

AD-A172 592

PRECONDITIONING FOR SINGULAR PERTURBATION PROBLEMS(U)
WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER
S V PARTER AUG 86 MRC-TSR-2949 DAAG29-80-C-0041

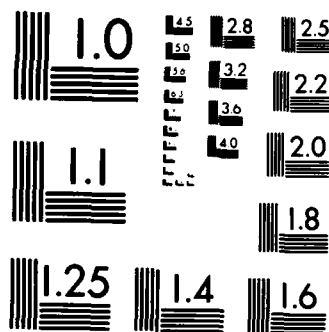
1/1

UNCLASSIFIED

F/G 12/1

NL





MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

AD-A172 592

MRC Technical Summary Report #2949

PRECONDITIONING FOR SINGULAR
PERTURBATION PROBLEMS

Seymour V. Parter

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

August 1986

(Received July 23, 1986)

DTIC
ELECTE
OCT 8 1986
S B
X

DTIC FILE COPY

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

Air Force Office of Scientific Research
Bolling AFB
Washington, DC 20332

86 10 7 164

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

PRECONDITIONING FOR SINGULAR PERTURBATION PROBLEMS*

Seymour V. Parter

Technical Summary Report #2949
August 1986

ABSTRACT

In this report we consider the concept of operators "equivalent in norm" and its use in understanding preconditioning for singularly perturbed elliptic problems.

AMS (MOS) Subject Classifications: 65F10, 65N10

Key Words: preconditioning, singular perturbation, elliptic, spectral equivalence, norm equivalence

Work Unit Number 3 (Numerical Analysis and Scientific Computing)

* This report was presented at the BAIL IV Conference, July 7-11, 1986
Novosibirsk, Akademgorodok, USSR.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and
the United States Air Force under Grant No. AFOSR-86-0163.

SIGNIFICANCE AND EXPLANATION

The classical iterative methods for the solution of the large systems of linear equations which arise in the numerical solution of elliptic boundary value problems become less efficient as the size of the system increases, i.e. as the numerical approximation gets better. This problem is particularly acute for singularly perturbed elliptic operators.

If one can find an appropriate "preconditioner" this difficulty can be either ameliorated or eliminated. Previous research in this direction has been limited to the special case where the symmetric part of the operator is positive definite. In this report we comment on current efforts to resolve the general problem.

[illegible]

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

PRECONDITIONING FOR SINGULAR PERTURBATION PROBLEMS*

Seymour V. Parter

1. Introduction

The numerical solution of elliptic boundary-value problems leads to the related problem of actually obtaining the solution of a large, sparse system of linear equations

$$A_n u = f \quad (1.1)$$

where A_n is an $n \times n$ matrix with $n = O(h^{-2})$. There is a large literature connected with the analysis of iterative methods for the solution of (1.1) - see [18], [19].

Almost all iterative methods, including the multigrid methods [14] can be cast in the framework of a preconditioning followed by iterative improvement. With the practical success of multigrid methods for uniformly elliptic problems with positive real part there is especial interest in preconditioned iterative methods for which the condition of the preconditioned system is bounded independent of the mesh spacing (i.e. the dimension n of (1.1)) for sufficiently fine meshes. In that case one can easily develop iterative methods that yield estimates of the form

$$\| \epsilon^j \| < K_0 \left[\frac{k-1}{k+1} \right]^j \| \epsilon^0 \|$$

where $k = c$ or $c^{1/2}$ ($c = \text{condition } (B_n^{-1} A_n)$) depending on the implementation. In particular, several authors [1], [3], [6], [8] have

* This report was presented at the BAIL IV Conference, July 7-11, 1986 Novosibirsk, Akademgorodok, USSR.

suggested preconditioned iterative methods where the preconditioner B_n^{-1} is the inverse of the discretization of a separable, positive definite elliptic operator. In the special case where the domain is a rectangle the separable problem can be solved by fast direct solvers [16], [17]. Others, e.g. [13], have considered using several iterations of a classical positive definite stationary one-step iterative method which is either consistent with the problem at hand or another (usually separable) elliptic problem.

