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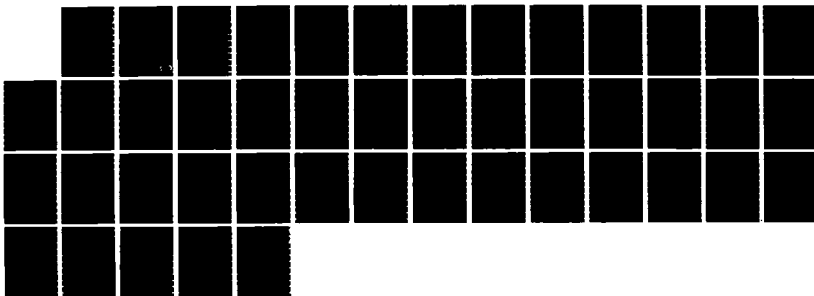
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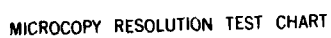
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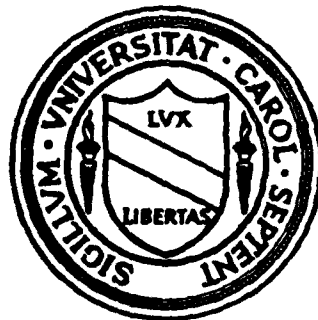
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SPREADING AND PREDICTABLE SAMPLING FOR
EXCHANGEABLE SEQUENCES AND PROCESSES

by

Olav Kallenberg

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SPREADING AND PREDICTABLE SAMPLING FOR EXCHANGEABLE SEQUENCES
AND PROCESSES

by Olav Kallenberg

Universities of Göteborg and North Carolina at Chapel Hill

Abbreviated title: Spreading and predictable sampling

Summary:

Pyll-Nardzewski (1957) proved that an infinite sequence of random variables is exchangeable, if every subsequence has the same distribution. We discuss some restatements and extensions of this result in terms of martingales and stopping times. In the other direction, we show that the distribution of a finite or infinite exchangeable sequence is invariant under sampling by means of a.s. distinct (but not necessarily ordered) predictable stopping times. Both types of result generalize to exchangeable processes in continuous time.

AMS 1980 subject classifications. Primary 60G99; secondary 60G40, 60G44.

Key words and phrases. Invariance in distribution, subsequences, thinning, stationarity, predictable stopping times, allocation sequences and processes, semimartingales, local characteristics, stochastic integrals.

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1. Introduction

A finite sequence of random variables $\xi = (\xi_1, \dots, \xi_n)$ is said to be exchangeable, if every permutation has the same distribution, i.e. if

$$(\xi_{k_1}, \dots, \xi_{k_n}) \stackrel{d}{=} (\xi_1, \dots, \xi_n) \quad (1)$$

for every permutation (k_1, \dots, k_n) of $(1, \dots, n)$. For infinite sequences, we require the same property for every finite subsequence.

It is easy to see that exchangeability of an infinite sequence

$\xi = (\xi_1, \xi_2, \dots)$ implies that

$$(\xi_{k_1}, \xi_{k_2}, \dots) \stackrel{d}{=} (\xi_1, \xi_2, \dots), \quad k_1 < k_2 < \dots \quad (2)$$

A sequence satisfying (2) is said to be spreadable. (Kingman (1978) calls (2) the selection property, while Aldous (1985) refers to (2) as the property of spreading-invariance.)

de Finetti's (1937) celebrated theorem states that an infinite exchangeable sequence is mixed i.i.d., in the sense that its distribution is a mixture of distributions of i.i.d. sequences. Ryll-Nardzewski (1957) noticed that the same conclusion follows from the weaker assumption of spreadability. Both results are in fact simple (though remarkable!) corollaries of the mean ergodic theorem. In proposition 2.1 below, we shall show that the same argument yields an even stronger result.

We proceed in Proposition 2.2 to restate the above results in terms of stopping times and martingales. In particular, a sequence ξ is spreadable iff $\theta_\tau \xi \stackrel{d}{=} \xi$ for every \mathbb{Z}_+ -valued stopping time τ (extensive use of this result was made in [11]), or equivalently, iff the prediction sequence

$$\pi_n = P[\theta_n \xi \in \cdot | \mathcal{F}_n], \quad n \in \mathbb{Z}_+, \quad (3)$$

is a measure valued martingale. Here $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \dots)$ is the filtration induced by ξ (so \mathcal{F}_0 is trivial), stopping times are defined with respect to \mathcal{F} , and $\theta_0, \theta_1, \dots$ denote the shift operators on \mathbb{R}^∞ .

The stopping time condition above characterizes exchangeability in terms of certain randomly selected subsequences. More generally, one may look for conditions on the random indices τ_1, τ_2, \dots , such that

$$(\xi_{\tau_1}, \xi_{\tau_2}, \dots) \stackrel{d}{=} (\xi_1, \xi_2, \dots) \quad (4)$$

implies that ξ is exchangeable. Another instant when (4) implies exchangeability (for stationary ξ) is that of thinning, where the elements of ξ are selected independently with a fixed probability $p \in (0,1)$. This result (Proposition 2.3) is closely related to a result in point process theory (cf. [12]), where mixed Poisson processes are characterized in terms of thinning.

Section 3 deals with the converse problem of finding general conditions on τ_1, τ_2, \dots , such that (4) holds for a given exchangeable sequence ξ . If ξ is infinite and i.i.d., we may e.g. take τ_1, τ_2, \dots to be any strictly increasing sequence of predictable stopping times. (Recall that a stopping time τ is predictable, if $\tau-1$ is a stopping time in the usual sense.) This result is well-known to gamblers (or at least it ought to be!), and the first formal proof appears in Doob (1936). Our main result in Section 3 states that (4) is true for arbitrary a.s. distinct predictable stopping times τ_1, τ_2, \dots , whenever ξ is a finite or infinite exchangeable sequence. Note in particular that the τ_j may form a random (but predictable) permutation of the indices of ξ , since no requirement is made on the order.

The above result, which generalizes Theorem 5.1 in [11], has the most surprising consequences for finite games (e.g. card games, lotteries, sampling from finite populations), as shown by examples in [13]. For the sake of applications (but also for the proof), it is useful to introduce the associated allocation sequence $\alpha_1, \alpha_2, \dots$, given by

$$\alpha_k = \inf\{j: \tau_j = k\}, \quad k=1,2,\dots \quad (5)$$

(Here $\inf \emptyset$ means ∞ , as usual.) Note that the finite values of $\alpha_1, \alpha_2, \dots$ are a.s. distinct, and that α_k is \mathcal{F}_{k-1} -measurable by assumption for each k . Informally, the element ξ_k is moved to a new position α_k , which is only allowed to depend on the past history $(\xi_1, \dots, \xi_{k-1})$. Note that ξ_k is discarded for the new sequence if $\alpha_k = \infty$.

Sections 4 and 5 deal with the corresponding problems in continuous time. A process X defined on $I=[0,1]$ or R_+ is said to be exchangeable, if $X_0=0$, if X is continuous in probability at every $t \in I$, and if the increments of X over an arbitrary set of disjoint intervals of equal length form an exchangeable sequence. In that case, we may (and will) choose a version of X which is right-continuous with left-hand limits. If $I=R_+$, the analogue of de Finetti's theorem states that X is a mixture (again in the distributional sense) of Lévy processes. For $I=[0,1]$, we have instead the more general representation (cf. [9])

$$X_t = \alpha t + \sigma B_t + \sum_{j=1}^{\infty} \beta_j (1\{\tau_j \leq t\} - t), \quad t \in [0,1], \quad (6)$$

($1\{\cdot\}$ denoting the indicator function of the event within brackets), where B is a Brownian bridge, while τ_1, τ_2, \dots are i.i.d. random variables uniformly distributed on $[0,1]$, and $\alpha, \sigma, \beta_1, \beta_2, \dots$ are arbitrary random variables satisfying $\sigma \geq 0$ and $\sum \beta_j^2 < \infty$, the three objects B , (τ_1, τ_2, \dots) and $(\alpha, \sigma, \beta_1, \beta_2, \dots)$ being independent. We shall write β for the point process $\sum \delta_{\tau_j}$, and say that X is directed by the triple $(\alpha, \sigma^2, \beta)$. Note that X is a mixture or ergodic exchangeable processes (6), where α , σ^2 and β are non-random.

Exchangeable processes will be seen to be semimartingales. In Section 4, we shall essentially characterize the exchangeability of a semimartingale X in terms of its local characteristics (as

defined in [7,8]). If X is exchangeable and integrable, the latter will be absolutely continuous, with densities which form martingales with respect to the filtration induced by X . Conversely, a semi-martingale X on P_+ with the above property can be shown to be exchangeable, provided that X has stationary increments, and a similar result (related also to Theorem 3.3 in [11]) will be proved for processes on $[0,1]$. A related characterization of mixed Poisson processes has been obtained, independently, by Heller and Pfeifer (1985).

The continuous time counterpart of the predictable sampling theorem of Section 3 is stated in Section 5 in terms of stochastic integrals. More precisely, the allocation sequence in (5) is now replaced by an allocation process V , which is predictable and a.s. measure preserving, at least on some suitable subinterval J of the index set I . (Thus $\lambda V^{-1} = \lambda$ on J a.s., where λ denotes Lebesgue measure.) Given X and V , we may define a new process XV^{-1} on J by

$$(XV^{-1})_t = \int_I 1\{V_s \leq t\} dX_s, \quad t \in J. \quad (7)$$

The main result of Section 5 (which generalizes Theorem 5.2 in [11]) states that X and (a suitable version of) XV^{-1} have the same distribution on J , whenever X is exchangeable. As in the discrete time case, there are some rather surprising applications of this result, which are discussed in [13]. The result has also proved useful in establishing representations of stable integrals, but this will be discussed elsewhere.

We now turn to discuss some technical extensions of the above. Our first point concerns the choice of filtration. For many purposes, one needs to introduce some more general filtration \mathcal{F} than the one generated by the sequence or process under consideration. Following [11], we shall then say that a sequence ξ is \mathcal{F} -exchangeable,

if ξ is adapted to \mathcal{F} , and if $\theta_n \circ \xi$ is conditionally exchangeable, given \mathcal{F}_n , for every $n \in \mathbb{Z}_+$. The latter condition means of course that the shifted sequence should a.s. be exchangeable under the conditional probability law. It is easy to check that an \mathcal{F} -exchangeable sequence is exchangeable, and that the two notions are equivalent in the case when \mathcal{F} is induced by ξ . Most of the results described above extend without effort to the above more general setting. In particular, this is true for the predictable sampling theorem, where one may hence allow for independent randomizations in each step, in the construction of (α_k) .

