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MRC Technical Summary Report #2944

NONLINEAR VISCOELASTIC MATERIALS
WITH FADING MEMORY

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AD-A172 587

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July 1986

(Received July 18, 1986)

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ABSTRACT

The equations governing the motion of viscoelastic materials with fading

memory incorporate a nonlinear elastic-type response with a natural

dissipative mechanism. Our purpose is to discuss the subtle effects of this

mechanism in viscoelastic materials of Boltzmann type. Recent results on the

global existence and decay of classical solutions for smooth and small data

(in one space dimension) are reviewed for smooth and singular memory kernels;

for smooth kernels a number of such results can be generalized to several

space dimensions. A recent result on the development of singularities in

finite time for large data is discussed; several open problems are

formulated. A program for studying weak solutions for such systems, including

the development of numerical algorithms, is outlined.

NONLINEAR VISCOELASTIC MATERIALS WITH FADING MEMORY

John A. Nohel

1. **Introduction.** The equations governing the motion of nonlinear elastic bodies are quasilinear hyperbolic systems for which smooth solutions generally lose regularity in finite time due to the formation of shock fronts. Some materials incorporate a nonlinear elastic-type response with a natural dissipative mechanism, and it is important to understand the effects of the dissipation on the behaviour of the solutions of the equations of motion.

The purpose of this lecture is to discuss the effects of the subtle dissipative mechanism due to memory effects in viscoelastic materials of Boltzmann type. This dissipation is more delicate than that exhibited by viscoelastic materials of the rate type for which globally defined smooth solutions exist, even for large smooth data.

The paper is organized as follows. In Section 2 we formulate mathematical models for the motion of nonlinear viscoelastic materials and we motivate the mathematical theory. In Section 3 we survey recent results on the global existence of smooth solutions for smooth and small data. In Section 4 we present a recent result on the breakdown of smooth solutions for large, smooth data and discuss briefly related open questions including those regarding weak solutions and numerical methods (Remarks 4.8). We restrict our attention throughout to one-dimensional problems and provide some references for multidimensional problems. Moreover, we consider only a purely mechanical theory, i.e. we neglect thermal effects.

2. **Mathematical Models and Dynamic Problems.** Consider the longitudinal motion of a homogeneous one-dimensional body (e.g. a bar of uniform cross-section) occupying an interval B in a reference configuration, which we assume to be an equilibrium state, and

having unit reference density. B may be bounded or unbounded. Let $u(x,t)$ denote the displacement at time t of a particle with reference position x (i.e. $x + u(x,t)$ is the position at time t of the particle at x). The strain which measures local stretching is defined by $\varepsilon = u_x(x,t)$. Let σ denote the stress at time t of the particle with reference position x (σ measures the contact force per unit area). The balance of linear momentum yields the equation of motion

$$u_{tt} = \sigma_x + f \quad , \quad x \in B, \ t > 0 \ , \tag{2.1}$$

where subscripts denote partial derivatives and where f is an external body force. In order to characterize the material, (2.1) is supplemented by a constitutive assumption which relates the stress to the motion. In addition, initial data, as well as suitable boundary data if B is not $\mathbb{R}$, are adjoined to (2.1). We remark that in a physical problem the cross-section does not generally remain uniform as the bar is stretched. More realistic problems can be treated by similar techniques.

If the body is homogeneous and purely elastic, the stress depends on the strain through the constitutive relation $\sigma(x,t) = \phi(\varepsilon(x,t))$, where ϕ is a given smooth function satisfying the assumptions (i) $\phi(0) = 0$, (ii) $\phi'(0) > 0$; (i) reflects the fact that the reference position is taken as an equilibrium state, and (ii) that the stress increases with the strain, at least near equilibrium. The equation of motion (2.1) becomes the familiar, one-dimensional, quasilinear wave equation

$$u_{tt} = \phi(u_x)_x + f \qquad (x \in B, \ t > 0) \ , \tag{2.2}$$

if B is bounded it is assumed that the assigned boundary data and initial data are compatible. For (2.2) there is no natural dissipative mechanism. Indeed, Lax [33], also MacCamy and Mizel [37] and Kleinerman and Majda [31] have shown that if ϕ is not linear, the Cauchy problem for (2.2) ($f \equiv 0$) does not generally possess globally defined smooth solutions, no matter how smooth and small one takes the initial data $u(x,0)$ and $u_t(x,0)$.

In a material with memory (such as certain polymers, suspensions, or emulsions) the stress at a material point x and at time t depends on the entire history of the strain at x. In 1874 Boltzmann [5] gave the following linear constitutive law for small

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deformations in such materials

$$\sigma(x,t) = \beta\epsilon(x,t) + \int_0^\infty m(s)[\epsilon(x,t) - \epsilon(x,t-s)]ds, \ x \in B, \ -\infty < t < \infty \ . \qquad (2.3)$$

In (2.3) $\beta > 0$ is a given constant and $m : (0,\infty) \to R$ is a given positive, smooth, nonincreasing function. We limit our discussion to the situation in which $m \in L^1(0,\infty)$, and we distinguish two cases:

$$(i) \ 0 < m(0) < \infty, \quad (ii) \ m(0^+) = +\infty \ . \qquad (2.4)$$

The function m is called a memory function. The fact that $m > 0$ and non-increasing on $(0,\infty)$ means that the stress "relaxes" as t increases and the memory term in (2.3) fades: deformations which occurred in the distant past have less influence on the present value of the stress than those which occurred in the recent past. In the applied literature m is often assumed to be a finite linear combination of decaying exponentials with positive coefficients (these expressions result from least squares approximations to experimental data). Such restrictions are neither desirable nor necessary. Moreover, kinetic theories for chain molecules [15,46,53] and certain experiments [32,28] suggest that there are materials for which m is singular as in (2.4)(ii), $m(t) \sim t^{\alpha-1}$ as $t \to 0^+$, $0 < \alpha < 1$, m is positive, nonincreasing on $0 < t < \infty$, and m decays rapidly at infinity. Stronger power singularities at zero ($\alpha < 0$) are also possible, but the resulting mathematical theory for nonlinear materials consistent with our objectives is incomplete at this time.

The assumption $m \in L^1(0,\infty)$ implies that (2.3) is equivalent to

$$\sigma(x,t) = c^2\epsilon(x,t) - \int_0^\infty m(s)\epsilon(x,t-s)ds, \ x \in B, \ -\infty < t < \infty \ , \qquad (2.5)$$

where $c^2 := \beta + \int_0^\infty m(s)ds > 0$ is a constant which measures the instantaneous response of stress to strain; $\beta > 0$ is the equilibrium stress modulus. If $\beta > 0$ the material acts like a solid, while if $\beta = 0$ it acts like a fluid.

A natural generalization of (2.5) to __nonlinear materials__ is the constitutive relation

$$\sigma(x,t) = \phi(\epsilon(x,t)) - \int_0^\infty m(s)\psi(\epsilon(x,t-s))ds, \ x \in B, \ -\infty < t < \infty \ , \qquad (2.6)$$

in which $\phi, \psi : R \to R$ are assigned, smooth material functions which satisfy

$$\phi(0) = \psi(0) = 0, \ \phi'(0) > 0, \ \psi'(0) > 0 \ . \qquad (2.7)$$

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The memory function m is positive, nonincreasing and integrable on $(0,\infty)$ as above. In the static case $\varepsilon(x,t) = \overline{\varepsilon}(x)$, $\sigma(x,t) = \overline{\sigma}(x)$, (2.6) reduces to

$$\overline{\sigma}(x) = \phi(\overline{\varepsilon}(x)) - (\int_0^\infty m(s)ds)\psi(\overline{\varepsilon}(x)), \quad x \in B \quad .$$

A natural assumption, appropriate for viscoelastic solids and crucial in the analysis of global existence results (section 3), is to require that ϕ, ψ also satisfy

$$\phi'(0) - (\int_0^\infty m(s)ds)\psi'(0) > 0 \quad ; \tag{2.8}$$

(2.8) states that the equilibrium stress modulus is positive. The constitutive assumption (2.6) is a particular case of a "simple material" [8] which retains many important qualitative properties of more general material models; moreover, the analysis of the resulting equation of motion is relatively simple and complete.

