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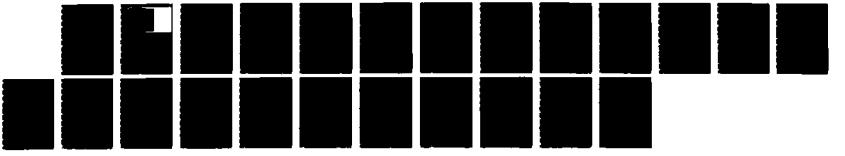

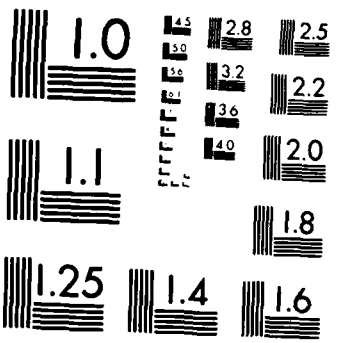

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CONVERGENCE THEOREMS FOR SEMI-GROUPS OF LINEAR OPERATORS OF CLASS (1,A)

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ABSTRACT

The subject of approximation of semi-groups of operators has been studied by many authors under the assumption that the sequence of semi-groups $\{T_n(t)\}_0^\infty$ *is uniformly bounded for all values of the parameter in the sense that* $\|T_n(t)\| \leq M$, *where* M *is independent of* n *and* t. *In particular this has restricted the results to semi-groups of class* c_0. *The purpose of the present paper is to investigate this problem for a more general class of semi-groups, namely the class (1,A) and thereby generalize some results of Trotter [9], Kurtz [4] and Kato [3].*

AMS (MOS) Subject Classification: 47D05

Key Words: approximation, semigroups, linear operators, resolvent operators, Banach space, limit, extended limit.

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CONVERGENCE THEOREMS FOR SEMI-GROUPS OF LINEAR OPERATORS OF CLASS (1,A)

Nazar H. Abdelaziz*

1. Introduction.

Approximation of semi-groups of linear operators has for quite somtime been a subject of interest to several authors (c.f. [3], [4], [9] and [10]), this is due to the importance of its role both in theory and applications. An instance where this becomes clear occurs e.g. when one tries to approximate the solution of an initial value problem in partial differential equations by a sequence of numerical solutions of some related finite difference equations. This leads in a natural way to the question of approximation of semi-groups of operators (c.f. [9]). However we find that in all the literature dealing with this problem, the following assumption was made:

If $\{T_n(\xi), \xi \geq 0\}$ is a sequence of c_0 semi-groups of bounded linear operators on some Banach space (or more generally a sequence of Banach spaces, c.f. [9]) then there exists constants $M > 0, \beta$ independent of n, ξ such that

$$\|T_n(\xi)\| \leq Me^{\beta\xi}, \qquad \forall n, \xi \geq 0 \tag{1}$$

In many cases the stronger requirement that all semi-groups be contractive was also used (i.e. $\|T_n(\xi)\| \leq 1, \forall n, \xi \geq 0$).

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It is perhaps worthwhile to comment at this point on the relation between (1) and the method of approximation by numerical solutions. It turns out that (1) is the functional analytic version of the von Neumann condition of stability for systems of finite difference equations. A condition which had resulted from the problem of 'error gross' that is associated with numerical solutions. On the other hand one finds (c.f. [6]) that there are unstable systems for which the numerical solutions do converge to the actual solution of the PDE.

The preceding paragraph suggests that one might investigate the problem of approximation of semi-groups of operators away from condition (1). Indeed, the purpose of the present paper is to carry on such an investigation. thus we replace (1) by the less restrictive assumption in def 2 below, (namely DIF) and thereby allowing semi-groups of a more general class to be considered.

Following the Trotter-Kurtz approach (c.f. [9], [4]), concerning a more general notion of convergence for sequences of vectors and operators, we obtain in propositions 1.2 and 3 necessary and sufficient conditions for the convergence of a sequence of semi-groups of class (1,A) (c.f. [2] for def.) in terms of the corresponding sequence of infinitesimal generators and also in terms of the resolvent operators. This extends the corresponding results of [3], [4] and [9].

Some remarks and examples are also discussed at the end. However we make here one final observation. While we noted before that according to [6], there are unstable systems for which convergence hold, yet it was indicated in [7] (see also [5]) that under appropriate conditions "stability is necessary and sufficient for convergence of the numerical

approximations. We clarify this from the functional analysis side in Remarks 2, 3 below.

2. Preliminaries.

In what follows $(X, \|\cdot\|)$ is a Banach space, $\mathcal{L}(X)$ the space of bounded linear operators of X. For an arbitrary linear operator A from X to itself we let $\mathcal{D}(A), \mathcal{R}(A), \rho(A)$ and $R(\lambda; A)$ denote respectively the domain, range, resolvent set and the resolvent operator of A, where $\lambda \in \rho(A)$. We begin by introducing the notion of limits for sequences of vectors and operators due to Trotter [9] and Kurtz [4].

A sequence $\{X_n, \|\cdot\|_n\}$ of Banach spaces is called an approximating sequence to the Banach space $(X, \|\cdot\|)$ iff there exists a sequence of linear operators $\{P_n\}$; $P_n : X \to X_n$ such that $lim_{n\to\infty}\|P_n x\|_n = \|x\|, \quad \forall x \in X$. In particular this $\Rightarrow \exists K > 0$, such that $\|P_n\|_n \leq K, \forall n$ where $\|P_n\|_n$ is the operator norm of P_n. Consider a sequence $\{x_n\}$ where $x_n \in X_n, \forall n$, we say that $\{x_n\}$ converges to $x \in X$, written

$$\hat{lim}_n x_n = x \quad \text{iff} \quad \lim_{n\to\infty} \|x_n - P_n x\|_n = 0.$$