The continuous time case is similar. For technical reasons we shall only consider standard filtrations \mathcal{F} , satisfying the usual conditions of right continuity and completeness (so that $\mathcal{F}_t = \mathcal{F}_{t+}$ for all t , while \mathcal{F}_0 contains all null-sets in a completion of $\mathcal{F}_\infty = \bigvee \mathcal{F}_t$). In particular, the filtration induced by X is defined as the smallest standard filtration making X adapted. Defining \mathcal{F} -exchangeability as before, we have the same relationship to the usual notion of exchangeability (cf. [11]).

A second point concerns the predictable sampling theorem discussed above. In many applications, the sample size is random, and there may be no obvious way of extending the given sequence of stopping times to a sequence of fixed length. In that case, we can still prove that the sampled sequence η can be embedded in distribution into the original sequence ξ (which we denote by $\eta \stackrel{d}{\subset} \xi$, cf. [11]). By this is meant that η can be continued, by randomization or otherwise, to a sequence η' of the same length as ξ and such that $\eta' \stackrel{d}{=} \xi$. A corresponding extension exists in the continuous time case, with a similar definition of embedding. Note that the above construction of η' may require an extension

of the original probability space.

A simple way of proving the embedding $\eta \stackrel{d}{\subset} \xi$ is to construct, on some suitable probability space, a sequence $\xi' \stackrel{d}{=} \xi$ and a \bar{Z}_+ -valued random variable ν' , such that

$$(\xi'_1, \dots, \xi'_{\nu'}) \stackrel{d}{=} \eta, \quad (8)$$

(where the left-hand side should be interpreted as ξ' when $\nu' = \infty$).

In continuous time, it is convenient first to extend the definition of the sampled process Y , originally given on some random interval $[0, \zeta)$, by putting $Y_t = \partial$ for $t \geq \zeta$, where ∂ denotes an auxiliary coffin state. We may further define the killing operators k_s by

$$(k_s f)_t = \begin{cases} f_t, & s < t \in I, \\ \partial, & s \geq t \in I, \end{cases}$$

defined for functions f on $I = [0, 1]$ or R_+ , and for numbers $s \in I \cup \{\infty\}$.

In order to prove that $Y \stackrel{d}{\subset} X$, it is then enough to construct, on some suitable probability space, a process $X' \stackrel{d}{=} X$ and a random variable $\zeta' \stackrel{d}{=} \zeta$, such that $k_{\zeta'} X' \stackrel{d}{=} Y$.

The above statements are simple consequences of the following randomization lemma.

Lemma 1.1. Let ξ and η be random elements on some probability space (Ω, P) and taking values in the spaces S and T , where S is separable metric while T is Polish. Assume that $\xi \stackrel{d}{=} f(\eta)$ for some measurable function $f: T \rightarrow S$. Then there exists some random element $\eta' \stackrel{d}{=} \eta$ on $(\Omega \times [0, 1], P \times \lambda)$, such that $\xi = f(\eta')$ a.s. $P \times \lambda$.

Proof. It is enough to prove the result for $T = R$, since it will then extend immediately to the case of linear Borel sets, and next, by Borel isomorphism (cf. [1], p.50), to arbitrary Polish spaces. For $T = R$, we may choose a regular version of the conditional probabilities

$$\mu_s = P[\eta \in \cdot | f(\eta) \in ds], \quad s \in S,$$

and define

$$\eta'(\omega, x) = \sup\{y: \mu_{\xi(\omega)}(-\infty, y] \leq x\}, \quad \omega \in \Omega, \quad x \in [0, 1].$$

It is easy to check that η' is measurable and satisfies $(\xi, \eta') \stackrel{d}{=} (f(\eta), \eta)$. Since S is separable, the diagonal in S^2 is measurable, so we get

$$1\{\xi = f(\eta')\} \stackrel{d}{=} 1\{f(\eta) = f(\eta)\} = 1,$$

which shows that $\xi = f(\eta')$ a.s. □

Let us conclude with some remarks on literature. Though the present paper is formally self-contained as far as exchangeability theory is concerned, we recommend Kingman's (1978) paper and Aldous' (1985) lecture notes for introductory reading. Some further background on the continuous time theory may be found in [9,10,11]. Standard results from stochastic calculus and weak convergence theory will often be used without explicit references, and for these the reader may e.g. consult Jacod (1979, 1985) and Billingsley (1968).

2. Spreading characterizations

Let us first show how de Finetti's and Ryll-Nardzewski's results follow easily from the mean ergodic theorem. So assume that $\xi = (\xi_1, \xi_2, \dots)$ is spreadable, and let the functions $f_1, f_2, \dots: R \rightarrow R$ be bounded and measurable. Write \mathcal{I} for the shift invariant σ -field in P^∞ , and let μ be a regular version of $P[\xi_1 \in \cdot | \xi^{-1}\mathcal{I}]$. Then

$$E \prod_{j=1}^k f_j(\xi_j) = E \prod_{j=1}^k \left\{ \frac{1}{n} \sum_{i=1}^n f_j(\xi_{j+n+i}) \right\} \rightarrow E \prod_{j=1}^k \mu f_j$$

as $n \rightarrow \infty$, by the L_2 ergodic theorem (where the convergence is clearly uniform under shifts) plus dominated convergence. Here and below, $\mu f = \int f d\mu$. The proof is completed by a monotone class argument. (Essentially the same proof yields the usual conditional forms of de Finetti's theorem, cf. [1].)

We shall use the same method to prove the following stronger result.

Proposition 2.1. Let $\xi = (\xi_1, \xi_2, \dots)$ be a stationary sequence of random variables satisfying

$$(\xi_1, \dots, \xi_n, \xi_{n+2}) \stackrel{d}{=} (\xi_1, \dots, \xi_n, \xi_{n+1}), \quad n \in \mathbb{Z}_+. \quad (1)$$

Then ξ is exchangeable.

First proof. Extend ξ to a doubly infinite stationary sequence $\dots, \xi_{-1}, \xi_0, \xi_1, \dots$, and conclude from (1) that

$$(\dots, \xi_n, \xi_{n+2}) \stackrel{d}{=} (\dots, \xi_n, \xi_{n+1}), \quad n \in \mathbb{Z}.$$

Iterating this result yields

$$(\dots, \xi_n, \xi_{n+k}) \stackrel{d}{=} (\dots, \xi_n, \xi_{n+1}), \quad n \in \mathbb{Z}, k \in \mathbb{N}.$$

Letting g and f_1, f_2, \dots be bounded measurable functions on P^∞ and R , and writing $\xi^- = (\dots, \xi_{-1}, \xi_0)$, we get by the L_2 ergodic theorem

$$\begin{aligned} E g(\xi^-) \prod_{j=1}^k f_j(\xi_j) &= E g(\xi^-) \prod_{j=1}^{k-1} f_j(\xi_j) \left(\frac{1}{n} \sum_{i=k}^{k+n-1} f_k(\xi_i) \right) \\ &\rightarrow E g(\xi^-) \prod_{j=1}^{k-1} f_j(\xi_j) \mu f_k. \end{aligned}$$

Since μ is ξ^- -measurable by the law of large numbers, we may continue recursively, until we get after k steps

$$E \left[g(\xi^-) \prod_{j=1}^k f_j(\xi_j) \right] = E \left[g(\xi^-) \prod_{j=1}^k \mu f_j \right]$$

Thus the conclusion follows as before. \square

We may also give a simple martingale proof, in the spirit of Aldous (1985), p. 22.

Second proof. It is convenient to reflect the index set in the origin, so we may assume instead that ξ is stationary and satisfies $\xi \stackrel{d}{=} (\xi_1, \theta_2 \circ \xi)$. By iteration and stationarity, we get

$$\theta_{k-1} \circ \xi \stackrel{d}{=} (\xi_k, \theta_n \circ \xi) \stackrel{d}{=} (\xi_1, \theta_n \circ \xi), \quad k \leq n,$$

so

$$E[f(\xi_k) | \theta_k \circ \xi] \stackrel{d}{=} E[f(\xi_k) | \theta_n \circ \xi] = E[f(\xi_1) | \theta_n \circ \xi], \quad k \leq n,$$

for any bounded and measurable function f . By Lemma 3.4 in [1], the left equality must also hold in the a.s. sense, and we get as $n \rightarrow \infty$

$$E[f(\xi_k) | \theta_k \circ \xi] = E[f(\xi_1) | \mathcal{T}] = \mu f \quad \text{a.s.},$$

where \mathcal{T} denotes the tail σ -field of ξ . Letting f_1, \dots, f_n be bounded and measurable, we hence obtain by iterated conditioning

$$E \left[\prod_{k=1}^n f_k(\xi_k) | \mathcal{T} \right] = E \left[\prod_{k=1}^n E[f_k(\xi_k) | \theta_k \circ \xi] | \mathcal{T} \right] = E \left[\prod_{k=1}^n \mu f_k | \mathcal{T} \right] = \prod_{k=1}^n \mu f_k,$$

which proves that ξ is conditionally i.i.d., given \mathcal{T} . \square

It is useful to restate the above conditions in terms of stopping times and martingales. For the sake of simplicity, these will here be defined with respect to the induced filtration

$$\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n), \quad n \in \mathbb{Z}_+.$$

Define the measure valued processes (π_n) and (λ_n) by

$$\pi_n = P[\theta_n \circ \xi \in \cdot | \mathcal{F}_n], \quad \lambda_n = P[\xi_{n+1} \in \cdot | \mathcal{F}_n], \quad n \in \mathbb{Z}_+, \quad (2)$$

and note that these formulas remain true when n is a finite stopping time. All functions below are assumed to be measurable.

Proposition 2.2. Let $\xi = (\xi_j)$ be an infinite sequence of random variables, and define (π_n) and (λ_n) by (2). Then (i)-(iii) and (i')-(iii') are sets of equivalent conditions.

- (i) ξ is spreadable,
- (ii) $\theta_\tau \cdot \xi \stackrel{d}{=} \xi$ for every finite stopping time τ ,
- (iii) $(\pi_n f)$ is a martingale for every bounded $f: P^\infty \rightarrow R$;
- (i') ξ satisfies (1),
- (ii') $\xi_{\tau+1} \stackrel{d}{=} \xi_1$ for every finite stopping time τ ,
- (iii') $(\lambda_n f)$ is a martingale for every bounded $f: R \rightarrow R$.