The balance of linear momentum and (2.6) yield the equation of motion

$$u_{tt} = \phi(u_x)_x - \int_{-\infty}^t m(t-\tau)\psi(u_x(x,\tau))_x d\tau + f, \; x \in B, \; -\infty < t < \infty \quad , \tag{2.9}$$

where f is a body force and where the change of variable $\tau := t-s$ was made in (2.6). The history of the motion is assumed to be known for $t < 0$ (the history may, but need not satisfy (2.9) for $t < 0$). An appropriate dynamic problem is to find a smooth function $u : B \times (-\infty,\infty) \to R$, satisfying (2.9) for $t > 0$, and such that

$$u(x,t) = \overline{u}(x,t) \quad , \quad x \in B, \; t < 0 \quad , \tag{2.10}$$

where the history $\overline{u} : B \times (-\infty,0] \to R$ is a given smooth function; (2.9), (2.10) will be referred to as a history value problem. If B is bounded or semibounded compatible boundary conditions are adjoined to (2.9), (2.10). Compatibility of the boundary conditions with the smooth data f and $\overline{u}$ is imposed in order to preclude the propagation of singularities from the boundary into the interior.

If $m \equiv 0$, (2.9) reduces to the quasilinear wave equation (2.2). At the other extreme, if one formally sets $m = -\delta'$, where δ is the Dirac mass at the origin, then (2.9) reduces to the parabolic equation

$$u_{tt} = \psi(u_x)_{xt} + \phi(u_x)_x + f \quad ;$$

the term $\psi(u_x)_{xt}$ represents viscosity of Newtonian type if ψ is smooth and $\psi'(\cdot) > 0$.

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This equation possesses globally defined smooth solutions even if the data are large [1,34].

Our objective is to discuss the strength of the dissipative mechanism induced by the memory in (2.9) under physically reasonable assumptions by studying the existence and the decay or growth of classical solutions of the history value problem (2.9), (2.10). To motivate the mathematical results, we follow Coleman and Gurtin [6] in their penetrating study on the growth and decay of acceleration waves propagating into a one-dimensional viscoelastic material with memory at rest. An acceleration wave solution u is similar to a shock wave; the difference is that second rather than first derivatives of u experience a jump acrosss the wave front. To apply the results of [6] to (2.9), (2.10), we assume that ϕ, ψ are smooth, satisfy (2.7), $f \hat{=} 0$, $B = R$, and m is a smooth, regular kernel satisfying (2.4)(i). The wave front is a smooth curve $t = \gamma(x)$, $\gamma(0) = 0$, and $u \equiv 0$ for $t < \gamma(x)$. In [6] the problem of existence of acceleration waves is not discussed. Assuming that they do, an easy but tedious calculation shows that for (2.9) $t = \gamma(x)$ is a straight line, of slope $(\phi'(0))^{-1/2}$, meaning that such waves propagate with constant speed although (2.9) is nonlinear. Let the amplitude of the wave be $q(t) := [u_{tt}]$, where $[u_{tt}]$ is the jump in u_{tt} across the line $t = \gamma(x)$. It follows from the computations in [6] that q evolves in accordance with the Ricatti-Bernoulli equation

$$\frac{d}{dt} q = Aq^2 - Bq \quad , \quad q(0) = q_0 \quad , \tag{2.11}$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + c \frac{\partial}{\partial x}$, $c^2 = \phi'(0)$, represents differentiation along the wave front and where

$$A = \frac{-\phi''(0)}{2[\phi'(0)]^{3/2}} \quad , \quad B = \frac{m(0)\psi'(0)}{\phi'(0)} \quad .$$

Thus if $\phi''(0) < 0$ (similar results hold for $\phi''(0) > 0$), and $q_0 < B/A$, then every solution of (2.11) tends to zero as $t \to +\infty$. By contrast, if $q_0 > + B/A$, then $q(t) \to +\infty$ as $t \to T_0^-$, where $T_0 = \frac{1}{B} \log \frac{Aq_0}{Aq_0 + B} > 0$. The corresponding jumps in u_{xt} and u_{xx} are given by $[u_{xt}] = -[\phi'(0)]^{1/2} q(t)$ and $[u_{xx}] = [\phi'(0)]^{-1} q(t)$.

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This resu't suggests the following conjectures regarding smooth solutions of the history value problem (2.9), (2.10):

(i) The problem (2.9), (2.10) should have globally defined classical solutions if the history v and the forcing term f are sufficiently smooth and small in appropriate norms. Moreover, such solutions should decay.

(ii) The smooth solutions of (2.9), (2.10) should develop singularities in second derivatives in finite time if the smooth data are chosen sufficiently large.

As will be summarized in Section 3, conjecture (i) has been established rigorously by a number of authors in a number of physically important cases of (2.9), (2.10) for regular kernels $(m(0) < \infty)$, as well as for singular kernels $(m(0^+) = +\infty)$. Conjecture (ii) has only been established for regular kernels (see Section 4). Moreover, based on the discussion in Section 4, Remark 4.6, singular kernels m strengthen the dissipative mechanism of the memory in (2.9) which suggests the possibility that for appropriate classes of singular kernels, global smooth solutions will exist even if the data are arbitrarily large; this interesting question is open.

Most of the results described in Sections 3 and 4 for smooth kernels satisfying (2.4a) apply to more general one-dimensional viscoelastic models with fading memory, e.g. a model for a solid, K-BKZ material [29,2]

$$u_{tt} = \phi(u_x)_x + \int_{-\infty}^{t} m(t-\tau)h(u_x(x,t),u_x(x,\tau))_x d\tau + f, \qquad (2.12)$$

$$x \in B, \quad -\infty < t < \infty \quad .$$

Here ϕ, m, and f are as in (2.9), while $h : R \times R \rightarrow R$ is a smooth material function, $h(p,p) = 0$ and the partial derivatives of h satisfy appropriate sign conditions, at least at $(0,0)$. If $\phi \equiv 0$, (2.12) models a K-BKZ fluid. Under suitable assumptions, the energy method for proving existence results in Section 3 and the method of characteristics used to prove blow-up results of Section 4 yield similar results for this case as well. The energy method can also be applied to prove existence for certain multidimensional viscoelastic problems with fading memory (e.g. [13, Sec. 4], [30]).

However, to our knowledge, the existence results described in Section 3 for singular kernels satisfying (2.4(ii)) depend crucially on the special form of equation (2.9).

