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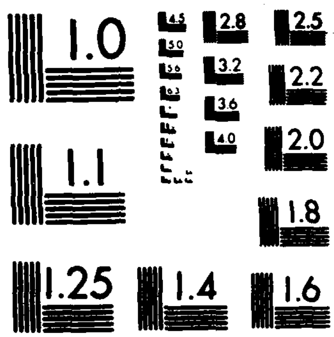

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On the Existence and Uniqueness of Invariant Measure for Continuous Time Markov Processes

by

Lukasz Stettner

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On the Existence and Uniqueness of Invariant Measure for Continuous Time Markov Processes

by

Lukasz Stettner*

Abstract

In this paper various conditions which guarantee the existence of unique invariant measure are formulated.

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1. Introduction

Let $(E,\mathcal{E})$ be a locally compact, separable state space endowed with a Borel σ-field $\mathcal{E}$. We consider a standard, nonterminating Markov process $X = (\Omega, F_t, F, x_t, \Theta_t, P_x, P_t)$ on $(E,\mathcal{E})$ (for definitions see [5] and [13]). If we want to apply any limit theorems to process X we have to impose first the assumptions for which there exists a unique finite or σ-finite invariant measure μ, i.e., a measure satisfying

$$\forall_{A \in \mathcal{E}}\ \mu(A) = \mu P_t(A) \stackrel{\text{def}}{=\!=} \int_E P_x(x_t \in A)\mu(dx) \quad \text{for } t > 0.$$

In the paper ~~we~~ attempt to find fairly general conditions under which the existence and uniqueness of invariant measure is guaranteed. The obtained results are new or generalize at least slightly known. The author introduces a terminology: weak, strong Harris, strong recurrence. Two ~~The~~ Sections ~~2 and 3~~ concern general standard processes. ~~In~~ Section ~~4~~ ~~we~~ restrict ourselves to Feller or strong Feller standard processes. Three examples are considered ~~in Section 5~~ to illustrate possible "unpleasant" situations one ~~we~~ can meet in general theory.

2. Weak Harris Condition

The studies of invariant measures for continuous time Markov processes can be reduced to discrete time case with the transition operator

$$U(x,A) \stackrel{\text{def}}{=\!=} \int_0^\infty e^{-t} P_t \chi_A(x)\,dt \quad \text{for } A \in \mathcal{E},$$

since from Proposition 2.1 [2] we have

Lemma 1. A measure μ is invariant for semigroup $(P_t)_{t \geq 0}$ if and only if it is invariant for U, i.e., $\mu U = \mu$.

The operator U and its powers satisfy the following identity

(1) $$\forall_{A\in\mathcal{E}}\ U_0(x,A) \overset{\text{def}}{=\!=} \int_0^\infty P_t\chi_A(x)dt = \sum_{n=1}^{\infty} U^n(x,A)$$

since one can calculate

(2) $$U^n(x,A) = \int_0^\infty e^{-t} \frac{t^{n-1}}{(n-1)!} P_t\chi_A(x)dt.$$

Definition 1. If there exists a probability measure m such that

(3) $$\forall_{A\in\mathcal{E}}\ m(A) > 0 \Rightarrow \forall_{x\in E}\ U_0(x,A) = \infty$$

then X is called weak Harris.

Comparing (3) with (1) we see that under (3) the operator U satisfies a discrete weak Harris condition, i.e., there exists a probability measure m such that

(4) $$\forall_{A\in\mathcal{E}}\ m(A) > 0 \Rightarrow \forall_{x\in E} \sum_{n=1}^{\infty} U^n(x,A) = \infty .$$

We will need an improvement of the Harris theorem [9] due to Foguel [6].

Proposition 1. Suppose there exists a probability measure m on a separable space $(E,\mathcal{E})$ such that for a Markov transition operator P

(5) $$\forall_{A\in\mathcal{E}}\ m(A) > 0 \Rightarrow \forall_{x\in E} \sum_{n=1}^{\infty} P^n(x,A) > 0.$$

Then there exists a σ-finite subinvariant measure μ i.e., $\forall_{A\in\mathcal{E}}\ \mu P(A) \leq \mu(A)$. Moreover m is absolutely continuous with respect to μ, what we denote $m << \mu$.

The following lemma provides a sufficient condition for a subinvariant measure μ to be invariant.