The fact that (ii) with a general filtration \mathcal{F} is equivalent to \mathcal{F} -exchangeability was noted with a direct proof in [11], Theorem 2.1. Condition (iii') is mainly interesting because of its analogy with the continuous time conditions of Section 4 below.

Proof. Condition (iii) means that

$$E[f(\theta_{n+1} \cdot \xi) : A] = E[f(\theta_n \cdot \xi) : A], \quad A \in \mathcal{F}_n, \quad n \in \mathbb{Z}_+,$$

for bounded $f: R^\infty \rightarrow R$. By a monotone class argument, this is equivalent to

$$(\xi_1, \dots, \xi_n, \xi_{n+2}, \xi_{n+3}, \dots) \stackrel{d}{=} \xi, \quad n \in \mathbb{Z}_+,$$

from which (i) follows by iteration. Thus (i) \Leftrightarrow (iii). Condition (iii) is further equivalent to

$$E\pi_\tau f = E\pi_0 f$$

for bounded $f: R^\infty \rightarrow R$ and for finite stopping times τ . This may be rewritten as

$$Ef(\theta_\tau \cdot \xi) = Ef(\xi),$$

which is equivalent to (ii). Thus (ii) \Leftrightarrow (iii), so (i)-(iii) are equivalent. A similar argument proves the equivalence of (i')-(iii'). \square

It should be noted that Proposition 2.1 is false without the hypothesis of stationarity. For a simple counterexample, let ξ_1, ξ_2, \dots take the values 0 or 1, and choose $P\{\xi_1=1\}=1/4$. Let us further assume that ξ_2, ξ_3, \dots are conditionally i.i.d., given ξ_1 , with

$$P[\xi_n=1|\xi_1] = \frac{1}{6} + \frac{1}{3} \xi_1, \quad n \geq 1.$$

Then

$$E[\xi_{n+1}|\mathcal{F}_n] = \begin{cases} \frac{1}{4}, & n=0, \\ \frac{1}{6} + \frac{1}{3} \xi_1, & n \geq 1, \end{cases}$$

is a martingale, and hence so is $(\lambda_n f)$ for every $f: \{0,1\} \rightarrow \mathbb{R}$.

Thus (1) holds by Proposition 2.2. But ξ is not exchangeable.

since $P\{\xi_1=\xi_2=1\}=1/8$ while $P\{\xi_2=\xi_3=1\}=1/12$.

We turn to the thinning characterization of exchangeability, mentioned in the introduction. For a formal definition of thinning, let ξ be an infinite random sequence, and let the random variables $\kappa_1, \kappa_2, \dots$ be i.i.d. and independent of ξ with

$$P\{\kappa_i=1\} = 1 - P\{\kappa_i=0\} = p, \quad i \in \mathbb{N},$$

for some $p \in (0,1]$. Then the random variables

$$\tau_j = \inf\{k \in \mathbb{N} : \sum_{i=1}^k \kappa_i = j\}, \quad j \in \mathbb{N},$$

are a.s. finite, so the sequence

$$\eta = (\xi_{\tau_1}, \xi_{\tau_2}, \dots)$$

is a.s. well-defined and will be called a p-thinning of ξ .

Proposition 2.3. Fix $p \in (0,1)$, and let ξ be a stationary sequence of random variables with p-thinning η . Then ξ is exchangeable iff $\xi \stackrel{d}{=} \eta$.

First proof. By iteration, we get the same property with p replaced by p^n , $n \in \mathbb{N}$, so we may take p arbitrarily small. Fix $m, n \in \mathbb{N}$ with $m \leq n$, and note that

$$P\left[\bigcap_{j=1}^m \{\tau_j = k_j\} \mid \tau_{m+1} = n+1\right] = \binom{n}{m}^{-1},$$

for all $k_1 < \dots < k_m \leq n$. Letting $f_1, \dots, f_m: R \rightarrow R$ be bounded, we get as in Kingman (1978), p. 188,

$$\lim_{n \rightarrow \infty} E\left[\prod_{j=1}^m f(\eta_j) \mid \tau_{m+1} = n+1\right] = E\left[\prod_{j=1}^m \mu f_j\right],$$

where $\mu f = E[f(\xi_1) \mid \mathcal{I}]$. Since $\tau_{m+1} \xrightarrow{P} \infty$ as $p \rightarrow 0$, it follows that

$$E\left[\prod_{j=1}^m f(\xi_j)\right] = E\left[\prod_{j=1}^m f(\eta_j)\right] = E\left[E\left[\prod_{j=1}^m f(\eta_j) \mid \tau_{m+1}\right]\right] \rightarrow E\left[\prod_{j=1}^m \mu f_j\right],$$

which implies that ξ is exchangeable. □

For readers acquainted with random measure theory, we shall outline an alternative proof, exhibiting the relationship with thinning of point processes. Here and for the remainder of this section, we shall use the terminology and notation of [12].

Second proof of Proposition 2.3. Introduce the marked point process

$$\tilde{\xi} = \sum_{j=1}^{\infty} \delta_{(1, \xi_j)},$$

and construct another point process $\tilde{\eta}_p$ from $\tilde{\xi}$ by a p -thinning followed by a scale contraction by a factor p^{-1} . Note that the successive marks of $\tilde{\eta}_p$ are given by $\eta = \eta_p$. Let us further construct $\tilde{\xi}_p$ by a p^{-1} -contraction of the random measure $p\tilde{\xi}$. As before, we may let $p \rightarrow 0$ along a sequence. By the ergodic theorem, we get $\tilde{\xi}_p \xrightarrow{v} \mu \times \lambda$ a.s., for some random probability measure μ on R , so Theorem 8.4 in [12] yields $\tilde{\eta}_p \xrightarrow{d} \tilde{\eta}'$, where $\tilde{\eta}'$ is a Cox process directed by $\mu \times \lambda$. It follows by continuous mapping that $\xi \stackrel{d}{=} \eta_p \xrightarrow{d} \eta'$, where η' is the sequence of successive marks of $\tilde{\eta}'$. It remains to notice that η' is conditionally i.i.d. μ . □

We conclude this section by stating an analogous point process result, which follows easily by a similar argument. Recall that a

marked point process on R_+ is exchangeable (in the sense of Chapter 9 in [12]), iff it is a mixture of stationary Poisson processes.

Proposition 2.4. Fix $p \in (0,1)$, let ξ be a stationary marked point process on R_+ , and let η be obtained from ξ by a p -thinning followed by a scale contraction by a factor p^{-1} . Then ξ is exchangeable iff $\xi \stackrel{d}{=} \eta$.

3. Predictable sampling

Here we shall prove the fact, already mentioned in the introduction, that the distribution of an exchangeable sequence is invariant under predictable sampling. To facilitate access, we begin with the special case when the sampled sequence has fixed length. Fix a filtration $\mathcal{F}=(\mathcal{F}_0, \mathcal{F}_1, \dots)$.

Theorem 3.1. Let ξ be a finite or infinite \mathcal{F} -exchangeable sequence with index set I , and let τ_1, \dots, τ_k be a.s. distinct I -valued predictable stopping times. Then

$$(\xi_{\tau_1}, \dots, \xi_{\tau_k}) \stackrel{d}{=} (\xi_1, \dots, \xi_k). \quad (1)$$

Proof. Let us first consider the case when $\xi=(\xi_1, \dots, \xi_n)$ is finite and $k=n$. Let $\alpha_1, \dots, \alpha_n$ be the allocation sequence associated with τ_1, \dots, τ_n , and note that the two sequences are inverse random permutations of the integers $1, \dots, n$. Define for each $m \in \{0, \dots, n\}$ another random permutation (α_{mj}) by putting $\alpha_{mj} = \alpha_j$ for $j \leq m$, and then recursively

$$\alpha_{mj} = \min(N \setminus \{\alpha_1, \dots, \alpha_{j-1}\}), \quad j=m+1, \dots, n.$$

Note that α_{mj} is \mathcal{F}_{m-1} -measurable for each j , and that $\alpha_{mj} = \alpha_{m-1, j} = \alpha_j$ whenever $j < m$. Using the \mathcal{F} -exchangeability of ξ , we get for any bounded measurable functions f_1, \dots, f_n

$$\begin{aligned} E \prod_{j=1}^n f_{\alpha_{mj}}(\xi_j) &= E E \left[\prod_{j=1}^n f_{\alpha_{mj}}(\xi_j) \middle| \mathcal{F}_{m-1} \right] \\ &= E \prod_{j < m} f_{\alpha_{mj}}(\xi_j) E \left[\prod_{j \geq m} f_{\alpha_{mj}}(\xi_j) \middle| \mathcal{F}_{m-1} \right] \\ &= E \prod_{j < m} f_{\alpha_{m-1, j}}(\xi_j) E \left[\prod_{j \geq m} f_{\alpha_{m-1, j}}(\xi_j) \middle| \mathcal{F}_{m-1} \right] \\ &= E \prod_{j=1}^n f_{\alpha_{m-1, j}}(\xi_j). \end{aligned}$$

Summing over $m=1, \dots, n$, and noting that $\alpha_{nj} = \alpha_j$ while $\alpha_{0j} = j$, we

hence obtain

$$E \prod_{j=1}^n f_j(\xi_{\tau_j}) = E \prod_{j=1}^n f_{\alpha_j}(\xi_j) = E \prod_{j=1}^n f_j(\xi_j).$$

The assertion now follows by a monotone class argument.

If instead $k < n$, we may extend the sequence (τ_j) by putting recursively

$$\tau_j = \min(N \setminus \{\tau_1, \dots, \tau_{j-1}\}), \quad j = k+1, \dots, n.$$

The assumptions are then fulfilled by the extended sequence, so

$$(\xi_{\tau_1}, \dots, \xi_{\tau_n}) \stackrel{d}{=} (\xi_1, \dots, \xi_n),$$

which implies the same result for the first k components.