In all those cases both operators have positive definite symmetric part and one can analyze the efficiency of the preconditioned iterative method by using the concept of "Spectrally Equivalent Operators" introduced by D'Yakanov [7].

In this report we are concerned with singularly perturbed elliptic operators of the form

$$L^\epsilon u := -\epsilon^2 \sum_{i,j=1}^2 \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} + \epsilon^r \sum_{i=1}^2 b_i(x) \frac{\partial u}{\partial x_i} + a_0(x)u \quad (1.2a)$$

and

$$\text{specified boundary conditions ,} \quad (1.2b)$$

where $0 < \epsilon \ll 1$, $x = (x_1, x_2)$ and $r = 0$ or 1 depending on the application/problem. We will either assume that

$$b_1(x) > b_0 > 0 , \quad (1.3a)$$

or that

$$b_1(x) \equiv b_2(x) \equiv 0 . \quad (1.3b)$$

A case of particular interest is the indefinite Helmholtz equation

$$Lu := -\epsilon^2 \Delta u - a_0(x)u, \quad a_0(x) > \bar{a} > 0 , \quad (1.4)$$

(one usually writes this equation in terms of $k^2 = \epsilon^{-2}$). On the other hand, the usual singular perturbation problems have $r = 0$ and (1.3a) holds. In

this case it is easy to show that [4] there is an $\varepsilon_0 > 0$ and for $0 < \varepsilon < \varepsilon_0$ the operator L has a positive definite symmetric part.

With all this in mind, we now outline the discussion in this paper. In Section 2 we discuss new results of Faber, Manteuffel and Parter [9] on (norm) Equivalence of Operators. These ideas and classical results of Nitsche and Nitsche [15] and Drya [5] allow one to discuss preconditioning of discrete elliptic operators without requiring positivity. One only requires that both the elliptic operators and their discretizations be invertible. In Section 3 we present some ideas and results of Bayliss, Goldstein and Turkel [2] and Goldstein [10] on preconditioning of singular perturbation problems by a "partial" multigrid cycle. Finally, in Section 4 we discuss current work of Goldstein and Parter [12] on preconditioners B_n^{-1} so that

$$c(B_n^{-1}A_n) = \|B_n^{-1}A_n\|_{\ell_2} \|A_n^{-1}B_n\|_{\ell_2} \quad (1.5)$$

is independent of n and has controlled, or at least known, dependence on ε .

2. Equivalence of Operators

Given two linear operators A and B on the Hilbert space H , we say that A is equivalent in norm to B on the set $D \subseteq H$ if there exist $0 < \alpha < \beta < \infty$ such that

$$\alpha < \frac{\|Ax\|}{\|Bx\|} < \beta, \quad \forall x \in D, \quad \|Ax\|, \|Bx\| \neq 0. \quad (2.1)$$

If A and B are positive, self-adjoint operators on H then, following D'Yakanov [7], we say that A is equivalent in spectrum to B on $D \subseteq H$ if there exist $0 < \alpha < \beta < \infty$ such that

$$\alpha < \frac{\langle Ax, x \rangle}{\langle Bx, x \rangle} < \beta, \quad \forall x \in D, \quad \langle Ax, x \rangle, \langle Bx, x \rangle \neq 0. \quad (2.2)$$

While there are many interesting relationships between these concepts there is one striking fact. In [9] one finds a simple example of an A and a B both

of which are compact, positive, self-adjoint operators and

(i) A is equivalent in spectrum to B , (ii) A is not equivalent in norm to B .

Let A_n and B_n be two sequences of operators defined on D . We say that $\{A_n\}$ is equivalent in norm to $\{B_n\}$ uniformly on D if there exist $0 < \alpha, \beta < \infty$ such that

$$\alpha < \frac{\|A_n x\|}{\|B_n x\|} < \beta, \quad \forall x \in D, \quad \|A_n x\|, \|B_n x\| \neq 0. \quad (2.3)$$

Or, we could say, let $S_n \subseteq D \subseteq H$ be a sequence of subspace of D and let A_n, B_n be operators defined on S_n . Then A_n is uniformly equivalent to B_n in norm if (2.3) holds for every $x \in S_n$.