We also consider sequences of operator $\{A_n\}$ where for each n, $\mathcal{D}(A_n)$ and $\mathcal{R}(A_n)$ are subspaces of X_n. An operator $A : X \to X$ is called the limit of $\{A_n\}$, written $A = \hat{lim}_n A_n$ (Trotter [9]) iff $x \in \mathcal{D}(A) \Rightarrow P_n x \in \mathcal{D}(A_n)$ for all n and there exists $y \in X$ such that $\hat{lim}_n A_n P_n x = y$ in which case $Ax = y$ by definition. A more general limit for sequences of operators is the following (T. Kurtz [4]); An operator $\mathcal{A}$ (possibly multivalued) from X to itself is called the extended limit of $\{A_n\}$, written $\mathcal{A} = ex - limA_n$ iff $x \in \mathcal{D}(\mathcal{A})$ implies that there exists a sequence $\{x_n\}, x_n \in \mathcal{D}(A_n)$ and $y \in X$ such that $\hat{lim}_n x_n = x$ and $\hat{lim}_n A_n x_n = y$, in this case we write $y \in \mathcal{A}x$. It is also noted that (c.f. [4]) $\mathcal{A}$ is closed and $\supseteq \hat{lim} A_n$.

Next, we review few facts from the theory of semi-groups of linear operators. Let $T(t)$: $(0,\infty) \to \mathcal{L}(X)$ be a strongly continuous semi-group of linear operators with infinitesimal operator A_0, where

$$A_0 x = s - \lim_{h \to 0^+} A_h x; \qquad A_h = h^{-1}(T(h) - I)$$

whenever the limit exists. In general A_0 is not closed but whenever the closure exists it will be denoted by A and called the infinitesimal generator (i.g.) of $T(t)$. We also denote the type of $T(t)$ by ω where

$$\omega = \liminf_{t \to \infty} t^{-1} ln \|T(t)\|$$

It is well known that $\omega < \infty$ (see e.g. [2]).

Definition 1. (c.f. Phillips [8]) $T(t)$ is said to be of class (1,A) iff the following conditions hold:

(a) (Integrability condition)

$$\int_0^1 \|T(t)\| \, dt < \infty, \tag{2}$$

this implies that for $\Re e\lambda > \omega$ the integral

$$J(\lambda)x := \int_0^\infty e^{-\lambda t} T(t)x \, dt \tag{3}$$

exists $\forall x \in X$; i.e. $J(\lambda) \in \mathcal{L}(X)$,

(b) (Abel-Summability)

$$\lim_{\lambda \to \infty} \lambda J(\lambda)x = x, \quad \forall x \in X. \tag{4}$$

In this case (c.f. [2]) the infinitesimal generator A exists and $R(\lambda; A) = J(\lambda)$; $\forall \Re e\lambda > \omega$.

A semi-group $T(t)$ is said to be of class c_0 if $s - \lim_{t\to 0^+} T(t)x = x, \forall x \in X$. Every c_0-semi-group is also of class (1,A) but the converse need not be true (c.f. [8]).

3. Convergence Theorems.

We begin by stating some conditions under which the results on convergence are established.

Definition 2. For each n, let $T_n(t), t > 0$ be a strongly continuous semi-group of bounded linear operators on X_n, we say that the sequence $\{T_n(t); t > 0\}$ satisfies the (DIF)- Condition, (DIF for dominance by integrable function) iff there exists a non-negative measurable function $\phi(t), t > 0$ and a positive constant γ_0 such that $e^{-\gamma_0 t}\phi(t)$ is integrable over $(0, \infty)$ and

$$\|T_n(t)\|_n \leq \phi(t) \qquad \text{for almost all } t \geq 0, \forall n. \tag{5}$$

To see how this may be compared with (1), we could replace (1) (WLOG) by

$$\|T_n(t)\|_n \leq M, \quad \forall n, t \geq 0.$$

This is done simply by replacing $\{T_n(t)\}$ with the equivalent sequence $\{e^{-\beta t}T_n(t)\}$. Thus we see (with $\phi(t) = M$) that (DIF) is also satisfied, the converse though need not be true.

Remark 1. It is clear that if $\{T_n(t)\}$ satisfies the (DIF) condition then (2) holds for each n, in particular the integral in (3) exists and defines a bounded linear operator $J_n(\lambda)$ on X_n for $\Re e\lambda > \gamma_0$. If in addition (4) is satisfied then $T_n(t)$ is of class (1,A). In this case we shall say that $\{T_n(t)\}$ is a proper sequence of class (1,A). It will be useful also to consider (4) in some uniform sense with respect to n, this is introduced in the following.

Definition 3. For each n assume that $T_n(t), t > 0$ is a strongly continuous semi-group of linear operators on X_n satisfying (2) of definition 1, and let $J_n(\lambda)$ be the corresponding

operator defined by (3), where $\Re e\lambda > \gamma_0$ then we say that the sequence $\{T_n(t); t > 0\}$ satisfies the (UAS) condition (Uniformly Abel Summable) if $\exists$ a positive constant L, independent of n and λ such that

$$\|\lambda J_n(\lambda)\|_n \leq L, \quad \forall n, \lambda \geq \gamma_0. \tag{6}$$

If $T_n(t), t > 0$ is of class (1,A) with i.g. A_n then as mentioned earlier, $\mathcal{A} = ex - limA_n$ may not be single valued in general. In lemma 1 we describe two instances in which it is single valued.

Lemma 1. Suppose $T_n(t), t > 0$ is of class (1,A) for all n, then each of the following $\Rightarrow \mathcal{A}$ is single valued; (a) $\{T_n(t)\}$ satisfies the (UAS) condition and $\mathcal{D}(\mathcal{A})$ is dense in X. (b) $A = \hat{lim}A_n$ has a closed extension (that is single valued and linear) such that for some λ_0; $(\lambda_0 - \mathcal{A})^{-1} \in \mathcal{L}(X)$ and $\mathcal{R}(\lambda_0 - A)$ is dense in X.

