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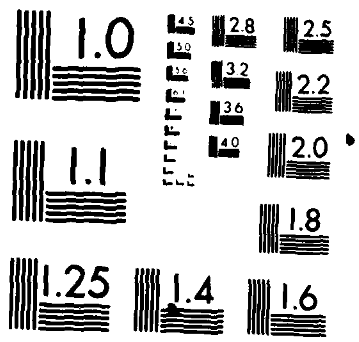

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A STABILITY PROPERTY OF CONDITIONAL EXPECTATIONS

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ABSTRACT

This paper is concerned with approximating a conditional expectation of a second order random variable given a random process defined over an interval by a conditional expectation of the random variable given distorted values of the random process at finitely many times. A sufficient condition which guarantees a good approximation is presented. Best estimates of more general fidelity criteria than mean square error are also considered, and the above situation is addressed for a wide class of fidelity criteria.

I. INTRODUCTION

Throughout this paper let $(\Omega,\mathscr{S},P)$ be a fixed probability space and (M,ρ) be a separable metric space. Suppose $\{X(t):t\in[0,T]\}$ is a stochastic process on $(\Omega,\mathscr{S},P)$ taking values in M that is continuous in probability and $Y\in L_2(\Omega,\mathscr{S},P)$. In many theoretical situations one is interested in $E(Y|X(t):t\in[0,T])$. This is the optimal $\sigma(X(t), t\in[0,T])$-measurable mean square estimate of Y given perfect knowledge of the process $\{X(t): t\in[0,T]\}$ at all times $t\in[0,T]$; that is, it is the unique solution [4, pp.43-45] to the problem: $\min\{\|Y-Z\|_{L_2(\Omega)}: Z\in L_2(\Omega,\sigma(X(t):t\in[0,T]),P)\}$.

However in many practical situations we are neither able to observe the process continuously nor do we have perfect knowledge about the process when we are able to observe it. Conventional measuring devices and computers can only handle finite data sets. Effectively, they partition M into finitely many disjoint subsets $E_1,\ldots,E_n$ and register a fixed value v_k of E_k if they observe $x\in E_k$, $1\le k\le n$. These devices are commonly unable to observe the process at all times $t\in[0,T]$. Our question becomes: How well can we estimate $E(Y|X(t):t\in[0,T])$ given our defective knowledge of $\{X(t):t\in[0,T]\}$ at only finitely many times $t_1,\ldots,t_n$ belonging to $[0,T]$?

More generally, we are tempted to ask this question about best estimates of more general fidelity criteria than mean square error. In this paper we will address this question for a very wide class of fidelity criteria.

II. ROUND OFF SCHEMES

Definition: Let $Q:M\to M$ be Borel measurable and have finite range, say $\{p_1,\ldots,p_n\}$. The map Q is said to be a round off map if $p_k=Q(p_k)$, $1\le k\le n$. The set $\{Q^{-1}(p_1),\ldots,Q^{-1}(p_n)\}$ is called the partition of M defined by Q.

Definition: Let $\{Q_n\}_{n=1}^{\infty}$ be a sequence of round off maps on M. The sequence $\{Q_n\}_{n=1}^{\infty}$ is called a round off scheme if

(i) $\forall x\in M \lim_{n\to\infty} \operatorname{dia} Q_n^{-1}(Q_n(x))=0$

and

(ii) the partition of M defined by Q_{n+1} refines that defined by Q_n, $n\in\mathbb{N}$. Note $\sigma(Q_n)\subset\sigma(Q_{n+1})$.

The action of these maps suggests a sequence of increasingly accurate measuring devices. We will show that, asymptotically, these distinguish Borel sets in M via the

Lemma 1: $\bigvee_{n=1}^{\infty}\sigma(Q_n)=\mathscr{B}(M)$, the Borel sets in M.

Proof: $\subset$: Obvious, since we require each Q_n to be Borel measurable.