Lemma 2. Suppose μ is subinvariant and for some $B \in \mathcal{E}$, $\mu(B) < \infty$

(6) $$\forall_{x\in E} \sum_{n=1}^{\infty} P^n(x,B) = \infty .$$

Then μ is invariant.

Proof. We have

$$0 \leq \langle \mu - \mu P, \sum_{n=1}^{N} P^n(\cdot, B)\rangle = \mu P(B) - \mu P^{N+1}(B) \leq \mu(B) < \infty .$$

Letting $N \to \infty$ we obtain $\mu = \mu P$.

Now we can adopt Proposition 1 and Lemma 2 to weak Harris processes, taking into account (4).

Corollary 1. For weak Harris process there exists an invariant measure μ and $m \ll \mu$.

Nevertheless Proposition 1 and Corollary 1 say nothing about the uniqueness of invariant measure μ. The remaining part of Section 2 will prove that under (3) there exists in fact unique invariant measure.

Let us start with the following:

Corollary 2. The measures $U^k(x,\cdot)$, $k = 1,2, \ldots$, and $U_0(x,\cdot)$ are equivalent, i.e., $U^k(x,\cdot) \ll U_0(x,\cdot) \ll U^k(x,\cdot)$.

Proof. Indeed, from (2),

$$\forall_{A \in \mathcal{E}} \; U^k(x,A) > 0 \Leftrightarrow U_0(x,A) > 0.$$

The next two lemmas are partially adapted from Lemma 1 and 2 of [2].

Lemma 3. Suppose B is an invariant set and ν is an invariant measure. Then under (3)

(7) $\qquad m(B^c) = \nu(B^c) = 0.$

Proof. If $m(B^c) > 0$, then from (3), $U_0(x,B^c) = \infty$ for $x \in B$ and B cannot be invariant. Thus $m(B^c) = 0$. Since B is invariable $\chi_B(x)P_tf(x) \leq P_t(f\chi_B)(x)$ for $x \in E$, and any nonnegative bounded measurable f. Therefore

$$\int P_tf(x)1_B(x)\nu(dx) \leq \int P_t(f\chi_B(x)\nu(dx)) = \nu(f\chi_B)$$

and measure $\eta(A) \stackrel{\text{def}}{==} \nu(A \cap B)$ is subinvariant for $(P_t)_{t\geq 0}$ and then also for U. From Lemma 2 the measure η is invariant. Since ν is invariant, also $\bar{\eta}(A) = \nu(A\backslash B)$ is invariant. But $m(B) > 0$ and from (3) and Corollary 2, $U(x,B) > 0$ for all $x \in E$ and $\bar{\eta}(U(\cdot,B)) = \bar{\eta}(B) = 0$. This means $\bar{\eta} \equiv 0$, and finally $\nu = \eta$, $\nu(B^c) = 0$.

Lemma 4. Under (3) each invariant measure μ is absolutely continuous with respect to mU.

Proof. Suppose $mU(A) = 0$. Then $U(x,A) = 0$ m a.e. . From Corollary 2 also $U_0(x,A) = 0$ m a.e. . The set $\Gamma = \{x: U_0(x,A) = 0\}$ is invariant. In fact, if $x \in \Gamma$, then $U_0(x,A) = E_x \int_0^t \chi_A(x_s)ds + E_x(U(x_t,A)) = 0$ and $U(x_t,A) = 0$ P_x a.s. Since $U(x_t,A)$ is P_x a.s. right continuous, then $P_x\{\exists_t, U(x_t,A) > 0\} = 0$. Now from Lemma 3, $\mu(\Gamma^c) = 0$. Thus

$$\mu(A) = \mu U(A) = \int_\Gamma U(x,A)\mu(dx) + \int_{\Gamma^c} U(x,A)\mu(dx) = 0.$$

We summarize the obtained results.

Corollary 3. For any invariant measure μ we have

(8) $\qquad m \ll \mu \ll mU.$

Proof. If $m(A) > 0$, then from (3) and Corollary 2, $U(x,A) > 0$ for each $x \in E$. Therefore for any invariant measure μ we have $\mu(A) = \mu(U(A)) > 0$. The relation $\mu \ll mU$ follows from Lemma 4.

To simplify notations we put $mU \stackrel{\text{def}}{=} \lambda$. We would like to consider U as a linear operator on $L_\infty(\lambda)$. But we have to know first that U is then well defined.

