0 \end{Bmatrix} \qquad (73)$$

where

$$P = -t * (1/M + \alpha^2 \, C_{ijkl} \delta_{kl} \delta_{ij}) \qquad (74)$$

$$L = \frac{1}{2} \, t * (\delta_{im} \frac{\partial}{\partial j} + \delta_{jm} \frac{\partial}{\partial i}) \qquad (75)$$

The governing function for the set of equations (73) along with the boundary conditions and internal jump discontinuity conditions, following Sandhu [11], is

$$J = \langle \rho u_i, u_i \rangle_R + 2\langle \rho_2 w_i, u_i \rangle_R - \langle t^* \tau_{ij,j}, u_i \rangle_R$$

$$+ \langle (\frac{\rho_2}{f} + 1^* \frac{1}{k}) w_i, w_i \rangle_R - \langle t^* \pi_{,i}, w_i \rangle_R + \langle t^* w_{i,i}, \pi \rangle_R$$

$$- \langle t^* (\frac{1}{M} + \alpha^2 C_{ijkl} \delta_{kl} \delta_{ij}) \pi, \pi \rangle_R$$

$$+ 2\langle t^* \alpha C_{ijkl} \delta_{kl} \quad _{ij}, \pi \rangle_R + \langle t^* u_{i,j}, \tau_{ij} \rangle_R$$

$$- \langle t^* C_{ijkl} \tau_{ij}, \quad _{kl} \rangle_R - 2\langle u_i, F_i \rangle_R - 2\langle w_i, G_i \rangle_R$$

$$- \langle \tau_{ij}, t^*(u_i - 2\hat{u}_i) n_j \rangle_{S_1} - \langle \pi, t^*(w_i - 2\hat{w}_i) n_i \rangle_{S_2}$$

$$+ \langle w_i, t^*(\pi - 2\hat{\pi}) n_i \rangle_{S_3} + \langle u_i, t^*(\tau_{ij} n_j - 2\hat{T}_i) \rangle_{S_4}$$

$$- \langle \tau_{ij}, t^*((u_i n_j)' - 2(g_1)_i n_j) \rangle_{S_{1i}} - \langle \pi, t^*((w_i n_i)' - 2g_2) \rangle_{S_{2i}}$$

$$+ \langle w_i, t^*((\pi n_i)' - 2g_3 n_i) \rangle_{S_{3i}} + \langle u_i, t^*((\tau_{ij} n_j)' - 2g_4 n_i) \rangle_{S_{4i}} \quad (76)$$

As in direct formulation, it can be shown that the Gateaux differential of (76) vanishes if and only if the field equations (72), the boundary conditons (54) and the jump condtions (55).

4.4.2 Extended Complementary Variational Principles

Following the principles and methodology presented in Section 4.3, it is possible to develop extended variational principles for the complementary form of the field equations (73) as well. Relations (56) and (57) can be used to eliminate some of the operators from (76). As a result, following extensions of (76) are possible.

Use of (56) an (57) to eliminate $\tau_{ij,j}$ and $\pi_{,i}$ from (76) gives

$$J_1 = \langle \rho u_i, u_i \rangle_R + 2\langle \rho_2 w_i, u_i \rangle_R + \langle (\frac{\rho_2}{f} + 1^* \frac{1}{k}) w_i, w_i \rangle_R$$

$$+ 2\langle t^* w_{i,i}, \pi \rangle - \langle t^* (\frac{1}{M} + \alpha^2 C_{ijkl} \delta_{kl} \delta_{ij}) \pi, \pi \rangle_R$$

$$+ 2\langle t^* \alpha C_{ijkl} \delta_{kl} \quad {}_{ij}, \pi \rangle_R + 2\langle t^* u_{i,j}, \tau_{ij} \rangle_R$$

$$- \langle t^* C_{ijkl} \tau_{ij}, \tau_{kl} \rangle_R - 2\langle u_i, F_i \rangle_R - 2\langle w_i, G_i \rangle_R$$

$$- 2\langle \tau_{ij}, t^* (u_i - \hat{u}_i) n_j \rangle_{S_1} - 2\langle \pi, t^* (w_i - \hat{w}_i) n_i \rangle_{S_2}$$

$$- 2\langle w_i, t^* \hat{\pi} n_i \rangle_{S_3} - 2\langle u_i, t^* \hat{T}_i \rangle_{S_4}$$

$$- 2\langle \tau_{ij}, t^* ((u_i n_j)' - (g_1)_i n_j) \rangle_{S_{1i}} - 2\langle \pi, t^* ((w_i n_i)' - g_2) \rangle_{S_{2i}}$$

$$- 2\langle w_i, t^* g_3 n_i \rangle_{S_{3i}} - 2\langle u_i, t^* g_4 n_i \rangle_{S_{4i}} \qquad (77)$$