Let us finally assume that ξ is infinite. We then define the predictable stopping times τ_{nj} , for $n \in \mathbb{N}$ and $j = 1, \dots, k$, by

$$\tau_{nj} = \begin{cases} \tau_j, & \tau_j \leq n, \\ n+j, & \tau_j > n. \end{cases} \quad (2)$$

Since $\tau_{n1}, \dots, \tau_{nk}$ are a.s. distinct and bounded by $n+k$, we may apply the result in the finite case to the subsequence $(\xi_1, \dots, \xi_{n+k})$, to obtain

$$(\xi_{\tau_{n1}}, \dots, \xi_{\tau_{nk}}) \stackrel{d}{=} (\xi_1, \dots, \xi_k).$$

But then the same relation must be true for τ_1, \dots, τ_k , since $\tau_{nj} \rightarrow \tau_j$ for each j , as we let $n \rightarrow \infty$. □

We turn to the general result, where the length of the sampled sequence is allowed to be random. Recall that the graph of a random time τ is the random set $\{t < \infty: t = \tau\}$. Recall also the definition of $\stackrel{d}{\subset}$ from Section 1.

Theorem 3.2. Let ξ be a finite or infinite \mathcal{F} -exchangeable sequence with index set I , and let τ_1, τ_2, \dots be $(I \cup \{\infty\})$ -valued predictable stopping times with a.s. disjoint graphs. Put $\nu = \inf\{j \geq 0: \tau_{j+1} = \infty\}$. Then

$$(\xi_{\tau_1}, \dots, \xi_{\tau_\nu}) \stackrel{d}{\subset} \xi. \quad (3)$$

Note that the left-hand side of (3) should be interpreted as the infinite sequence $(\xi_{\tau_1}, \xi_{\tau_2}, \dots)$ when $\nu = \infty$. The above result was obtained for increasing (τ_j) in Theorem 5.1 of [11], by a cumbersome direct argument.

Proof. We may clearly assume that $\tau_j = \infty$ for $j > \nu$. Consider first the case when $\xi = (\xi_1, \dots, \xi_n)$ is finite. Define a new allocation sequence (α'_k) recursively by

$$\alpha'_k = \begin{cases} \alpha_k, & \alpha_k < \infty, \\ \max(\{1, \dots, n\} \setminus \{\alpha'_1, \dots, \alpha'_{k-1}\}), & \alpha_k = \infty. \end{cases}$$

The inverse permutation $(\tau'_1, \dots, \tau'_n)$, given by

$$\{\tau'_j = k\} = \{\alpha'_k = j\}, \quad j, k = 1, \dots, n,$$

will then satisfy the requirements of Theorem 3.1, and moreover

$\tau'_j = \tau_j$ for $j \leq \nu$, so we get

$$(\xi_{\tau_1}, \dots, \xi_{\tau_\nu}, \xi_{\tau'_\nu+1}, \dots, \xi_{\tau'_n}) \stackrel{d}{=} (\xi_1, \dots, \xi_n),$$

proving (3).

It remains to consider the case when ξ is infinite. Defining τ_{nj} as in (2), we get by Theorem 3.1

$$\eta_n \equiv (\xi_{\tau_{n1}}, \xi_{\tau_{n2}}, \dots) \stackrel{d}{=} \xi. \quad (4)$$

Let us further write

$$\nu_n = \inf\{j \geq 0: \tau_{j+1} > n\}, \quad n \in \mathbb{N},$$

and note that $\nu_n \rightarrow \nu$. Note also that

$$(\eta_{n1}, \dots, \eta_{n\nu_n}, \partial, \partial, \dots) = (\xi_{\tau_1}, \dots, \xi_{\tau_{\nu_n}}, \partial, \partial, \dots), \quad (5)$$

since $\tau_{nj} = \tau_j$ for $j \leq \nu_n$. The sequence of pairs (η_n, ν_n) is trivially tight in $R^{\infty} \times \bar{\mathbb{N}}$, so $(\eta_n, \nu_n) \xrightarrow{d}$ some (η', ν') along a suitable subsequence, where $\eta' \stackrel{d}{=} \xi$ by (4). Letting $n \rightarrow \infty$ in (5), and noting that

$$(x_1, x_2, \dots, x_k) \rightarrow (x_1, \dots, x_k, \partial, \partial, \dots)$$

defines a continuous mapping from $R^{\infty} \times \bar{\mathbb{N}}$ to $(R \cup \{\partial\})$, we get in the limit

$$(\eta_1', \dots, \eta_{\nu'}', 2, 2, \dots) \stackrel{d}{=} (\xi_{\tau_1}, \dots, \xi_{\tau_{\nu'}}, 2, 2, \dots),$$

with the usual interpretation in case of infinite ν or ν' . Thus

(3) holds by Lemma 1.1. □

4. Martingale characterizations in continuous time

The main purpose of the present section is to discuss a continuous time counterpart of condition (iii') in Proposition 2.2, and its bearing on the exchangeability of a process on $[0,1]$ or R_+ . Recall from Propositions 2.1 and 2.2 that (iii') is equivalent to exchangeability, for a stationary sequence of random variables.

Before indulging in the main theme, we remark that most methods and results related to the notion of spreadability carry over rather easily to the context of processes on R_+ . In particular, the continuous time ergodic theorem yields an easy direct approach to the continuous time analogue of de Finetti's theorem (though under the assumption of measurability). Much deeper is the spreading characterization of ergodic exchangeable processes on $[0,1]$ in Theorem 3.3 of [11], whose proof employed some martingale techniques akin to those below.

Recall (e.g. from [7]) that a process X on some interval I is a semimartingale (with respect to a standard filtration \mathcal{F}), if X is right-continuous and adapted, and if $X=M+V$ for some local martingale M and some process V with locally finite variation and $V_0=0$. Moreover, X is a special semimartingale, if V can be chosen to be predictable, and in that case the above decomposition is unique and will be called the canonical decomposition of X . Associated with a semimartingale is marked point process ξ_t and a continuous increasing process σ_t^2 , given (for Borel sets $A \subset R$ with $0 \notin A$) by

$$\xi_t A = \sum_{s \leq t} 1_A(\Delta X_s), \quad \sigma_t^2 = \langle X^C, X^C \rangle_t, \quad t \in I, \quad (1)$$

where $X^C = M^C$ is the unique continuous component of the martingale part M . The compensator (dual predictable projection) of ξ will be denoted by $\hat{\xi}$. For special semimartingales, the processes V , σ^2

and $\hat{\xi}$ will be called the local characteristics of X . (Note the slight deviation from common practice, in our definition of the first characteristic V .)

The continuous time counterpart of condition (iii') above is to assume that X is a special semimartingale with absolutely continuous local characteristics, such that the associated densities may be chosen to be martingales. Here absolute continuity is understood to be in the time parameter and with respect to Lebesgue measure λ . In case of $\hat{\xi}$, this means that

$$\hat{\xi}_t A = \int_0^t \mu_s A \, ds, \quad t \in I, \quad (2)$$

for some measure valued process μ , such that $\mu_t A$ is a martingale in t for every fixed A . All martingales in this section are with respect to a fixed standard filtration \mathcal{F} , and we shall always choose their right-continuous versions.

Our plan for this section is first to show in Theorem 4.1 that the above condition is fulfilled for an exchangeable process, under suitable moment conditions. (We shall actually prove slightly more, in preparation for the next section.) We then show in Theorems 4.3 and 4.4 that the stated condition is also sufficient, under appropriate additional assumptions, for a process on R_+ or $[0,1]$ respectively to be exchangeable. As in case of Proposition 2.1, the sufficiency assertion fails without such extra conditions.

In what follows, we shall avoid to use the explicit representation of exchangeable processes stated in Section 1, since the results of this section will then provide a martingale approach to the basic representation formula, at least under moment restrictions.

Theorem 4.1. Any \mathcal{F} -exchangeable process X on $[0,1]$ is a semimartingale, such that σ^2 and $\hat{\xi}$ are absolutely continuous. If moreover $E|X_t| < \infty$, then X is a special semimartingale on $[0,1)$, such that $X-V$ is a martingale on $[0,1)$, while V is absolutely continuous with a martingale density on $[0,1)$. If $EX_t^2 < \infty$, then even σ^2 and $\hat{\xi}$ have martingale densities on $[0,1)$, and $X-V$ is an L_2 -martingale on $[0,1)$, while $\int_0^t \sigma^2 < \infty$.

We shall need the following simple lemma.

Lemma. If X is \mathcal{F} -exchangeable on $[0,1]$, we have for any Borel set $A \subset \mathbb{R}$ with $0 \in A$

$$E[\xi_t | \mathcal{F}_t] < \infty \text{ a.s.}, \quad t \in (0,1).$$

Proof. This is trivial for $t=1$, so we may fix a $t \in (0,1)$.

Letting $n \in \mathbb{Z}_+$ be arbitrary, we get

$$\begin{aligned} E[\xi_t A; \xi_t A = k] &= \sum_{n \geq k} n P\{\xi_1 A = n\} \binom{n}{k} t^k (1-t)^{n-k} \\ &\leq \sum_{n \geq k} n \binom{n}{k} (1-t)^{n-k} = \sum_{n \geq k} a_n < \infty, \end{aligned}$$

since

$$\frac{a_n}{a_{n-1}} = \frac{n^2(1-t)}{(n-1)(n-k)} \rightarrow 1-t < 1.$$

Hence

$$E[E[\xi_1 A | \mathcal{F}_t] \xi_t A] = E[\xi_1 A \xi_t A] < \infty \text{ a.s.},$$

so

$$P[E[\xi_1 A | \mathcal{F}_t] < \infty | \xi_t A] = 1 \text{ a.s.},$$

and the assertion follows by taking expectations on both sides. \square

Proof of Theorem 4.1. Let us first assume that $E|X_t| < \infty$.

Write M for a right-continuous version of the process

$$M_t = E[X_1 - X_t | \mathcal{F}_t] / (1-t), \quad t \in [0,1).$$

Letting $s \leq t$ with $1-s$ and $1-t$ rationally dependent, and using the exchangeability of X , we get

$$E[M_t | \mathcal{F}_s] = E[X_1 - X_t | \mathcal{F}_s] / (1-t) = E[X_1 - X_s | \mathcal{F}_s] / (1-s) = M_s,$$

which extends by right-continuity to arbitrarily related s and t .