Lemma 5. If f_n are bounded measurable and $f_n(x) \to 0$ as $n \to \infty$, λ a.e., then also $Uf_n \to 0$, λ a.e.

Proof. It is enough to consider $f_n(x)$ nonnegative. Let $\mathfrak{X} = \{x\colon f_n(x) \not\to 0\}$. By assumption $\lambda(\mathfrak{X}) = mU(\mathfrak{X}) = 0$. Therefore $U(x,\mathfrak{X}) = 0$ m a.e. and using Corollary 2, $U^2(x,\mathfrak{X}) = 0$ m a.e. . But this means $U(x,\mathfrak{X}) = 0$, $mU = \lambda$ a.e. Thus

$$U(x,f_n) = U(x,f_n\chi_{\mathfrak{X}}) + U(x,f_n\chi_{\mathfrak{X}^c}) \leq \|f_n\|U(x,\mathfrak{X}) + U(x,f_n\chi_{\mathfrak{X}^c}).$$

First term is equal to 0, λ a.e., while the second converges to 0 from the definition of the set $\mathfrak{X}$. This means $U(x,f_n) \to 0$, λ a.e.

The next lemma tells that the operator U satisfies discrete weak Harris condition with measure m replaced by λ.

Lemma 6. If (3) is satisfied, then

$$(9) \qquad \forall_{A\in\mathcal{E}}\ \lambda(A) > 0 \Rightarrow \forall_{x\in E} \sum_{n=1}^{\infty} U^n(x,A) = \infty .$$

Proof. If $mU(A) > 0$, then $m(\{x\colon U(x,A) > 0\}) > 0$, and there exists $\delta > 0$ such that $m(\{x\colon U(x,A) \geq \delta\}) > 0$. Put $B = \{x\colon U(x,A) \geq \delta\}$.

From (4), $\sum_{n=1}^{\infty} U^n(x,B) = \infty$ for $x \in E$. Thus

$$U^{1+n}(x,A) = \int_E U(y,A)U^n(x,dy) \geq \delta \int_B U^n(x,dy) = \delta U^n(x,B)$$

and for any $x \in E$, $\sum_{n=1}^{\infty} U^n(x,A) = \infty$ as well.

The operator U is Markov on $L_\infty(\lambda)$ i.e. positive, linear, and $U1 = 1$. Since we want to apply ergodic theory on $L_\infty(\lambda)$ we need two notions associated with so-called Hopf decomposition of E (see [7]).

Namely consider a family of functions $G = \{f \in L_\infty(\mu),\ 0 \leq f \leq 1,\ Uf \leq f,\ \lim_{n\to\infty} U^n f = 0\}$. The set $D = \sup\{\chi_{f>0}: f \in G\}$ is called dissipative part of E, while $C = E \setminus D$ is conservative. If C = E, then U is called *conservative*.

If a family of invariant sets $\Sigma = \{A \in \mathcal{E}: U(x,A) = \chi_A(x),\ \lambda \text{ a.e.}\}$ contains only the sets that differ from $\emptyset$ or E on the set of λ measure 0, then U is called *ergodic*.

Lemma 7. Under (3), the operator U is conservative and ergodic.

Proof. From Theorem 1.2.1 [7], $D = \bigcup_{n=1}^{\infty} D_k$, where $D_k \subset D_{k+1}$ and $\Sigma\, U^n(x,D_k)$ is bounded for any k. If $\lambda(D) > 0$ then there exists $\bar{k}$ such that $\lambda(D_{\bar{k}}) > 0$. But from (9) this implies $\sum_{n=1}^{\infty} U^n(x,D_k) = \infty$, a contradiction, Thus $\lambda(D) = 0$ and U is conservative.

Suppose now $U(x,A) = \chi_A(x)$ λ a.e. and $\lambda(A) > 0$. Then using (9) again, $\sum_{n=1}^{\infty} U^n(x,A) = \infty$ for $x \in E$, and by Corollary 2, $U(x,A) > 0$ for each $x \in E$. So $\chi_A(x) = 1$, λ a.e. and U is ergodic.

We are in position now to prove the main result of this section.

Theorem 1. If the semigroup $(P_t)_{t\geq 0}$ satisfies (3), then there exists a unique σ-finite invariant measure μ and $m << \mu$.

