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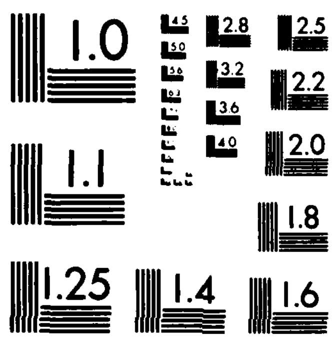
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Program Engineering &
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Characteristics of Altitude Error at Reduced Quantization

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16. Abstract

This report describes an analysis of the impact of reducing the quantization level on the altitude error produced by the aircraft's altimetry system. Specifically, the mean and the standard deviation of the altitude error at the output of the aircraft's onboard quantization device are derived assuming altitude error at its input is normally distributed. The formulae derived are very general, exact, and applicable to all ranges of input statistics and quantization levels. A computer program has been written to evaluate the impact of reduced quantization for a wide range of input error standard deviation and several quantization levels. Additionally, a comparison is made between the commonly used and approximate uniform distribution model method and the exact formulae derived in this report for calculating the quantized altitude error.

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1. INTRODUCTION

The pressure altitude information reported by the aircraft plays an important role in the enhancement of safety and efficiency of flight in the National Airspace System. The altitude information serves as vital input for a number of today's and future ground-based automation functions including surveillance, vertical separation assurance, conflict alert, and minimum safe altitude system (MSAW). The accuracy of the reported altitude depends heavily on the aircraft altimetry system. The altimeter senses the ambient pressure, converts the pressure into electrical signal, quantizes the signal into discrete level, and sends it to the transponder (Mode C) for digital transmission. Currently, the Mode C data are reported at 100 ft increments.

With the planned implementation of the Mode S ground stations and in particular, the capability of the Mode S transponders to report altitude at a higher resolution (25 ft increments), there have been suggestions from both the international and U.S. aviation communities to examine the potential benefits of reduced quantization for altitude data [1,2]. For the ground-based Air Traffic Control (ATC) system, the primary interest lies in the design of an effective altitude tracker aimed at improving aircraft surveillance and other safety-related automation functions.

In the airborne segment, the FAA has developed a family of Traffic Alert and Collision Avoidance Systems (TCAS) as backup to the ground ATC system. In particular, the TCAS II system provides escape maneuver advisories automatically in the vertical plane. This system tracks the altitude and altitude rate of the intruder and own TCAS-equipped aircraft to determine the hazard of mid air collision. The tracking of the intruder's altitude is based on data quantized to 100 ft (Mode C) or in the future, an option for 25 ft (Mode S) increments.

For own TCAS altitude tracking, the input data can be either at 100 ft increments or finely quantized. Since TCAS II is expected to be installed in air carriers, the onboard altimeter is generally of high precision, including an air data computer (ADC). Depending on the ADC type, digital altitude data of 1 ft increment or analog synchro outputs are available. The synchro outputs can be quantized to small increments such as under 10 ft for TCAS application.

Efforts are also underway by the FAA to collect altimetry data with the objectives of quantifying altimetry errors at high and low altitude regimes. The purpose of the data collection program for high altitude is to evaluate the feasibility of reduced vertical separation standard above flight level 290. On the other hand, the objective of the altimetry data collection program at low altitude is to determine the distribution of the altimetry error for the general aviation aircraft population. Both of these two data collection programs involve the analysis of reported altitude (Mode C) error relative to the true altitude. Highly accurace ground based systems are used to measure the true aircraft altitude.

Common to the aforementioned efforts of altimetry data collections and the design of the altitude trackers is the need to characterize the altitude error statistically at the output of the quantizer. This error is defined herein as the difference between the altitude at the output of the quantizer and the aircraft's true altitude. Thus, this error consists of the quantization error plus the error prior to quantization. The latter error includes the error associated with the onboard altimetry system (before quantization) and the deviation from the standard atmospheric model for pressure-to-altitude conversion.

In calculating the mean and the standard deviation of the altitude error at the output of the quantizer, frequently it has been assumed that the standard deviation of the altimetry error (σ_n) before quantization is comparable or greater than the quantization level (Q). This assumption, coupled with the additional assumption that the input error is Gaussian, leads to the well-known result that the output quantization error is uniformly distributed within the quantization bin. However, the standard deviation of the altimetry error before quantization can be quite small (such as a few feet) relative to the quantization level; the consequence is that the uniform distribution assumption may not be valid except for small Q.

Another common assumption is that the mean of the error prior to quantization is zero. This assumption is not always satisfied since the static system of the altimeter or the use of the standard atmospheric model for pressure-to-altitude conversion can introduce a constant error or bias.

This report presents a mathematical analysis of the mean and standard deviation of the altitude error at the output of the quantizer. The formulae derived are very general in that they do not assume an uniform distribution for the quantization error. Furthermore, the results are exact and applicable to all values of σ_n and Q. The only assumption made is that the distribution at the input of the quantizer is Gaussian. Numerical results are also presented for Q=100 ft, 25 ft, and 6.25 ft, and σ_n ranging from 2.5 ft to 25 ft.

The next section discusses the error model used in the analysis and the previous work in this area. Section 3 derives the theoretical results for the mean and standard deviation of the output altitude error. Section 4 provides the numerical results, as well as a comparison with the uniform distribution model. The last section summarizes the results of the analysis.

2. ERROR MODEL AND PREVIOUS WORK

Figure 1 depicts the model used in the analysis of the altitude error at the output of the quantizer. The aircraft's true altitude as a function of time is represented by h(t), which may include aircraft sinusoidal motion. The true altitude is corrupted by an additive term, n(t), representing the altimetry error and other sources of error prior to quantization. This error includes those attributable to static source error and transducer error. The degraded altitude information x(t) is quantized at a level Q to produce the output y(t). The quantized altitude, y(t), can be thought of as composed of the true altitude and an error term, e(t). It is the determination of the mean and standard deviation of the output error term e(t) that is of primary interest.

It is noted that the analysis of the impact of e(t) on altitude tracking is outside the scope of this report since it depends on the specific tracker of interest. However, the results in characterizing the first-order statistics of e(t) are applicable to the analysis of nonlinear and linear tracking algorithms.

The additive error term, n(t), is modeled as a Gaussian process of arbitrary mean, b(t), and standard deviation $\sigma_n(t)$. These two parameters can be a function of time to account for their time variation as a function of altitude and atmospheric condition. For economy of notation, b(t) and $\sigma_n(t)$ are denoted as b and σ_n respectively.

The parameter b represents the constant altimetry error or bias error before quantization, whereas σ_n denotes the random component of the error or jitter. For air carrier type of aircraft such as the L-1011, the σ_n before quantization is on the order of several feet [3]. Preliminary analysis of actual flight data from B-747 and Airbus 300 also shows the same order of magnitude for σ_n.

Since n(t) is a Gaussian random process, x(t), a linear function of n(t), is also a Gaussian process with the following mean and variance:

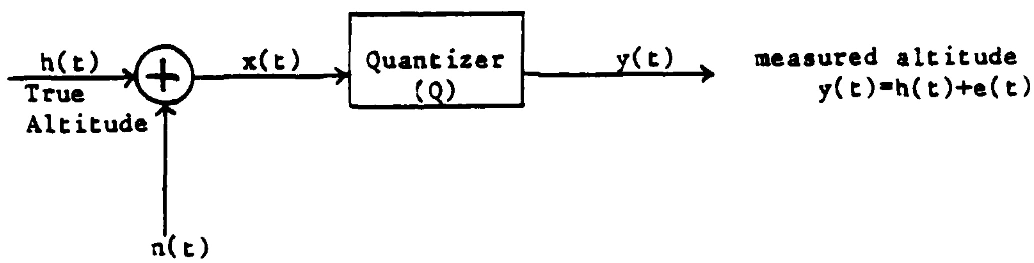
$$\overline{x(t)}=E[x(t)] \tag{1}$$

$$=E[h(t)+n(t)]=h(t)+b$$

where E denotes the mathematical expectation operator

$$\sigma^2_{x(t)} = \sigma^2_n$$

The analysis of the output error e(t) is generally quite complicated because of the nonlinear nature of the quantizer. The altitude quantizer under consideration is a round-off type, as shown in Figure 2. For simplicity, it is assumed that quantization occurs at the pre-selected thresholds with negligible error. Since the quantizer performs a nonlinear transformation of input x(t), the output altitude y(t) is generally non-Gaussian.



h(t): true aircraft altitude
n(t): altimetry error before quantization; Gaussian distribution with mean b and variance σ_n^2
x(t): corrupted altitude input to quantizer
y(t): output altitude
e(t): altitude error at quantizer output; defined as y(t)-h(t)