$\supset$: Choose any open $U\subset M$. Pick $x\in U$; $\lim_{n\to\infty} Q_n^{-1}(Q_n(x))=0$ so there is $n\in\mathbb{N}$ s.t. $Q_n^{-1}(Q_n(x))\subset U$. Thus U may be written as a union of point inverses of the Q_n. Since there are only countably many of these, the union is countable so $U\in\bigvee_{n=1}^{\infty}\sigma(Q_n)$. Since $\bigvee_{n=1}^{\infty}\sigma(Q_n)$ is a σ-algebra on M containing every open subset of M, we conclude $\mathscr{B}(M)\subset\bigvee_{n=1}^{\infty}\sigma(Q_n)$. QED

Lemma 2: Let $X:\Omega\to M$ be Borel measurable. Then $\sigma(X)=\bigvee_{n=1}^{\infty}\sigma(Q_n(X))$.

Proof: This is an easy application of the "good sets" principle described in [3, p.5].

Theorem 3: Let $X:\Omega\to M$ be Borel measurable, $1\le p<\infty$, and $Y\in L_p(\Omega,\mathscr{S},P)$. Then $E(Y|Q_n(X)) \xrightarrow{L_p,\ a.s.} E(Y|X)$.

Proof: [3, p.301] demonstrates that if $\{\mathscr{F}_n\}_{n=1}^{\infty}$ is an increasing collection of σ-algebras on Ω contained in $\mathscr{S}$ and $\mathscr{F}_\infty=\bigvee_{n=1}^{\infty}\mathscr{F}_n$, then $E(Y|\mathscr{F}_n)\xrightarrow{L_p,\ a.s.}E(Y|\mathscr{F}_\infty)$ QED

Martingale convergence theorems allow us to asymptotically reconstruct $E(Y|X)$ from $E(Y|Q_n(X))$; see, for instance, [7, Chap. 7].

III. THE L_2 CASE

Notation: Henceforth for convenience we will assume $\mathscr{S}$ is complete. If $\mathscr{F}\subset\mathscr{S}$ is a σ-algebra, we denote its P-completion by $\bar{\mathscr{F}}$.

First we dispose of a technicality.

Lemma 4: Let $\{X(t):t\in[0,T]\}$ be a process on $(\Omega,\mathscr{S},P)$ continuous in probability and $D\subset[0,T]$ be dense. Then

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$\overline{\sigma(X(t):t\epsilon[0,T])} = \overline{\sigma(X(t):t\epsilon D)}$.

Proof: $\supset$: Obvious.

$\subset$: Fix an open $U\epsilon M$, set $U_m = \{x\epsilon M:\rho(x,U^c) > \frac{1}{m}\}$, and let $\bar{U}_m$ denote the topological closure of U_m. It is easy to see that $\{U_m\}_{m=1}^{\infty}$ is a non-decreasing collection of open subsets of M and $U = \bigcup_{m=1}^{\infty} U_m$. Pick $t\epsilon[0,T]$. Since $\{X(t),t\epsilon[0,T]\}$ is continuous in probability, there exists a sequence $\{t_n\}_{n=1}^{\infty}$ in D such that $t_n \to t$ and $X(t_n) \to X(t)$ a.s. Pick $\omega\epsilon X(t)^{-1}(U)$; there exists $m\epsilon\mathbb{N}$ such that $X(t)(\omega)\epsilon U_m$. Suppose $\lim_{n\to\infty} X(t_n)(\omega) = X(t)(\omega)$; then U_m is a neighborhood of $X(t)(\omega)$ so that there exists $N\epsilon\mathbb{N}$ such that for all $n \geq N$, $X(t_n)(\omega)\epsilon U_m$. Thus, $\omega\epsilon \bigcup_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} X(t_n)^{-1}(U_m) = \bigcup_{m=1}^{\infty} \liminf_{n\to\infty} X(t_n)^{-1}(U_m)$, which we will define as A. Conversely, suppose $\omega\epsilon A$ and that $\lim_{n\to\infty} X(t_n)(\omega) = X(t)(\omega)$. Then there exists $n\epsilon\mathbb{N}$ such that $\omega\epsilon \liminf_{n\to\infty} X(t)^{-1}(U_m)$ and $X(t)(\omega)\epsilon\bar{U}_m \subset U$. We have just shown $A\Delta X(t)^{-1}(U) \subset \Omega - \{\omega\epsilon\Omega:\lim_{n\to\infty} X(t_n)(\omega) = X(t)(\omega)\}$. Since $X(t_n) \to X(t)$ a.s., we see that $A\Delta X(t)^{-1}(U)$ has zero probability; and since $A\epsilon \overline{\sigma(X(t),t\epsilon D)}$, we see that $X(t)^{-1}(U)\ \epsilon\ \overline{\sigma(X(t),t\epsilon D)}$. Thus for any Borel $B\subset M$, $X(t)^{-1}(B)\ \epsilon\ \overline{\sigma(X(t):t\epsilon D)}$. It follows $\sigma(X(t):t\epsilon[0,T]) \subset \overline{\sigma(X(t):t\epsilon D)}$. QED