Proof. (a) Let $x_n \in \mathcal{D}(A_n)$, then $\exists y_n \in X_n$ such that $x_n = R(\lambda; A_n)y_n$,. Thus making use of (6) we see that $\lambda\|x_n\|_n \leq L\|\lambda x_n - A_n x_n\|_n$ for $\lambda > \gamma_0$ and all n. Now the proof that $\mathcal{A}$ is single valued follows exactly as in lemma 1.1, of [4].

(b) Since $\mathcal{A}$ is an extension of A and $\lambda_0 \in \rho(\mathcal{A})$, the inverse $(\lambda_0 - A)^{-1}$ of $(\lambda_0 - A)$ exists as a bounded operator on its domain $\mathcal{R}(\lambda_0 - A)$. Since this is dense by assumption, $(\lambda_0 - \mathcal{A})^{-1}$ is the unique bounded extension of $(\lambda_0 - A)^{-1}$ to all X. Now assume $x, y \in X$ such that $(\lambda_0 - \mathcal{A})^{-1}x = (\lambda_0 - \mathcal{A})^{-1}y = z$, and let $\{x_n\}, \{y_n\} \subseteq \mathcal{R}(\lambda_0 - A)$ be such that $x_n \to x$ and $y_n \to y$, so that $v_n = (\lambda_0 - A)^{-1}x_n \to z$ and $u_n = (\lambda_0 - A)^{-1}y_n \to z$. Therefore, with $w_n = v_n - u_n$, we have $w_n \to 0$ and $(\lambda_0 - A)w_n \to (x - y)$. Thus x = y since A admits a closed extension, showing that $\lambda_0 - \mathcal{A}$ (hence $\mathcal{A}$) is single valued.

The following lemmas will be needed in the sequel.

Lemma 2. Assume that $\{T_n(t)\}$ is a proper sequence of class (1,A), then there exists a constant $M > 0$, independent of n and λ such that

$$\|R(\lambda; A_n)\|_n \leq M, \quad \forall n, \Re e\lambda > \gamma_0.$$

In particular if $\hat{lim} x_n = 0$ where $x_n \in X_n, \forall n$ then $\hat{lim}_n R(\lambda; A_n)x_n = 0$, uniformly with respect to λ, in the half plane $\Re e\lambda > \gamma_0$.

Proof. By hypothesis, each $T_n(t), t > 0$ is of class (1,A) in particular $R(\lambda; A_n) = J_n(\lambda), \Re e\lambda > \gamma_0$. therefore in view of (3),

$$\begin{aligned}\|R(\lambda; A_n)x\|_n &\leq \int_0^\infty e^{-\gamma t}\|T_n(t)x\|_n\, dt \\ &\leq \|x\|_n \int_0^\infty e^{-\gamma_0 t}\phi(t)\, dt = M \cdot \|x\|_n, \quad \forall x \in X_n.\end{aligned}$$

Where $\gamma = \Re e\lambda > \gamma_0$. Hence $\|R(\lambda; A_n)\|_n \leq M$ for all n. The rest follows easily from this.

To further simplify the notations we let $L(\gamma_0) := \{\lambda / \Re e\lambda > \gamma_0\}$ and $S(\gamma_0) := \rho(\mathcal{A}) \cap L(\gamma_0)$.

Lemma 3. Assume the hypothesis of lemma 2 and that $\mathcal{A}$ is single valued. If $\lambda \in S(\gamma_0)$ then

$$R(\lambda; \mathcal{A}) = ex - limR(\lambda; A_n), \tag{7}$$

moreover:

$$\|R(\lambda; \mathcal{A})\| \leq M, \quad \forall \lambda \in S(\gamma_0). \tag{8}$$

Proof. Let $\lambda \in S(\gamma_0), x \in X$, then $\exists y \in \mathcal{D}(\mathcal{A})$ such that $x = \lambda y - \mathcal{A}y$. By definition $\exists$ a sequence $\{y_n\}$, $y_n \in \mathcal{D}(A_n)$ such that $\hat{lim}_n y_n = y$ and $\hat{lim}_n A_n y_n = \mathcal{A}y$. Put $x_n = \lambda y_n - A_n y_n$ and note by remark 1 that $\Re e\lambda > \gamma_0 \Rightarrow \lambda \in \rho(A_n)$ so that $y_n = R(\lambda; A_n)x_n$.

Thus we find that $\hat{lim}_n x_n = \lambda y - Ay = x$ and $\hat{lim}_n R(\lambda; A_n)x_n = \hat{lim}_n y_n = y = R(\lambda; A)x$, this proves (7). Now (8) follows from (7) and lemma 2 in a straight forward manner.

The following lemma presents a slightly stronger version of a result used in the proof of theorem 2.1 of [4]. We present the proof for convenience.

Lemma 4. Assuming the hypotheses of lemma 2 and that A is single valued. If $\mathcal{R}(\lambda_0 - A)$ is dense in X for some $\lambda_0, \Re e\lambda_0 > \gamma_0$, then $L(\gamma_0) \subseteq \rho(A)$.