Figure 1- Error Model

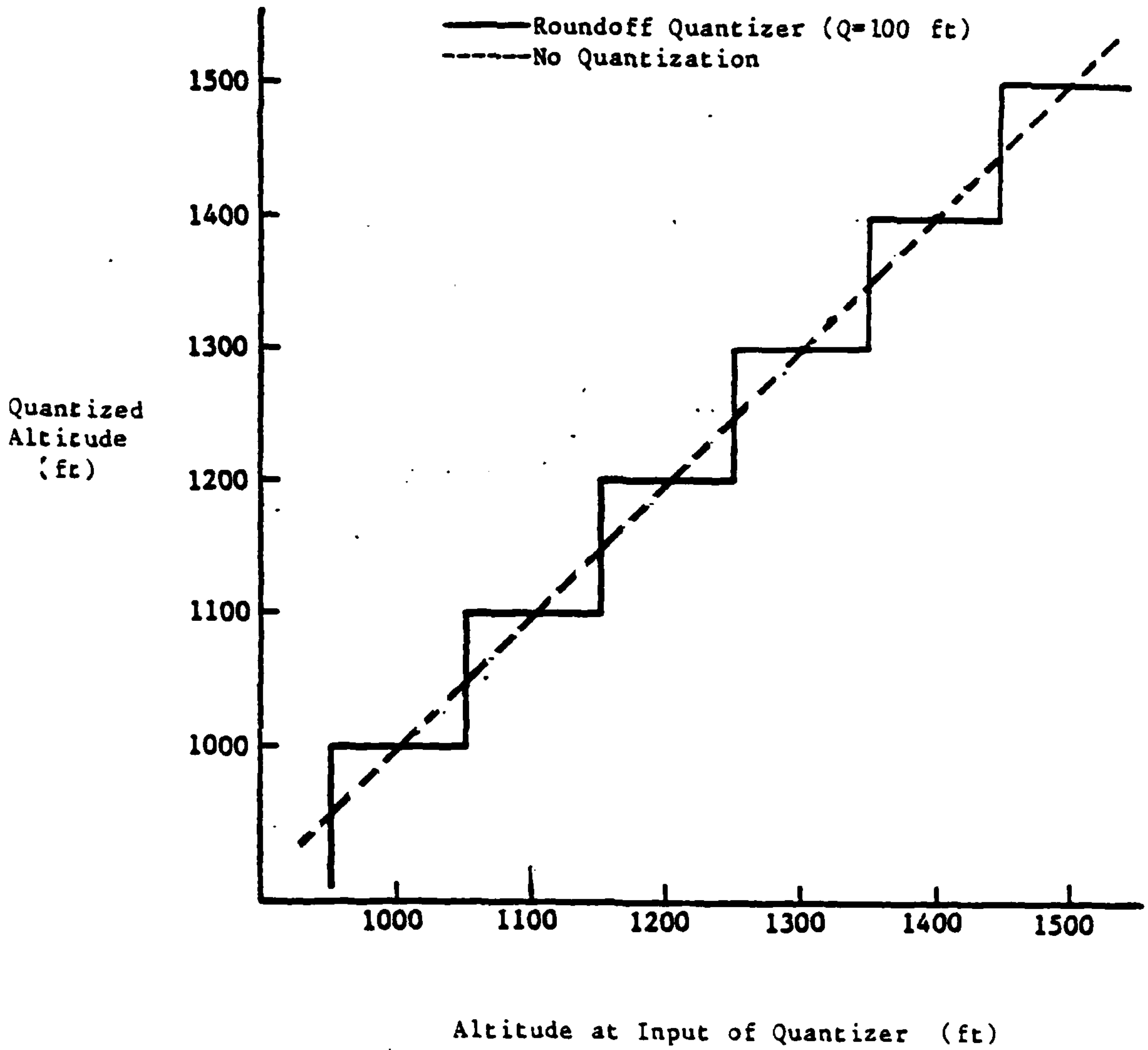


Figure 2- Example Input/Output Transfer Function of Altitude Quantizer (Q=100 ft)

The statistical effect of quantization has been widely studied in many fields. Quantization error is unavoidable in digital systems when analog-to-digital conversion is required. Perhaps the earliest work in establishing the relationship between the moments at the output of the quantizer and those at the input is known as Sheppard's corrections for group data [4]. The pioneering work in the rigorous analysis of the quantization effect was performed by Widrow [5], who first established the statistical distribution at the output of the quantizer and the necessary conditions for the output to be uniform and white. His results are also known as the quantization theorem. Extensions of his work, as well as previous efforts, include those of references [6,7,8].

The following summarizes the previous key results relevant to the determination of the mean and variance of the quantized output error, e(t), assuming the input to the quantizer, x(t), is Gaussian :

i) For "large" σ_n/Q ratio (e.g., $\sigma_n/Q \gg 1$), it is well-known that the quantization error, defined by y(t)-x(t), is uniformly distributed over the quantization interval and that the input x(t) and the quantization error are uncorrelated. This leads to the folowing relationship between the variance of the quantized altitude error to the variance at the input:

$$\sigma_e^2 = \sigma_n^2 + \frac{Q^2}{12} \tag{2}$$

In the above equation, $Q^2/12$ represents the quantization error due to the uniform distribution model.

ii) For $\sigma_n/Q \geq 1/3$ the following approximation has been derived [4,5]:

For $\overline{x(t)} \neq 0$,

$$\overline{e(t)} = E[e(t)] = -\frac{Q}{\pi} \exp\left(-2\pi^2 \sigma_n^2/Q^2\right) \sin\left(\frac{2\pi \overline{x(t)}}{Q}\right) \tag{3}$$

For $\overline{x(t)} = 0$,

$$\sigma_e^2 = \sigma_n^2 + \frac{Q^2}{12} - 2\sigma_n^2 \left[2 + \frac{Q^2}{2\pi^2 \sigma_n^2} \right] \exp\left(-2\pi^2 \sigma_n^2/Q^2\right) \tag{4}$$

iii) For all range of σ_n/Q and $\overline{x(t)}=0$, the variance of e(t) is given by [8] :

$$\sigma_e^2 = \sigma_n^2 + \frac{Q^2}{12} + 4\sigma_n^2 \sum_{i=1}^{\infty} (-1)^i \exp(-2\pi^2 i^2 \sigma_n^2/Q^2)$$
$$+ \frac{Q^2}{\pi^2} \sum_{i=1}^{\infty} \frac{(-1)^i}{i^2} \exp(-2\pi^2 i^2 \sigma_n^2/Q^2) \qquad (5)$$

It can be noted in (ii) above for moderate σ_n/Q ratio and for nonzero mean x(t) case, the variance of e(t) has not been derived. Similarly, in (iii) the mean and the variance of the error at the quantizer output are not available when the mean of x(t) is not zero. For ATC applications, the mean of x(t) is only zero when both the aircraft altitude and the bias of the altimetry error are zero.

3. DERIVATION OF THE OUTPUT ALTITUDE ERROR

The derivation of the mean and the variance of the altitude error at the output of the quantizer is relatively straight-forward, albeit algebraically cumbersome. The altitude error at the output of the quantizer is defined as the difference between the output altitude and the true aircraft altitude. That is, the output error e(t) is,

$$e(t)=y(t) - h(t) \tag{6}$$

The output error e(t) defined above is related to the commonly known quantization error by an additive noise term n(t). This can be seen by noting that the quantization noise is defined as :

$$\begin{aligned} e_Q(t)&=y(t) - x(t) \\ &=y(t) - h(t) - n(t) \end{aligned}$$

or,

$$y(t)=e_Q(t) + h(t) + n(t) \tag{7}$$

Substituting the above into equation (6), the output error e(t) is related to the quantization error $e_Q(t)$ by:

$$e(t)=e_Q(t) + n(t) \tag{8}$$

In general, $e_Q(t)$ and n(t) are correlated, i.e., $E[e_Q(t)n(t)] \neq E[e_Q(t)]*E[n(t)]$. The correlation between the two error terms on the right-hand side of equation (8) decreases as σ_n/Q increases [8]

The mean and the variance of e(t) are:

$$\begin{aligned} E[e(t)]&=E[y(t)] - h(t) \\ \overline{e(t)}&=\overline{y(t)} - h(t) \end{aligned} \tag{9}$$

$$\begin{aligned} \sigma_e^2&=\text{Variance } [y(t) - h(t)] \\ &=\text{Variance } [y(t)] \\ &=E[y^2(t)] - [\ \overline{y(t)}]^2 \end{aligned} \tag{10}$$

It can be observed from equations 9,10 that the mean and the variance of the altitude error at the output of the quantizer are related directly to the first and second moment of the quantized output y(t). To calculate the moments, the technique of the characteristic function is adopted.