Thus M is a martingale on $[0,1)$. In particular,

$$E[X_t - X_s | \mathcal{F}_s] = (1-s)M_s - (1-t)E[M_t | \mathcal{F}_s] = (t-s)M_s, \quad 0 \leq s \leq t < 1.$$

Writing

$$V_t = \int_0^t M_s ds, \quad t \in [0,1), \quad (3)$$

and noting that

$$E \int_0^t |M_s| ds = \int_0^t E|M_s| ds \leq tE|M_t| < \infty, \quad t \in [0,1),$$

since $|M|$ is a submartingale, it is further seen that

$$E[V_t - V_s | \mathcal{F}_s] = E \left[\int_s^t M_u du | \mathcal{F}_s \right] = (t-s)M_s, \quad 0 \leq s \leq t < 1.$$

Thus $X-V$ is a martingale on $[0,1)$. Since V is predictable, this shows that X is a special semimartingale on $[0,1)$ with canonical decomposition $X = (X-V) + V$.

Let us next assume that $EX_t^2 < \infty$. By Jensen's inequality,

$$EM_t^2 \leq E(X_1 - X_t)^2 / (1-t)^2 = E\alpha^2 + \frac{t}{1-t} E(\sigma^2 + \sum \beta_j^2),$$

so by Schwarz' inequality,

$$E \left(\int_0^1 |dV| \right)^2 = E \int_0^1 \int_0^1 |M_s M_t| ds dt \leq \left(\int_0^1 (EM_t^2) dt \right)^2 < \infty.$$

Thus

$$\sup_t E(X_t - V_t)^2 \leq 2EX_1^2 + 2E \left(\int_0^1 |dV| \right)^2 < \infty,$$

so $X-V$ is uniformly integrable and extends to an L_2 -martingale on $[0,1]$. In particular,

$$E[\sigma_1^2 + \int x^2 \xi_1(dx)] = E[X-V, X-V]_1 = E(X_1 - V_1)^2 < \infty,$$

which implies that $E\xi_1 A < \infty$ for Borel sets A with $0 \notin \bar{A}$.

The $\bar{\mathcal{F}}$ -exchangeability of X is clearly inherited by the processes σ_t^2 and ξ_t . We may thus conclude as above that there exists a martingale M' on $[0,1)$ making the process

$$M_t'' = \sigma_t^2 - \int_0^t M_s' ds, \quad t \in [0,1).$$

a martingale. Since M'' is continuous with locally finite variation, it follows that $M''=0$, which proves the desired representation for σ_t^2 . Similarly, ξ_t is compensated by the process $\hat{\xi}_t$ in (2), with μ_t chosen as the measure valued martingale

$$\mu_t A = E[\xi_1 A - \xi_t A | \mathcal{F}_t] / (1-t), \quad t \in [0,1). \quad (4)$$

Let us finally turn to the general case, when there are no moment restrictions. Let us then assume that X is directed by some triple $(\alpha, \sigma^2, \beta)$, and define a new filtration \mathcal{G} by $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{I}$, where $\mathcal{I} = \sigma(\alpha, \sigma^2, \beta)$. (The threefold meaning of σ shouldn't cause any confusion.) Then X remains \mathcal{G} -exchangeable, and moreover $E[X_t^2 | \mathcal{I}] < \infty$ for all t , so it may be seen as above that X is a special \mathcal{G} -semimartingale on $[0,1]$ with canonical decomposition $X = (X-V) + V$, where V is now given by (3) with

$$M_t = E[X_1 - X_t | \mathcal{G}_t] / (1-t), \quad t \in [0,1).$$

Since $\mathcal{F}_t \subset \mathcal{G}_t$ for all t , we may conclude from Theorem 9.19 of Jacod [7] that X remains a semimartingale with respect to \mathcal{F} . Note also that the process σ_t^2 is absolutely continuous, since this is conditionally true, given \mathcal{I} .

To see that $\hat{\xi}$ remains absolutely continuous in the general case, fix a Borel set $A \subset \mathbb{R}$ with $0 \notin \bar{A}$, and let $\mu_t A$ be given by (4) for $t > 0$. Then $\mu_t A < \infty$ a.s. by Lemma 4.2, and we get as before

$$E[\xi_t A - \xi_s A | \mathcal{F}_s] = E\left[\int_s^t \mu_u A du \mid \mathcal{F}_s\right] = (t-s) \mu_s A, \quad 0 < s \leq t \leq 1,$$

which shows that

$$\hat{\xi}_t A - \hat{\xi}_s A = \int_s^t \mu_u A du, \quad 0 < s \leq t \leq 1.$$

Letting $s \rightarrow 0$ and noting that $\hat{\xi}_0 = 0$, we obtain the representation (2). \square

We turn to the results in the opposite direction and begin with the case of processes on \mathbb{R}_+ .

Theorem 4.3. Let X be a special semimartingale on \mathbb{R}_+ with stationary increments and with $X_0=0$ and $E[X, X]_1 < \infty$, and assume that V , σ^2 and $\hat{\xi}$ are absolutely continuous with martingale densities. Then X is exchangeable.

Proof. Assume that X is compensated by the process V in (3), for some martingale M . Then V is integrable, and $X-V$ is a martingale since $E[X, X]_1 < \infty$, so we get for $s \leq t$

$$E[X_t - X_s | \mathcal{F}_s] = E[V_t - V_s | \mathcal{F}_s] = E\left[\int_s^t M_u du | \mathcal{F}_s\right] = \int_s^t E[M_u | \mathcal{F}_s] du = (t-s)M_s.$$

In particular,

$$E[X_{s+h} - X_s | \mathcal{G}_s] = h E[X_{t+1} - X_t | \mathcal{G}_s], \quad 0 \leq s \leq t, h > 0, \quad (5)$$

where $\mathcal{G} = (\mathcal{G}_s)$ denotes the standard filtration generated by X .

Let us now extend X to a right-continuous process X' on \mathbb{R} with stationary increments, and define

$$\mathcal{H}_s = \sigma\{X'_s - X'_{s-h}, h > 0\}, \quad s \geq 0.$$

Let us further write \mathcal{I} for the σ -field generated by all shift-invariant functions of the increments of X' , and note that $\mathcal{I} \subset \mathcal{H}_s$ a.s. for all s . From the stationarity of the increments, it is easily seen that (5) remains true with \mathcal{G}_s replaced by $\mathcal{G}_s \vee \mathcal{H}_s$, and hence also by $\mathcal{G}'_s = \mathcal{G}_s \vee \mathcal{I}$.

Applying the ergodic theorem to the right-hand side of (5) yields

$$E[X_{s+h} - X_s | \mathcal{G}'_s] = h E[X_1 | \mathcal{I}], \quad s, h \geq 0.$$

A similar argument shows that

$$E[\xi_{s+h} A - \xi_s A | \mathcal{G}'_s] = h E[\xi_1 A | \mathcal{I}], \quad s, h \geq 0,$$

$$E[\sigma_{s+h}^2 - \sigma_s^2 | \mathcal{G}'_s] = h E[\sigma_1^2 | \mathcal{I}], \quad s, h \geq 0.$$

Thus the processes

$$X_t - tE[X_1 | \mathcal{I}], \quad \xi_t A - tE[\xi_1 A | \mathcal{I}], \quad t \geq 0,$$

are \mathcal{G}' -martingales, while

$$\sigma_t^2 = t E[\sigma_1^2 | \mathcal{F}], \quad t \geq 0.$$

This means that X is a special \mathcal{G}' -martingale with linear and \mathcal{F} -measurable local characteristics. It then follows as in Theorem 3.57 of Jacod [7] (cf. [8]) that X is conditionally a Lévy process, given \mathcal{F} . Hence X is a mixture of Lévy processes, and therefore exchangeable. \square

We turn to the case of processes on $[0,1]$. Here the stationarity assumption in Theorem 4.3 will be replaced by a suitable constraint at the terminal point. The following result, in conjunction with Theorem 4.1 above, yields a complete martingale characterization of ergodic exchangeable processes on $[0,1]$. The corresponding characterization of finite exchangeable sequences is the martingale version of Proposition 2.3 in [11].

Theorem 4.4. Let X be a uniformly integrable special semi-martingale on $[0,1]$ with $X_0=0$ and non-random X_1 , σ_1^2 and ξ_1 , and such that V , σ^2 and $\hat{\xi}$ are absolutely continuous with martingale densities on $[0,1)$. Then X is ergodic $\bar{\mathcal{F}}$ -exchangeable.

Two lemmas will be needed for the proof.

Lemma 4.5. Let B be a Brownian $\bar{\mathcal{F}}$ -martingale and ξ an $\bar{\mathcal{F}}$ -adapted marked point process with an $\bar{\mathcal{F}}$ -compensator which depends predictably on ξ . Then B and ξ are independent.

Proof. It is clearly enough to show that $E f(B) g(\xi) = 0$ for any bounded measurable functions f and g . By Theorems 11.16 and 12.23 in Jacod [7], there exist predictable processes V and W with

$$\int_0^t V_s^2 ds < \infty, \quad \iint_0^t W_{s,x}^2 d\hat{\xi}_{s,x} < \infty,$$

and such that $f(B) = M_\infty$ while $g(\xi) = N_\infty$, where M and N denote the martingales

$$M_t = \int_0^t V_s dB_s, \quad N_t = \int_0^t \int W_{s,x} d(\xi - \hat{\xi})_{s,x}, \quad t \geq 0.$$

In fact, this is all true with respect to the filtrations generated by B and ξ respectively. But by assumptions, $\hat{\xi}$ remains the \mathcal{F} -compensator of ξ while B remains an \mathcal{F} -martingale. Moreover, M is continuous while N is purely discontinuous, so $M \perp N$, and we get

$$E f(B) g(\xi) = E M_{\infty} N_{\infty} = 0.$$

□

Lemma 4.6. Let X be an exchangeable process on $[0,1]$ directed by $(\alpha, \sigma^2, \beta)$. Then $[X, X]_1 = \sigma^2 + \sum \beta_j^2$.

This result was obtained in [10] by cumbersome arguments. Here is a simple martingale proof.