Proof. The proof consists of showing that $S(\gamma_0)$ is a nonvoid closed subset of $L(\gamma_0)$ and since it is also open by definition, it must coincide with $L(\gamma_0)$. To show that $S(\gamma_0) \neq \emptyset$, it suffices to show that $\lambda_0 \in \rho(A)$. As before $\lambda_0 \in \rho(A_n)$, thus for every $w \in X_n$ $\exists u \in \mathcal{D}(A_n)$ such that $u = R(\lambda_0; A_n)w$ or equivalently $\lambda_0 u - A_n u = w$. Making use of lemma 2 we see that

$$\|u\|_n \leq M\|\lambda_0 u - A_n u\|_n.$$

Now if $x \in \mathcal{D}(A)$ and $y = Ax$ then by definition $\exists$ a sequence $\{x_n\}, x_n \in \mathcal{D}(A_n)$ such that $\hat{lim}_n x_n = x$ and $\hat{lim}_n A_n x_n = Ax$; applying the preceding inequality we have that $(1/M)\|x_n\|_n \leq \|\lambda_0 x_n - A_n x_n\|_n$, and by passing to the limit we obtain $(1/M)\|x\| \leq \|\lambda_0 x - Ax\|$. This shows that $(\lambda_0 - A)$ has a bounded inverse whose domain is dense in X (therefore equal to X since A is closed). Consequently $\lambda_0 \in \rho(A)$. Next to show that $S(\gamma_0)$ is closed in $L(\gamma_0)$, let $\{\lambda_n\} \subseteq S(\gamma_0)$, $\lambda_n \to \lambda \in L(\gamma_0)$. By Lemma 3. we see that for each $x \in X$

$$\|(\lambda - A)R(\lambda_n; A)x - x\| = \|(\lambda - \lambda_n)R(\lambda_n; A)x\| \leq |\lambda - \lambda_n| \cdot M \cdot \|x\|,$$

which means that $\mathcal{R}(\lambda - A)$ is dense in X. Hence $\lambda \in \rho(A)$ by the 1st part of the proof.

Our first result is concerned with the equivalence between the convergence of a sequence of operators and that of its resolvent operators.

Proposition 1. Assume that $\{T_n(t), t > 0\}$ is a proper sequence of class (1,A) with infinitesimal generators $\{A_n\}$ respectively, such that $ex - limA_n$ is single valued. Further let $\breve{A}$ be a linear operator with domain and range in X, then the following assertions are equivalent:

(i) $\breve{A} = ex - limA_n$ and $\mathcal{R}(\lambda_0 - \breve{A})$ is dense in X, for some $\lambda_0 \in L(\gamma_0)$

(ii) $R(\lambda; \breve{A})x = \hat{lim}_n R(\lambda; A_n)x_n$, whenever $\hat{lim} x_n = x$ and $\lambda \in L(\gamma_0)$.

Proof. (i) $\Rightarrow$ (ii): This follows directly from lemma 4 and eq. (7).

(ii) $\Rightarrow$ (i): Let $\lambda \in L(\gamma_0)$ and $x \in \mathcal{D}(\breve{A})$ then there exists $y \in X$ such that $\lambda x - \breve{A}x = y$. Put $y_n = P_n y$ and $x_n = R(\lambda; A_n)y_n$, then by assumption $\hat{lim}_n x_n = R(\lambda; \breve{A})y = x$, and since $y_n = \lambda x_n - A_n x_n$ we see that $\hat{lim}_n A_n x_n = \breve{A}x$. Thus $\breve{A} \subseteq ex - limA_n =: \mathcal{A}$ and in particular $\mathcal{R}(\lambda - \breve{A}) \subseteq \mathcal{R}(\lambda - \mathcal{A})$.

Now $\mathcal{R}(\lambda - \breve{A})$ is dense in X since $\lambda \in \rho(\breve{A})$ by assumption, thus we find in view of lemma 4 that $\lambda \in \rho(\mathcal{A})$. Next let $x \in X$ and $y = R(\lambda; \mathcal{A})x \in \mathcal{D}(\mathcal{A})$, and let $\{y_n\}, y_n \in \mathcal{D}(A_n)$ be a sequence satisfying $\hat{lim}_n y_n = y$, $\hat{lim}_n A_n y_n = \mathcal{A}y$. Put $x_n = \lambda y_n - A_n y_n$ then $y_n = R(\lambda; A_n)x_n$ since $\lambda \in L(\gamma_0) \subseteq \rho(A_n)$, so we have that $\hat{lim} x_n = \lambda y - \mathcal{A}y = x$, $\hat{lim}_n R(\lambda; A_n)x_n = y = R(\lambda; \mathcal{A})x$. On the other hand we know by assumption that $\hat{lim}_n R(\lambda; A_n)x_n = R(\lambda; \breve{A})x$. Thus $R(\lambda; \mathcal{A})x = R(\lambda; \breve{A})x$, $\forall x \in X$. This $\Rightarrow \mathcal{D}(\mathcal{A}) = \mathcal{D}(\breve{A})$ and $\mathcal{A} = \breve{A}$, which completes the proof.

Lemma 5. Suppose that $\{T_n(t), t > 0\}$ is a proper sequence of class (1,A) such that $\mathcal{A} = ex - limA_n$ is single valued. where A_n is the i.g. of $T_n(t)$. Further assume that

$\mathcal{R}(\lambda_0 - \mathcal{A})$ is dense in X, for some $\lambda_0 \in L(\gamma_0)$ then the following assertions hold:

(i) If $\hat{lim}_n x_n = x$, $x_n \in X_n, x \in X$ and $\lambda \in L(\gamma_0)$ then $\hat{lim}_n [R(\lambda; A_n)]^k x_n = [R(\lambda; \mathcal{A})]^k x$, for all positive integers k.

(ii) $ex - limA_n^2$ is an extension of $\mathcal{A}^2$ furthermore; $\forall z \in \mathcal{D}(\mathcal{A}^2)$ there exists a sequence $\{z_n\}, z_n \in \mathcal{D}(A_n^2)$ such that $\hat{lim}_n z_n = z$, $\hat{lim}_n A_n z_n = \mathcal{A}z$ and $\hat{lim}_n A_n^2 z_n = \mathcal{A}^2 z$.