Denoting y(t) and x(t) at a particular time τ as Y and X respectively, the k<u>th</u> moment of Y is related to its characteristic function $\Phi_Y(p)$ by:

$$E[\,Y^k\,] = \left(\frac{1}{j}\right)^k \frac{d^k}{d\,p^k} \Phi_Y(p) \Bigg|_{p=0} \qquad ; k=1,2,3,\ \ldots$$

(11)

Only the first two moments are derived, i.e., k=1,2. The relationship between the characteristic function $\Phi_Y(p)$ and $\Phi_X(p)$, where $\Phi_X(p)$ is the characteristic function of X, is given by [5,7]:

$$\Phi_Y(p) = \sum_{i=-\infty}^{\infty} \Phi_X(p - 2\pi i/Q) \cdot \left[\frac{\sin\,[(p-2\pi i/Q)\cdot Q/2]}{(p-2\pi i/Q)\cdot Q/2}\right]$$

(12)

The characteristic function of the Gaussian random variable X with mean $\bar{X}$ and variance σ_n^2 is given by:

$$\Phi_X(p) = \exp\,(j\bar{X}p - \sigma_n^2 p^2/2)$$

(13)

Performing the differentiation in accordance with equation (11) and making use of equation (12) and (13), the first two moments of the output altitude error have been derived. The details of the derivation are provided in Appendix A. From the appendix, the first and second moment of Y are given by:

$$E(Y) = \bar{X} + \frac{Q}{\pi} \sum_{i=1}^{\infty} \frac{(-1)^i}{i} \sin\,(2\pi i\bar{X}/Q)\ \exp\,(-2\pi^2 i^2 \sigma_n^2/Q^2)$$

(14)

$$E(Y^2) = \sigma_n^2 + (\bar{X})^2 + \frac{Q^2}{12} + \frac{Q^2}{\pi^2} \sum_{i=1}^{\infty} \frac{(-1)^i}{i^2} \cos(2\pi i\bar{X}/Q)\ \exp\,(-2\pi^2 i^2 \sigma_n^2/Q^2)$$

$$+ \frac{Q}{\pi} \sum_{i=1}^{\infty} \frac{(-1)^i}{i} \left\{ \frac{4\pi i \sigma_n^2}{Q} \cos\left(\frac{2\pi i\bar{X}}{Q}\right) + 2\bar{X} \sin\left(\frac{2\pi i\bar{X}}{Q}\right) \right\} \exp\,(-2\pi^2 i^2 \sigma_n^2/Q^2)$$

(15)

Substituting the mean of X from equation (1) into the above two equations and using equations (9,10), the formulae for the mean and the variance of the altitude error at the output of the quantizer can be shown to be:

$$E[e(t)] = \bar{e}$$

$$= E(Y) - h$$

$$= b + \frac{Q}{\pi} \sum_{i=1}^{\infty} \frac{(-1)^i}{i} \exp\,(-2\pi^2 i^2 \sigma_n^2/Q^2)\ \sin\,[2\pi i(h+b)/Q] \qquad (16)$$

$$\sigma_e^2 = E(Y^2) - (\overline{Y})^2$$

$$= \sigma_n^2 + \frac{Q^2}{12} + \frac{Q^2}{\pi^2} \sum_{i=1}^{\infty} \frac{(-1)^i}{i^2} \exp(-2\pi^2 i^2 \sigma_n^2/Q^2) \cos[2\pi i(h+b)/Q]$$

$$+ 4\sigma_n^2 \sum_{i=1}^{\infty} (-1)^i \exp(-2\pi^2 i^2 \sigma_n^2/Q^2) \cos[2\pi i(h+b)/Q]$$

$$- \left\{ \frac{Q}{\pi} \sum_{i=1}^{\infty} \frac{(-1)^i}{i} \exp(-2\pi^2 i^2 \sigma_n^2/Q^2) \sin[2\pi i(h+b)/Q] \right\}^2 \tag{17}$$

For $\sigma_n/Q \geq 1/3$, equations (16,17) can be approximated by the first term of the infinite series. Hence,

For $\sigma_n/Q \geq 1/3$,

$$\overline{e} = b - \frac{Q}{\pi} \exp(-2\pi^2\sigma_n^2/Q^2) \sin[2\pi(h+b)/Q]$$

$$\sigma_e^2 = \sigma_n^2 + \frac{Q^2}{12} - \left(\frac{Q^2}{\pi^2} + 4\sigma_n^2\right) \exp(-2\pi^2\sigma_n^2/Q^2) \cos[2\pi(h+b)/Q]$$

$$- \frac{Q^2}{\pi^2} \exp(-4\pi^2\sigma_n^2/Q^2) \sin^2[2\pi(h+b)/Q]$$

4. NUMERICAL EVALUATION

It can be seen from equations (16) and (17) that the mean and the variance of the altitude error at the output of the quantizer are a function of four variables: h (aircraft's true altitude), b (altimetry bias), σ_n and Q. Furthermore, even if the input bias b and σ_n do not vary with time, the output error parameters of $\bar{e}$ and σ_e can be time dependent since the altitude of the aircraft generally changes with time unless it is flying level.

To reduce the number of variables for graphical display purposes, the following modified altitude error is defined:

$$\bar{e}^* = \bar{e} - b$$
$$= \frac{Q}{\pi} \sum_{i=1}^{\infty} \frac{(-1)^i}{i} \exp\left(-2\pi^2 i^2 \sigma_n^2 / Q^2\right) \sin\left[2\pi i (h+b)/Q\right] \qquad (18)$$

The above simply subtracts the bias from the mean output altitude error. The $\bar{e}^*$ and σ_e are function of three variables in $(h+b)/Q$, σ_n and Q. Since the variation of $\bar{e}^*$ and σ_e^2 is sinusoidal with respect to $(h+b)/Q$, it is only sensitive to the fractional remainder of $(h+b)/Q$, or $R[(h+b)/Q]$. The function R is defined as,

$$R(z) = z - \text{Integer}(z)$$

where Integer (z) denotes the integer part of z.
(e.g., Integer(10.5)=10; Integer(-1.5)=-1)

Therefore, $0 \le |R(z)| < 1$. It can be noted that two aircraft at different altitudes can result in the same fractional remainders and therefore the same output mean and variance if the bias (b) and Q are the same. For example, if $h_1+b=20{,}025$ ft and $Q_1=100$ ft, and $h_2+b=50{,}025$ ft and $Q_2=100$ ft, then

$$R(20025/100)=1/4$$

$$R(50025/100)=1/4$$

A computer program has been written to compute the output altitude error parameters, $\bar{e}^*$ and σ_e^2, based on equations 17 and 18. The specific Q levels evaluated are Q=100, 25, 6.25 ft. The input altimetry error σ_n was varied from 2.5 ft to 25 ft.

The plotting of the output altitude error vs. $R[(h+b)/Q]$, σ_n, and Q are given in two forms. The first form presents the errors vs. $R[(h+b)/Q]$ with σ_n and Q as parameters in order to heighten the sensitivity of errors with respect to altitude level of the aircraft and Q. The second form emphasizes the variation of error with respect to σ_n, with $R[(h+b)/Q]$ and Q as plotting parameters. These two forms are somewhat redundant in information data base, but with different emphasis.

Without loss of generality, only the positive values of R(h+b/Q) will be illustrated in the figures herein. That is, the emphasis is on the important case in which the the aircraft's true altitude plus bias is higher than zero feet.

4.1 Mean Altitude Error at the Output of Quantizer

The mean altitude error at the output of the quantizer, $\bar{e}^*$, as a function of the fractional remainder, R[(h+b/Q], is shown in Fig. (3) for Q=100 ft and in Fig. (4) for Q=25 ft. It can be observed from these two figures that the mean output altitude error $\bar{e}^*$ is an odd function of R[(h+b)/Q] about R[(h+b)/Q] = 1/2. It is also noted that $\bar{e}^*$ is zero at R[(h+b)/Q]=0,1/2, regardless of the quantization levels.

Another observation is that, for the same input σ_n, the peak $\bar{e}^*$ decreases with reduced Q. For σ_n=2.5, the peak $\bar{e}^*$ values for Q=100 ft and Q=25 ft are approximately 43 ft and 7.3 ft respectively.

It can also be noted that, as σ_n increases, the peak $\bar{e}^*$ decreases regardless of Q. In fact, from Figure (3), as σ_n/Q increases to 1/4, corresponding to σ_n=25 ft, $\bar{e}^*$ approaches a sinusoid with respect to R[(h+b)/Q]. This agrees well with the known result that for moderate values of σ_n/Q, only the first term of the infinite series in equation (18) needs to be used. As σ_n/Q becomes very large, $\bar{e}^*$ approaches zero.

It has often been mentioned that it is the σ_e that has the most impact on altitude rate estimation, rather than $\bar{e}^*$. This is true when the mean error $\bar{e}^*$ does not fluctuate significantly with time. For high Q case such as Q=100 ft, and for some unique combination of aircraft altitude profile and σ_n, the contribution of $\bar{e}^*$ to rate error estimation can be appreciable. This aspect needs further investigation.