Proof. We may clearly take $(\alpha, \sigma^2, \beta)$ to be non-random with $\alpha=0$. In that case,

$$M(t) = \frac{X_t}{1-t} = \sigma \frac{B_t}{1-t} + \sum_{j=1}^{\infty} \beta_j \frac{1\{\tau_j \leq t\} - t}{1-t} = \sum_{j=0}^{\infty} M_j(t)$$

is an orthogonal decomposition of the L_2 -martingale on the left, and we get

$$[M, M]_t = \sum_{j=0}^{\infty} [M_j, M_j]_t,$$

which yields a corresponding decomposition of $[X, X]_t$. It remains to notice that $[B, B]_t = t$, since $B_t = W_t - tW_1$ for some Brownian motion W . □

Proof of Theorem 4.4. Let N be a right-continuous version of the martingale density of σ_t^2 . Fixing $s \in [0,1]$, we get a.s.

$$\int_s^1 N_t dt = \sigma_1^2 - \sigma_s^2 = E[\sigma_1^2 - \sigma_s^2 | \mathcal{F}_s] = E\left[\int_s^1 N_t dt \middle| \mathcal{F}_s\right] = \int_s^1 E[N_t | \mathcal{F}_s] dt = (1-s)N_s.$$

Hence N is a.s. continuously differentiable and satisfies the differential equation

$$-N_s = (1-s)N'_s - N_s, \quad 0 \leq s \leq 1,$$

so $N'_s = 0$ a.s., and we get

$$\sigma_t^2 = t\sigma_1^2, \quad t \in [0,1], \quad \text{a.s.} \quad (6)$$

This shows that X^C is a Brownian motion with diffusion rate σ_1^2 .

Let us next assume that ξ is compensated by the process $\hat{\xi}$ in (2) for some measure valued martingale μ . Letting $A \subset \mathbb{R}$ be a Borel set with $0 \notin \bar{A}$, we get a.s. for any $s \in [0, 1]$

$$\xi_1 A - \xi_s A = E[\xi_1 A - \xi_s A | \mathcal{F}_s] = E\left[\int_s^1 \mu_t A dt | \mathcal{F}_s\right] = \int_s^1 E[\mu_t A | \mathcal{F}_s] dt = (1-s)\mu_s A,$$

so by right-continuity,

$$\mu_s = (\xi_1 - \xi_s)/(1-s), \quad s \in [0, 1], \quad \text{a.s.} \quad (7)$$

By Lemma 4.5, it follows in particular that ξ and X^C are independent.

Let us finally assume that X is compensated by the process V in (3), for some martingale M on $[0, 1]$. Then V has integrable variation on compact subintervals of $[0, 1]$, and $X - V$ is a martingale on $[0, 1]$ since

$$[X - V, X - V]_1 = \sigma_1^2 + \int x^2 \xi_1(dx) < \infty,$$

so we get a.s., for any $0 \leq s \leq t < 1$,

$$E[X_t - X_s | \mathcal{F}_s] = E[V_t - V_s | \mathcal{F}_s] = E\left[\int_s^t M_u du | \mathcal{F}_s\right] = \int_s^t E[M_u | \mathcal{F}_s] du = (t-s)M_s.$$

By the continuity of X at 1, the uniform integrability of X , and the right-continuity of X and M , it follows that

$$X_1 - X_s = (1-s)M_s, \quad s \in [0, 1], \quad \text{a.s.},$$

so

$$dX_s = -(1-s)dM_s + M_s ds = d(X_s - V_s) + dV_s,$$

and therefore

$$V_t = \int_0^t M_s ds = -\int_0^t ds \int_0^s \frac{d(X_u - V_u)}{1-u}, \quad t \in [0, 1], \quad \text{a.s.} \quad (8)$$

Let us now consider instead an ergodic exchangeable process X' on $[0, 1]$ directed by (X_1, σ_1^2, ξ_1) . Theorem 4.1 shows that X is a special semimartingale with respect to the induced standard filtration, and that the local characteristics of X' are absolutely continuous with martingale densities on $[0, 1]$. Since X' is further L_2 -bounded and hence uniformly integrable, everything said above for X applies equally to X' . In particular, (6)-(8) remain true

for the processes σ'^2 , μ' , ξ' and V' associated with X' .

As for X above, it is seen that X'^C is a Brownian motion independent of ξ' , and since $\sigma_1'^2 = \sigma_1^2$ by Lemma 4.6, the diffusion rate is the same as for X^C . Since the functional dependence in (7) is the same for ξ and ξ' , it may further be seen from Theorem 3.42 of Jacod [7] that $\xi' \stackrel{d}{=} \xi$, so $\xi' - \hat{\xi}' \stackrel{d}{=} \xi - \hat{\xi}$, and hence $X' - V' \stackrel{d}{=} X - V$. We may next infer from the two versions of (8) that $(X' - V', V') \stackrel{d}{=} (X - V, V)$, which implies that $X' \stackrel{d}{=} X$. Thus X is exchangeable.

To reach the stronger conclusion of \mathcal{F} -exchangeability, it suffices to fix an arbitrary $s \in [0, 1]$, and to check that the preceding arguments apply to the conditional distribution of X on the interval $[s, 1]$, given the σ -field \mathcal{F}_s . We omit the details of this verification. \square

We conclude this section with some remarks. First we show by an example that the last two lemmas are false without the additional assumptions of stationarity of the increments or of non-randomness of the local characteristics at the terminal point. Let us then take ξ to be a simple point process on R_+ , such that the restriction to $[0, 1]$ is a mixture of Poisson processes with intensities 1 or 0, where each possibility is chosen with probability $\frac{1}{2}$. On the remaining interval, we choose ξ to be Poisson with intensity 1 or $(1+e)^{-1}$, depending on whether $\xi_1 > 0$ or not. It is then easy to verify that the density of $\hat{\xi}$ is a martingale. But ξ fails to be exchangeable, since $P\{\xi_1 = 0\} = \frac{1}{2}(1+e^{-1})$, while

$$P\{\xi_2 - \xi_1 = 0\} = \frac{1}{2}(1-e^{-1})e^{-1} + \frac{1}{2}(1+e^{-1})e^{-(1+e)^{-1}}.$$

As a second remark, we shall sketch how the above results may be combined to yield a simple martingale approach to the representation theorem for exchangeable processes on $[0, 1]$. Let us then assume that the process X on $[0, 1]$ is exchangeable, integrable, and continuous

in probability at every fixed point. Then $Y_t = (X_1 - X_t)/(1-t)$ is seen to be a martingale on $[0,1)$, so X must have a version in $D[0,1]$. Note also that $[X, X]_1 < \infty$ a.s., since the exchangeability of X carries over to $[X, X]$. Since X remains conditionally exchangeable, given the triple $(X_1, [X, X]_1, \xi_1)$, we may assume that X_1 , $[X, X]_1$ and ξ_1 are all fixed. Then $EY_t^2 < \infty$, so even $EX_t^2 < \infty$. It may hence be seen as in Theorem 4.1 that X is a special semimartingale on $[0,1]$, whose local characteristics are absolutely continuous with martingale densities on $[0,1)$. Note also that X is uniformly integrable on $[0,1]$, since EX_t^2 is bounded. The hypotheses of Theorem 4.4 are then fulfilled, so the desired representation formula follows as in the proof of that theorem.

5. Predictable transformations in continuous time

Our aim in the present section is to prove continuous time versions of Theorems 3.1 and 3.2. Let us then fix a standard filtration \mathcal{F} , and recall from Section 1 the definition of an \mathcal{F} -exchangeable process. Recall also our definition of the transformed processes XV^{-1} . The stochastic integrals occurring in the definition exist by Lemma 5.2 below.

Theorem 5.1. Let X be an \mathcal{F} -exchangeable process on $I=[0,1]$ or \mathbb{R}_+ , and let the process V on I be \mathcal{F} -predictable with values in $I \cup \{\infty\}$, and such that $\lambda V^{-1} = \lambda$ a.s. on some interval $J \subset I$ containing 0. Then

$$XV^{-1} \stackrel{d}{=} X \text{ on } J. \quad (1)$$

Since $(XV^{-1})_t$ is only defined a.s. for each t , (1) should be interpreted as a relation between the finite-dimensional distributions. However, (1) implies that XV^{-1} has a right-continuous version with left hand limits, and for the latter there is clearly equality between the distributions on the Skorohod space $D(J)$.

Two lemmas will be needed for the proof.

Lemma 5.2. Let X be an \mathcal{F} -exchangeable process on $I=[0,1]$ or \mathbb{R}_+ , and let $A \subset I$ be predictable with $\lambda A < \infty$ a.s. Then the stochastic integral $\int_A dX$ exists. Moreover $\int_{A_n} dX \xrightarrow{P} 0$ whenever $A_1, A_2, \dots \subset I$ are predictable with $\lambda A_n \xrightarrow{P} 0$.

Proof. Let us first consider the case of processes on $[0,1]$. Changing the filtration, as in the proof of Theorem 4.1, and applying Theorem 9.26 of Jacod [7], we may reduce the discussion to the case when X is conditionally ergodic exchangeable, given \mathcal{F}_0 . But then Theorem 4.1 shows that X is a special semimartingale on $[0,1]$, with a canonical decomposition $X=M+V$ such that both $\langle M, M \rangle$ and V are absolutely continuous. The existence of the stochastic

integrals $\int 1_A dX$ follows immediately from this. To prove the convergence assertion, consider an arbitrary subsequence such that $\lambda A_n \rightarrow 0$ a.s. Then

$$\lim_{n \rightarrow \infty} \int 1_{A_n} |dV| = \lim_{n \rightarrow \infty} \int 1_{A_n} d\langle M, M \rangle = 0 \text{ a.s.,}$$

which yields the desired conclusion.