Here, the stress field and the pore pressure need not be differentiable. Similarly, elimination of $\tau_{ij,j}$ and $w_{i,i}$ from (76) leads to

$$J_2 = \langle \rho u_i, u_i \rangle_R + 2\langle \rho_2 w_i, u_i \rangle_R + \langle (\frac{\rho_2}{f} + 1^* \frac{1}{k}) w_i, w_i \rangle_R$$

$$- 2\langle t^* \pi_{,i}, w_i \rangle - \langle t^* (\frac{1}{M} + \alpha^2 C_{ijkl} \delta_{kl} \delta_{ij}) \pi, \pi \rangle_R$$

$$+ 2\langle t^* \alpha C_{ijkl} \delta_{kl} \tau_{ij}, \pi \rangle_R + 2\langle t^* u_{i,j}, \tau_{ij} \rangle_R$$

$$- \langle t^* C_{ijkl} \tau_{ij}, \tau_{kl} \rangle_R - 2\langle u_i, F_i \rangle_R - 2\langle w_i, G_i \rangle_R$$

$$- 2\langle \tau_{ij}, t^* (u_i - \hat{u}_i) n_j \rangle_{S_1} + 2\langle \pi, t^* \hat{w}_i n_i \rangle_{S_2}$$

$$+ 2\langle w_i, t^* (\pi - \hat{\pi}) n_i \rangle_{S_3} - 2\langle u_i, t^* \hat{T}_i \rangle_{S_4}$$

$$- 2\langle \tau_{ij}, t^* ((u_i n_j)' - 2(g_1)_i n_j) \rangle_{S_{1i}} + 2\langle \pi, t^* g_2 \rangle_{S_{2i}}$$

$$+ 2\langle w_i,\ t^*((\pi n_i)' - 2g_3 n_i)\rangle_{S_{3i}} - 2\langle u_i,\ t^* g n_i\rangle_{S_{4i}} \tag{78}$$

which does not involve derivatives of τ_{ij} and w_i. Eliminating $u_{i,j}$ and $\pi_{,i}$, (76) is

$$\begin{aligned} J_3 = {} & \langle \rho u_i, u_i\rangle_R + 2\langle \rho_2 w_i, u_i\rangle_R - 2\langle t^* \tau_{ij,j},\ u_i\rangle_R \\ & + \langle(\frac{\rho_2}{f} + 1^*\frac{1}{k})w_i, w_i\rangle_R + 2\langle t^* w_{i,i}, \pi\rangle_R \\ & - \langle t^*(\frac{1}{M} + \alpha^2 C_{ijkl}\delta_{kl}\delta_{ij})\pi, \pi\rangle_R \\ & + 2\langle t^*\alpha C_{ijkl}\delta_{kl}\tau_{ij}, \pi\rangle_R \\ & - \langle t^* C_{ijkl}\tau_{ij}, \tau_{kl}\rangle_R - 2\langle u_i, F_i\rangle_R - 2\langle w_i, G_i\rangle_R \\ & + 2\langle \tau_{ij}, t^*\hat{u}_i n_j\rangle_{S_1} - 2\langle \pi, t^*(w_i - \hat{w}_i)n_i\rangle_{S_2} \\ & - 2\langle w_i,\ t^*(\pi - \hat{\pi})n_i\rangle_{S_3} + 2\langle u_i,\ t^*(\tau_{ij}n_j - \hat{T}_i)\rangle_{S_4} \\ & + 2\langle \tau_{ij}, t^*(g_1)_i n_j\rangle_{S_{1i}} - 2\langle \pi, t^*((w_i n_i) - g_2)\rangle_{S_{2i}} \\ & - 2\langle w_i,\ t^* g_3 n_i\rangle_{S_{3i}} + 2\langle u_i,\ t^*((\tau_{ij}n_j)' - g_4 n_i)\rangle_{S_{4i}} \end{aligned} \tag{79}$$

in which the solid displacement and the pore pressure need not be differentiable. Finally, relaxing the differentiability requirements of $u_{i,j}$ and $w_{i,i}$ from (76) by using (56) and (57), we have

$$\begin{aligned} J_4 = {} & \langle \rho u_i, u_i\rangle_R + 2\langle \rho_2 w_i, u_i\rangle_R - 2\langle t^* \tau_{ij,j},\ u_i\rangle_R \\ & + \langle(\frac{\rho_2}{f} + 1^*\frac{1}{k})w_i, w_i\rangle_R - 2\langle t^*\ \pi_{,i}, w_i\rangle_R \end{aligned}$$