Proof. The existence follows from Corollary 1. From Lemma 4 any invariant measure is absolutely continuous with respect to λ. Lemmas 5 and 7 tell, that U is a well defined Markov conservative ergodic operator on $L_\infty(\lambda)$. Therefore we can apply Theorem 3.4.7 [7], which guarantees the uniqueness of invariant measure.

Similarly an analogous result can be proved for discrete time Markov processes. Since to the best of the author's knowledge such result seems to be unknown (see [14], [15]), we formulate and point out the only changes in the proof.

Theorem 1'. If the transition operator $P(x,\cdot)$ of a discrete time Markov process satisfies condition:

There exists a probability measure m such that

$$(10) \qquad \forall_{A\in\mathcal{E}}\ m(A) > 0 \Rightarrow \forall_{x\in E} \sum_{n=1}^{\infty} P^n(x,A) = \infty$$

then there exists unique σ-finite invariant measure μ and $m << \mu$.

Proof. Consider the operator $\bar{U} = \sum_{n=1}^{\infty} (\frac{1}{2})^n P^n$. Then

$$\bar{U}_0 \overset{\text{def}}{==} \sum_{n=1}^{\infty} \bar{U}^n = \frac{1}{2} \sum_{n=1}^{\infty} P^n$$

and (10) can be reformulated

$$(11) \qquad \forall_{A\in\mathcal{E}}\ m(A) > 0 \Rightarrow \forall_{x\in E} \sum_{n=1}^{\infty} \bar{U}^n(x,A) = \infty .$$

The operators $\bar{U}^k$ are Markov and for each $x \in E$, the measures $\bar{U}^k(x,\cdot)$ and $\bar{U}_0(x,\cdot)$ are equivalent.

Finally μ is P invariant if and only if $\bar{U}$ is invariant. The remaining steps of the proof are exactly the same as in the proof of Theorem 1.

3. Strong Harris Condition

In this section we will recall first so-called Harris condition which we call strong Harris comparing with its weak version from Section 2.

Definition 2. If there exists a probability measure m such that

$$(12) \qquad \forall_{A\in\mathcal{E}}\ m(A) > 0 \Rightarrow \forall_{x\in E}\ P_x\left\{ \int_0^\infty \chi_A(x_s)ds = \infty \right\} = 1$$

then the continuous time Markov process is called strong Harris. Analogously for discrete time Markov process with transition operator $P(x,\cdot)$ if

$$(13) \qquad \forall_{A\in\mathcal{E}}\ m(A) > 0 \Rightarrow \forall_{x\in E}\ P_x\left\{ \sum_{n=1}^{\infty} \chi_A(x_n) = \infty \right\} = 1$$

then discrete strong Harris condition is satisfied.

To get an unique invariant measure it is imposed usually either (12) for continuous time Markov processes ([2]) or (13) for discrete time Markov processes ([9], [14], [15]). Since strong Harris process is also weak Harris we can formulate a Corollary of *Theorem 1*.

Corollary 4. Under (12) in continuous time or (13) in discrete time case, there exists a unique invariant measure μ and $m << \mu$.

Below, we consider several situations in which strong Harris condition is satisfied. Define first the most natural topology for standard Markov processes, so called fine topology.

Definition 3. A Borel set A is a fine neighborhood of x (denote $A \in \mathcal{O}_f(x)$) if $P_x[\sigma_A > 0] = 1$, where $\sigma_A = T_{A^c}$, and for any $B \in \mathcal{E}$, $T_B = \inf\{t > 0, x_t \in B\}$. If $\forall_{x\in A}$, $A \in \mathcal{O}_f(x)$, then A is finely open and we denote $A \in \mathcal{O}_f$.

The family of open sets in original topology of $(E,\mathcal{E})$ will be denoted by $\mathcal{O}$ and its basis at point x by $\mathcal{O}(x)$.

Definition 4. A point $x \in E$ is finely strongly recurrent for standard Markov process X if and only if

$$(14) \qquad \forall_{y \in E} \forall_{A \in \mathcal{O}_f(x)} L(y,A) \stackrel{def}{=\!=} P_y\{\exists_{t \geq 0}, x_t \in A\} = 1.$$

A point $x \in E$ is called strongly recurrent if

$$(15) \qquad \forall_{y \in E} \forall_{A \in \mathcal{O}(x)} L(y,A) = 1.$$

The next lemma provides an equivalent characterization of finely strongly and strongly recurrent points.