Proof. (i) follows directly using proposition 1 and the induction on k. To verify (ii), we note as before that $R(\lambda_0; \mathcal{A}) \in \mathcal{L}(X)$ so that $\mathcal{D}(\mathcal{A}^2) = R^2(\lambda_0, \mathcal{A})X$. Let $z \in \mathcal{D}(\mathcal{A}^2)$ and let $x \in X$ be such that $z = R^2(\lambda_0; \mathcal{A})x$. Further put $y := R(\lambda_0; \mathcal{A})x$, $x_n := P_n x$, $y_n := R(\lambda_0; A_n)x_n$ and $z_n := R(\lambda_0; A_n)y_n$. By the first part we see that $\hat{lim}_n y_n = R(\lambda_0; \mathcal{A})x = y$ and $\hat{lim}_n z_n = R(\lambda_0; \mathcal{A})y = z$. This yields $\hat{lim}_n(\lambda_0 z_n - A_n z_n) = \hat{lim}_n y_n = y = \lambda_0 z - \mathcal{A}z$ and hence

$$\hat{lim}_n A_n z_n = \mathcal{A}z. \tag{9}$$

Moreover $\hat{lim}_n(\lambda_0 y_n - A_n y_n) = \hat{lim}_n x_n = x = \lambda_0 y - \mathcal{A}y$ which implies that $\hat{lim} A_n y_n = \mathcal{A}y$, i.e. $\hat{lim}_n A_n(\lambda_0 z_n - A_n z_n) = \mathcal{A}(\lambda_0 z - \mathcal{A}z)$. This together with (9) $\Rightarrow \hat{lim}_n A_n^2 z_n = \mathcal{A}^2 z$, showing that $z \in \mathcal{D}(ex - limA_n^2)$ and so that $\mathcal{A}^2 \subset ex - limA_n^2$. This concludes the proof.

The proposition which we now present establishes the equivalence between the convergence of a sequence of semi-groups and that of the corresponding sequence of infinitesimal generators. This extends a result of Kurtz [4].

Proposition 2. Assume that $\{T_n(t), t > 0\}$ is a proper sequence of class (1,A) with infinitesimal generators $\{A_n\}$, such that $ex - limA_n$ is single valued. Further let $\check{A}$ be a linear operator from X to itself with dense domain, then the following are equivalent:

(i) $\check{A} = ex - limA_n$ and $\mathcal{R}(\lambda_0 - \check{A})$ is dense in X for some $\lambda_0 \in L(\gamma_0)$.

(ii) There exists a strongly continuous semi-group $T(t), t > 0$ of bounded linear operators on X such that:

(a) $\hat{lim}_n T_n(t)x_n = T(t)x,\ t > 0$ whenever $\hat{lim}_n x_n = x,\ x_n \in X_n$ and $x \in X$,

(b) $R(\lambda; \breve{A})x = \int_0^\infty e^{-\lambda t}T(t)xdt, \qquad \forall x \in X, \lambda \in L(\gamma_0)$.

In particular it follows from (a) that $\|T(t)\| \leq \phi(t)$ for almost all $t > 0$.

Proof. (i) $\Rightarrow$ (ii):

In view of lemmas 4, 3 and the fact that $ex - limA_n$ is closed we find that $\breve{A}$ is a closed operator whose resolvent $R(\lambda; \breve{A})$ exists and is bounded in the half plane $L(\gamma_0)$. Thus according to a lemma of Hille and Phillips (c.f. [2]) we can choose $\gamma > \gamma_0$ such that

$$(10) \qquad \mathcal{Y}(t; z) = z + t\breve{A}z + \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}R(\lambda; \breve{A})\breve{A}^2 z\frac{d\lambda}{\lambda^2} \qquad , \forall z \in \mathcal{D}(\breve{A}^2),$$

defines a strongly continuous function in t for all $t \geq 0$ with $\mathcal{Y}(0; z) = z$. Moreover, since each $T_n(t)$ is of class (1,A), with $L(\gamma_0) \subseteq \rho(A_n)$ we have by the same reasoning that

$$(11) \qquad T_n(t)w = w + tA_n w + \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\gamma t}R(\lambda; A_n)A_n^2 w\frac{d\lambda}{\lambda^2}, \quad \forall w \in \mathcal{D}(A_n^2),$$

Here γ is chosen to be the same as in (10). Now let $z \in \mathcal{D}(\breve{A}^2)$ then according to assertion (ii) of lemma 5 there exists a sequence $\{z_n\}, z_n \in \mathcal{D}(A_n^2)$ such that

$$\hat{lim}_n z_n = z, \quad \hat{lim}_n A_n z_n = \breve{A}z \quad \text{and} \quad \hat{lim}_n A_n^2 z_n = \breve{A}^2 z$$

thus replacing w by z_n in (11) we see that the first and second terms in the right hand side converge (in the sense of §2 above) to the corresponding terms in (10). As for the integrals we note upon recalling lemma 5-(i) that

$$\hat{lim}_n R(\lambda; A_n)A_n^2 z_n = R(\lambda; \breve{A})\breve{A}^2 z.$$

Now the integrand in (11) is dominated in norm by $\alpha M e^{\gamma t}/ \mid \lambda \mid^2$, where $\alpha = max\{(\|A_i^2 z_i\|_i : 1 \leq i \leq N), \varepsilon + K\|\check{A}^2 z\|\}$, here N depends on ε and K is the bound on $\|P_n\|_n$ (see §2.) Since this is integrable over the line γ, the integrals in (11) converge to that in (10). Summing up we find that

$$\hat{lim}_n T_n(t) z_n = \mathcal{Y}(t; z), \qquad \forall t \geq 0, z \in \mathcal{D}(\check{A}^2), \tag{12}$$

moreover it can be verified directly from this, that

$$\|\mathcal{Y}(t; z)\| \leq \phi(t) \cdot \|z\| \quad \text{for almost all} \quad t \geq 0, z \in \mathcal{D}(\check{A}^2) \tag{13}$$