For processes on R_+ , we may reduce as above to the case when X is conditionally a Lévy process, given \mathcal{F}_0 . In this case we get a decomposition $X=M+V+J$, where V is linear, while M is a local martingale such that $\langle M, M \rangle$ is linear, and J is conditionally a compound Poisson process. For integrals with respect to $M+V$, the existence and convergence assertions follow as before, so it remains only to consider integrals with respect to J . Letting N denote the associated mixed Poisson process, it is seen from the results for $M+V$ that $\int 1_A dN$ exists and that $\int 1_{A_n} dN \xrightarrow{P} 0$. Since the integrals $\int 1_A dN$ and $\int 1_A |dJ|$ are simultaneously finite and simultaneously zero, the corresponding statements then follow for J . \square

Lemma 5.3. Let A_1, \dots, A_n be disjoint predictable sets in $[0, 1]$ of equal length n^{-1} , and fix an $\varepsilon > 0$. Then there exist some integer $m \in \mathbb{N}$ and some disjoint predictable sets A'_1, \dots, A'_n of equal length n^{-1} , such that each A'_j is a union of intervals $((j-1)m^{-1}, jm^{-1}]$, and such that moreover

$$\sum_{j=1}^n (\lambda \wedge P)(A_j \Delta A'_j) < \varepsilon. \quad (2)$$

Proof. Recall that the restriction of the predictable σ -field to the interval $(0, 1]$ is generated by the stochastic intervals of the form $(\sigma, \tau]$, where σ and τ are rational valued stopping times in $[0, 1]$. From this it follows easily by a monotone class argument that any predictable set in $[0, 1]$ can be approximated arbitrarily closely in measure $\lambda \wedge P$ by a predictable union of intervals

$I_j = ((j-1)m^{-1}, jm^{-1}]$, with m a fixed multiple of n . This implies in particular that the process $\sum j 1_{A_j}$ can be approximated in $L_1(\lambda \times P)$ by a process of the form $\sum j 1_{U_j}$, where U_1, \dots, U_n are disjoint predictable interval unions as above with union $(0, 1]$. Taking the error to be less than ϵ/n , we get

$$E \sum_{j=1}^n |\lambda U_j - n^{-1}| \leq \sum_{j=1}^n (\lambda \times P)(A_j \Delta U_j) < \epsilon/n. \quad (3)$$

Let us now define the variables $\alpha_1, \dots, \alpha_m$ by the condition

$$\alpha_j = k \quad \text{if} \quad I_j \subset U_k, \quad j=1, \dots, m, \quad k=1, \dots, n,$$

and put recursively

$$\alpha'_j = \begin{cases} \alpha_j, & \text{if } \sum_{i < j} 1\{\alpha'_i = \alpha_j\} < m/n, \\ \min\{k: \sum_{i < j} 1\{\alpha'_i = k\} < m/n\}, & \text{otherwise.} \end{cases}$$

It is then easily seen that the sets

$$A'_k = \bigcup_{j=1}^m \{I_j: \alpha'_j = k\}, \quad k=1, \dots, n,$$

are disjoint predictable unions of I_1, \dots, I_m of equal length n^{-1} .

Moreover, (2) follows from (3) and the fact that, by construction

$$\sum_{j=1}^n \lambda(U_j \Delta A'_j) \leq (n-1) \sum_{j=1}^n |\lambda U_j - n^{-1}|. \quad \square$$

Proof of Theorem 5.1. Let us first assume that $I=J=[0, 1]$. By the right-continuity of X and by dominated convergence for stochastic integrals, it is then enough to prove that, for fixed n ,

$$(\xi_{n1}, \dots, \xi_{nn}) \stackrel{d}{=} (\eta_{n1}, \dots, \eta_{nn}), \quad (4)$$

where ξ_{nj} and η_{nj} denote the increments of X and XV^{-1} respectively over the interval $I_{nj} = ((j-1)n^{-1}, jn^{-1}]$. Note that

$$\eta_{nj} = \int 1_{A_{nj}} dX, \quad j=1, \dots, n,$$

where A_{nj} denotes the predictable random set

$$A_{nj} = \{t \in I: V_t \in I_{nj}\}, \quad j=1, \dots, n.$$

Consider first the case when each set A_{nj} is a union of m/n randomly selected intervals I_{mk} , for some multiple m of n . Then

$$\eta_{nj} = \sum_{k=1}^{m/n} \xi_{m, \tau_{jk}}, \quad j=1, \dots, n,$$

for some functions

$$\tau_{j1} < \dots < \tau_{j, m/n}, \quad j=1, \dots, n.$$

Write \mathcal{G} for the discrete filtration $\mathcal{G}_j = \mathcal{F}_{j^{m-1}}$, $j=0, \dots, m$, and note that $(\xi_{m1}, \dots, \xi_{mn})$ is \mathcal{G} -exchangeable, while the τ_{jk} are \mathcal{G} -predictable stopping times. Hence Theorem 3.1 yields

$$(\xi_{m, \tau_{11}}, \dots, \xi_{m, \tau_{n, m/n}}) \stackrel{d}{=} (\xi_{m1}, \dots, \xi_{mn}),$$

and (4) follows by a suitable summation on each side.

In the case of general sets A_{nj} , it is seen from Lemma 5.3 that A_{n1}, \dots, A_{nn} can be approximated in $(\lambda \times P)$ -measure by disjoint predictable sets B_{m1}, \dots, B_{mn} of equal length n^{-1} , and such that each B_{mj} is a union of randomly selected intervals I_{mk} . As shown above, we get for each m

$$(\int_{B_{m1}} dX, \dots, \int_{B_{mn}} dX) \stackrel{d}{=} (\xi_{n1}, \dots, \xi_{nn}). \quad (5)$$

Moreover, it is seen from Lemma 5.2 that

$$\int_{B_{mj}} dX \xrightarrow{P} \int_{A_{nj}} dX \quad \text{as } m \rightarrow \infty, \quad j=1, \dots, n.$$

Hence (5) remains true with the sets B_{mj} replaced by A_{nj} , and the assertion follows.

Retaining $I=[0,1]$, we turn to the case when $J=[0,p]$ for some $p < 1$. We may then construct another predictable process U on I by putting

$$U_t = \begin{cases} V_t, & V_t \leq p, \\ 1 - \lambda \{s \leq t: V_s > p\}, & V_t > p. \end{cases}$$

Noting that $\lambda U^{-1} = \lambda$, we may conclude as above that $XU^{-1} \stackrel{d}{=} X$. Since moreover $XU^{-1} = XV^{-1}$ on J , the assertion follows.

If instead $I = R_+$ while $J = [0, 1]$, say, we may define the processes

$$U_n(t) = \begin{cases} V_t, & t \leq n, \\ \inf\{s \in J: s - \lambda\{r \leq n: V_r \leq s\} = t - n\}, & t > n. \end{cases}$$

Then each U_n is predictable with $\lambda U_n^{-1} = \lambda$ on J and $U_n^{-1}J \subset [0, n+1]$, so the result for processes on finite intervals yields $XU_n^{-1} \stackrel{d}{=} X$ on J . Since moreover

$$\lambda\{s > n: U_n(s) \leq t\} = \lambda\{s > n: V_s \leq t\} \rightarrow 0 \text{ a.s.}$$

by dominated convergence, as $n \rightarrow \infty$ for fixed $t \in J$, we get by

Lemma 5.2

$$(XU_n^{-1})_t - (XV^{-1})_t = \int_n^\infty (1\{U_n(s) \leq t\} - 1\{V_s \leq t\}) dx \xrightarrow{P} 0,$$

so the finite-dimensional distributions of XU_n^{-1} on J tend weakly to those of XV^{-1} , and the assertion follows again. \square

With regard to applications, it is useful to extend Theorem 5.1 to the case when V is only measure preserving on some interval of random length. Recall from Lemma 5.2 that XV^{-1} is defined at t if $\lambda\{s: V_s \leq t\} < \infty$ a.s. In general, it can be defined by localization on the random set $A_t = \{\lambda\{s: V_s \leq t\} < \infty\}$.

Theorem 5.4. Let X be an \mathcal{F} -exchangeable process on $I = [0, 1]$ or R_+ , and let the process V on τ be \mathcal{F} -predictable with values in $I \cup \{\infty\}$. Put

$$\zeta = \sup\{t \geq 0: \lambda\{s: V_s \leq t\} = t\},$$

and let Y denote the restriction of XV^{-1} to $[0, \zeta)$. Then $Y \stackrel{d}{=} X$.

Note that the process Y is well-defined on $[0, \zeta)$, since $\{\zeta > t\} \subset A_t$ for each t . The theorem states that Y can be extended to a process on R_+ with the same finite-dimensional distributions as X . As before, this yields the existence on $[0, \zeta)$ of a right-continuous version with left-hand limits.

The core of our proof consists in constructing a measure preserving process V' , to replace V in the definition of V . This will essentially be accomplished by the next two lemmas.

Lemma 5.5. Let $f: R_+ \rightarrow \bar{R}_+$ be measurable with $\lambda f^{-1} \leq \lambda$ on some interval $[0, p)$. Define

$$g_t = \sup\{x \geq 0: \lambda\{s \leq t: f_s \in \cdot\} \leq \lambda \text{ on } [0, x)\}, \quad t \geq 0,$$

and put $h = f + \infty \cdot 1_{\{f > g\}}$. Then $\lambda h^{-1} \leq \lambda$ on R_+ , and we have $h = f$ on the set where $f < p$. If f is a predictable process, then so is h .

Proof. By a monotone class argument, $\lambda h^{-1} \leq \lambda$ follows if we can prove that $\lambda h^{-1} I \leq \lambda I$ for every finite interval $I = [a, b)$ with dyadic endpoints. For this purpose, define

$$g_n(t) = 2^{-n} [2^n g_t], \quad t \geq 0, n \in \mathbb{N},$$

and note that $g_n \uparrow g$. Hence

$$\lambda\{s: f_s \in I, f_s < g_n(s)\} \uparrow \lambda\{s: f_s \in I, f_s < g_s\} = \lambda h^{-1} I,$$

so it is enough to show, for large n , that the left-hand side is bounded by λI . We may then assume that $I = [(k-1)2^{-n}, k2^{-n})$ for some k .

Put

$$t = \inf\{s \geq 0: g_s < k2^{-n}\},$$

and note that $g_t \geq k2^{-n}$, since g is left-continuous. Note also that $g_n(s) \leq (k-1)2^{-n}$ for all $s > t$. Using the definition of g , we hence obtain

$$\lambda\{s: f_s \in I, f_s < g_n(s)\} = \lambda\{s \leq t: f_s \in I\} \leq \lambda I,$$

as required. The second statement follows immediately from the fact that $g > p$.

If f is predictable, then $\lambda\{s \leq t: f \in I\}$ is \mathcal{F}_t -measurable for every t and I , by Fubini's theorem, and hence so is the event

$$\{g_t \geq x\} = \bigcap \{\lambda\{s \leq t: f_s \in I\} \leq \lambda I\},$$

where the intersection extends over all rational intervals I in

$[0, x)$. Thus g is adapted, and since it is also left-continuous, it must be predictable, like f . Hence so is h . \square

Lemma 5.6. Let $f: [0, 1] \rightarrow [0, 1] \cup \{\infty\}$ be measurable with $\lambda f^{-1} \leq \lambda$ on $[0, 1]$ and $\lambda f^{-1} = \lambda$ on some interval $[0, p]$. Define

$$g_t = \inf\{x \in [0, 1]: x = 1 - \lambda\{s \leq t: f_s > x\}\}, \quad t \in [0, 1],$$

and put $h = f \wedge g$. Then $\lambda h^{-1} = \lambda$ on $[0, 1]$, and $h \wedge p = f \wedge p$ a.e. λ . If f is a predictable process, then so is $h_t^- = f_t \wedge g_{t-}$.