The variation of $\bar{e}^*$ with input altitude error σ_n is shown on Fig. 5 and 6 for Q=100 ft and Q=25 ft respectively. It can be observed from Fig. 6 for Q=25 ft that for $\sigma_n \geq$10 ft, the output $\bar{e}^*$ is zero. However, for Q=100 ft, the peak $\bar{e}^*$ varies between 27 ft to -27 ft for the same range of σ_n.

4.2 Standard Deviation of Altitude Error at The Output of the Quantizer

The standard deviation of the output altitude error as a function of R[(h+b)/Q] for Q=100 ft is shown in Fig. 7. It can be observed that σ_e is an even function of R[(h+b)/Q] about R[(h+b)/Q]=1/2. The peak σ_e is 50 ft, occuring at R[(h+b)/Q]=1/2. That is, if b=0, the maximum σ_e would occur at the round-off threshold point. This agrees with the intuition that at low σ_n/Q, any small jitter about the round-off threshold point would push the quantized altitude (up or down) to the next quantized altitude, resulting in σ_e of approximately Q/2. As will be noted later, this observation is not true for high σ_n/Q case.

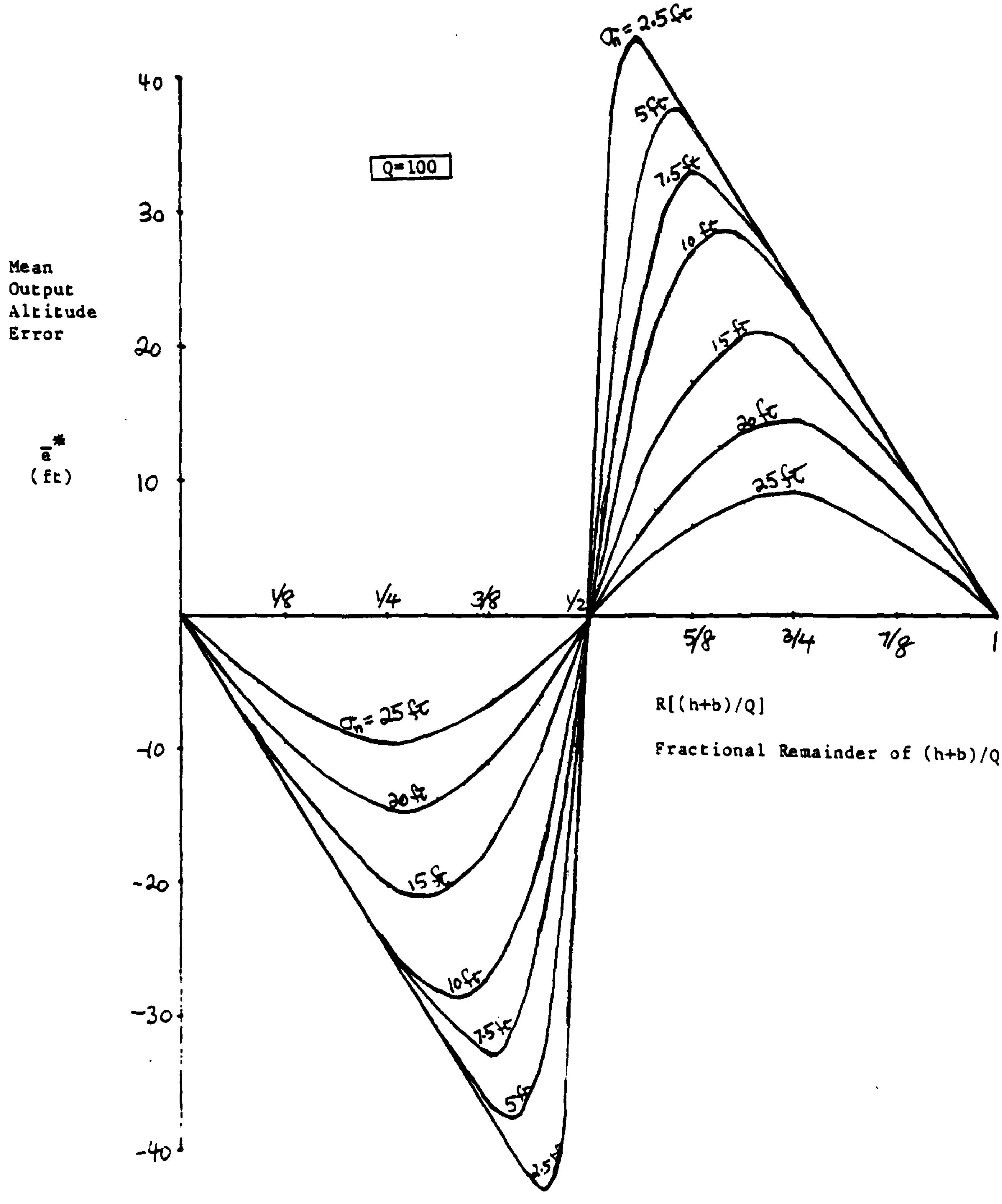


Figure 3- Mean Output Error Vs. R[(h+b)/Q] for Q=100 ft

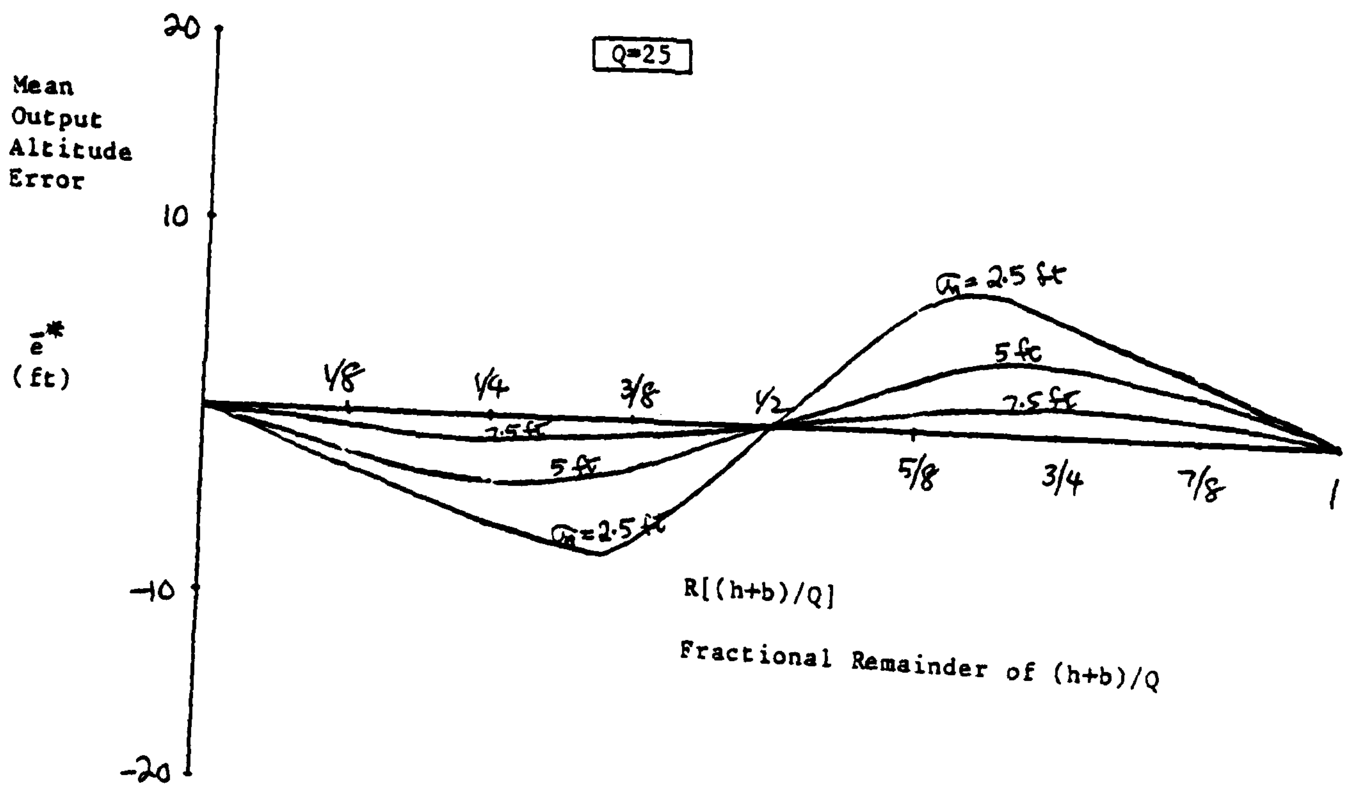


Figure 4- Mean Output Error Vs. R[(h+b)/Q] for Q=25 ft

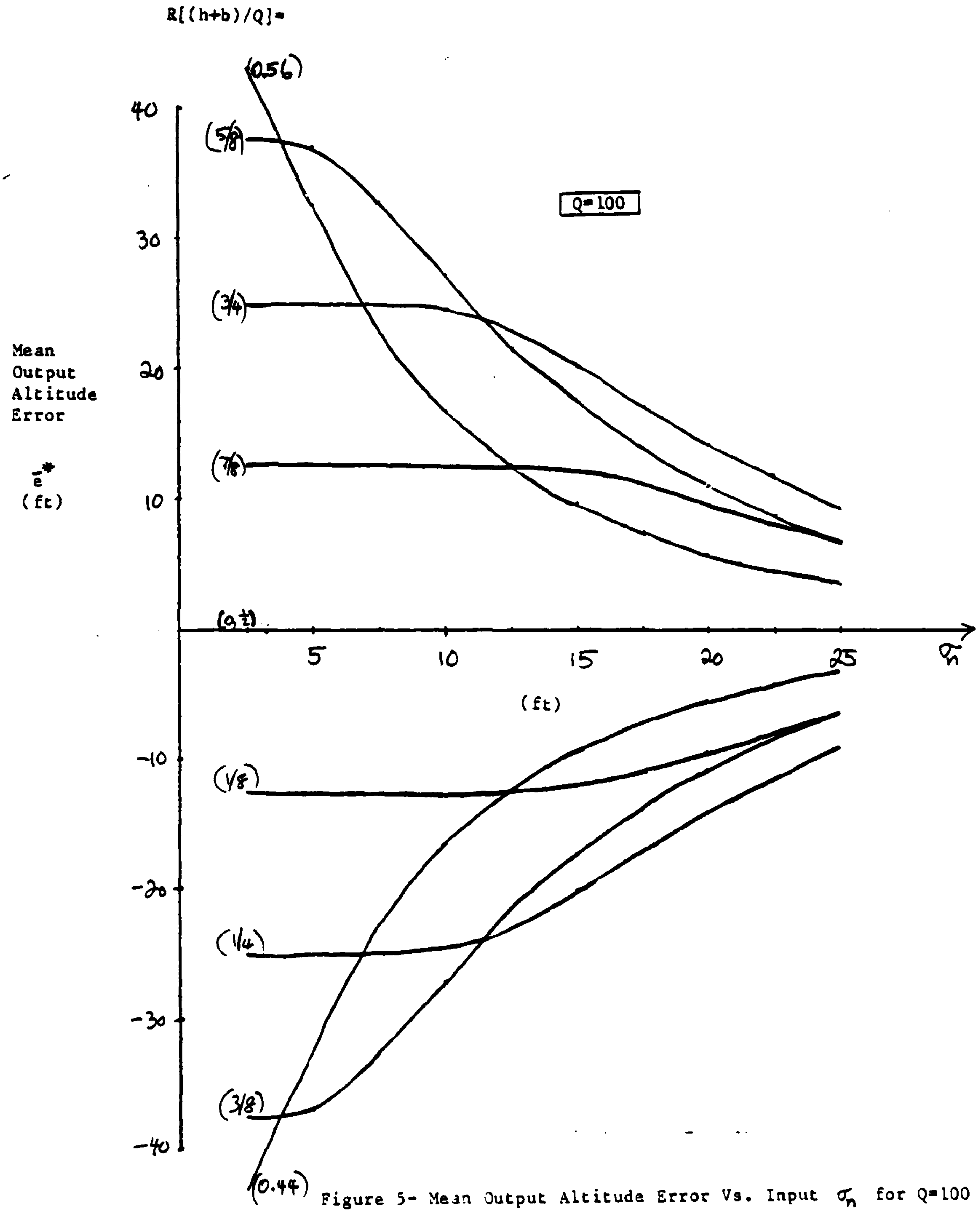


Figure 5- Mean Output Altitude Error Vs. Input σ_n for Q=100 ft

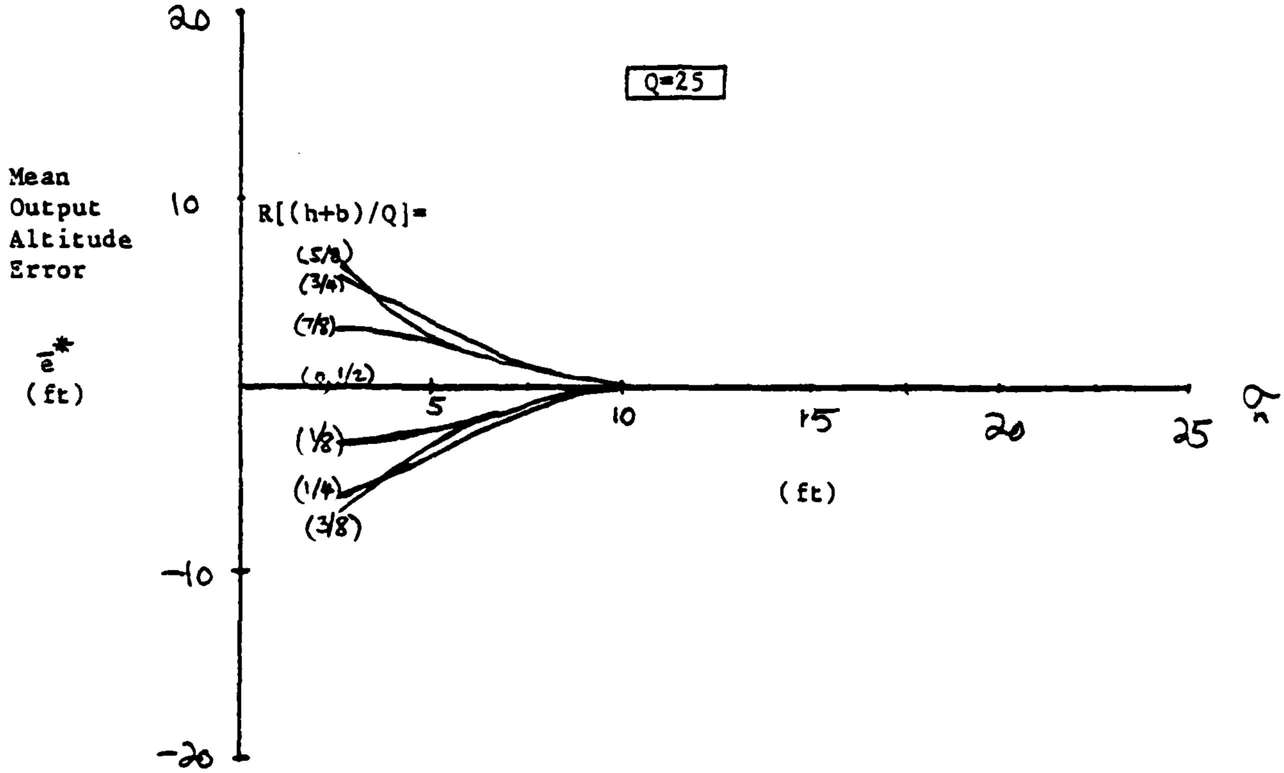


Figure 6- Mean Output Altitude Error Vs. Input σ_n for Q=25 ft