Since $\mathcal{D}(\check{A})$ is dense, $\mathcal{D}(\check{A}^2)$ is also dense and therefore $\mathcal{Y}(t; z)$ has a unique bounded extension to all of X (we denote it again by $\mathcal{Y}(t; z)$) which satisfies (13). To finish the proof of (ii)-(a), we show that for any $z_0 \in X$

$$\hat{lim}_n T_n(t) P_n z_0 = \mathcal{Y}(t; z_0). \tag{14}$$

Let $\varepsilon > 0$ be given, then $\exists z \in \mathcal{D}(\check{A}^2)$ such that $\|z_0 - z\| < \varepsilon$, further let $\{z_n\}, z_n \in \mathcal{D}(A_n)$ be a sequence such that $\hat{lim}_n z_n = z$ and for which (12) is satisfied. Choose N such that $n \geq N \Rightarrow \|z_n - P_n z\|_n < \varepsilon$ and $\|T_n(t) z_n - P_n \mathcal{Y}(t; z)\|_n < \varepsilon$. We find

$$\|T_n(t) P_n z_0 - P_n \mathcal{Y}(t; z_0)\|_n \leq \|T_n(t) P_n z_0 - T_n(t) z_n\|_n +$$

$$\|T_n(t) z_n - P_n \mathcal{Y}(t; z)\|_n + \|P_n \mathcal{Y}(t; z) - P_n \mathcal{Y}(t; z_0)\|_n$$

$$\leq \phi(t)\{\|P_n z_0 - P_n z\|_n + \|P_n z - z_n\|_n\} + \varepsilon + \|P_n\|_n \cdot \phi(t)\|z - z_0\|$$

$$\leq \varepsilon\{\phi(t)[2K + 1] + 1$$

and (14) is now established.

Using (5) one can show that (14) indeed holds with $P_n z_0$ replaced by an arbitrary $z_n \in X_n$ for which $\hat{lim} z_n = z_0$. Finally, that $y(t;x)$ has the semi-group property is a direct consequence of the fact that it is a limit (in the above sense) of a sequence of semi-groups. We shall write $T(t)x$ for $y(t;x)$.

To verify (b) we note from above that $\|T(t)\| \leq \phi(t)$ a.e. and since $e^{-\gamma_0}\phi(t)$ is integrable over $(0,\infty)$, the integral

$$J(\lambda)x := \int_0^\infty e^{-\lambda t}T(t)x\,dt$$

exists for all $x \in X, \Re e\lambda > \gamma_0$. Now let $x \in X, x_n \in X_n$ be such that $\hat{lim}_n x_n = x$, then using (a) we find in view of the dominated convergence that

$$\lim_{n\to\infty} \|R(\lambda;A_n)x_n - P_n J(\lambda)x\|_n \tag{15}$$

$$\leq \lim_{n\to\infty} \int_0^\infty e^{-\gamma t}\|T_n(t)x_n - P_n T(t)x\|_n\,dt = 0,$$

where $\gamma = \Re e\lambda > \gamma_0$. This together with lemma 5-(i) imply that $R(\lambda;\check{A}) = J(\lambda)$.

(ii) $\Rightarrow$ (i)

We note from (b) and (15) that

$$\hat{lim}_n R(\lambda;A_n)x_n = R(\lambda;\check{A})x, \quad \text{whenever} \quad \hat{lim}_n x_n = x,$$

so that (i) follows by proposition 1.

Between propositions 1 and 2 it is possible now to deduce a generalization of a result of [3] (c.f. theorem 2.16, chapter IX).

Corollary. Let $\{T_n(t), t > 0\}$ be a proper sequence of class (1,A) with respective generators $\{A_n\}$, futher let $T(t)$ be a semi-group of class (1,A) with i.g. $\check{A}$. If $ex - limA_n$ is single valued, then for every sequence $\{x_n\}$ and $x \in X$ for which $\hat{lim}_n x_n = x$

$$\hat{lim}_n R(\lambda; A_n)x_n = R(\lambda; \check{A})x \qquad \text{for some} \quad \lambda \in L(\gamma_0)$$

if and only if

$$\hat{lim}_n T_n(t)x_n = T(t)x, \qquad \forall t > 0.$$

It is worthwhile to note from proposition 2 above that although each semi-group $T_n(t)$ is of class (1,A) the limit $T(t)$ need not be Abel summable. We also find (see examples in §20.3, [2]) that this situation persists even when members of the approximating sequence belong to class c_0. On the other hand we know (c.f. [4], [9]) that if a sequence of semi-groups of linear contractions converge then the limit is again a semi-group of linear contractions. The question therefore arises as whether one could say more about the limit semi-group, perhaps by imposing some extra conditions on the approximating sequence. An instance of this is shown in the following proposition.

Proposition 3. Assume that $\{T_n(t), t > 0\}$ is a proper sequence of class (1,A) satisfying the (UAS) condition and with respective generators $\{A_n\}$. Then the following assertions are equivalent:

$(i)'$ $\mathcal{D}(\mathcal{A})$ and $\mathcal{R}(\lambda - \mathcal{A})$ are both dense in X for some $\lambda \in L(\gamma_0)$,

$(ii)'$ $\mathcal{D}(\mathcal{A})$ is dense and $R(\lambda; \mathcal{A}) = ex - limR(\lambda; A_n), \lambda \in L(\gamma_0)$

$(iii)'$ There exists a semi-group $T(t), t > 0$ of class (1,A), defined on X, such that

$$\hat{lim}_n T_n(t)x_n = T(t)x, \qquad \text{whenever} \quad \hat{lim}_n x_n = x, \quad t > 0.$$

Furthermore, if $(iii)'$ holds then $\mathcal{A}$ is the i.g. of $T(t)$.

