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Suresh P. Sethi and Michael Taksar

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The paper of Merton's "Optimum Consumption and Portfolio Rules in a Continuous-Time Model" now became classical and is one of the most cited papers in the field. It was among the ones which initiated a new brunch of consumption/investment stochastic models. However, certain fundamental errors in formulation and solution of continuous time models were overlooked there.

The present paper indicates when these errors can be corrected and suggests alternative ways of treating consumption investment models with possible bankruptcy.

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A Note on Merton's "Optimum Consumption and Portfolio Rules in a Continuous-Time Model"

by

Suresh P. Sethi
Faculty of Management Studies
University of Toronto
Toronto, Ontario

and

Michael Taksar
Department of Statistics
The Florida State University
Tallahassee, Florida

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A Note on Merton's "Optimum Consumption and Portfolio Rules in a Continuous-Time Model"

Suresh Sethi and Michael Taksar

1. INTRODUCTION

In the area of consumption and portfolio problem in continuous time, Merton [2] is the most widely cited paper. It is an important paper because of its many significant contributions. Among these was the provision of explicit solutions for utility functions in the HARA family specified in equation (43) of Merton [2]. These solutions in the form of lengthy formulas were simply stated without any derivation. Perhaps, because of this, some errors went undetected. While some minor errors were corrected in Merton [3], the purpose of this note is to delineate the subfamily of HARA utility functions for which the explicit solution obtained in Section 6 of Merton [2] are correct and the remaining subfamily for which they are not. In Merton's notation, the HARA family is given by

$$V(C) = \frac{1-\gamma}{\gamma}\left[\frac{\beta C}{1-\gamma} + \eta\right]^{\gamma} \qquad (43M)$$

with $\beta > 0$, $\gamma \neq 1$, $\eta \geq 0$ when $\gamma < 1$, and $\eta > 0$ when $\gamma > 1$.[1] Now, more specifically, the solutions in Section 6 of Merton [2] are correct only when $V'(0) = \infty$, i.e., when $\gamma < 1$ and $\eta = 0$. On the

[1]Equation (43M) refers to equation (43) in Merton [2]. Hereafter, we shall refer to equations in Merton [2] by their numbers followed by the letter "M".

other hand, when $V'(0) < \infty$, i.e., when $\eta > 0$, the solutions obtained in Section 6 of [2] violate the feasibility conditions $W(t) > 0$, $0 \le t < T$ and $C(t) \ge 0$, $0 \le t \le T$, where $W(t)$ is the wealth and $C(t)$ is the rate of consumption at time t. These conditions are specified in Merton [4] and, we believe, are also assumed in [2], although not explicitly stated there. We remark that the condition $W(T) \ge 0$ is not violated.

2. FEASIBILITY VIOLATIONS

The solution for the value function $J(W,t)$ obtained in equation (47) of Merton [2] and corrected in Merton [3] has a printing error, which requires the replacement of the term $\rho - \delta\nu$ in the denominator by $\rho - \gamma\nu$. We reproduce this solution

$$J(W,t) = \frac{\delta}{\gamma}\beta^{\gamma} e^{-\rho t}\left[\frac{\delta\left(1-e^{-\frac{\rho-\gamma\nu}{\delta}(T-t)}\right)}{\rho-\gamma\nu}\right]^{\delta}\left[\frac{W}{\delta}+\frac{\eta}{\beta r}\left(1-e^{-r(T-t)}\right)\right]^{\gamma} \tag{47M}$$

as the starting point for this note.

If we substitute $\eta = 0$ in (47 M), it provides us with the correct value function for the HARA cases with $\gamma < 1$.[2]

Furthermore,

$$\hat{J}(W) = \lim_{T\to\infty} e^{\rho t} J(W,t) = \frac{1-\gamma}{\gamma}\left[\frac{1-\gamma}{\rho-\gamma\nu}\right]^{1-\gamma}\left[\frac{\beta W}{1-\gamma}\right]^{\gamma}, \tag{1}$$

gives the infinite horizon value function in current value terms for HARA cases with $\gamma < 1$ and $\eta = 0$. To prevent this solution

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[2] Note, as has already been indicated in Section 1, that for $\gamma > 1$, the value $\eta = 0$ is not admissible [2].

from blowing up, it appears that we should have $\rho - \gamma\nu > 0$. This growth condition agrees with the condition (14.4) derived in [1], and it is weaker than (41) imposed in [4].

2.1 HARA CASES WITH $\eta > 0$ AND $\gamma < 1$

For these cases, we note that there is no finite consumption satiation level and remind that $V'(0) < \infty$.