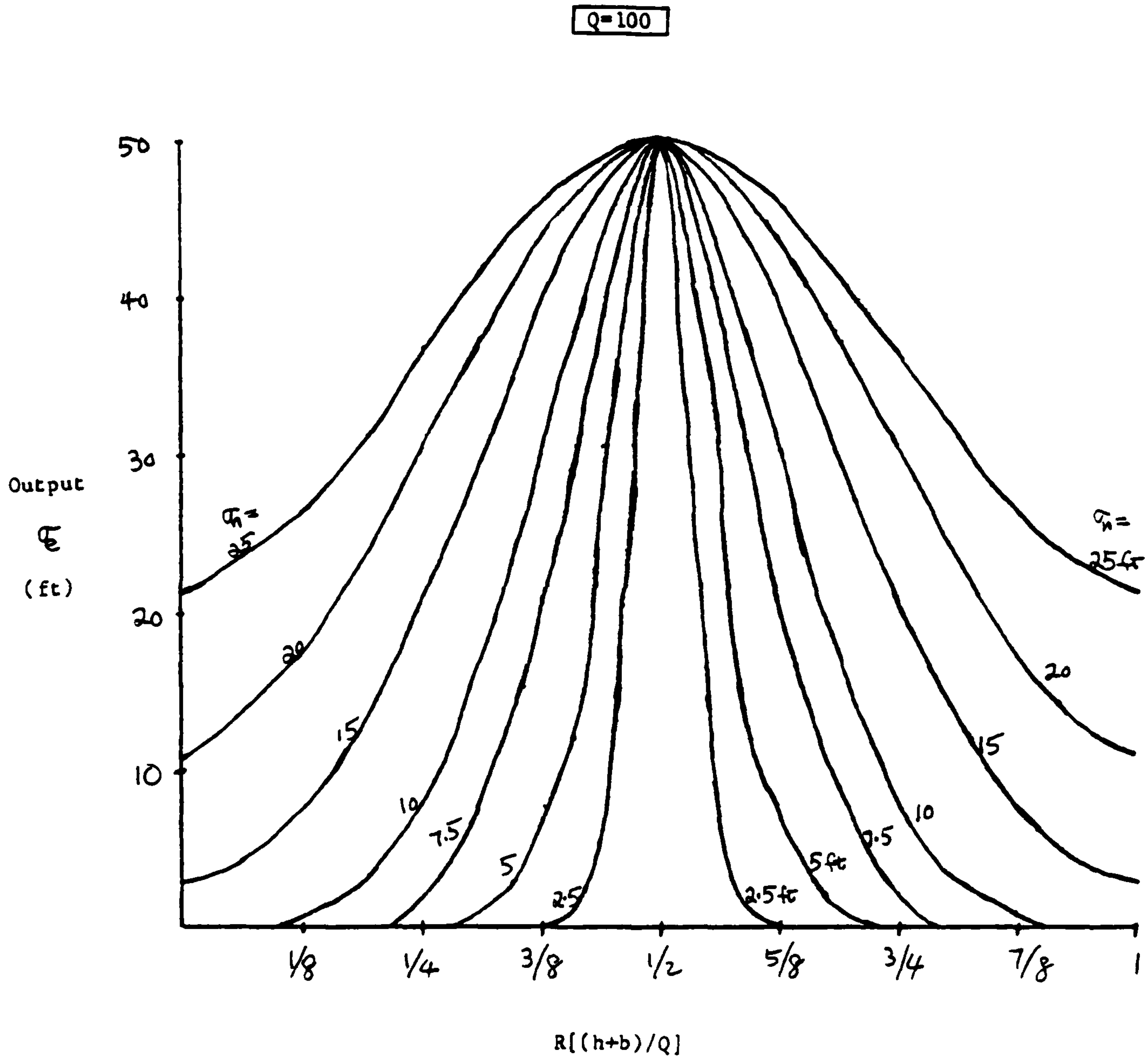


Figure 7- Output σ_e Vs. R[(h+b)/Q] for Q=100 ft

It can be noted from this figure that σ_e decreases as R[(h+b)/Q] moves away from R[(h+b)/Q]=1/2 and reaches a minimum at R[(h+b)/Q]=0. Furthermore, for the smallest noise case of σ_n=2.5 ft, the spread of σ_e about R[(h+b)/Q]=1/2 is very small. As σ_n increases, σ_e becomes more dispersed.

Figure 8 displays the variation of σ_e for Q=25 ft. In comparison with the Q=100 ft and $\sigma_n \leq 7.5$ ft cases, the same observations about σ_e also applies to Q=25 ft. However, for $\sigma_n \geq 10$ ft, σ_e is constant and does not depend on R[(h+b)/Q].

Figures 9, 10, and 11 display the standard deviation of the quantized error as a function of the input altitude error (σ_n) for Q=100 ft, 25 ft, and 6.25 ft respectively. The dashed lines in these figures represent the ideal case of Q approaching zero or no quantization case.

For a specific σ_n , it can be seen from these figures that as Q decreases, the spread of the σ_e decreases and the maximum σ_e also decreases. It can be noted that as Q drops to 6.25 ft, σ_e almost approaches that for the non-quantization case. Furthermore, σ_e is almost independent of the aircraft altitude.