Lemma 8. A point x is finely strongly recurrent if and only if

$$(16) \qquad \forall_{y \in E} \forall_{A \in \mathcal{O}_f(x)} Q(y,A) \stackrel{def}{=\!=} P_y\{\omega: \overline{\lim_{t \to \infty}} \ \chi_A(x_t(\omega)) = 1\} = 1.$$

Similarly a point x is strongly recurrent if and only if

$$(17) \qquad \forall_{y \in E} \forall_{A \in \mathcal{O}(x)} Q(y,A) = 1.$$

Proof. It suffices to show (14) $\Rightarrow$ (16) and (15) $\Rightarrow$ (17). We will prove the first implication since the proof of the second is a particular case of the first one. Let for $A \in \mathcal{O}_f(x)$

$$T_A^N = \inf\{s > N, x_s \in A\}$$

If for each N, $T_A^N < \infty$, P_y a.s., then $Q(y,A) = 1$. Suppose there exists N such that $P_y\{T_A^N = \infty\} > 0$. Then $E_y[P_{x_N}\{T_N = \infty\}] > 0$ what contradicts (14) since $P_z\{T_A = \infty\} = 1 - L(z,A) = 0$ for any $z \in E$.

Now we can formulate sufficient assumptions for strong Harris condition to be satisfied.

Proposition 2. Suppose there exists a probability measure m such that

(18) $\qquad \forall_{A \in \mathcal{O}_f}\, m(A) > 0 \Rightarrow \forall_{y \in E}\, Q(y,A) = 1$

then a continuous strong Harris condition with measure $\lambda = mU$ is satisfied.

Proof. Let $B \in \mathcal{E}$. Denote $A_t = \int_0^t \chi_B(x_s)ds$. If $\lambda(B) > 0$, then $\int_E U(x,B)m(dx) > 0$ and because of Corollary 2, $P_m(A_\infty > 0) > 0$. Therefore there exists $\epsilon, \delta > 0$ such that for $T_\epsilon = \inf\{s > 0: A_s > \epsilon\}$, $m(\{x: P_x\{T_\epsilon < \infty\} > \delta) > 0$. Since

$$u_{T_\epsilon}(x) \overset{\text{def}}{=\!=} P_x\{T_\epsilon < \infty\}$$

is excessive, then the set $\{x: u_{T_\epsilon}(x) > \delta\}$ is finely open and we can apply (18). But from Lemma 2a [1], $u_{T_\epsilon}(x_t) \to 0$ on $\{T_\epsilon = \infty\}$ as $t \to \infty$, P_y a.s. for $y \in E$. Thus for each $y \in E$, $P_y(T_\epsilon = \infty) = 0$. Define $T_1 = T_\epsilon$, $T_{n+1} = T_n + T_\epsilon \circ \theta_{T_n}$. By induction $P_y(T_\epsilon < \infty) = 1$ for $y\ ^n \in E$, and since $A_{T_n} \geqslant n\epsilon$, then

$$P_y\left\{\int_0^\infty \chi_B(x_s)ds = \infty\right\} = 1$$

for $y \in E$.

Corollary 5. If x is finely strongly recurrent, then a strong Harris condition with measure $U(x,\cdot)$ is satisfied.

Proof. Put $m = \delta_{\{x\}}$, Dirac measure at x, then taking into account (16) we obtain (18).

Corollary 6. Assume x is strongly recurrent and any excessive function is l.s.c. at x. Then the process X is strong Harris with measure $U(x,\cdot)$.

Proof. Consider $m = \delta_{\{x\}}$. Then (18) holds for any open set. Define similarly as in the proof of Proposition 2 the additive functional A_t, $\epsilon,\delta > 0$ and excessive function $u_{T_\epsilon}(y)$. Since $u_{T_\epsilon}(y)$ is l.s.c. at x, there exists an open set $C \in \mathcal{O}(x)$ such that $P_y(T_\epsilon < \infty) > \delta$ for $y \in C$. Since $Q(z,C) = 1$ for each $z \in E$, then using the same arguments as in the proof of Proposition 2 we obtain $P_z\{T_\epsilon < \infty\} = 1$ for $z \in E$, and finally strong Harris condition (12).

Corollary 7. If for some $\Delta > 0$, the process $X_\Delta = (x_{n\Delta})$ satisfies discrete strong Harris condition (13), then the continuous time process $X = (x_t)_{t \geq 0}$ also satisfies strong Harris condition (12).