Using (47M) as the optimal value function, Merton obtains the wealth equation (54M). From this, he derives equations (55M) and (56M), which because they contain some minor errors, are rewritten here as:

$$\frac{dX}{X} = \left[r - \frac{\mu}{1-e^{-\mu(T-t)}} + \frac{(\alpha-r)^2}{\delta\sigma^2}\right] dt + \frac{\alpha-r}{\sigma\delta}\, dz \quad , \tag{55M}$$

and

$$X(t) = X(0) \exp\left\{\left[r-\mu + \frac{(\alpha-r)^2}{\delta\sigma^2} - \frac{(\alpha-r)^2}{2\delta^2\sigma^2}\right] t + \frac{\alpha-r}{\delta\sigma}\int_0^t dz\right\} \cdot \frac{1-e^{-\mu(T-t)}}{1-e^{-\mu T}} \tag{56M}$$

where $\mu = (\rho - \gamma\nu)/\delta$ and

$$X(t) = W(t) + \frac{(1-\gamma)\eta}{\beta r}\left[1 - e^{-r(T-t)}\right]$$

Thus, $X(t)$ is a geometric Brownian motion. It is, therefore obvious that $X(t)$, $t > 0$ can be arbitrarily close to zero with a positive probability. Thus for $\eta > 0$, there is a positive probability that $W(t) \le 0$, or for that matter, $W(t) < 0$, for some

$t \in (0,T)$.[3]

We have now shown that while $J(W,t)$ given in (47M) solves the H-J-B equation (44M), it is not the value function. In fact its computation, since $W(t)$ could fall to zero, would require an additional boundary condition, say, the specification of the behavior of the function $J(W,t)$, $t\in[0,T]$ in the neighborhood of the line $W = 0$ in addition to $J(W,T) = 0$ already imposed in [2].[4] Moreover, it would require additional machinery to deal with the possibility of the boundary consumption.

The value functions $J(W)$ for general concave utility functions, including the HARA cases with $\eta > 0$ and $\gamma < 1$, have already been obtained by Karatzas, Lehoczky, Sethi and Shreve [1] when the horizon is infinite. When $V'(0) < \infty$, which is the case with $\eta > 0$, there are three cases depending on the value of P stipulated in the boundary condition $J(0) = P$. In what follows, we let

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[3]Note that $C^*(t) = \frac{\mu X(t)}{1-e^{-\mu(T-t)}} - \frac{\delta\eta}{\beta}$, given by (48M), becomes negative when $X(t)$ is close to zero. Moreover, for some values of the parameters, $C^*(W,t)$ expressed also by (48M) becomes negative even for small positive wealth levels. Specifically, for $\mu < r$, there exists a $\underline{W}(t) > 0$ for every $t < T$, such that $C^*(\underline{W}(t),t) < 0$. For example, with $\rho = 0.20$; $r = 0.16$, $(\alpha - r)^2/2\sigma^2 = 0.05$, (therefore, $\mu = .14$), $\gamma = 0.5$, $\beta = \eta = 1$, $t = T - 1$, and $\underline{W}(t) = 1$, we have $C^*(1,T-1) \approx - 0.26$.

[4]If one assumes that zero wealth results in bankruptcy and the problem stops, then one should specify $J(0,t)$, $t\in[0,T]$. One particular specification

$$J(0,t) = \frac{V(0)}{\rho}\left[e^{-\rho t} - e^{-\rho T}\right] = \frac{1-\gamma}{\gamma}\eta^{\gamma}\left[e^{-\rho t} - e^{-\rho T}\right],$$

is associated with zero consumption from the time of bankruptcy t to the terminal time T. If, however, one considers a model in which it is possible to start a "new life" after bankruptcy (e.g. see [5]), then the required boundary condition would involve $J(0,t)$ and $J_W(0,t)$, $t\in[0,T]$.

$C^{**}(W,t)$ denote the optimal feedback consumption rate obtained in [1].

For $P < V(0)/\rho$, there exists a wealth level $\bar{W}(P) > 0$ such that

$$C^{**}(W,t) \begin{cases} = 0, & W \in [0, \bar{W}(P)] \\ > 0, & W \in (\bar{W}(P), \infty) \end{cases}$$

and $W(t) > 0$, almost surely, for all $t > 0$.

For $P \in (V(0)/\rho, P^*]$, where $P^* < \frac{1}{\rho} \lim_{C\to\infty} V(C)$, there exists a wealth level $\bar{W}(P) > 0$ (except when $P = P^*$ in which case $\bar{W}(P^*)=0$) such that $C^{**}(W,t)$ has the above form, but the optimal investment policy gives rise to a positive probability of bankruptcy, i.e., of

$W(t) = 0$ for some t.

Finally, for $P > P^*$, there exists $\xi > 0$ such that

$$C^{**}(W,t) > \xi, \quad W \in (0,\infty)$$

and there is a positive probability of bankruptcy.

An important conclusion, therefore, for the purpose of this note is that whatever the value of P, there is either a boundary consumption at low wealth levels or a positive probability of bankruptcy, or both. This conclusion will also hold for finite horizon problems.