Proof. Since $\lambda\{s \leq t: f_s > x\}$ is continuous in x for fixed t , with values ≤ 1 at 0 and $\lambda\{s \leq t: f_s = \infty\} \geq 0$ at 1, the set of solutions to the equation

$$x = 1 - \lambda\{s \leq t: f_s > x\} \quad (6)$$

forms a non-empty closed set. In particular, g_t solves (6) at t . Note also that g_t decreases from 1 to 0. Substituting $x = g_t$ in (6) and letting $t \rightarrow t'$ from above and below, it follows easily that both $g_{t',+}$ and $g_{t',-}$ solve (6) at t' , and the same must then be true for every intermediate value. This shows the existence, for every $x \in [0, 1]$, of some $t = t_x \in [0, 1]$, such that x solves (6) at t and moreover $g_{t,+} \leq x \leq g_{t,-}$.

Let us now assume that x is such that $\lambda\{s: g_s = x\} = 0$. Then

$$\lambda\{s: h_s > x\} = \lambda\{s < t: h_s > x\} = \lambda\{s < t: f_s > x\} = 1 - x.$$

Since the set of x 's with the above property is dense in $[0, 1]$, it follows that $\lambda h^{-1} = \lambda$. In particular we get $\lambda(h \wedge p)^{-1} = \lambda(f \wedge p)^{-1}$, so $\lambda(f \wedge p - h \wedge p) = 0$. Since the integrand is non-negative, it follows that $h \wedge p = f \wedge p$ a.e. λ .

If f is predictable, then $\lambda\{s \leq t: f_s > x\}$ is \mathcal{F}_t -measurable for every t and x , and hence so is the event

$$\{g_t \geq y\} = \bigcap \{x < 1 - \lambda\{s \leq t: f_s > x\}\},$$

where the intersection extends over all rational numbers x in $[0, y)$.

Thus g is adapted, so g_{t-} is predictable, like f . Hence so is h^- . \square

Next one needs to verify that the new predictable process V obtained through the last two lemmas gives rise to the same process Y on the random interval $[0, \zeta)$.

Lemma 5.7. Let U be another predictable process on I , and assume that

$$\lambda\{s: U_s \neq V_s, U_s \wedge V_s < \zeta\} = 0 \quad \text{a.s.}$$

Then XU^{-1} and XV^{-1} represent the same process on $[0, \zeta)$.

Proof. Fix $t \in I$, and define the stopping time

$$\tau = \inf\{r \geq 0: \lambda\{s \leq r: U_s \neq V_s, U_s \wedge V_s \leq t\} > 0\}.$$

Then

$$\lambda(\{U_s \leq t, s \leq \tau\} \Delta \{V_s \leq t, s \leq \tau\}) = 0 \quad \text{a.s.},$$

so by Lemma 5.2 we get for all $n \in \mathbb{N}$

$$\begin{aligned} \int_0^{\tau \wedge n} 1_{\{V_s \leq t\}} dX_s &= \int_0^{\tau \wedge n} 1_{\{V_s \leq t, s \leq \tau \wedge n\}} dX_s \\ &= \int_0^{\tau \wedge n} 1_{\{U_s \leq t, s \leq \tau \wedge n\}} dX_s = \int_0^{\tau \wedge n} 1_{\{U_s \leq t\}} dX_s, \end{aligned}$$

which shows that $(XU^{-1})_t = (XV^{-1})_t$ a.s. on the set $\{\tau = \infty, \zeta > t\}$. It remains to notice that a.s.

$$\{\tau = \infty\} = \{\lambda\{s: U_s \neq V_s, U_s \wedge V_s \leq t\} = 0\} \supset \{\zeta > t\}. \quad \square$$

We shall also need an extension of Lemma 5.2, to deal with convergence of our specific stochastic integrals on events of the form $\{\zeta > t\}$. Note that the result is trivial when ζ is \mathcal{F}_n -measurable for some n . Write \mathcal{B}_n for the Borel σ -field on the interval (n, ∞) .

Lemma 5.8. Let $I = \mathbb{R}_+$, and assume $A_n \in \mathcal{F}_n \times \mathcal{B}_n$, $n \in \mathbb{N}$, and $F \in \mathcal{F}_\infty$ to be such that

$$1_F \lambda_{A_n} \xrightarrow{P} 0.$$

Then

$$1_F \int 1_{A_n} dX \xrightarrow{P} 0.$$

Proof. Let \mathcal{T} denote the tail σ -field for the increments of X , and write

$$P_n = P[\cdot | \mathcal{F}_n \vee \mathcal{T}], \quad E_n = E[\cdot | \mathcal{F}_n \vee \mathcal{T}], \quad n \in \mathbb{N}.$$

Assuming without loss that $\lambda_{A_n} \rightarrow 0$ a.s., and writing $g(x) = 1 - e^{-|x|}$, we get

$$1_F E_n g(\lambda_{A_n}) = 1_F g(\lambda_{A_n}) \rightarrow 0 \quad \text{a.s.},$$

since λ_{A_n} is \mathcal{F}_n -measurable. Hence $\lambda_{A_n} \xrightarrow{P_n} 0$ a.s. on F , where $\xrightarrow{P_n}$ denotes convergence in probability with respect to the conditional law. Now $X_n = \theta_n \circ X - X(n)$ is conditionally a Lévy process independent of n (cf. [11]), and moreover

$$I_n = \int 1_{A_n} dX = \int 1_{A_n - n} dX_n, \quad n \in \mathbb{N}.$$

where the definitions of stochastic integrals are the same under P and P_n (cf. Theorem 9.26 in [7]). Thus Lemma 5.2 shows that even $I_n \xrightarrow{P_n} 0$ a.s. on F , so by dominated convergence

$$E[P_n F \cdot g(I_n)] = E[1_F E_n g(I_n)] \rightarrow 0,$$

which means that $1_F I_n \xrightarrow{P} 0$. It remains to notice that $P_n F \rightarrow 1_F$ a.s. by martingale theory. □

We shall finally need an elementary result on weak convergence in the function space $D(\mathbb{R}_+)$ with the Skorohod-Stone topology (cf. [8]). Recall that k_t denotes killing at t . The coffin state ∂ is regarded as isolated in $R \cup \{\partial\}$.

Lemma 5.9. Let X, X_1, X_2, \dots and $\tau, \tau_1, \tau_2, \dots$ be random elements in $D(R_+)$ and R_+ respectively, and assume that $(X_n, \tau_n) \xrightarrow{d} (X, \tau)$ with respect to the Skorohod-Stone topology on $D(R_+)$. Then

$$k_{p\tau_n} \circ X_n \xrightarrow{d} k_{p\tau} \circ X \quad (7)$$

for $p \in [0, 1]$ a.e. λ .

Proof. It is easy to check that the mapping $(x, t) \rightarrow k_t x$ from $D(R_+) \times \bar{R}_+$ to $D(R_+, R_+ \setminus \{0\})$ is continuous at $t = \infty$, and also at every (x, t) with $t < \infty$ and such that x is continuous at t . Thus (7) is true for every $p \in [0, 1]$ such that X is continuous at $p\tau$ a.s. on $\{\tau < \infty\}$. But conditionally on that event, the process $Y_p = X_{p\tau}$, $p \in [0, 1]$, has paths in $D[0, 1]$, so Y has an at most countable set of fixed discontinuities (cf. [2]). It remains to notice that X is continuous at $p\tau$ iff Y is continuous at p . \square

Proof of Theorem 5.4. Let us first assume that $I = [0, 1]$. By Lemmas 5.5 and 5.6, there exists a predictable process V' with $\lambda V'^{-1} = \lambda$ a.s., and such that $V'_s = V_s$ a.e. $\lambda \times P$ on the set $\{V_s \wedge V'_s < \zeta\}$. Putting $Y' = X V'^{-1}$, it is seen from Lemma 5.7 that Y and Y' represent the same process on $[0, \zeta)$. Since moreover $Y' \stackrel{d}{=} X$ by Theorem 5.1, it follows that $Y \stackrel{d}{=} X$.

Let us turn to the case when $I = R_+$. By Lemmas 5.5 and 5.7, we may assume that $\lambda V^{-1} \stackrel{d}{=} \lambda$ a.s. on P_+ . For every $n \in \mathbb{N}$, we define a predictable process U_n by

$$U_n(t) = \begin{cases} V_t, & t \leq V_t \leq n, \\ \infty, & t \leq n < V_t, \\ \inf\{s \geq 0: s - \lambda\{r \leq n: V_r \leq s \wedge n\} = t - n\}, & t > n. \end{cases}$$

Since clearly $\lambda U_n^{-1} = \lambda$ a.s., we have $Y_n = X U_n^{-1} \stackrel{d}{=} X$ by Theorem 5.1.

It follows in particular that the sequence of pairs (Y_n, ζ) is tight

in $D(R_+) \times \bar{R}_+$, so $(Y_n, \zeta) \xrightarrow{d} (X', \zeta')$ along some subsequence $N' \subset N$, for some process $X' \stackrel{d}{=} X$ and some random variable $\zeta' \stackrel{d}{=} \zeta$. By

Lemma 5.9 it follows that

$$k_{p\zeta} \circ Y_n \xrightarrow{d} k_{p\zeta'} \circ X', \quad p \in [0,1] \text{ a.e. } \lambda \quad (n \in N'). \quad (8)$$

On the other hand, we have for fixed $t \geq 0$

$$\lambda\{s > n: U_n(s) \leq t\} = \lambda\{s > n: V_s \leq t\} \rightarrow 0 \quad \text{a.s. on } \{t < \zeta'\},$$

so Lemmas 5.2 and 5.8 yield

$$1_{\{t < \zeta'\}}(Y(t) - Y_n(t)) = \int_n^\infty (1_{\{V_s \leq t\}} - 1_{\{U_n(s) \leq t\}}) dX_s \xrightarrow{P} 0,$$

which shows that

$$(k_{p\zeta} \circ Y_n)_t \xrightarrow{P} (k_{p\zeta} \circ Y)_t, \quad p \in [0,1], \quad t \in R_+.$$

Comparing this with (8) yields

$$k_{p\zeta} \circ Y \stackrel{d}{=} k_{p\zeta'} \circ X', \quad p \in [0,1] \text{ a.e. } \lambda.$$

so the same relation must be true for $p=1$. But then $Y \stackrel{d}{=} X' \stackrel{d}{=} X$ by Lemma 1.1. □

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