3. **Existence of Classical Solutions.** For discussion of the mathematical results it is convenient to renormalize the memory function m. Define the relaxation function a by

$$a(t) := \int_t^\infty m(s)ds, \quad 0 < t < \infty \quad ; \tag{3.1}$$

observe that if m is smooth, positive, decreasing and integrable on $[0,\infty)$ then $a'(t) = -m(t)$ and

$$a \text{ is smooth, positive, decreasing and convex on } (0,\infty) \quad . \tag{3.2}$$

Analogous to (2.4) we distinguish two classes of kernels a:

$$(i) \quad 0 < -a'(0^+) < \infty \quad , \quad (ii) \quad -a'(0^+) = +\infty \quad . \tag{3.3}$$

Other normalizations of the memory m are possible; for example, the relaxation function

$$G(t) := \phi'(0) - a(0)\psi'(0) + a(t)\psi'(0) \quad , \quad 0 < t < \infty \quad , \tag{3.4}$$

where ϕ, ψ are the material functions in (2.6), is consistent with the applied literature. Observe that $G(\infty) = \phi'(0) - a(0)\psi'(0)$ and $G(0) = \phi'(0)$.

Returning to the history value problem (2.9), (2.10), let the history $\bar{u}$ be identically zero for $t < 0$. One then seeks a solution of the initial value problem

$$u_{tt} = \phi(u_x)_x + \int_0^t a'(t-\tau)\psi(u_x(x,\tau))_x d\tau + f, \quad x \in B, \, t > 0 \quad , \tag{3.5}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in B \quad , \tag{3.6}$$

together with suitable and compatible boundary conditions if B is not R. If the history $\bar{u}$ is not zero for $t < 0$, the part of the integral in (2.9) on $(-\infty,0)$ is incorporated in f.

Global Existence of Classical Solutions. We next discuss global existence and asymptotic behaviour for the Cauchy problem (3.5), (3.6) with $B = R$, for smooth, small data, and for regular kernels a satisfying (3.2), (3.3)(i). To simplify the exposition, we make the hypothesis

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$$a \in C^3[0,\infty), \quad (-1)^k a^{(k)}(t) > 0 \ (0 < t < \infty; \ k = 0,1,2,3) \quad,$$

$$a' \not\equiv 0, \quad \text{and} \quad \int_0^\infty ta(t)dt < \infty \quad . \tag{3.7}$$

The results hold under assumptions on a considerably weaker than (3.7). The interested reader is referred to [13], [22], [24], and the survey paper [23] for the generalizations. The essential point is that kernel a satisfies a, a', $a'' \in L^1(0,\infty)$, the moment condition in (3.7), and is "strongly positive" on $[0,\infty)$. The result for B bounded [13] is somewhat simpler than for the Cauchy problem (3.5), (3.6); in particular, the moment condition in (3.7) is only needed for the Cauchy problem (see remarks following Theorem 3.1 and the outline of its proof).

Concerning ϕ, ψ assume

$$\phi, \ \psi \in C^3(\mathbb{R}), \ \phi(0) = \psi(0) = 0, \ \phi'(0) > 0, \ \psi'(0) > 0, \ \phi'(0) - a(0)\psi'(0) > 0 \ ; \tag{3.8}$$

the latter is the analogue of (2.8) in the present normalization. Assume that

$$\text{(i)} \ f, \ f_x, \ f_t \in C([0,\infty); \ L^2(\mathbb{R})) \cap L^\infty([0,\infty); \ L^2(\mathbb{R})) \quad \text{and}$$

$$\tag{3.9}$$

$$\text{(ii)} \ f \in L^1([0,\infty); \ L^2(\mathbb{R})); \ f_x, \ f_t, \ f_{xt} \in L^2([0,\infty); \ L^2(\mathbb{R})) \quad,$$

and let u_0, u_1 satisfy

$$u_0 \in L^2_{loc}(\mathbb{R}), \quad \text{and} \quad u_0', \ u_1 \in H^2(\mathbb{R}) \quad . \tag{3.10}$$

To measure the size of the data define the quantities

$$U_0(u_0,u_1) := \int_{-\infty}^\infty \{u_0'^2 + u_0''^2 + u_0'''^2 + u_1^2 + u_1'^2 + u_1''^2\}(x)dx, \quad \text{and} \tag{3.11}$$

$$F(f) := \sup_{t>0} \int_{-\infty}^\infty \{f^2 + f_x^2 + f_t^2\}(x,t)dx + \left(\int_0^\infty \left(\int_{-\infty}^\infty f^2(x,t)dx\right)^{1/2} dt\right)^2 \tag{3.12}$$

$$+ \int_0^\infty \int_{-\infty}^\infty \{f_x^2 + f_t^2 + f_{xt}^2\}(x,t)dxdt \quad .$$

The following result is a special case of Theorem 1.1 of [24].

Theorem 3.1. Let assumptions (3.7) - (3.10) be satisfies. There exists a constant $\mu > 0$ such that for each u_0, u_1, f satisfying

$$U(u_0,u_1) + F(f) < \mu^2 \quad, \tag{3.13}$$

the Cauchy problem (3.5), (3.6) <u>has a unique solution</u> $u \in C^2(\mathbb{R} \times [0,\infty))$, <u>and</u>

$$u_x, \ u_t, \ u_{xx}, \ldots, u_{ttt} \in C([0,\infty); \ L^2(\mathbb{R})) \cap L^\infty([0,\infty); \ L^2(\mathbb{R})) \ . \qquad (3.14)$$

<u>Moreover</u>,

$$u_{xx}, \ u_{xt}, \ldots, u_{ttt} \in L^2([0,\infty); \ L^2(\mathbb{R}) \ , \qquad (3.15)$$

$$u_{xx}, \ u_{xt}, \ u_{tt} \to 0 \quad \text{in} \quad L^2(\mathbb{R}) \quad \underline{\text{as}} \quad t \to \infty \ , \qquad (3.16)$$

$$u_x, \ u_t, \ u_{xx}, \ u_{xt}, \ u_{tt} \to 0 \ \underline{\text{uniformly on}} \ \mathbb{R} \ \underline{\text{as}} \ t \to \infty \ . \qquad (3.17)$$

A similar result holds for the history value problem (2.9), (2.10) with $B = \mathbb{R}$. The

special case $a(t) = \alpha e^{-\lambda t}$, $\alpha > 0$, $\lambda > 0$, studied by Greenberg [18] for B bounded, is

carried out in [23] in the more complicated case when $B = \mathbb{R}$.

<u>Remark 3.2</u>. Theorem 3.1 is a generalization of Theorems 1.1 and 4.1 of [13] establishing

small-data global existence results for analogous initial boundary value problems

corresponding to motions of **bounded viscoelastic bodies**; Neumann, Dirichlet and mixed

boundary conditions are treated. The principal difficulty in proving Theorem 3.1 is that

various Poincaré inequalities, not applicable to (3.5), (3.6) when $B = \mathbb{R}$, are used in

an essential way in [13] to establish an a priori estimate similar to (3.26) from (3.32)

(see outline of proof following Proposition 3.4); the estimate (3.26) is essential for

completing the proof. The reader is referred to Hrusa [22] for a discussion of general

history value problems on a bounded interval. Although technically extremely complicated,

the generalization of the results in [22] to the Cauchy problem is relatively straight-

forward.

<u>Remark 3.3</u>. If $\psi \equiv \phi$ equations of the form (3.5) have been studied by MacCamy [35],

Dafermos and the author [12], and Staffans [49] for bounded and unbounded bodies. If

$\psi \equiv \phi$, (3.5) admits certain estimates which do not carry over to the general case $\psi \neq \phi$

(see [23]); there does not appear to be any physical motivation for the restriction $\psi \equiv \phi$

for solids.