Let A, B be two invertible elliptic operators of the same order, say $2m$, defined on the same domain Ω with homogeneous boundary conditions. Clearly

$$A : H_{2m}(\Omega) \rightarrow L_2(\Omega), \quad B : H_{2m}(\Omega) \rightarrow L_2(\Omega) \quad (2.4)$$

where $H_{2m}(\Omega)$ is the Sobolev space of functions having $2m$ 'th derivatives in $L_2(\Omega)$. In many cases one also has

$$A^{-1} : L_2(\Omega) \rightarrow H_{2m}(\Omega), \quad B^{-1} : L_2(\Omega) \rightarrow H_{2m}(\Omega). \quad (2.5)$$

When this happens

$$AB^{-1} : L_2 \rightarrow L_2, \quad BA^{-1} : L_2 \rightarrow L_2 \quad (2.6)$$

and the operators A and B are equivalent in norm. Suppose $\{A_n\}, \{B_n\}$ with A_n, B_n defined on $S_n \subseteq D \subseteq H$ are discretizations of A and B . One can ask, is it possible to prove analogs of (2.4), (2.5) and (2.6)? If one uses a finite element approach with $S_n \subseteq H_{2m}$, then it is easy to show that these analogs all hold. In the case of finite-difference discretizations the problem is more subtle. For second order equations and when Ω is a

rectangle one can extend the results of [15] to obtain an appropriate definition of $H_2(\Omega;h)$ and obtain estimates of the following form:

(i) Let

$$Lu = -\{(au_x)_x + (bu_x)_y + (bu_y)_x + (cu_y)_y\} + Du_x + Eu_y + \sigma u. \quad (2.7)$$

(ii) Let the boundary conditions be of the form

$$u \text{ or } \frac{\partial u}{\partial n} = 0 \text{ on a full side.} \quad (2.8)$$

(iii) Let $L_h u$ be a standard finite-difference discretization of (2.7).

Then, if there is a constant $k_0 = k_0(L_h)$ independent of h , $0 < h < h_0$ such that $\|u\|_{\ell_2} < k_0 \|L_h u\|_{\ell_2}$ then there are constants K_0, K_1 depending only on the coefficients of L and k_0 such that $\|L_h u\|_{\ell_2} < K_1 \|u\|_{H_2(\Omega;h)}$,

$\|u\|_{H_2(\Omega;h)} < K_0 \|L_h u\|_{\ell_2}$. Thus, if A and B are two such elliptic operators and $\{A_n\}$ and $\{B_n\}$ are their finite-difference discretizations we have

$$\|A_n B_n^{-1}\|_{\ell_2} < K_1(A_n) \cdot K_0(B_n). \quad (2.9)$$

At the same time, since A_n^*, B_n^* are finite-difference approximation to A^* and B^* we obtain

$$\|B_n^{-1} A_n\|_{\ell_2} = \|A_n^* (B_n^*)^{-1}\|_{\ell_2} < K_1(A_n^*) K_0(B_n^*). \quad (2.10)$$

These results show that, in the rectangle with boundary conditions (2.8) one may precondition any invertible discrete elliptic operator with any other such discrete elliptic operator and obtain

$$c(B_n^{-1} A_n) < K_1(A_n^*) K_0(A_n^*) K_1(B_n^*) K_0(B_n^*) \quad (2.11)$$

which is independent of n . Therefore, the important, and as yet unresolved, question is: how to choose "good" preconditioners and how to measure the "goodness". Some preliminary results and ideas are presented in [9].

Remark 1: If one has only Dirichlet boundary conditions one may make a simple extension of the results of Drya [4] to obtain similar results when Ω is a

convex, polygonal region whose sides have slope $0, \infty$ or ± 1 and whose vertices lie on points of a regular mesh $h = \Delta x = \Delta y$.