Definition: A partition P of the closed interval $[0,T]$ is a finite point set $\{0=t_0 < t_1 \ldots < t_n = T\}$. The mesh of P is defined by $\mu(P) = \max\{t_k - t_{k-1}: 1 \leq k \leq n\}$.

Lemma 5: Let $Y\epsilon L_2(\Omega,\mathcal{A},P)$ and $\{P_m\}_{m=1}^{\infty}$ be an increasing sequence of partitions of $[0,T]$ with $\mu(P_m) \to 0$. If $\{X(t),t\epsilon[0,T]\}$ is a process on $(\Omega,\mathcal{A},P)$ continuous in probability, then $E(Y|X(t):t\epsilon P_m) \to E(Y|X(t):t\epsilon[0,T])$ in L_2 and a.s.

Proof: Set $D = \bigcup_{m=1}^{\infty} P_m$; $\mu(P_m) \to 0$ so D is dense in $[0,T]$. Thus $E(Y|X_t:t\epsilon P_m) \xrightarrow{L_2,a.s.} E(Y|X_t:t\epsilon D)$. Lemma 4 implies $E(Y|X(t):t\epsilon D) = E(Y|X(t):t\epsilon[0,T])$ a.s. QED

Theorem 6: Let Lemma 5 set notation and $\{Q_n\}_{n=1}^{\infty}$ be a round off scheme on M. Then

$$\lim_{m,n\to\infty} E(Y|Q_n(X(t)):t\epsilon P_m) = E(Y|X(t):t\epsilon[0,T])$$

in L_2.

Proof: For $m,n\epsilon\mathbb{N}$ put

$$a_{mn} = \|E(Y|Q_n(X(t)):t\epsilon P_m) - E(Y|X(t):t\epsilon[0,T]\|_{L_2(\Omega)}$$

$$\mathcal{F}_{mn} = \sigma(Q_n(X(t)).t\epsilon P_m).$$

Then it is clear $\mathcal{F}_{mn} \subset \mathcal{F}_{m+1,n}$ and $\mathcal{F}_{m,n} \subset \mathcal{F}_{m,n+1}$ $m,n\epsilon\mathbb{N}$. For each m we have

$$E(Y|Q_n(X(t)):t\epsilon P_m) \xrightarrow{L_2,\ a.s.} E(Y|X(t):t\epsilon P_m)$$

as $n\to\infty$. Now letting $m\to\infty$ and applying Lemma 4, $\lim_{m\to\infty}\lim_{n\to\infty} a_{mn} = 0$. For each $n\epsilon\mathbb{N}$,

$$E(Y|Q_n(X(t)):t\epsilon P_m) \xrightarrow{L_2,a.s.} E(Y|Q_n(X(t)):t\epsilon[0,T]).$$

An easy extension of Theorem 3 shows

$$E(Y|Q_n(t)):t\epsilon[0,T]) \xrightarrow{L_2,a.s.} E(Y|X(t):t\epsilon[0,T]).$$

Thus $\lim_{n\to\infty}\lim_{m\to\infty} a_{mn} = 0$.

Now turn to the L_2 minimization property of the conditional expectation operator to see that $a_{m+1,n} \leq a_{mn}$ and $a_{m,n+1} \leq a_{mn}$, $m,n\epsilon\mathbb{N}$. It immediately follows $\lim_{m,n\to\infty} a_{mn} = 0$.