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Outline of the proof of Theorem 3.1. An essential ingredient of any global result is an appropriate local existence theorem. For regular kernels a satisfying (3.2),(3.3)(i), the idea is to iterate the sequence of linear problems which treat the memory as a lower-order perturbation:

$$u_{tt} = \phi'(w_x)u_{xx} + \int_0^t a'(t-\tau)\psi(w_x(x,\tau))_x d\tau + f, \quad x \in R, \quad 0 < t < T \quad , \qquad (3.18)$$

where $T > 0$, u satisfies the initial conditions (3.6), and where w is an element of a suitably chosen function space X. By using fairly standard enery estimates deduced from (3.18), requiring only very simple estimates of the convolution term which do not use any sign information on the memory, it is shown that the mapping S which carries w into a solution of (3.18) has a unique fixed point for $T > 0$ sufficiently small. The proof is almost identical with that of Theorem 2.1 of [13]. The only significant difference is that the proof in [13] is for $x \in [0,1]$ with Neumann boundary conditions satisfied at $x = 0$ and $x = 1$; thus the Poincaré inequality enables one to deduce estimates for lower order derivatives of u in $L^\infty([0,T); L^2(0,1))$ from higher order derivative estimates. As far as local existence is concerned when $B = R$, this causes no serious difficulties. One simply expresses the lower order derivatives of the solution in terms of initial conditions and time integrals of the higher order derivatives, yielding time dependent bounds which, however, cannot be used for obtaining global estimates. The result is:

Proposition 3.4. Let a, a', a" $\in L^1_{\ell oc}[0,\infty)$ and assume that $\phi, \psi \in C^3(R)$, $\phi'(0) > 0$, and that there exists a number $\underline{\phi}$ such that

$$\phi'(\xi) > \underline{\phi} \quad \text{for every } \xi \in R \quad . \qquad (3.19)$$

Concerning the data, let u_0, u_1 satisfy (3.10), f satisfy (3.9)(i) and assume that $f_{xt} \in L^1_{\ell oc}([0,\infty); L^2(R))$. Then the Cauchy problem (3.5), (3.6) has a unique solution u defined on a maximal time interval $[0,T_0)$ satisfying

$$u_x, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, u_{xxt}, u_{xtt}, u_{ttt} \in C([0,T_0); L^2(R)) \quad . \qquad (3.20)$$

Moreover, if

$$\sup_{t \in [0,T_0)} \int_{-\infty}^{\infty} \{u_x^2 + u_t^2 + \cdots + u_{xtt}^2 + u_{ttt}^2\}(x,t)dx < \infty \quad , \qquad (3.21)$$

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<u>then</u> $T_0 = +\infty$. <u>By Sobolev embedding</u> $u \in C^2(\mathbb{R} \times [0,T_0))$.

In outline, the proof of the global result then proceeds as follows. Define the equilibrium stress χ by

$$\chi(\xi) := \phi(\xi) - a(0)\psi(\xi), \quad \forall \xi \in \mathbb{R} \ ; \tag{3.22}$$

observe that $\chi \in C^3(\mathbb{R})$ and that $\chi'(0) > 0$ (by 3.8). Choose a sufficiently small number $\delta > 0$ and modify ϕ, ψ, and χ outside $[-\delta,\delta]$ such that ϕ'', ψ'', χ'' vanish outside $[-2\delta,2\delta]$, and choose positive constants $\underline{\phi}$, $\underline{\psi}$, $\underline{\chi}$ such that

$$\phi'(\xi) > \underline{\phi}, \ \psi'(\xi) > \underline{\psi}, \ \chi'(\xi) > \underline{\chi} \quad \forall \xi \in \mathbb{R} \ . \tag{3.23}$$

It is shown a posteriori that $|u_x(x,t)| < \delta$ for all $x \in \mathbb{R}$, $t > 0$. By Proposition 3.4 the Cauchy problem (3.5), (3.6), $B = \mathbb{R}$ has a unique solution u on a maximal interval $[0,T_0)$. The objective is to show that if (3.13) holds with $\mu > 0$ sufficiently small, then (3.21) is bounded independent of T_0; a standard continuation procedure implies $T_0 = +\infty$. Define

$$E(t) := \max_{s \in [0,t]} \int_{-\infty}^{\infty} \{u_t^2 + u_x^2 + \cdots + u_{ttt}^2\}(x,s)dx$$

$$+ \int_0^t \int_{-\infty}^{\infty} \{u_{xx}^2 + u_{xt}^2 + \cdots + u_{ttt}^2\}(x,s)dxds \ , \tag{3.24}$$

where $\cdots$ represent the sum of the second and third derivatives not explicitly written down. It is shown that if (3.13) holds for $\mu > 0$ sufficiently small, then $E(t)$ is bounded. For this purpose define

$$\nu(t) := \sup_{\substack{x \in \mathbb{R} \\ s \in [0,t]}} \{u_x^2 + u_{xx}^2 + u_{xt}^2\}^{1/2}(x,s), \quad \forall t \in [0,T_0) \ . \tag{3.25}$$

To prove the result one establishes the following key estimate

$$E(t) < \Gamma(U_0(u_0,u_1) + F(f)) + \Gamma(\nu(t) + \nu^3(t))E(t), \ 0 < t < T_0 \ , \tag{3.26}$$

where here and below Γ is a generic constant, possibly large, independent of u_0, u_1, f, and T_0. We shall comment below only briefly how this is accomplished.

Once (3.26) is established, the conclusions of Theorem 3.1 are obtained as follows. Choose $\overline{E}$, $\mu > 0$ such that

$$\overline{E} < \delta^2, \ \Gamma\{(2\overline{E})^{1/2} + (2\overline{E})^{3/2}\} < \frac{1}{2}, \ \Gamma\mu^2 < \frac{1}{4}\overline{E} \ . \tag{3.27}$$

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Select the data u_0, u_1, f such that (3.13) holds for μ chosen in accordance with (3.27). The Sobolev embedding theorem implies that

$$\nu(t) \le (2E(t))^{1/2} \quad \forall t \in [0,T_0) \quad . \tag{3.28}$$

Therefore, it follows from (3.26), (3.27), (3.28) that for any $t \in [0,T_0)$ with $E(t) \le \overline{E}$, we actually have $E(t) \le \frac{1}{2}\overline{E}$. By continuity $E(t) \le \frac{1}{2}\overline{E}$, $\forall t \in [0,T_0)$, provided $E(0) < \frac{1}{2}\overline{E}$; the latter is insured by choosing μ^2 smaller if necessary so that (3.13) will imply $E(0) < \frac{1}{2}\overline{E}$. Then $E(t) \le \frac{1}{2}\overline{E}$, $\forall t \in [0,T_0)$, and (3.24), Proposition 3.1, and a standard continuation method yield $T_0 = +\infty$. One also has that (3.14), (3.15) hold, and conclusions (3.16), (3.17) follow by standard embedding inequalities. Moreover, (3.25), (3.27), (3.28) yield

$$|u_x(x,t)| \le \nu(t) \le (2E(t))^{1/2} \le (\overline{E})^{1/2} \le \delta, \quad \forall x \in \mathbb{R}, \ t \in [0,\infty)$$

and the proof is complete.