$$- \langle t*(\frac{1}{M} + \alpha^2 C_{ijkl}\delta_{kl}\delta_{ij})\pi, \pi\rangle_R$$

$$+ 2\langle t*\alpha C_{ijkl}\delta_{kl} \tau_{ij}, \pi\rangle_R$$

$$- \langle t*C_{ijkl} \tau_{ij}, \tau_{kl}\rangle_R - 2\langle u_i, F_i\rangle_R - 2\langle w_i, G_i\rangle_R$$

$$+ 2\langle \tau_{ij}, t*\hat{u}_i n_j\rangle_{S_1} + 2\langle \pi, t*\hat{w}_i n_i\rangle_{S_2}$$

$$+ 2\langle w_i, t*(\pi - \hat{\pi})n_i\rangle_{S_3} + 2\langle u_i, t*(\tau_{ij}n_j - \hat{T}_i)\rangle_{S_4}$$

$$+ 2\langle \tau_{ij}, t*(g_1)_i n_j\rangle_{S_{1i}} + 2\langle \pi, t*g_2\rangle_{S_{2i}}$$

$$+ 2\langle w_i, t*((\pi n_i)' - g_3 n_i)\rangle_{S_{3i}} + 2\langle u_i, t*((\tau_{ij}n_j)' - g_4 n_i)\rangle_{S_{4i}} \quad (80)$$

Derivatives of u_i and w_i are not involved in (80).

4.4.3 Specializations of Complementary Extended Variational Principles

As in direct formulation, some specializations on the complementary extended variational principles are possible by requiring that certain field equations and/or boudary conditions and jump conditions be identically satisfied. Here, we present some secializations.

If we assume that $(73)_4$ is identically satisfied, J_1 results in

$$J_5 = \langle \rho u_i, u_i\rangle_R + 2\langle \rho_2 w_i, u_i\rangle_R + \langle(\frac{\rho_2}{f} + 1*\frac{1}{k})w_i, w_i\rangle_R$$

$$+ 2\langle t*w_{i,i}, \pi\rangle - \langle t*(\frac{1}{M} + \alpha^2 C_{ijkl}\delta_{kl}\delta_{ij})\pi, \pi\rangle_R$$

$$+ \langle t*C_{ijkl} \tau_{ij}, \tau_{kl}\rangle_R - 2\langle u_i, F_i\rangle_R - 2\langle w_i, G_i\rangle_R$$

$$- 2\langle \tau_{ij}, t*(u_i - \hat{u}_i)n_j\rangle_{S_1} - 2\langle \pi, t*(w_i - \hat{w}_i)n_i\rangle_{S_2}$$

$$- 2\langle w_i,\ t*\hat{\pi} n_i\rangle_{S_3} - 2\langle u_i,\ t*\hat{T}_i\rangle_{S_4}$$

$$- 2\langle \tau_{ij}, t*((u_i n_j)' - (g_1)_i n_j)\rangle_{S_{1i}} - 2\langle \pi, t*((w_i n_i)' - g_2)\rangle_{S_{2i}}$$

$$- 2\langle w_i,\ t*g_3 n_i\rangle_{S_{3i}} - 2\langle u_i,\ t*g_4 n_i\rangle_{S_{4i}} \tag{81}$$

Specializing J_1 to satisfy $(73)_3$,

$$J_6 = \langle \rho u_i, u_i\rangle_R + 2\langle \rho_2\, w_i, u_i\rangle_R + \langle(\frac{\rho_2}{f} + 1*\frac{1}{k})w_i, w_i\rangle_R$$

$$+ \langle t*(\frac{1}{M} + \alpha^2\, C_{ijkl}\delta_{kl}\delta_{ij})\pi, \pi\rangle_R + 2\langle t*u_{i,j}, \tau_{ij}\rangle_R$$

$$- \langle t*C_{ijkl}\,\tau_{ij}, \tau_{kl}\rangle_R - 2\langle u_i, F_i\rangle_R - 2\langle w_i, G_i\rangle_R$$

$$- 2\langle \tau_{ij}, t*(u_i - \hat{u}_i)n_j\rangle_{S_1} - 2\langle \pi, t*(w_i - \hat{w}_i)n_i\rangle_{S_2}$$

$$- 2\langle w_i,\ t*\hat{\pi} n_i\rangle_{S_3} - 2\langle u_i,\ t*\hat{T}_i\rangle_{S_4}$$

$$- 2\langle \tau_{ij}, t*((u_i n_j)' - (g_1)_i n_j)\rangle_{S_{1i}} - 2\langle \pi, t*((w_i n_i)' - g_2 n_i)\rangle_{S_{2i}}$$

$$- 2\langle w_i,\ t*g_3 n_i\rangle_{S_{3i}} - 2\langle u_i,\ t*g_4 n_i\rangle_{S_{4i}} \tag{82}$$

It is unlikely from the above two specializations that further specialization on J_5 or J_6 through requiring the other field equations to be satisfied would make them simpler. Since, moreover, specializations of the extended functionals, J_1 to J_4 would show little differences, we do not present every possible specializations.