QED

IV. AN ABSTRACT PRINCIPLE OF BANACH SPACES

Definition: A Banach space B is uniformly convex if for all $\varepsilon > 0$ there exists $\delta > 0$ s.t. for all $x,y\epsilon B$ with $\|x\| = \|y\| = 1$, $\|x-y\| > \varepsilon$ implies $\|x+y\| > 2(1-\delta)$. A Banach space B is locally uniformly convex if for any sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ with $\|x_n\| = \|y_n\| = 1$, $n\epsilon\mathbb{N}$, $\|x_n+y_n\| \to 2$ implies $\|x_n-y_n\| \to 0$. A Banach space B is strictly convex if each point of the unit sphere is an extreme point of the closed unit ball.

It is well known that uniform convexity implies local uniform convexity, which in turn implies strict convexity.

We denote the metric of B as d.

Theorem 7: Let B be a reflexive Banach space and $K\subset B$ be closed and convex. Then for any $x\epsilon B$ the set $L = \{y\epsilon K:d(x,y) = d(x,K)\}$ is closed and nonvoid.

Proof: See [12, sections 38 and 39].

Theorem 8: Let B be a reflexive Banach space. Then B is strictly convex if and only if for all $x\epsilon B$ and for all closed and convex $K\subset B$ there exists a unique $y\epsilon K$ such that $d(x,y) = d(x,K)$.

Proof: This is an easy consequence of Theorem 7.

Note that if B is any Banach space so that for any $x\epsilon B$ and any closed and convex $K\subset B$ there exists a unique $y\epsilon K$ s.t. $d(x,y) = d(x,K)$, then B is strictly convex and reflexive. For a proof see [9, p.161].

Theorem 9: Let $\{K_n\}_{n=1}^{\infty}$ be an increasing collection of closed convex subsets of a strictly convex reflexive Banach space B and let K_∞ be the norm closure of $\bigcup_n K_n$. Note that K_∞ is closed and convex. Let P_n denote minimum norm projection on K_n, $n\epsilon\mathbb{N}\cup\{\infty\}$; this is well defined by Theorems 7 and 8. Then for all $x\epsilon B$ we have

$$\|x-P_\infty(x)\| = \lim_{n\to\infty}\|x-P_n(x)\|.$$

Proof: Fix $x\in B$. By the minimality of the projections P_n, $n\in\mathbb{N}\cup\{\infty\}$, we have $\|x-P_\infty(x)\| \leq \|x-P_{n+1}(x)\| \leq \|x-P_n(x)\|$, $n\in\mathbb{N}$. Thus $\lim_{n\to\infty}\|x-P_n(x)\|$ exists and is not less than $\|x-P_\infty(x)\|$. Conversely, choose $\varepsilon>0$. Note that $P_\infty(x)\in K_\infty$ implies that there exists $n\in\mathbb{N}$ and $y\in K_n$ s.t. $\|x-P_n(x)\| \leq \|x-y\|+\|y-P_\infty(x)\| \leq \|x-P_\infty(x)\|+\varepsilon$. The arbitrariness of ε implies that $\lim_{n\to\infty}\|x-P_n(x)\| \leq \|x-P_\infty(x)\|$. QED

Theorem 10: Let B be a reflexive strictly convex Banach space and let $\{K_n\}_{n=1}^\infty$, K_∞, $\{P_n\}_{n=1}^\infty$, and P_∞ be as in Theorem 9. Suppose $x,z\in B$ and $P_n(x) \rightharpoonup z$. Then $z=P_\infty(x)$.

Proof: Note that $P_n(x)\in K_n$, $n\in\mathbb{N}$, and K_∞ is weakly closed, so $z\in K_\infty$. By the weak lower semicontinuity of the norm and Theorem 9, $\|x-z\| \leq \liminf_{n\to\infty}\|x-P_n(x)\| = \|x-P_\infty(x)\|$. Thus we conclude $z=P_\infty(x)$. QED

Theorem 11: Let the previous theorem set notation. Then $P_n(x) \rightharpoonup P_\infty(x)$.

Proof: Choose any subsequence $\{P_{n_k}(x)\}$ of $\{P_n(x)\}_{n=1}^\infty$. By the Smul'lyan theorem [9, pp.145-156] there exists a further subsequence $\{P_{n_{k(j)}}(x)\}$ of $\{P_{n_k}(x)\}$ and $z\in B$ s.t. $P_{n_{k(j)}}(x) \rightharpoonup z$. Theorem 10 implies $z=P_\infty(x)$. We conclude that $P_n(x) \rightharpoonup P_\infty(x)$. QED

Proposition 12: Let B be a locally uniformly convex Banach space and $\{x_n\}_{n=1}^\infty$ be a sequence in B with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$. Then $x_n \to x$ in norm.