Proof. The fact that X_Δ satisfies (13) implies (18) and then (12).

Remark 1. An inverse result to Corollary 7 can be obtained under an additional assumption that $X = (x_t)_{t \geq 0}$ is regular Harris. Let μ be the unique invariant measure associated with X and $P_t^s(x,\cdot)$ denotes a singular part from the decomposition of $P_t(x,\cdot)$ with respect to μ. If $\lim_{t\to\infty} P_t^s(x,E) = 0$ for each $x \in E$, then X is called regular Harris. Every strong Feller Harris process is regular. For proofs and details see [4].

4. Feller Markov Processes

Throughout this section almost everywhere we will assume at least that standard Markov process X is Feller, i.e., the semigroup $(P_t)_{t\geq 0}$ associated with X transforms the space of all continuous bounded functions C into C.

Let us generalize first two existence theorems due to Foguel [8] and Lin [12].

Theorem 2. Let X be Feller Markov and K a compact set. Then either

$$\sup_{x\in E}\left|\frac{1}{t}\int_0^t P_s\chi_K(x)ds\right| \to 0 \quad \text{as} \quad t\to\infty$$

or there exists an invariant probability measure μ.

Proof. In Foguel [8], Section 4, one can find an identical result formulated for strongly continuous on C semigroup $(P_t)_{t\geq 0}$. The analysis of proof shows that in fact only Feller property was applied.

Theorem 3. Assume X is Feller Markov and there exists a continuous nonnegative, with compact support function g such that

$$(19) \qquad U_0 g(x) = \int_0^\infty P_s g(x)ds = \infty \quad \text{for} \quad x\in E.$$

Then there exists an invariant σ-finite measure μ.

Proof. We follow again Foguel [8]. The proof for discrete time case of Section 5 [7] can be easily adapted to continuous time Feller processes, or we can consider discrete time Feller process with the transition operator $U(x,\cdot)$ for which Theorem 5.3 [8] can be exactly applied.

Theorems 2 and 3 do not provide any information about possible uniqueness of invariant measure. To obtain this we have to assume more. In Section 3 we presented several uniqueness assumptions which led to Harris condition. Below we formulate an uniqueness result exploiting different arguments.

Proposition 3. Suppose the assumptions of Theorem 3 are satisfied and moreover the measures $U(x,\cdot)$ for $x \in E$ are equivalent. Then there exists a unique σ-finite invariant measure.

Proof. Since $U(x,\cdot)$, $x \in E$ are equivalent, for fixed $\bar{x}$ put $m(\cdot) = U(\bar{x},\cdot)$. An analysis of the proofs of Lemmas 3 and 4 shows that they still hold for m defined above. Thus from Corollary 3, for any invariant measure μ, $m << \mu << mU = U^2(\bar{x},\cdot)$. But m is equivalent to $U^2(\bar{x},\cdot)$ (Corollary 2). Therefore any invariant measure is equivalent to m. Let μ_1, μ_2 be σ-finite invariant measures such that $\mu_1(g) = \mu_2(g)$, and f_1, f_2 denote their densities with respect to measure m. Then the measure $(\mu_1 - \mu_2)^+$ is well defined as a measure with density $(f_1 - f_2)^+$, and one can easily see $(\mu_1 - \mu_2)^+ \leq (\mu_1 - \mu_2)^+P_r$ for any $r \geq 0$. Now

$$
\begin{aligned}
& 0 \leq \langle(\mu_1 - \mu_2)^+P_r - (\mu_1 - \mu_2)^+, \int_0^t P_s g\, ds\rangle \\
(20) \qquad & = (\mu_1 - \mu_2)^+ \int_t^{t+r} P_s g\, ds - \int_0^r P_s g\, ds \leq \mu_1 \int_t^{t+r} P_s g\, ds \\
& = r\mu_1(g) < \infty .
\end{aligned}
$$

Letting $t \to \infty$, from (19) we obtain $(\mu_1 - \mu_2)^+ = (\mu_1 - \mu_2)^+P_r$ i.e., the measure $(\mu_1 - \mu_2)^+$ is invariant. Then also $(\mu_1 - \mu_2)^-$ is invariant, and $(\mu_1 - \mu_2)^+$, $(\mu_1 - \mu_2)^-$ form a pair of singular invariant measures. Since all invariant measures are equivalent to m, and $\mu_1(g) = \mu_2(g)$, this can happen only when $\mu_1 = \mu_2$.