Noise suppression or reduction can be observed for Q=100 ft and Q=25 ft (Fig. 9 &10) for specific input altimetry error (σ_n , b), quantization level (Q), and true aircraft altitude (h). This means that the output σ_e is less than the input σ_n because of altitude quantization. In general, this phenomenon occurs at low σ_n/Q levels and a subset of R[(h+b)]/Q values. For example, for Q=100 ft and input $\sigma_n \leq 10$ ft, Fig. (9) shows that the output σ_e is less than the input σ_n for R[(h+b)/Q]=0,1/8,1/4,3/4,7/8.

Another interesting example is the case in which Q=100 ft, b=0, and the true aircraft altitude is exactly divisible by 100 (coinciding with the flight level notation). In this case, R(h/Q)=0, Fig. (9) shows that the output σ_e is the lowest, as compared to other R(h/Q) values. Furthermore, the mean output error $(\bar{e})$* is also the lowest (see Fig. 5). However, when the last two digits of the aircraft altitude are 50, the output σ_e is the largest while the mean output error is the smallest.

In general, an aircraft in flight will experience the full spectrum of σ_e variation as its altitude changes with time.

4.3 Comparison with the Uniform Altitude Error Model

For ease of analysis, it has frequently been assumed that the error (variance) at the output of a quantizer device can be calculated by summing the input error and the quantization error using the RSS method (see equation 2). The variance of the quantization error is assumed to be $Q^2/12$ based on the uniform distribution of this error. However, the uniform distribution model is not universally valid, especially in low σ_n/Q cases.