Establishing the estimate (3.26) is lengthy, delicate, and relies on the correct sign of the memory [under assumption (3.7) or certain generalizations]. The energy method, combined with relevant properties of Volterra operators and their resolvents, is employed. The estimates of derivatives of u appearing in (3.24) are deduced from energy identities obtained directly from (3.5), (3.6), and from the equation equivalent to (3.5):

$$u_{tt} = \chi(u_x)_x + \int_0^t a(t-\tau)\psi(u_x)_{xt}(x,\tau)d\tau$$
$$+ a(t)\psi(u_0(x)_x)_x + f \quad (x \in \mathbb{R}, \ 0 < t < T) \quad , \tag{3.29}$$

where $T < T_0$; (3.29) is obtained from (3.5), (3.6) by an integration by parts and use of (3.22). Useful identities for derivatives of u can only be obtained by multiplying the equations by quantities which make it possible to estimate the memory terms. A crucial role is played by the "quadratic integral form"

$$Q(w,t,b) := \int_0^t \int_{-\infty}^{\infty} w(x,s) \int_0^s b(s-\tau)w(x,\tau)d\tau \, dx \, ds, \quad t > 0 \quad ,$$

defined for $b \in L^1_{loc}[0,\infty)$ and for every $w \in C([0,t]; L^2(\mathbb{R}))$. In the first energy identity, which is obtained by multiplying (3.29) by $\psi(u_x)_{xt}$ and integrating the equation

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over $\mathbb{R} \times [0,T]$, Q arises with $w = \psi(u_x)_{xt}$ and $b = a$. It is an important fact that kernels a satisfying (3.7) (indeed much weaker assumptions) are positive definite on $[0,\infty)$. To obtain the second energy identity, one needs to take the forward time-difference of (3.29) and integrate the resulting equation over $\mathbb{R} \times [0,T]$. To estimate the relevant derivatives of u from a combination of the first two identities one needs the following technical estimate: It is shown in [24; Lemma 2.5] that if a satisfies (3.7), there exists a constant $\kappa > 0$ such that

$$\int_0^t \int_{-\infty}^\infty w_t^2(x,t)dxdt \leqslant \kappa \int_{-\infty}^\infty w_t(x,0)dx + \kappa Q(w_t,t,a)$$

$$+ \kappa \liminf_{h \to 0} \frac{1}{h^2} Q(\Delta_h w_t, t, a) \quad , \quad \forall t \in [0,T] \quad , \tag{3.30}$$

where $v \quad C^1([0,T]; L^2(\mathbb{R})) \ \forall \ T > 0$, and where the forward difference operator $\Delta_h w$ is defined by $\Delta_h w(x,t) := w(x,t+h) - w(x,t)$. In the application of (3.30), $w = \psi(u_x)_{xt}$ and the forward difference operator Δ_h is applied to equations (3.29). The proof of (3.30) also makes use of a result of Staffans ([49, Lemma 4.2]). Using the two energy identities, and (3.30), it is relatively straightforward to estimate all of the terms and arrive at:

$$\int_{-\infty}^\infty \{u_{xx}^2 + u_{xt}^2 + u_{xxt}^2 + u_{xtt}^2\}(x,t)dx + \int_0^t \int_{-\infty}^\infty u_{xxt}^2(x,s)dxds$$

$$\leqslant \Gamma(U_0 + F) + \Gamma(v(t) + v^3(t))E(t) + \Gamma(\sqrt{U_0} + \sqrt{F})\sqrt{E(t)}, \ \forall \ t \in [0,T] \quad . \tag{3.31}$$

Estimates of $\int_{-\infty}^\infty u_{tt}^2(x,t)dx$, $\int_{-\infty}^\infty u_{ttt}^2(x,t)dx$, $\int_0^t \int_{-\infty}^\infty u_{ttt}^2(x,\tau)dxdt$, $\forall \ t \in [0,T]$ in terms of the right side of (3.31) are obtained from (3.5). A bound for $\int_0^t \int_{-\infty}^\infty u_{xtt}^2(x,s)dxds$ can then be obtained by interpolation. Using the fact that a certain resolvent kernel of a' in (3.5) is in $L^1[0,\infty)$, Lemma 3.2 of [13] makes it possible to estimate $\int_{-\infty}^\infty u_{xxx}^2(x,t)dx$ and $\int_0^t \int_{-\infty}^\infty u_{xxx}^2(x,s)dxds$. Combining these with (3.31) yields the estimate

$$\int_{-\infty}^{\infty} \{u_{xx}^2 + u_{xt}^2 + u_{tt}^2 + u_{xxx}^2 + u_{xxt}^2 + u_{xtt}^2 + u_{ttt}^2\}(x,t)dx \qquad (3.32)$$

$$+ \int_0^t \int_{-\infty}^{\infty} \{u_{xxx}^2 + u_{xxt}^2 + u_{xtt}^2 + u_{ttt}^2\}(x,s)ds$$

$$< \Gamma(U_0 + F) + \Gamma(\nu(t) + \nu^3(t))E(t)$$

$$+ \Gamma(\sqrt{U_0} + \sqrt{F})\,\sqrt{E(t)}, \ \forall\, t \in [0,T] \quad .$$

The estimate (3.32) is implicit in the argument of [13]. It should be observed that for problems on bounded intervals (Remark 3.2), it is a simple matter to apply the Poincaré inequality to deduce the remaining estimates of derivatives of u appearing in (3.24) and arrive at the final estimate (3.26) directly from (3.32). However, to accomplish this task for (3.5), (3.6) when $B = R$ is quite tricky and involves additional properties of Volterra operators and certain other of their resolvents. The reader is referred to Lemmas 2.3 and 2.4, as well as the argument on pages 405-410 of [24] for details. This part of the proof makes essential use of the assumption $a'' \in L^1[0,\infty)$ which is automatic when a satisfies (3.7), but cannot be satisfied by singular kernels.

For _singular kernels_ satisfying (3.2) and (3.3)(ii), it is simpler to restrict the analysis to the history value problem, (2.9), (2.10), with a defined by (3.1), in which that history $\bar{u}$ satisfies the equation (and the boundary conditions if B is bounded). This ensures that the compatibility conditions between the history and boundary data, as well as compatibility conditions between the derivatives of the history and the solution for $t > 0$ are satisfied. If u is a smooth solution of (2.9) and the kernel a is singular, the integral in (2.9) is also a smooth function, but the integrals $\int_{-\infty}^0$ and $\int_0^t$ have singularities at $t = 0$ which cancel. Thus if formulated as an initial value problem the results would involve a singular forcing term. For reasons explained below, global existence results for singular kernels only hold for B bounded.

The principal difficulty when dealing with singular kernels is establishing a suitable local existence result. In Proposition 3.4 for regular kernels no hypothesis is made concerning the sign of the memory and the size of the data. In the proof the memory is treated as a perturbation of the elastic term $\phi(u_x)_x$ in (3.5). However, the proof makes

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crucial use of the hypothesis $a'' \in L^1_{loc}[0,\infty)$ which rules out singular kernels a satisfying (3.2), (3.3)(ii).