Proof. Lemma 1-(a) shows that $\mathcal{A}$ is single valued, thus the equivalence of $(i)'$ and $(ii)'$ follow by proposition 1. Also in $(iii)'$, if we take $\breve{A}$ to be the infinitesimal generator of $T(t)$ then the preceding corollary shows that $(iii)' \Rightarrow (ii)'$. The implication $(i)' \Rightarrow (iii)'$ follows from proposition 2 with the only exception of showing that $T(t)$ is Abel summable. For this, let C denote the continuity set of $T(t)$, i.e. the set of all $x \in X$ for which $T(\eta)x \to x$ as $\eta \to 0^+$, and note that $\lim_{\lambda\to\infty} \lambda J(\lambda)x = x$, $\forall x \in C$ where $J(\lambda)$ is defined as in (3). To conclude that this limit holds for all $x \in X$, we need to show that C is dense and that $\|\lambda J(\lambda)\|$ is bounded for all real $\lambda > \gamma_0$, for in this case the conclusion follows by the Banach-Steinhauss theorem. Now as in proposition 2, $(i)' \Rightarrow J(\lambda) = R(\lambda; \mathcal{A})$. Thus if $x \in \mathcal{D}(\mathcal{A}), \exists y \in X$ such that $x = \int_0^\infty e^{-\lambda t}T(t)y\,dt, \quad \lambda > \gamma_0$. Consequently $T(\eta)x \to x$ as $\eta \to 0^+$ showing that $\mathcal{D}(\mathcal{A}) \subseteq C$, hence C is dense. The boundedness of $\|\lambda J(\lambda)\|$ follows directly from (6) and (7).

As a consequence of Proposition 2 we also obtain the following generalization of a result of Trotter (c.f. [9]).

Proposition 4. Let $\{T_n(t)\}$ be a proper sequence of class (1,A). Let A_n be the i.g. of $T_n(t)$ and $A = \hat{lim}_n A_n$. Further, suppose that $\mathcal{D}(A)$ and $\mathcal{R}(\lambda - A)$, $(\lambda > \gamma_0)$ are dense subspaces of X. If $\bar{A}$ (the closure of A) exists, then there exists a strongly continuous semi-group T(t) of linear operators of X such that

$$\hat{lim}_n T_n(t) = T(t)$$

and

$$R(\lambda; \bar{A})x = \int_0^\infty e^{-\lambda t}T(t)x\,dt, \quad \Re e\lambda > \gamma_0, x \in X.$$

Moreover, if $T(t)$ is Abel summable then $\bar{A}$ is the i.g. of $T(t)$.

Proof. By lemma 2 above and lemma 2.4 of ([9]) we see that

$$lim_n R(\lambda; A_n) = J(\lambda) \in \mathcal{L}(X), \quad \lambda > \gamma_0.$$

We show that $J(\lambda)$ is an inverse of $\lambda - \mathcal{A}$.

Let $x \in X$, $y = J(\lambda)x$ then $lim_n \|R(\lambda; A_n)P_n x - P_n y\|_n = 0$, also let $y_n = R(\lambda; A_n)P_n x \in \mathcal{D}(A_n)$, so that $\lambda y_n - A_n y_n = P_n x$ therefore

$$\|A_n y_n - P_n(\lambda y - x)\|_n = \lambda \|y_n - P_n y\|_n \to 0.$$

This means that $\hat{lim}_n A_n y_n = \lambda y - x$ and since $\hat{lim} y_n = y$, we see that $\mathcal{A}y = \lambda y - x$, which shows that $(\lambda - \mathcal{A})J(\lambda)x = (\lambda - \mathcal{A})y = x$, $\forall x \in X$. Thus $J(\lambda)$ is a right inverse. To see that it is also a left inverse, let $y \in \mathcal{D}(\mathcal{A})$ and put $x = \lambda y - \mathcal{A}y$, $z = J(\lambda)x$, then with y_n as before $\|y_n - P_n z\|_n \to 0$. Also, by definition $\exists u_n \in \mathcal{D}(A_n)$ such that $\hat{lim}_n u_n = y$ and $\hat{lim}_n A_n u_n = \mathcal{A}y$, put $v_n = \lambda u_n - A_n u_n$ then $\hat{lim}_n v_n = \lambda y - \mathcal{A}y = x$. Therefore

$$\|P_n z - P_n y\|_n \leq \|P_n z - R(\lambda; A_n)P_n x\|_n + \|R(\lambda; A_n)P_n x - R(\lambda; A_n)v_n\|_n$$

$$+ \|u_n - P_n y\|_n \to 0 \qquad \text{as} \quad n \to \infty.$$

Hence, $y = z$ and $J(\lambda)(\lambda - \mathcal{A})y = J(\lambda)x = z = y$.

Now, since $\mathcal{A}$ is single valued (lemma 1-(b)), we see from the above argument and lemma 2.4 of [9] that $\bar{A} = \mathcal{A}$. The conclusion of the theorem now follows from proposition 2, with $\breve{A}$ replaced by $\bar{A}$. The last part follows from a result of Phillips (c.f. [8]).

As noted in the introduction it was shown in [5] that under appropriate conditions " stability is equivalent to the convergence of the solutions " where everything was considered in the same space, i.e. $X_n = X, P_n = I$ for all n. We explain this in a functional analytic setting in the following remarks:

Remark 2. Assuming $X_n = X, P_n = I$, $\forall n$, we note that if a sequence $\{T_n(t)\}$ of c_0 semi-groups converges to a c_0 semi-group $T(t)$, then $\|T_n(t)\| \leq M$, $\forall n$, and $t \in [0,\infty)$ iff the convergence is uniform on $[0,1]$, (hence on any compact t interval of $[0,\infty)$). That the above condition $\Rightarrow$ uniform convergence, was shown in the theorems of Trotter [9] and Kurtz [4]. For the converse, assume not, then $\exists x \in X$ and subsequences $\{n_j\}$ and $\{t_j\}$, $t_j \in [0,1]$ such that $\|T_{n_j}(t_j)x\| \to \infty$ as $j \to \infty$ (this is due to the uniform boundedness theorem). Let $t \in [0,1]$ be such that $\lim_{j\to\infty} t_j = t$ (using a subsequence), then we have

$$\|T_{n_j}(t_j)x\| \leq \|T_{n_j}(t_j)x - T(t_j)x\| + \|T(t_j)x - T(t)x\| + \|T(t)x\|$$

where the first and second terms on the right side tend to 0 as $j \to \infty$, while the third is a finite constant, which is a contradiction.