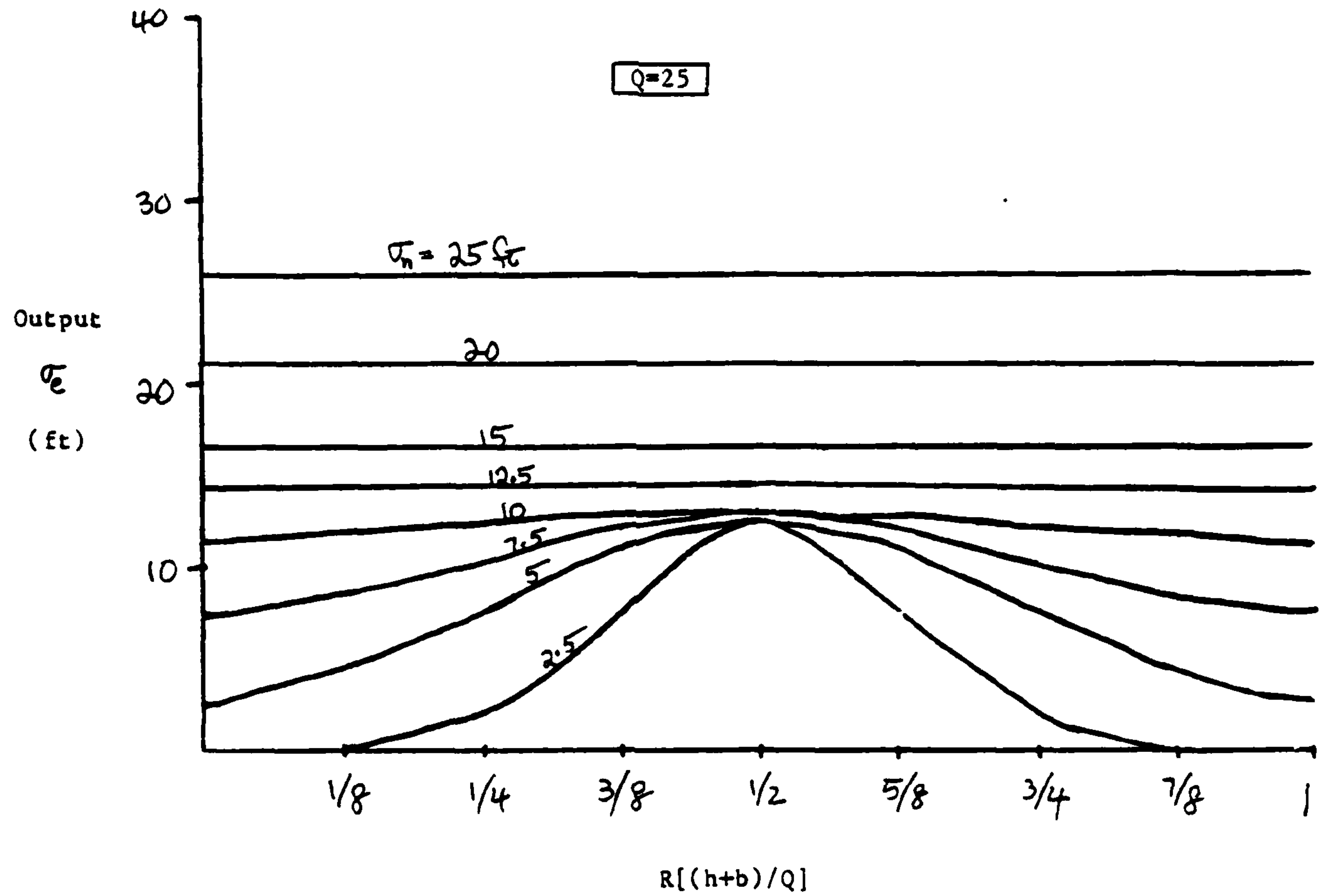


Figure 8- Output σ_e Vs. R[(h+b)/Q] for Q=25 ft

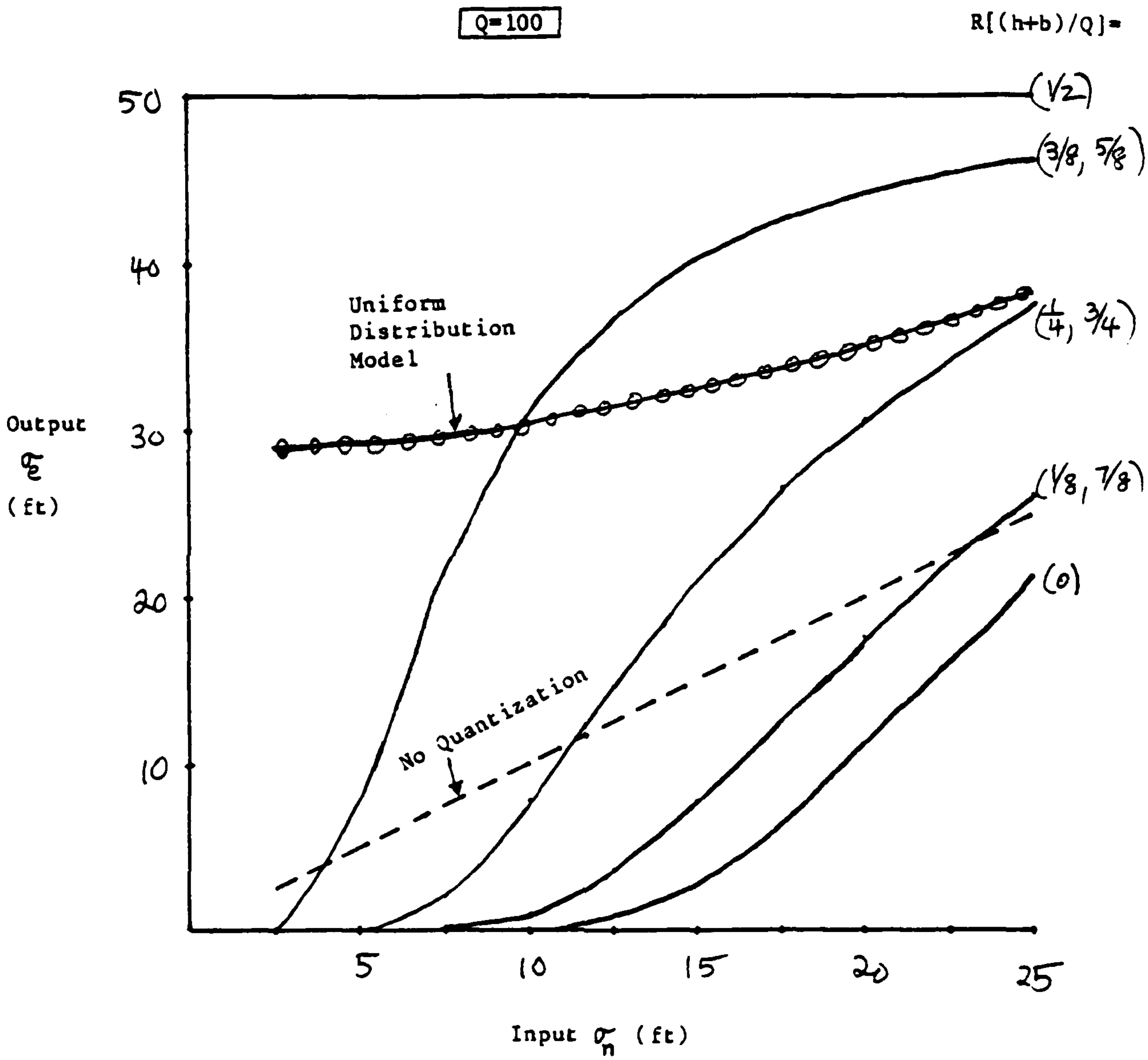


Figure 9- Output σ_ε Vs. Input σ_n for Q=100 ft

Q=25

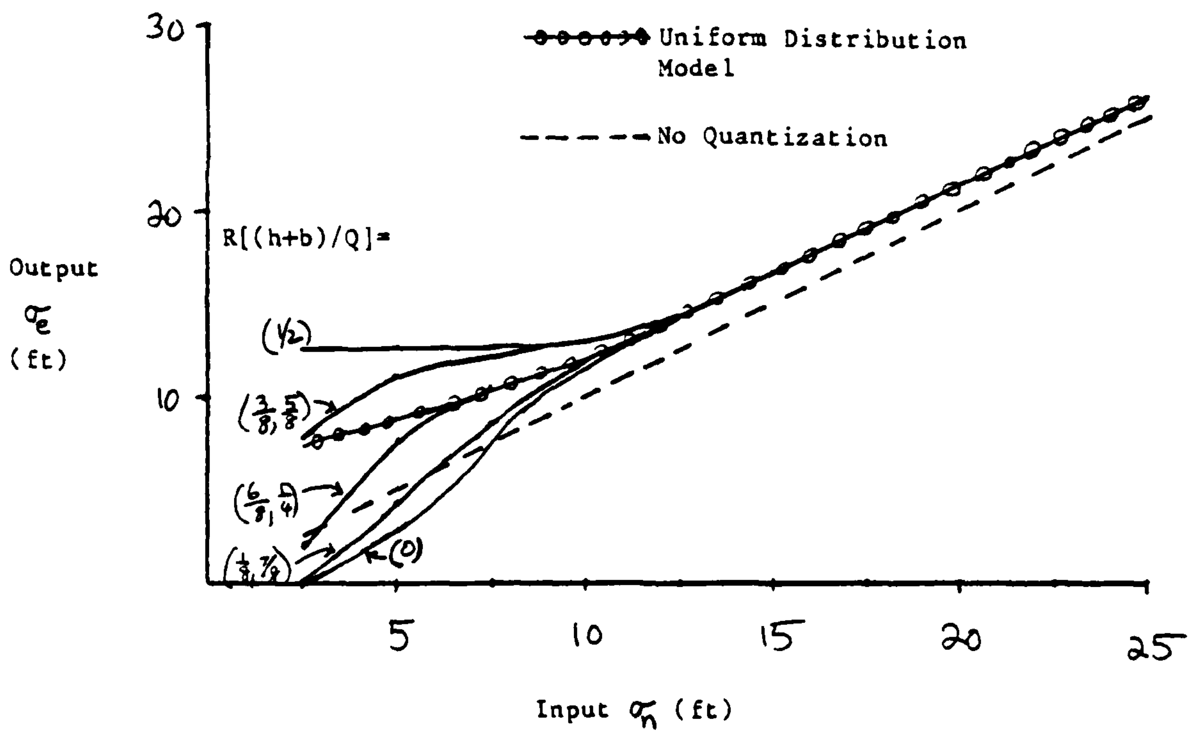


Figure 10- Output σ_e Vs. Input σ_n for Q=25 ft

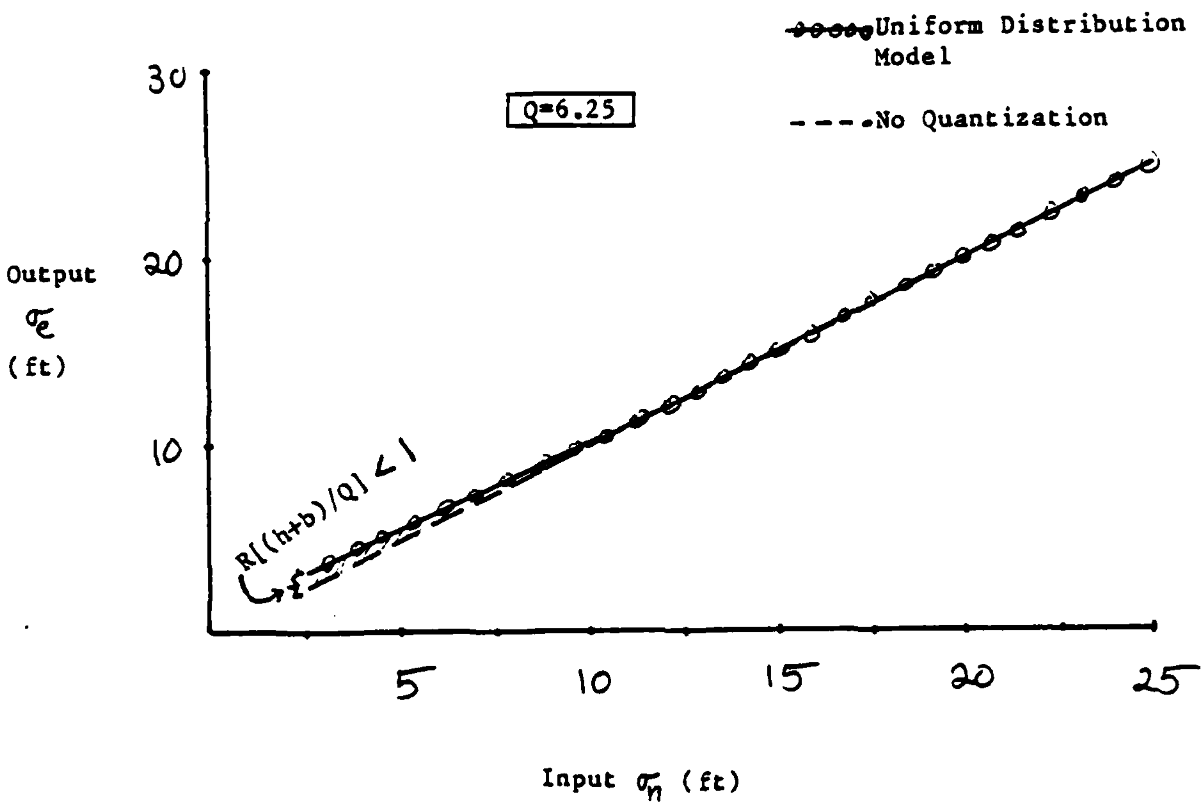


Figure 11- Output σ_ϵ Vs. Input σ_η for Q=6.25 ft

The accuracy of this model is compared with the results given in Figures 9, 10, 11. It can be seen from these figures that the σ_e values predicted by the uniform distribution model fall in between the maximum and the minimum of the true theoretical values. As Q decreases, the uniform model becomes more accurate for the range of input noise examined. As Q approaches 6.25 ft, negligible difference exists between the computed (exact) values and those based on the uniform distribution model.

For quantization levels of 100 ft and 25 ft, the validity of the uniform distribution model depends on the σ_n of interest. For example, for Q=25 ft, it can be observed from Fig. 10 that when $\sigma_n \geq 10$ ft approximately, there is little difference between the values provided by the uniform distribution model and the exact calculation.

The appropriateness of the uniform distribution model is probably more dependent on the application at hand. If approximate analysis is the desired goal, then the uniform distribution model would suffice.

5. CONCLUSIONS

The analysis and numerical results provided herein show that reduced altitude quantization has a positive impact in reducing the errors at the output of the quantizer. Specific conclusions reached include the following:

-The mean and the variance of the altitude error at the quantizer output vary with the fractional remainder of (h+b)/Q, the input altitude error variance, and the quantization level Q. Since an aircraft in flight generally has a dynamic altitude profile, the quantized altitude error also changes with time.

-Reducing the quantization level also reduces the peak mean output altitude error. Reducing Q from 100 ft to 25 ft diminishes the peak mean altitude error at the quantizer output from 43 ft to 7.3 ft. for the range of input error statistics considered.

-As quantization level is reduced, the peak of the σ_e and the spread of σ_e(max and min of σ_e) are also reduced. Figures 9,10,11 contain the numerical results for various Q and input statistics. As altitude is finely quantized such as to 6.25 ft, there is negligible difference between the quantized altitude error and the altitude error without quantization.

-A comparison of the approximate uniform distribution model with the exact formulae shows that the former method produces a value σ_e in between the maximum and the minimum of the true value. For a specific standard deviation of input altitude error, as Q decreases, the uniform distribution model becomes more accurate.

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APPENDIX

First and Second Moment of Quantizer Output

The first and second moment of the quantizer output are derived in this appendix assuming the input distribution is Gaussian. The input and the output of the quantizer of level Q are denoted by X and Y respectively. A round-off type of quantizer with input and output characteristic shown in Fig. (2) is assumed.