Hrusa and Renardy [25, Theorem 4.1] recently obtained an elegant extension of Proposition 3.4 for such singular kernels. They consider the history value problem with the history satisfying the equation and the boundary conditions for $t < 0$. The singular kernel a satisfies the assumptions

$$a, a' \in L^1(0,\infty); \ a(t) > 0, \ a'(t) < 0, \ a''(t) > 0, \ 0 < t < \infty \qquad (3.33)$$

in the sense of measures, and a'' is not a purely singular measure; a certain assumption on the Laplace transform of a is imposed in order to guarantee that the third derivatives of u are continuous with values in $L^2(0,1)$. The material function ψ is also required to satisfy $\psi'(0) > 0$, and the technical assumptions regarding the forcing function f are strengthened. The sign of the memory now plays a crucial role in the local analysis in which one iterates a sequence of linear integrodifferential equations (compare with (3.18))

$$u_{tt} = \phi'(w_x)u_{xx} + \int_{-\infty}^{t} a'(t-\tau)\psi'(w_x)u_{xx}(x,\tau)d\tau + f \qquad (3.34)$$

where $u(x,t) = \bar{u}(x,t)$ for $t < 0$, and where w is an element of an apropriately chosen function space. The singular kernel a satisfying (3.33) is replaced in (3.34) by regular kernels a_δ defined by

$$a_\delta(t) := \int_{-\delta}^{\delta} p_\delta(\tau)a(t+\delta+\tau)d\tau, \ 0 < t < \infty, \ \delta > 0 \ ,$$

where p_δ is a standard mollifier supported in $[-\delta/2,\delta/2]$. The analysis with singular kernels is far more complicated because $a'' \notin L^1_{loc}[0,\infty)$, and $|a_\delta''|_{L^1}$ does not necessarily remain bounded as $\delta \to 0$. The energy estimates are also considerably more delicate and to obtain them certain technical lemmas concerning Volterra operators with kernels a satisfying (3.33) are required (such kernels are known to be strongly positive definite [43]). It is first shown that each linear problem (3.34) has a unique solution having the required regularity by justifying passage to the limit as $\delta \to 0$. Then a contraction mapping argument for (3.34) is used in [25] to obtain the analogue of Proposition 3.4 for w belonging to an appropriate function space. The proof in [25] is carried out for $B = [0,1]$ with Dirichlet boundary conditions satisfied at $x = 0$ and

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x = 1; it is straightforward to obtain a similar local result for B = R, because the local existence proof in [25] avoids the use of Poincaré inequalities.

Using their local result, Hrusa and Renardy then obtain an analogue of Theorem 3.1 for the history value problem (2.9), (2.10) and the (singular) kernel a, defined by (3.1), satisfying (3.33) on bounded intervals. They impose the requirement that the history and the solution satisfy Dirichlet boundary conditions at x = 0 and x = 1 and that the history and forcing term be suitably small. Their result ([25, Theorem 5.1]) is then a simple extension of the proof of [13, Theorem 1.1] involving the modification of only one estimate in [13]; the modification uses a refinement of Lemma 4.2 in [49], because $a" \notin L^1[0,\infty)$ whenever a is singular. The fact that $a" \notin L^1[0,\infty)$ makes it difficult to prove Theorem 3.1 for singular kernels using the analysis in [25]. It is a challenging open problem to prove such a result for singular kernels on all of space.

4. Development of Singularities and Related Problems. In this section we consider the Cauchy problem (3.5), (3.6) for regular kernels a, and we discuss the development of singularities in smooth solutions in finite time for smooth but large data by using the method of characteristics. To avoid technical complications we assume that the forcing term $f \equiv 0$ in (3.5), and we study

$$u_{tt} = \phi(u_x)_x + a'*\psi(u_x)_x, \quad x \in R, \; t > 0 \;, \tag{4.1}$$

$$u(x,0) = u_0(x), \; u_t(x,0) = u_1(x), \; x \in R \;, \tag{4.2}$$

where * denotes the time convolution on [0,t]. The following result was recently established by M. Renardy and the author [42], and independently by Dafermos [10] for general memory functionals using a somewhat different proof. The result can also be established by extending techniques of F. John [27] to quasilinear, first-order hyperbolic systems with lower order source terms; however, the approach outlined below is more direct.

Theorem 4.1. Let $\phi, \psi \in C^3(\mathbb{R})$ let ϕ satisfy (3.19) and let a be smooth with $a, a', a'' \in L^1_{loc}[0,\infty)$. In addition, let $\phi''(0) \neq 0$. Then for every $T_1 > 0$, there exists initial data $u_0^1, u_1 \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that the maximal interval of existence of the smooth solution u of the Cauchy problems (4.1), (4.2) cannot exceed T_1. More precisely, if $\sup_{x \in \mathbb{R}} |u_0'(x)|$ and $\sup_{x \in \mathbb{R}} |u_1(x)|$ are sufficiently small, while $u_0''(x)$ and $u_1'(x)$ are sufficiently large (with appropriate signs), then there exists a number $t^* < T_1$ such that

$$\sup_{\mathbb{R} \times [0,t^*)} \{|u_{xx}(x,t)| + |u_{xt}(x,t)|\} = \infty \ , \tag{4.3}$$

while

$$\sup_{\mathbb{R} \times [0,t^*)} \{|u_x(x,t)| + |u_t(x,t)|\} < \infty \ , \tag{4.4}$$

For the special case $\psi \equiv \phi$, Hattori [21] has shown that if $\phi'' \neq 0$ and if the body B is bounded, then there exist data u_0, u_1 such that the initial-boundary value problem (consisting of (4.1), (4.2) and compatible Dirichlet boundary conditions) does not have a globally defined smooth solution. However, his method does not enable him to characterize the data. Ramaha [45] has recently obtained a blow-up result when $\psi \equiv \phi$.

For first-order model problems with fading memory, blow-up results similar to Theorem 4.1 have been obtained by a number of authors ([38], [36], [9]) by the method of characteristics. Existence of classical solutions for small data for such models is discussed in [41]. The elegant method of Dafermos [9] avoids use of characteristics; instead a maximum principle is obtained and used.

Remark 4.2. The reader should observe that in Theorem 4.1 only the additional hypothesis $\phi''(0) \neq 0$ is added to the assumptions guaranteeing the existence of a local smooth solution of (4.1), (4.2) (Proposition 3.4). No sign information on the kernel a is required. Assumption (3.19) is not restrictive because it is shown that the supremum in (4.4) is in fact small.

The proof of Theorem 4.1 generalizes the approach of Lax [33] using the method of characteristics and generalized Riemann invariants. We transform (4.1), (4.2) to an

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equivalent first-order system as follows. Let $w = u_x$, $v = u_t$, define

$$\sigma := \phi(w) - z \quad , \quad z := -a'*\psi(w) \quad , \tag{4.5}$$

and observe that σ is the stress-strain functionals (2.6). Since $\phi'(\cdot) > 0$, equation (4.5) can be solved for w, $w = \phi^{-1}(\sigma+z) := g(\sigma,z)$, and g is a smooth function on $R \times R$. As long as the solution u of (4.1), (4.2) remains smooth, (4.1), (4.2) is equivalent to the system

$$v_t = \sigma_x$$

$$\sigma_t = C^2(\sigma,z)v_x + a'(0)\psi(g(\sigma,z)) + a''*\psi(g(\sigma,z)) \quad , \tag{4.6}$$

$$z_t = -a'(0)\psi(g(\sigma,z)) - a''*\psi(g(\sigma,z)) \quad ,$$

$$v(x,0) = u_1(x), \; \sigma(x,0) = \phi(u_0'(x)), \; z(x,0) \equiv 0 \quad , \tag{4.7}$$

where the wave speed $C(\sigma,z) := [\phi'(g(\sigma,z))]^{1/2}$ is a smooth function. The system (4.6) is hyperbolic with eigenvalues C, $-C$, 0. We define generalized Riemann invariants r, s by

$$r = r(v,\sigma,z) := v + \Phi(\sigma,z); \; s = s(v,\sigma,z) := v - \Phi(\sigma,z); \; \Phi(\sigma,z) := \int^\sigma \frac{d\xi}{C(\xi,z)} \quad .$$