Remark 2: While we have not shown the details here, an important fact which emerges from the analysis of [9] is the observation that in these cases the finite-difference calculus allows one to completely mimic certain techniques of estimation normally used in the analysis of elliptic differential operators. Hence, in Section 4 we will obtain all estimates in the computationally easier case of the continuous elliptic operators.

Remark 3: Obviously, equivalence in norm is transitive. Hence, so is the concept "the condition number of $(B_n^{-1}A_n)$ is bounded independent of n ".

3. Multigrid as a Preconditioner

In [2] Bayliss, Goldstein and Turkel consider the indefinite Helmholtz equation $\Delta u + k^2 a_0(x,y)u = 0$ with a suitable radiation condition at infinity. After discretization via a finite-element method (and truncation of the region) one has a system of the form (1.1) where A_n is not symmetric (because of the radiation condition at infinity) and the symmetric part is not positive definite (because $k^2 \gg 1$ and $a_0 > 0$).

Let A_n^0 be the symmetric positive definite operator obtained by setting $k^2 = 0$. Let M_r^{-1} denote the application of a symmetric "partial" multigrid operator for the solution of $A_n^0 x = b$ based on r grids. We say a partial multigrid operator because we do not "solve" on the coarsest grid. Rather, we only "smooth" on the coarsest grid. Consider the preconditioned normal equation

$$(M_r^{-1}A_n)^*(M_r^{-1}A_n)u = (M_r^{-1}A_n)^*M_r^{-1}f. \quad (3.1)$$

These equations are now solved by the conjugate gradient method. While the authors of [2] give no rigorous treatment of the effectiveness of their

method, their experimental results and heuristic arguments are extremely interesting.

In [10] Goldstein makes a complete analysis of a simpler problem. Consider the operator L^ε given by (1.2a) with $r = 1$ and boundary conditions $u|_{\partial\Omega} = 0$, where Ω is the unit square. The coefficients are real and smooth and $a_i(x) > c_0 > 0$, for $i = 0, 1, 2$. Let L_h^ε be a standard finite-difference discretization of L^ε where $\varepsilon \sim h^\sigma$ with $0 < \sigma < 1$. Let M_σ^{-1} denote the application of a partial symmetric multigrid for the Laplace operator $Mu : -\Delta u$. In this notation the subscript σ reminds us that Goldstein chooses the coarsest grid so that $h_{\text{coarsest}} \sim \varepsilon \sim h^\sigma$. He now solves this problem by applying conjugate gradient to the normal equations $(M_\sigma^{-1} L_h^\varepsilon)^* (M_\sigma^{-1} L_h^\varepsilon) u = (M_\sigma^{-1} L_h^\varepsilon)^* M_\sigma^{-1} f$. In [11] he applies this method to some interesting applied singular perturbation problems. In this definite case he shows that $c(M_\sigma^{-1} L_h^\varepsilon) < c_1$ where c_1 is independent of h and ε !

4. General Estimates

Let Ω be a smooth domain. Let $|u|_r$ be the seminorm

$$|u|_r^2 = \sum_{|\alpha|=r} \iint |D^\alpha u|^2 dx. \quad (4.1)$$

Let L be a second order elliptic operator of the form given by (1.2a). We assume that the boundary conditions are homogeneous and yield no boundary terms. That is

$$\begin{aligned} \iint_\Omega u [Lu] dx dy &= \varepsilon^2 \iint_\Omega [\Sigma a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}] dx \\ &+ \varepsilon^r \Sigma \iint b_i \frac{\partial u}{\partial x_i} u dx dy + \iint a_0 u^2 dx. \end{aligned} \quad (4.2)$$

Further, we assume the basic estimate of [15] holds. That is, let

$$Au = -\sum a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} \quad (4.3)$$

with the same boundary conditions as hold for L . Then there is a constant $K_0 = K_0(A)$ such that $|u|_2^2 \leq K_0 \|Lu\|_{L_2}^2$.