The method of characteristic function is used to derive the first two moments of Y. The kth moment of the random variable Y is given by:

$$E[\,Y^k\,] = \left(\frac{1}{j}\right)^k \frac{d^k}{d\,p^k}\Phi_Y(p)\Bigg|_{p=0} \quad ; k=1,2,3,\ \ldots \tag{A-1}$$

where $\Phi_Y(p)$ is the characteristic function of the quantizer output Y

Denoting the characteristic function of X as $\Phi_X(p)$, the relationship between the the input and output characteristic functions is given by the following series [7]:

$$\Phi_Y(p) = \sum_{i=-\infty}^{\infty} \Phi_{X_i}(p) \tag{A-2}$$

where $\Phi_{X_i}(p) = \Phi_X(p - 2\pi i/Q)\cdot\left[\frac{\sin[(p-2\pi i/Q)\cdot Q/2]}{(p-2\pi i/Q)\cdot Q/2}\right]$

The expected value of Y, or first moment, is

$$E(Y) = \frac{1}{j}\ \frac{d}{dp}\Phi_Y(p)\Bigg|_{p=0} = \frac{1}{j}\sum_{i=-\infty}^{\infty}\frac{d}{dp}\Phi_{X_i}(p)\Bigg|_{p=0} \tag{A-3}$$

Differentiating $\Phi_{X_i}(p)$ with respect to p, we get,

$$\frac{d}{dp}\Phi_{X_i}(p)\Bigg|_{p=0} = \frac{d}{dp}\left\{\Phi_X(p - 2\pi i/Q)\cdot\left[\frac{\sin[(p-2\pi i/Q)\cdot Q/2]}{(p-2\pi i/Q)\cdot Q/2}\right]\right\}\Bigg|_{p=0}$$

Carrying out the above operations and making use of equation (A-3), the first moment of Y is given by:

$$E(Y) = \bar{X} - \frac{Q}{2\pi j} \sum_{i \neq 0}^{\infty} \frac{(-1)^i \Phi_X(-2\pi i/Q)}{i} \tag{A-4}$$

From equation (A-1), the second moment of Y is,

$$E(Y^2) = -\sum_{i=-\infty}^{\infty} \frac{d^2 \Phi_Y(p)}{d\,p^2}\Bigg|_{p=0}$$

$$= -\sum_{i=-\infty}^{\infty} \Bigg\{ \Phi_X''(p-2\pi i/Q)\left[\frac{\sin\,(p-2\pi i/Q)Q/2}{(p-2\pi i/Q)Q/2}\right]\Bigg|_{p=0}$$

$$+ 2\Phi_X'(p-2\pi i/Q).\frac{d}{d\,p}\left[\frac{\sin\,(p-2\pi i/Q)Q/2}{(p-2\pi i/Q)Q/2}\right]\Bigg|_{p=0}$$

$$+ \Phi_X(p-2\pi i/Q).\frac{d^2}{d\,p^2}\left[\frac{\sin\,(p-2\pi i/Q)Q/2}{(p-2\pi i/Q)Q/2}\right]\Bigg|_{p=0}\Bigg\}$$

where Φ' and Φ'' denote the first and second derivatives respectively

After considerable algebraic manipulations, the second moment is given by,

$$E(Y^2) = E(X^2) + \frac{Q^2}{12} + \frac{Q}{\pi} \sum_{i \neq 0}^{\infty} \frac{(-1)^i}{i}\Phi_X'(-2\pi i/Q) + \frac{Q^2}{2\pi^2}\sum_{i \neq 0}^{\infty} \frac{\Phi_X(-2\pi i/Q)}{i^2} \tag{A-5}$$

It should be noted that the above expression is valid regardless of the distribution of the input X. That is, the normality assumption for the input to the quantizer has not been used.

Substituting the characteristic function of a Gaussian distribution with mean X and variance σ^2 into equations (A-4, A-5), the first two moments of Y can be shown to be:

$$E(Y)=\bar{X} + \frac{Q}{\pi} \sum_{i=1}^{\infty} \frac{(-1)^i}{i} \sin(2\pi i\bar{X}/Q) \exp(-2\pi^2 i^2 \sigma^2/Q^2)$$

$$E(Y^2)= \sigma^2 + (\bar{X})^2 + \frac{Q^2}{12} + \frac{Q^2}{\pi^2} \sum_{i=1}^{\infty} \frac{(-1)^i}{i^2} \cos(2\pi i\bar{X}/Q) \exp(-2\pi^2 i^2 \sigma^2/Q^2)$$

$$+ \frac{Q}{\pi} \sum_{i=1}^{\infty} \frac{(-1)^i}{i} \left\{ \frac{4\pi i \sigma^2}{Q} \cos\left(\frac{2\pi i\bar{X}}{Q}\right) + 2\bar{X} \sin\left(\frac{2\pi i\bar{X}}{Q}\right) \right\} \exp(-2\pi^2 i^2 \sigma^2/Q^2)$$

END

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