In view of the above, it is clear that there is no easy way to fix (47M), (48M) and (49M). More specifically, $C^*(t)$ will not have the form $C^* = aW + b$ at least when the boundary consumption is possible. In the other case, when $P > P^*$ or when $J(0,t)$ is sufficiently large in the finite horizon case, there is no a priori reason to believe that C^*, although an interior solution, will have the form $C^* = aW + b$. This implies that Theorems III, IV and V, based on the assumption of interior consumption and no bankruptcy are correct only for $\eta = 0$.

Before leaving this section, let us try to find a meaning of the expression obtained in (47M). First we note that (43M) is defined for $C \geq -(1-\gamma)\eta/\beta$. Also, $J(W,t)$ in (47M) is defined for $W \geq \frac{-(1-\gamma)\eta}{\beta r}[1-e^{-r(T-t)}]$, $t \in [0,T]$.

Finally, we know that $W(T) = 0$, almost surely. It is possible therefore, to say that $J(W,t)$ in (47M) is the value function for the fictitious problem, in which consumption is constrained as $C \geq -(1-\gamma)\eta/\beta$ and the agent's bequest function is:

$$B(W,T) = \begin{cases} 0, & \text{if } W \geq 0 \\ -\infty, & \text{if } W > 0 \end{cases}$$

One then solves (44M) with the boundary condition $J(W,T) = B(W,T)$ and obtains (47M).

We now turn to HARA cases with $\gamma > 1$; note that η must be strictly positive in these cases [2].

2.2 HARA CASES WITH $\eta > 0$ AND $\gamma > 1$

In these cases, there exists a consumption satiation level $(\gamma-1)\eta/\beta$. Feasible consumption levels are given by

$$0 \leq C \leq (\gamma - 1)\eta/\beta. \tag{2}$$

In this consumption range,

$$V(C) \leq 0 \quad \text{with } V[(\gamma - 1)\eta/\beta] = 0. \tag{3}$$

As mentioned in Merton [3], the investor with wealth

$$\hat{W}(t) = \frac{(\gamma-1)\eta}{\beta r}\quad[1 - e^{-r(T-t)}]$$

at time t can ensure with certainty a program of the maximal level of consumption by simply holding the riskless asset.

Clearly, the initial wealth $W(0)$ must satisfy

$$0 < W(0) < \hat{W}(0) \tag{4}$$

for the problem to be nontrivial. Thus $-\hat{W}(0) < X(0) < 0$ in (56M). This implies that there is a positive probability that $X(t) < -\hat{W}(t)$ for some $t \in (0,T)$ and, therefore, that

$$W(t) = X(t) + \hat{W}(t) < 0$$

for some $t \in (0,T)$.

Once again, the solution in (47M) does not provide us with a feasible wealth trajectory and is, therefore, not the value function.

It is interesting to note that (56M) imply $W(t) < \hat{W}(t)$, $t \in [0,T]$, almost surely. Moreover, (43M) satisfies $J(\hat{W}(t),t) = 0$. We believe that the correct value functions for these problems should satisfy these properties. Thus, in order to solve the H-J-B equation (44M), we need only impose $J(W,T) = 0$ and another boundary condition, say, on $J(0,t)$.

The solution of the infinite horizon problems with $\gamma > 1$ can be obtained from Karatzas et.al. [1]. We need only to define the utility function $U(C)$ in the notation of [1] as

$$U(C) = \begin{cases} \dfrac{1-\gamma}{\gamma} \left[\dfrac{\beta C}{1-\gamma} + \eta\right]^{\gamma}, & 0 \leq C \leq (\gamma-1)\eta/\beta \\ 0, & C > (\gamma-1)\eta/\beta \end{cases} \tag{5}$$

We note that $U(C)$ satisfies all the conditions imposed in Section 2 of [1] except at $C = (\gamma - 1)\eta/\beta$, where we interpret $U' = 0$ and U'' and U''' as the left-hand derivatives. With this proviso, the formulas in [1] can be used to obtain the solution for the case $\gamma > 1$.

3. CONCLUDING REMARKS

By showing that wealth in the solutions obtained in Section 6 of Merton [2] could, when $\eta > 0$, become negative with a positive probability, it is noted that his solution does not provide the value function for the problem. As a result, Theorems III, IV and V in Section 6 of [2] are not correct for $\eta > 0$. Furthermore, solutions (70M) and (71M) in Section 7 of [2], based on the results of Section 6 will not hold for $\eta > 0$.

It should be noted before concluding this paper that the erroneous solutions in Sections 6 and 7 were obtained because of the erroneous assumption of the interiority of consumption used in (19M). Boundary consumption is possible when $V'(0) < \infty$. As a result, (22M), (28M) and (29M) cannot be assumed to hold for all levels of wealth. This would imply that several other problems treated in Sections 8 and 9 of [2], that do not satisfy the condition $V'(0) = \infty$, should be reexamined.

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