SECTION V

DISCUSSION

A systematic development of variational principles for linear elastodynamics of fluid-saturated solids have been presented. Nonhomogeneous boundary conditions and internal jump discontinuities were explicitly incorporated in general variational principles. Allowance of jump discontinuity terms in variational formulation is meaningful in the context of direct approximation in finite element spaces, since the space of approximants may not be sufficiently smooth. Based on the direct and the complementary formulations, extensions of the variational principles through elimination of certain field operators and specializations by restricting some of the field equations and boundary conditions to be identically satisfied have been proposed. These formulations should provide a basis for development of approximate solution procedurse and also approximation theories governing the problem. Fig.1 diagramatically depicts the possible extensions of the general variational principle based on the direct formulation. Fig.2 shows the same for the complementary formulation. In either case the specializations listed in this report are shown. Evidently, other extended forms could be used as starting points for specialization. In the interests of brevity we desist from attempting to catalog all the possibilities.

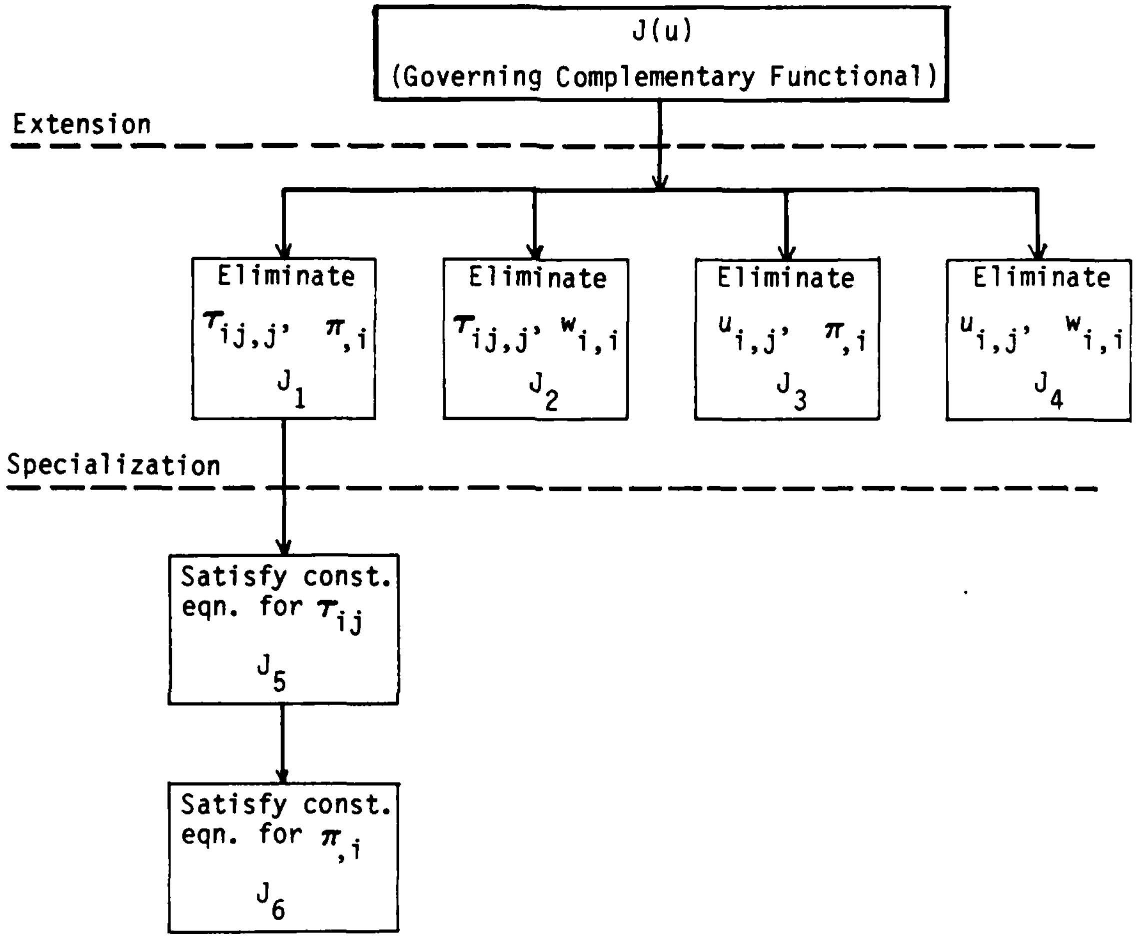

Fig.2; Family of Complementary Variational Principles

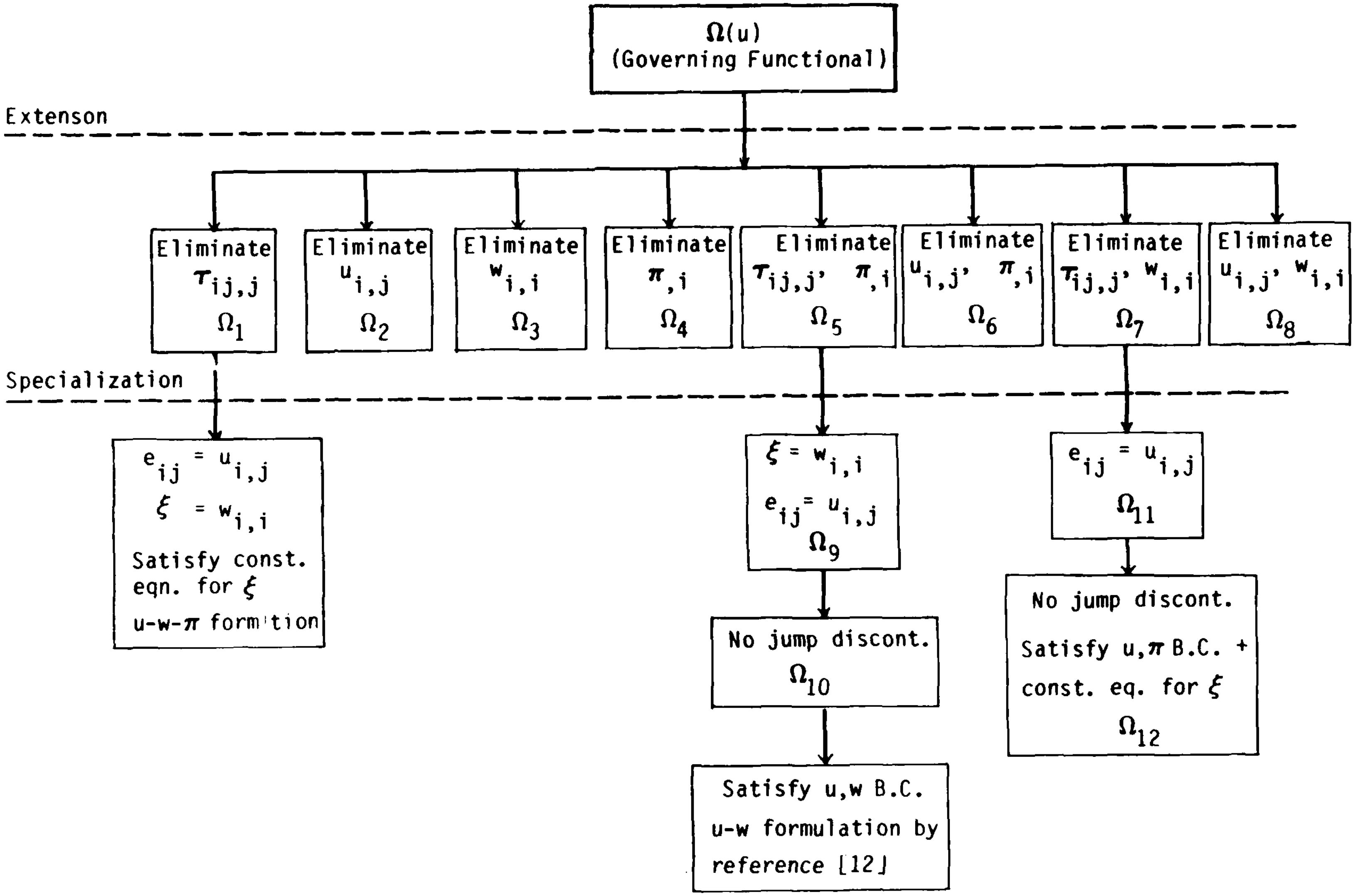

Fig.1 ; Family of Variational Principles by Direct Formulation

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