In the above proof the Feller property of X and the continuity of g was only important to guarantee the existence of an invariant measure. Therefore we have the following corollary:

Corollary 8. If for standard Markov process X with semigroup $(P_t)_{t\geq 0}$ there exists a nonnegative bounded measurable function g with compact support such that $U_0g(x) =$

∞ for $x \in E$, and the measures $U(x,\cdot)$ for $x \in E$ are equivalent, then there exists at most one σ-finite invariant measure.

There is a large family of standard Markov processes with semigroup $(P_t)_{t\geq 0}$ transforming the space of bounded measurable functions into C. We call such processes strong Feller Markov. The next proposition generalizes an existence and uniqueness result for continuous strong Feller processes due to Khasminskii [11] to right continuous strong Feller processes.

Proposition 4. Suppose X is strong Feller Markov such that

(21) the semigroup $(P_t)_{t\geq 0}$ transforms the space of continuous functions vanishing at infinity C_0 into C_0

(22) $$\forall_{A\in\mathcal{O}}\ \forall_{t>0,x\in E}\ P_x\{\chi_A(x_t)\} \overset{\text{def}}{=\!=} p(x,t,A) > 0$$

(23) there exists a compact recurrent set K, i.e.,

$$\forall_{x\in E}\ P_x\{T_K < \infty\} = 1.$$

Then there exists a unique σ-finite invariant measure.

Proof. Let $L = \{x \in E, \rho(x,K) \leq R\}$, where ρ denote a metric compatible with topology of $(E,\mathcal{E})$ in which every closed ball is compact. We will prove first that for a sufficiently large R and fixed $a > 0$ there exists $\delta > 0$ such that $\inf_{x\in K} P_x(\sigma_L > a) \geq \delta$. In fact, from Proposition 1 [18], because of (21)

$$\sup_{x\in K} P_x(\sigma_L \leq a) = \sup_{x\in K} P_x\left[\sup_{0\leq s\leq a} \rho(x_s,x) > R\right] \to 0 \quad \text{as } R \to \infty .$$

Define now the following sequence of Markov times

$$T_1 = T_K,\ T_2 = T_1 + \sigma_L \circ \theta_{T_1},\ \dots\ T_{2n} = T_{2n-1} + \sigma_L \circ \theta_{T_{2n-1}},$$
$$T_{2n+1} = T_{2n} + T_K \circ \theta_{T_{2n}},\ \dots .$$

For each $x \in E$, if T_{2i} is finite, then also T_{2i+1} is P_x a.s. finite.

Let for $n = 1,2, \dots,$

$$S_n = \{\omega: T_{2n}(\omega) - T_{2n-1}(\omega) > a,\ T_{2n-1}(\omega) < \infty\}.$$

Then

$$(24) \qquad P_x[S_n \mid F_{T_{2n-1}}] = P_{x_{T_{2n-1}}}[\sigma_L > a] \cdot \chi_{T_{2n-1}<\infty} \geq \delta\chi_{T_{2n-1}<\infty}$$

From generalized Borel Cantelli lema

$$(25) \qquad \sum_{n=1}^{\infty} \chi_{S_n} = \infty \Leftrightarrow \sum_{n=1}^{\infty} P_x[S_n \mid F_{T_{2n-1}}] = \infty \quad P_x \text{ a.s.}$$

Therefore if $T_{2n-1} < \infty$ for each $n \in N$, then from (24), (25),

$$(26) \qquad \sum_{n=1}^{\infty} \chi_{S_n} = \infty \quad P_x \text{ a.s.} .$$

But if $T_{2n-1} = \infty$ for some $n \in N$, the process X remains in L forever. Thus for any continuous function g with compact support, equal 1 on L, we have $U_0 g(x) = \infty$ for all $x \in E$.

The proof will be finished if we show that measures $U(x,\cdot)$, $x \in E$ are equivalent, since then we can apply Proposition 3.

For $B \in \mathcal{E}$ put $\Gamma = \{x: p(t,x,B) > 0\}$ for some $t > 0$. Since $p(t,x,B)$ is x-continuous, then Γ is open and from (22) for any $s > 0$, $y \in E$, $p(s,y,\Gamma) = p(s + t, y, B) > 0$. Because B and t were chosen arbitrarily, for any set $B \in \mathcal{E}$, we have either $p(s,x,B) > 0$ for $s > 0$, $x \in E$ or $p(s,x,B) = 0$ for all $s > 0$, $x \in E$. But then of course the measures $U(x,\cdot)$, $x \in E$ are equivalent. The proof of Proposition 4 is finished.