Thus $v = \frac{r+s}{2}$, $\Phi = \frac{r-s}{2}$; the correspondence is smoothly invertible because $\Phi_\sigma = C^{-1} > 0$. Observe that if $a' \equiv 0$ in (4.1), $z \equiv 0$ and g, C are independent of z. In this situation r and s reduce to the Riemann invariants for the system

$$v_t = \sigma_x, \; \sigma_t = \phi'(\phi^{-1}(\sigma))v_x$$

which can be transformed to the quasilinear wave equation. In the proof r, s, z are introduced as dependent variables and (4.6) is replaced by an equivalent system obtained by differentiating r, s, z along the characteristics C, $-C$, 0 respectively. One then differentiates the quantities

$$\rho := v_x + \frac{\sigma_x}{C(\sigma,z)} \quad , \quad \tau := v_x - \frac{\sigma_x}{C(\sigma,z)} \quad , \quad \text{and} \quad z_x$$

along the C, $-C$, 0 characteristics respectively (observe that if $a' \equiv 0$, $\rho = r_x$, $\tau = x_x$). It is shown (see [42] for details) that to leading order the characteristic derivatives of $\sqrt{C}\,\rho$, $\sqrt{C}\,\tau$ satisfy a coupled system of Ricatti equations in ρ and τ with coefficients which are smooth functions of r, s, z. The differential equation for

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z_x is linear in ρ, τ, z_x, and it is shown that z_x grows at most logarithmically.
Blow-up in finite time is established by showing that r, s, z remain in a neighborhood
U of zero up to the blow-up time, if they are small initially (i.e. if
$\sup_{\mathbb{R}}\{|v(x,0)| + |\sigma(x,0)|\}$ is small), while $v'(x,0)$ and $\sigma'(x,0)$ are sufficiently large
(with appropriate signs). Moreover, the hypothesis $\phi''(0) \neq 0$ provides upper and lower
nonzero bounds for the coefficients ρ^2 and τ^2 in the Ricatti equations when r, s, z
are in U.

Remark 4.3. A physical interpretation of conclusions (4.3), (4.4), coupled with examples
of Coleman, Gurtin, and Herrera [7], is that the strain remains bounded but its first
derivatives become infinite as $t \to t_-^*$. Thus Theorem 4.1 suggests, but does not prove,
the development of a shock front in finite time.

Remark 4.4. Certain models for shearing flows of viscoelastic fluids can be analyzed by
the technique of Theorem 4.1. With $v(x,t)$ denoting the velocity of the fluid in simple
shear, Slemrod [48] studies the problem

$$v_t = a*\phi(v_x)_x, \quad x \in \mathbb{R}, \quad t > 0$$
$$v(x,0) = v_0(x) \quad , \quad x \in \mathbb{R}$$

(4.8)

in the special case $a(t) = e^{-t}$. Differentiation of the equation leads to a Cauchy problem
of the form (4.1), (4.2). Global existence for smooth, small data follows from [12,
Theorem 4.1]; see also Remark 3.3. Development of singularities for large data is an easy
application of Theorem 4.1 above. Other popular models for viscoelastic fluids can be
discussed by a similar analysis. Slemrod [47] and Gripenberg [20] established similar
results for a different model of shearing flows for a viscoelastic fluid. If $a = e^{-t}$,
(4.8) as well as the problem studied in [47], can be transformed to the quasilinear wave
equation with linear frictional damping for which finite time blow-up for large data can be
established by the method of Lax [33].

We close this section by discussing a number of open problems.

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Remark 4.5. The techniques of proof of Theorem 4.1 and that of [10] depend crucially on the hypothesis $\phi''(0) \neq 0$. The physically important situation $\phi''(0) = 0$, permitted in the finite time blow-up result for the quasilinear wave equation (2.2) (with $f \equiv 0$) in [37], constitutes an interesting open problem for (4.1), (4.2).

Remark 4.6. Singular kernels a satisfying (3.2) and (3.3)(ii) $-a'(0^+) = +\infty$ violate the hypothesis $a'' \in L^1_{loc}[0,\infty)$ which is crucial to the technique of proof of Theorem 4.1 and that of the similar result in [10]. Indeed, there is strong evidence based on the following arguments, that there may exist singular kernels a such that (4.1) would have globally defined smooth solutions, even if the data are arbitrarily large. These arguments suggest that singular kernels strengthen the dissipation induced by the memory. Thus far it has not been possible to resolve this important open problem.

First, for smooth kernels with $-a'(0^+)$ finite, it follows from (2.11) and the definition of the constant B that the diameter of the set of points $q_0 > B/A$ for which $q(t) \rightarrow +\infty$ in finite time shrinks as $m(0) = -a'(0^+) > 0$ is increased. However, the derivation of (2.11) rests on the assumption that $m(0) = -a'(0^+)$ remains finite. Second, there are interesting results of Hrusa and Renardy [26] in their analysis of wave propagation in linear visco-elasticity. They study the linear history value problem (2.9), (2.10) with $\phi'(\cdot) \equiv c^2 = \beta + \int_0^\infty m(\tau)d\tau$ and $\psi'(\cdot) \equiv 1$, $\overline{u}(x,t) \equiv 0$, $t < 0$, $B = R$, and they adjoin step jump initial data $u(x,0)$, $u_t(x,0)$, $x \in R$. They prove that if the memory m is smooth on $[0,\infty)$, the solution has discontinuities propagating along characteristics of the linear wave equation $u_{tt} = c^2 u_{xx}$ and a stationary discontinuity of higher order at the initial step-jumps. For singular memory kernels the propagating waves are smoothed out. The degree of smoothing increases as the kernel becomes more singular; the stationary discontinuities remain.

Remark 4.7. There is numerical evidence concerning the development of singularities in finite time for regular kernels a and large smooth data. Markowich and Renardy [39] used the Lax-Wendroff method to discretize the hyperbolic part in (4.1) and the trapezoidal rule

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to discretize the integral. They show that the method is second-order convergent and stable on any finite time interval on which smooth solutions exist. For spatially periodic and small Cauchy data, and for kernels a which are finite sums of decaying exponentials, they prove second order convergence on $[0,\infty)$. They also carry out numerical experiments in the special case $\psi \equiv \phi$ which exhibit the formation of a singularity in finite time for particularly chosen ϕ, a, and suitably large u_0 and u_1. Their numerical solution exhibits but does not prove the formation of shock fronts in u_x and u_t at the critical time. Other numerical schemes merit investigation.

Remark 4.8. <u>Weak Solutions</u>. Remarks 4.3 and 4.7 motivate the study of weak solutions for equations such as (4.1), (4.2) governing the motion of materials with memory. Except for certain special situations valid for steady viscoelastic fluid flows (Pipkin [44] and Greenberg [17]), there is no rigorous theory for the existence of shock waves and acceleration waves. MacCamy [36], Greenberg and Hsiao [19] have studied several aspects of weak solutions but only for a single first-order conservation law with memory in one space dimension. Dafermos and Hsiao [11] proved the existence of weak solution of one-dimensional first-order quasilinear hyperbolic systems with memory using Glimm's modified random choice method [16] with fractional steps. However, their method requires assumptions of "diagonal dominance" which are not satisfied in the case of the Cauchy problem (4.1), (4.2) modelling a viscoelastic solid. They are satisfied for certain models of heat flow (see [12]) and the specific model (4.8) for viscoelastic fluid flow).