Proof: See [8, p.233].

Theorem 13: Let B be a locally uniformly convex Banach space and $\{x_n\}_{n=1}^\infty$, $\{P_n\}_{n=1}^\infty$, K_∞, and P_∞ be as in Theorem 9. Then for any $x\in B$, $P_n(x) \to P_\infty(x)$ in norm.

Proof: Recall that local uniform convexity implies strict convexity, so minimum norm projections are defined. Pick $x\in B$; $P_n(x) \rightharpoonup P_\infty(x)$ implies $x-P_n(x) \rightharpoonup x-P_\infty(x)$. But $\|x-P_n(x)\| \to \|x-P_\infty(x)\|$, so the theorem follows from Proposition 12. QED

V. THE CASE OF L_Φ

The basic facts about Orlicz spaces we use here may be found in [6] and [10]. Henceforth we stipulate that $(\Omega,\mathscr{S},P)$ be nonatomic. Throughout we will assume that our Young function Φ: $[0,\infty)\to[0,\infty)$ has strictly increasing first derivatives on $[0,\infty)$ and that Φ and its complementary Young function Ψ satisfy the Δ_2 or doubling condition. Recall the Luxemburg norm of $Y\in L_\Phi(\Omega,\mathscr{S},P)$ is defined by

$$N_\Phi(Y) = \inf\left\{\lambda>0: \int_\Omega \Phi\left(\frac{|f|}{\lambda}\right)dP \leq \Phi(1)\right\}$$

and that for a sequence $\{Y_n\}_{n=1}^\infty$ in L_Φ, $N_\Phi(Y_n-Y)\to 0$ iff

$$\lim_{n\to\infty}\int_\Omega \Phi(|Y_n-Y|)dP = 0.$$

Furthermore, this norm makes L_Φ a reflexive uniformly convex Banach space. Thus the machinery of the last section applies. Note however that in general these minimum norm projections are nonlinear.

Now let $\mathscr{F}$ be any sub σ-algebra of $\mathscr{S}$ and $Y\in L_\Phi(\Omega,\mathscr{S},P)$. The set $L_\Phi(\Omega,\mathscr{F},P)$ is a closed subspace of $L_\Phi(\Omega,\mathscr{S},P)$ so Y has a unique minimum norm projection into $L_\Phi(\Omega,\mathscr{F},P)$ which we will denote by $E_\Phi(Y|\mathscr{F})$. The primary tool used in the L_2 case was the martingale convergence theorem; we will obtain an analog of it here.

Lemma 14: Let $Y\in L_\Phi(\Omega,\mathscr{S},P)$, $\{\mathscr{F}_n\}_{n=1}^\infty$ be an increasing collection of sub σ-algebras, of $\mathscr{S}$ and $\mathscr{F}_\infty = \bigvee_{n=1}^\infty \mathscr{F}_n$. Then $\overline{\bigcup_{n=1}^\infty L_\Phi(\Omega,\mathscr{F}_n,P)} = L_\Phi(\Omega,\mathscr{F}_\infty,P)$.

Proof: Put $Z_n = E(Y|\mathscr{F}_n)$, $n\in\mathbb{N}\cup\{\infty\}$. Repeated application of Jensen's inequality yields:

$$\begin{aligned}0 \leq \Phi(|Z_n|) &= \Phi(|E(Y|\mathscr{F}_n)|)\\ &\leq \Phi(E(|Y|\,|\mathscr{F}_n|)\\ &\leq E(\Phi(|Y|)|\mathscr{F}_n),\end{aligned}$$

dominating $\{\Phi(|Z_n|)\}_{n=1}^\infty$ by a uniformly integrable sequence of functions. Thus, $\{\Phi(|Z_n|)\}_{n=1}^\infty$ is uniformly integrable. Now apply convexity and the doubling condition, yielding