We remind the reader that the estimates in this section are for these elliptic operators. However, in the case of rectangles or certain convex polygonal domains (with zero Dirichlet boundary conditions) these estimates go over for finite-difference operators. Moreover, because of the transitivity of certain concepts, (e.g. "uniformly bounded condition number") these results also apply to the partial multigrid operators discussed in Section 3.

Case 1: $a_0(x) \geq c_0 > 0$ $r = 1$. (in (1.2a))

In this case we easily find that there are constants $\bar{c}_0, c_1, c_2$ so that

$$\|u\|_{L_2} \leq \bar{c}_0 \|Lu\|_{L_2} \quad (4.3a)$$

$$|u|_1 \leq \frac{c_1}{\epsilon} \|Lu\|_{L_2} \quad (4.3b)$$

$$|u|_2 \leq \frac{c_2}{\epsilon^2} \|Lu\|_{L_2} . \quad (4.3c)$$

These estimates lead immediately to the result

Theorem 1: Let $L, \tilde{L}$ be two elliptic operators which satisfy the hypothesis of Case 1. Then there are constants depending only on the coefficients $\bar{K}_1, \bar{K}_2$, but not on $\epsilon > 0$ such that

$$\|\tilde{L}L^{-1}\|_{L_2} \leq \bar{K}_1 \quad (4.4a)$$

$$\|\tilde{L}L^{-1}\|_{L_2} \leq \bar{K}_2 . \quad (4.4b)$$

Case 2: $r = 0$ and $b_1(x) > b_0 > 0$.

Once $\varepsilon < \varepsilon_0$, we may assume that the symmetric part of L is positive definite. This is done by setting $u = e^{\alpha x} v$, $\alpha \sim 0(1)$. Hence we obtain

$$(4.3a), (4.3b) \text{ and } \|u\|_2 < (c_2/\varepsilon^3) \|Lu\|_{L_2}.$$

Theorem 2: Let $L, \tilde{L}$ be two operators which satisfy the hypotheses of Case 1 or Case 2. There are constants $\bar{K}_1, \bar{K}_2$ such that

$$\|L\tilde{L}^{-1}\|_{L_2} < \bar{K}_1/\varepsilon, \quad \|\tilde{L}L^{-1}\|_{L_2} < \bar{K}_2/\varepsilon. \quad (4.5)$$

Case 3: $a_0(x) < -c_0 < 0$ and $r = 1$. We assume that for this value of ε , L^{-1} exists. That is, there are constants c_0', c_1, c_2 depending on ε but, essentially of order 1 and

$$\|u\|_{L_2} < \frac{c_0'}{\varepsilon^2} \|Lu\|_{L_2}, \quad \|u\|_1 < \frac{c_1}{\varepsilon} \|Lu\|_{L_2}, \quad \|u\|_2 < \frac{c_2}{\varepsilon} \|Lu\|_{L_2}. \quad (4.6)$$

Theorem 3: Let L be an operator of Case 3. There are constants $\bar{K}_1, \bar{K}_2$ such that

- (i) if $\tilde{L}$ is of Case 3 we have $\|L\tilde{L}^{-1}\|_{L_2} < \bar{K}_1/\varepsilon^2$.
- (ii) if $\tilde{L}$ is of Case 1 we have $\|L\tilde{L}^{-1}\|_{L_2} < \bar{K}_1/\varepsilon^2$, $\|\tilde{L}L^{-1}\|_{L_2} < \bar{K}_2$.
- (iii) if $\tilde{L}$ is of Case 2 we have $\|L\tilde{L}^{-1}\|_{L_2} < \bar{K}_1/\varepsilon^3$, $\|\tilde{L}L^{-1}\| < \bar{K}_2/\varepsilon$.