In order to address the problem of weak solutions which would include one-dimensional problems for viscoelastic solids of the form (4.1), (4.2), a program has been initiated involving analytical techniques, the design of numerical algorithms and numerical experiments. We consider the Cauchy problem (4.1), (4.2) in the form of a first-order equivalent system. Let $w = u_x$, $v = u_t$. For classical solutions, (4.1), (4.2) is equivalent to the system

$$w_t = v_x$$
$$v_t = \phi(w)_x + a'*\psi(w)_x$$

$$(4.9)$$

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satisfying the initial conditions

$$w(x,0) = w_0(x), \quad v(x,0) = v_0(x) \quad . \tag{4.10}$$

It is easy to show that a weak solution (in the sense of distributions) of (4.1), (4.2) is a weak solution of (4.9), (4.10). It is straightforward that the Rankine-Hugoniot jump conditions for elastic shocks ($a \equiv 0$ in (4.1)) are also necessary for viscoelastic shocks.

The Riemann problem is only partially understood for scalar first-order conservation laws with memory [36], but not at all for the viscoelastic problem (4.9), (4.10). Therefore it is difficult to use the random choice method [16]. If $\psi \neq \phi$, define $z = a' * \psi(w)$. Then (4.9) transforms to the hyperbolic system with lower order source terms:

$$\begin{aligned}
w_t &= v_x \\
v_t &= \phi(w)_x + z_x \\
z_t &= a'(0)\psi(w) + a''*(w) \quad ,
\end{aligned} \tag{4.11}$$

with $w(x,0)$, $v(x,0)$ satisfying (4.10) and $z(x,0) \equiv 0$. If $\phi'(\cdot) > 0$ (4.11) has the eigenvalues $\pm (\phi'(\cdot))^{1/2}$ and 0. If $\phi'(\cdot) - a(0)\psi'(\cdot) > 0$, (4.11) has a uniquely determined steady state solution. Observe that initially $z_x \equiv 0$; one can solve the first two equations in (4.11) by various techniques for conservation laws on the first time step, update z using the last equation and proceed forward in time. Jointly with B. Plohr we have initiated a study of various numerical algorithms for (4.11) in the special case $a(t) = \sum_{i=1}^{n} a_k \exp(-\lambda_k t)$, $a_k > 0$, $\lambda_k > 0$, including the Glimm scheme with fractional steps. One objective is to establish existence of weak solutions for small BV data. Another is to obtain implementable numerical algorithms which can be tested on concrete problems.

Boldrini [3, 4] used techniques of compensated compactness to study elastic and viscoelastic problems including the system (4.9), (4.10). These techniques were developed by Tartar [50,51,52], Murat [40] and DiPerna [14]; in [14] DiPerna succeeded to extend these techniques and apply them to establish the existence of weak solutions of the purely elastic one-dimensional problem (i.e. (4.9), (4.10) with $a \equiv 0$) on $R \times [0,T]$ for any $T > 0$, without restricting the size of the data. Boldrini [4] assumes that the memory in (4.9) is small in the sense that

$$a := a(\delta,t), \quad \psi(\cdot) := \phi(\cdot) + \mu g(\cdot) \quad , \tag{4.12}$$

where $\delta > 0$, $\mu > 0$ are small parameters, g is a smooth function satisfying the growth condition $|g(w)| < \kappa|w|$, $\kappa > 0$, and $a'(\delta,t) = O(\delta)$, $a''(\delta,t) = O(\delta)$ uniformly in t. In place of (4.9) he considers the regularized system

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$$w_t = v_x$$

$$v_t = \phi(w)_x + a'(\delta,\cdot) * (\phi(w) + \mu g(w))_x + \epsilon v_{xx},$$

(4.13)

with initial data (4.10) (the Newtonian viscosity can be more general than ϵv_{xx}), where $\epsilon > 0$ is a small parameter. Let $w_{\epsilon,\delta,\mu}, v_{\epsilon,\delta,\mu}$ be a solution of (4.9), (4.10) on $R \times [0,T]$ for any $T > 0$. Boldrini gives sufficient conditions which insure that there is a subsequence such that $w_{\epsilon,\delta,\mu} \to w, v_{\epsilon,\delta,\mu} \to v$ on $R \times [0,T]$ as $\epsilon, \delta, \mu \to 0^+$, where $\mu = O(\epsilon^{1/2}\delta^{-1})$. Moreover, w, v is a weak solution of the purely elastic problem on $R \times [0,T]$. The most serious of his assumptions is the crucial hypothesis requiring the solutions $w_{\epsilon,\delta,\mu}, v_{\epsilon,\delta,\mu}$ of (4.13) to lie in L^∞ uniformly in the parameter ϵ, δ, μ. Since the memory is a nonlocal operator, this assumption is difficult to verify.

Jointly with W. Rogers and T. Tzavaras, we are using compensated compactness techniques to establish the existence of weak solutions of (4.9), (4.10). The special case $\psi \equiv \phi$, but with the memory not small (i.e. a independent of δ) is tractable by these methods and the case $\psi \neq \phi$ appears doable. However, obtaining an invariant region in order to show that solutions of the relevant regularized system lie in L^∞ is extremely difficult. It is of interest to note that the existence of weak solutions of the Cauchy problem for the model first-order scalar equation with memory

$$u_t + \phi(u)_x + a'*\psi(u)_x = 0, \quad x \in R, \ t > 0$$

$$u(x,0) = u_0(x), \quad x \in R,$$

(4.14)

where a, ϕ, ψ have the same meaning as in (4.9), can be solved completely by using the method of compensated compactness. The maximum principle proved by Dafermos in [9] for classical solution of (4.14) makes it possible to prove the needed L^∞ estimates for solutions of the regularized problem (i.e. (4.14) with ϵu_{xx} on the right side in place of zero). This problem was recently solved by Dafermos (oral communication). Unfortunately, it does not appear that this approach can be extended to coupled two by two systems with memory.

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1. REPORT NUMBER #2944	2. GOVT ACCESSION NO. *AD-A172 587*	3. RECIPIENT'S CATALOG NUMBER
4. TITLE *(and Subtitle)* NONLINEAR VISCOELASTIC MATERIALS WITH FADING MEMORY		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) John A. Nohel		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53705		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE July 1986
		13. NUMBER OF PAGES 28
14. MONITORING AGENCY NAME & ADDRESS*(If different from Controlling Office)*		15. SECURITY CLASS. *(of this report)* UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT *(of this Report)*

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT *(of the abstract entered in Block 20, If different from Report)*

18. SUPPLEMENTARY NOTES

19. KEY WORDS *(Continue on reverse side if necessary and identify by block number)*

Nonlinear viscoelasticity, hyperbolic equations, Volterra equations, smooth kernels, singular kernels, global smooth solutions, energy estimates, development of singularities, Riemann invariants, weak solutions, compensated compactness

20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

The equations governing the motion of viscoelastic materials with fading memory incorporate a nonlinear elastic-type response with a natural dissipative mechanism. Our purpose is to discuss the subtle effects of this mechanism in viscoelastic materials of Boltzmann type. Recent results on the global existence and decay of classical solutions for smooth and small data (in one space dimension) are reviewed for smooth and singular memory kernels; for smooth kernels a number of such results can be generalized to several space dimensions. A recent result on the development of singularities in finite time for large

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20. ABSTRACT (continued)

data is discussed; several open problems are formulated. A program for studying weak solutions for such systems, including the development of numerical algorithms, is outlined.

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