$$\begin{aligned}\Phi(|Z_n-Z_\infty|) &\leq \Phi(|Z_n|+|Z_\infty|)\\ &\leq \tfrac{1}{2}\Phi(2|Z_n|)+\tfrac{1}{2}\Phi(2|Z_\infty|)\\ &\leq \tfrac{c}{2}\Phi(|Z_n|)+\tfrac{c}{2}\Phi(|Z_\infty|),\end{aligned}$$

where c is a constant from the doubling condition, independent of n. It follows now $\{\Phi(|Z_n-Z_\infty|)\}_{n=1}^\infty$ is uniformly integrable. Since $\Phi(|Z_n-Z_\infty|)\to 0$ a.s., $\int_\Omega \Phi(|Z_n-Z_\infty|)dP\to 0$ and $N_\Phi(Z_n-Z_\infty)\to 0$. The lemma follows immediately. QED

Theorem 15: Let $\{\mathscr{F}_n\}_{n=1}^\infty$ be an increasing collection of sub σ-algebras of $\mathscr{S}$ and $\mathscr{F}_\infty$ =

$\bigvee_n \mathscr{F}_n$. Then if $Y \in L_\Phi(\Omega,\mathscr{S},P)$, $E(Y|\mathscr{F}_n) \xrightarrow{L_\Phi} E(Y|\mathscr{F}_\infty)$.

Proof: Apply Lemma 14 and Theorem 13. QED

Remark: Consulting [1],[2],[5], and [11] it is possible to see this convergence is almost sure.

Now we extend Lemma 5:

Lemma 16: Let $Y \in L_\Phi(\Omega,\mathscr{S},P)$ and $\{P_m\}_{m=1}^\infty$ be an increasing sequence of partitions of $[0,T]$ with $\mu(P_m) \to 0$. If $\{X(t): t \in [0,T]\}$ is a process on $(\Omega,\mathscr{S},P)$ taking values in M that is continuous in probability then

$$E_\Phi(Y|X(t): t \in P_m) \xrightarrow{L_\Phi,\ a.s.} E_\Phi(Y|X(t): t \in [0,T])$$

as $m \to \infty$. Furthermore,

$$\lim_{m\to\infty} \int_\Omega \Phi(|E_\Phi(Y|X(t): t \in P_m) - E_\Phi(Y|X(t): t \in [0,T])|)\, dP = 0 .$$

Proof: Imitate Lemma 5. QED

Theorem 17: Let the previous lemma set notation and $\{Q_n\}_{n=1}^\infty$ be a round off scheme on M. Then

$$E_\Phi(Y|Q_n(X(t)): t \in P_m) \xrightarrow{L_\Phi} E_\Phi(Y|X(t): t \in [0,T])$$

and

$$\int_\Omega \Phi(|E_\Phi(Y|Q_n(X(t): t \in P_m) - E_\Phi(Y|X(t): t \in [0,T])|)dP \to 0$$

as $m,n \to \infty$.

Proof: For $m,n \in \mathbb{N}$ set $\mathscr{F}_{mn} = \sigma(Q_n(X(t)): t \in P_m)$. Choose sequences $\{m_k\}_{k=1}^\infty$ and $\{n_k\}_{k=1}^\infty$ so that $m_k, n_k \uparrow \infty$. Put $\mathscr{F}_k = \mathscr{F}_{m_k n_k}$, $k \in \mathbb{N}$. Then $\bigvee_k \mathscr{F}_k = \bigvee_{m,n} \mathscr{F}_{mn}$. Apply Theorem 15. QED

Finally, for icing on the cake we get a similar result for ordinary conditional expectation:

Theorem 18: Let Theorem 17 set notation. Then as $m,n \to \infty$,

$$E(Y|Q_n(X(t)): t \in P_m) \xrightarrow{L_\Phi} E(Y|X(t): t \in [0,T])$$

and

$$\int_\Omega \Phi(|E(Y|Q_n(X(t)): t \in P_m) - E(Y|X(t): t \in [0,T])|)dP \to 0.$$

Proof: Mimic Theorem 17 using the fact that

$$E(Y|\mathscr{F}_k) \xrightarrow{L_\Phi} E(Y|X(t): t \in [0,T])$$

derived in Theorem 13.

QED

Remark: If $\Phi(x) = x^p/p$, $x \in [0,\infty)$ and $p > 1$, $L_\Phi = L_p$ and the nonatomicity assumption may be dropped.

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