REFERENCES

- [1] Bank, Randolph E., "An automatic scaling procedure for a D'Yakanov-Gunn iteration scheme", *Linear Algebra and Appl.* 28 (1979), 17-33.
- [2] Bayliss, A., Goldstein, C. I. and Turkel, E., "The numerical solution of the Helmholtz equation for wave propagation problems in underwater acoustics", *Comp. Math. Appl.* 11 (1985), 655-665.
- [3] Concus, Paul and Golub, Gene H., "A Generalized Conjugate Gradient Method for Nonsymmetric Systems of Linear Equations", in *Lecture Notes in Economics and Mathematical Systems* 134, R. Glowinski and J. L. Lions, eds., Springer-Verlag, Berlin (1976), 56-65.
- [4] Dorr, F. W., Parter, S. V. and Shampine, L. F., "Applications of the maximum principle to singular perturbation problems", *SIAM Rev.* 15 (1973), 43-88.
- [5] Drya, M., "Priori estimates in W_2^2 in a convex domain for systems of difference elliptic equations", *Z. Vychisl. Mat. i Mat. Fiz.* 12 (1972), 1595-1601. Translated in *U.S.S.R. Computational Math. and Math. Phys.* 6, 291-300.
- [6] D'Yakanov, E. G., "On an iterative method for the solution of a system of finite-difference equations", *Dokl. Akad. Nauk* 138 (1961), 522.
- [7] D'Yakanov, E. G., "The construction of iterative methods based on the use of spectrally equivalent operators", *U.S.S.R. Computational Math. and Math. Phys.* 6 (1965), 14-46.
- [8] Elman, H. C. and Schultz, M. H., "Preconditioning by fast direct methods for non-self-adjoint nonseparable elliptic equations", *SIAM J. Numer. Anal.* 23 (1986), 44-57.

- [9] Faber, V., Manteuffel, T. A. and Parter, S. V., "On the equivalence of operators and the implications to preconditioned iterative methods for elliptic problems", to appear.
- [10] Goldstein, C. I., "Multigrid preconditioners applied to the iterative solution of singularly perturbed elliptic boundary value problem", BNL Report 51916.
- [11] Goldstein, C. I., "Multigrid preconditioners applied to three dimensional parabolic type models", BNL Report.
- [12] Goldstein, C. I. and Parter, S. V., work in progress.
- [13] Gunn, James E., "The solution of elliptic difference equations by semi-explicit iterative techniques", SIAM J. Numer. Anal., Ser. B. 2, 1 (1964).
- [14] McCormick, S. F., ed., "Multigrid Methods", SIAM series on Frontiers of Applied Mathematics 5 (1986).
- [15] Nitsche, J. and Nitsche, J. C. C., "Error estimates for the numerical solution of elliptic differential equations", Arch. Rational Mech. Anal. 5 (1960), 293-306.
- [16] Swarztrauber, Paul N., "A direct method for the discrete solution of separable elliptic equations", SIAM J. Numer. Anal. 11 (1974), 1136-1150.
- [17] Swarztrauber, Paul N., "The methods of cyclic reduction, Fourier analysis and the FACR algorithm for the discrete solution of Poisson's equation on a rectangle", SIAM Rev. 19 (1977), 190-501.
- [18] Varga, R. S. "Matrix Iterative Analysis", Prentice-Hall, Englewood Cliffs, NJ (1962).
- [19] Young, D. M., "Iterative Solution of Large Linear Systems", Academic Press, New York (1971).

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2949	2. GOVT ACCESSION NO. AD-A172	3. RECIPIENT'S CATALOG NUMBER 592
4. TITLE (and Subtitle) PRECONDITIONING FOR SINGULAR PERTURBATION PROBLEMS		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Seymour V. Parter		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041 AFOSR-86-0163
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Madison, Wisconsin 53705		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 3 - Numerical Analysis and Scientific Computing
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 below.		12. REPORT DATE August 1986
		13. NUMBER OF PAGES 11
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES U. S. Army Research Office Air Force Office of Scientific Research P. O. Box 12211 Bolling AFB Research Triangle Park Washington, DC 20332 North Carolina 27709		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) preconditioning singular perturbation elliptic spectral equivalence norm equivalence		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) In this report we consider the concept of operators "equivalent in norm" and its use in understanding preconditioning for singularly perturbed elliptic problems.		

END

11-86

DT/C