Remark 3. In contrast with remark 2, we have the following situation (c.f. [2]); if $T(t), t > 0$ is of class (1,A) and of negative type, with i.g. A, and if $B_n = n^2R(n;A) - nI$, $n = 1,2,....$ Then

$$lim_n exp(tB_n)x = T(t)x, \quad \forall x \in X, t > 0.$$

Since B_n is a bounded operator the semi-group $T_n(t) = exp(tB_n)$ is of class c_0. Thus a sequence of c_0 semi-groups may converge to a semi-group that need not be of class c_0 (see example below); in particular (1) or equivalently $\|T_n(t)\| \leq M$, $\forall n, t \geq 0$ cannot hold in such a case, according to ([4] or [9]).

Example. We close our discussion by considering the example of Phillips, [8]. Let X be the space consisting of all sequence pairs $\{(\chi_n, \eta_n) : n = 1,2,...\}$ such that $\lim_{n\to\infty} \chi_n =$

0 and $\sum_{n=1}^{\infty} n^{\frac{1}{2}}|\eta_n| < \infty$. The norm is defined as follows: for $x \in X$, $\|x\| = \sup_n |\chi_n| + \sum_{n=1}^{\infty} n^{\frac{1}{2}}|\eta_n|$. The semi-group is defined by $T(t)x = \acute{x} = (\acute{\chi}_n, \acute{\eta}_n)$ where

$$\acute{\chi}_n = exp[-(n^{\frac{3}{2}} + in^2)t]\{\chi_n \cos(n^{\frac{1}{2}}t) - \eta_n \sin(n^{\frac{1}{2}}t)\},$$

$$\acute{\eta}_n = exp[-(n^{\frac{3}{2}} + in^2)t]\{\chi_n \sin(n^{\frac{1}{2}}t) + \eta_n \cos(n^{\frac{1}{2}}t)\}.$$

It is shown in [8] that $T(t)$ is of class (1,A) but not a c_0 semi group. Now if we let X_n be the space of all sequence pairs $\{(\chi_k, \eta_k) : \chi_k = \eta_k = 0, \forall k > n\}$, with norm $\|x_n\| = \sup_{1 \le k \le n} |\chi_k| + \sum_{k=1}^{n} k^{\frac{1}{2}}|\eta_k|$, $x_n \in X_n$, then it is easy to see that $\{X_n, \|\cdot\|\}$ is an approximating sequence of Banach spaces to the space $(X, \|\cdot\|)$. For the corresponding sequence of semi groups we put $T_n(t)x_n = \acute{x}_n = \{(\acute{\chi}_k, \acute{\eta}_k)_n\}$, where $x_n \in X_n$ and $\acute{\chi}_k, \acute{\eta}_k$ are given as above, i.e. $T_n(t) = T(t)\,|_{X_n}$.

One can verify directly that $T_n(t)$ is of class c_0 $\forall n$ and that the sequence satisfies the DIF condition moreover ; if $\lim x_n = x$, $x_n \in X_n$, $x \in X$ then $\lim_n T_n(t)x_n = T(t)x$. A routine calculation also shows that the i.g. A_n of $T_n(t)$ is given by

$$A_n = \begin{pmatrix} H_{11} & 0 & \dots & 0 \\ 0 & H_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & H_{nn} \end{pmatrix},$$

where

$$H_{kk} = (k^{\frac{3}{2}} + ik^2) \begin{pmatrix} -1 & \sqrt{k} \\ \sqrt{k} & -1 \end{pmatrix}.$$

We conclude in view of the above results that $\mathcal{A} = ex - limA_n$ is the i.g. of $T(t)$. Likewise, it is easy to check that $R(\lambda; A_n)$ is the restriction to X_n of the Laplace transform of $T(t)$ as computed in [8], so that $\lim_n R(\lambda; A_n) = R(\lambda; \mathcal{A})$.

References

[1] M. Hasegawa , A note on the convergence of semi-groups of operators. Proc. Japan Acad. 40 (1964), 262-266.

[2] E. Hille and R.S. Phillips, Functional analysis and semi-groups, rev. ed., Amer. Math. SOc. Colloq. Publ., Vol. 31, Providence, R.I., 1957.

[3] T. Kato, "Purturbation Theory for linear operators", Springer-Verlag, New York, 1984.

[4] T.G. Kurtz, Extensions of Trotter's operator semi-group approximation theorems, Jour. of Funl. Anal., Vol. 3, No. 3 (1969), 354-375.

[5] P.D. Lax and R.D. Richtmyer, Survey of the stability of linear finite difference equations, Comm. Pure and Appl. Math. 9 (1956), 267-293.

[6] W. Leutert, On the convergence of approximate solutions of the heat equation to the exact solution, Proc. Amer. Math. Soc. 2 (1951), 433-439.

[7] G.G. O'Brien, M.A. Hyman and S. Kaplan, A study of the numerical solutions of partial differential equations, J. Math. Phys. 29 (1951), 223-251.

[8] R.S. Phillips, An inversion formula for the Laplace transforms and semi-groups of linear operators, Annals of Math. Vol. 59, No. 2 (1954), 325-356.

[9] H.F. Trotter, Approximation of semi-groups of operators, Pac. Jour. Math. ,8 (1958), 887-919.

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The subject of approximation of semi-groups has been studied by many authors under the assumption that the sequence of semi-groups $\{T_n(t)\}$ is uniformly bounded for all values of the parameter in the sense that $\|T_n(t)\| \leq M$, where M is independent of n and t. In particular this has restricted the results to semi-groups of class c_0. The purpose of the present paper is to investigate this problem for a more general class of semi-groups, namely the class (1,A) and thereby generalize some results of Trotter [9], Kurtz [4] and Kato [3].

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