The remaining part of this section is devoted to standard Markov processes with semigroup $(P_t)_{t\geq 0}$ satisfying (21).

It is proved in [17] that if $(P_t)_{t\geq 0}$ in addition is quasicompact on the space of bounded measurable functions, then there exists a finite disjoint family of invariant sets, with each of them is associated a unique invariant probability measure, and any invariant measure is a combination of these measures.

Moreover a quasicompactness of $(P_t)_{t\geq 0}$ is equivalent to so-called Doeblin condition satisfied for some $\Delta > 0$ which tells, that there exists a finite measure m, a positive integer k and $\epsilon > 0$ such that

(27) $$\forall_{B\in \mathcal{E}}\ m(B) < \epsilon \Rightarrow \forall_{x\in E}\ P_x(x_{k\Delta} \in B) < 1 - \epsilon.$$

One can easily see that if we impose any condition which guarantees the existence of unique invariant set, then we automatically obtain the existence and uniqueness of invariant measure.

Proposition 5. Suppose the semigroup $(P_t)_{t\geq 0}$ of standard Markov process is quasicompact and satisfies (21). If moreover the measures $U(x,\cdot)$, $x \in E$ are equivalent, then there exists a unique invariant probability measure.

Proof. It suffices to notice that the equivalence of $U(x,\cdot)$ for $x \in E$ implies that there are no disjoint invariant sets.

5. Examples

We finish the paper with three examples which should explain some assumptions imposed in the paper as well as difficulties we meet.

Example 1. Consider a continuous version of Horowitz example [10]. Let $E = R^1$, and μ be a measure concentrated on a countable set, that is not contained in a discrete subgroup of R^1. Let X be a right continuous Markov process with semigroup

$$P_t f(x) = e^{-t} f(x) + (1 - e^{-t}) \int f(x - y)\mu(dy).$$

Following [10] we can show that Lebesgue measure λ is a unique invariant measure for P_t. The measures μ and λ are singular and invariant sets are of λ measure 0. Moreover X is Feller. Nevertheless it is not weak Harris. In fact, every weak Harris process, because of Corollary 3 should satisfy the condition (3) with m replaced by invariant measure. But in our case $U_0(x,\cdot)$ is concentrated on the set of λ measure 0. Thus there are processes possessing unique invariant measure which are not weak Harris.

Example 2. Consider an example from [3], p. 289. Let E = (0,1) and X be a right continuous Markov process with the semigroup

$$P_t f(x) = e^{-t} f(x) + (1 - e^{-t}) \int_0^1 f(y)dy.$$

Then X is Feller and $\{x\} \in \mathcal{O}_f(x)$. There are no finely recurrent points, but any point is strongly recurrent. The functions $f(y) = \chi_{\{x\}}(y)$ are excessive and are not l.s.c. Thus Corollary 6 cannot be applied. But X is strong Harris, and for $m = \lambda$ (12) is satisfied, and λ is in fact a unique invariant measure.

Example 3. Let E = [0,1]. We will apply the following fact from Lebesgue measure theory.

Proposition 6. There exists a measurable set $C \subset [0,1]$, such that $\lambda(C) = 1/2$ and for any open interval $I \subset [0,1]$

$$\lambda(C \cap I) = \lambda(I \setminus C) = \frac{1}{2}\lambda(I).$$

The proof is based on filling in the holes in a Cantor set by another smaller Cantor set. For details see Exercise 5, p. 244 [16].

Put $\lambda_1(B) = \lambda(C \cap B)$, $\lambda_2(B) = \lambda(B \setminus C)$

$$P(x,B) = \begin{cases} \lambda_1(B) & \text{if } x \in C \\ \lambda_2(B) & \text{if } x \in C \end{cases}$$

for measurable $B \subset [0,1]$.

A right continuous process with the semigroup

$$P_t f(x) = e^{-t} f(x) + (1 - e^{-t}) \int_0^1 f(y) P(x,dy)$$

is Feller Markov. Every point is strongly recurrent and λ_1, λ_2 are singular invariant measures for $(P_t)_{t \geq 0}$.

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