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Conditions are formulated which guarantee the existence of positive solutions for systems of the form:

$$-\Delta u_1 + f_1(u_1,\ldots,u_n) = m_1$$
$$\vdots$$
$$-\Delta u_n + f_n(u_1,\ldots,u_n) = m_n$$

where Δ is the Laplacian (with Dirichlet boundary conditions) on an open domain in R^d, and where each m_i is a positive measure. The main tools used are probabilistic potential theory, Markov processes, and an iterative scheme which is not a generalization of the one used for

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POSITIVE SOLUTIONS OF SYSTEMS OF SEMILINEAR ELLIPTIC EQUATIONS: THE PENDULUM METHOD

by

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Summary: Conditions are formulated which guarantee the existence of positive solutions for systems of the form

$$-\Delta u_1 + f_1(u_1,...,u_n) = \mu_1$$

$$-\Delta u_2 + f_2(u_1,...,u_n) = \mu_2$$

$$\vdots \qquad \vdots \qquad\qquad \vdots \quad \vdots$$

$$-\Delta u_n + f_n(u_1,...,u_n) = \mu_n$$

where Δ is the Laplacian (with Dirichlet boundary conditions) on an open domain in R^d, and where each μ_i is a positive measure. The main tools used are probabilistic potential theory, Markov processes, and an iterative scheme which is not a generalization of the one used for quasi-monotone systems. Quasi-monotonicity is not assumed and new results are obtained even for the case where $\partial f_k/\partial x_j > 0$ for every k and j.

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0. INTRODUCTION

In this article, we introduce conditions which are sufficient to guarantee existence of a solution for a system of semilinear equations of the form

(0.1)
$$-\Delta u_1 + f_1(u_1,\dots,u_n) = \mu_1$$
$$-\Delta u_2 + f_2(u_1,\dots,u_n) = \mu_2$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$-\Delta u_n + f_n(u_1,\dots,u_n) = \mu_n$$

on an open (bounded or unbounded) domain E in R^d, with Dirichlet boundary conditions. We assume each measure μ_1 is positive, and we show that the solution is positive; i.e. $u_1 \geq 0$ for every i. In fact, we give a constructive procedure for solving (0.1) with general elliptic operators in place of the Laplacian, and our methods even allow us to replace the Laplacian with certain integro-differential operators. See section 4 for other extensions. In (0.1), the (μ_k) are assumed to be positive measures, and each $f_k:R^d \to R$ is continuous. The functions f_k need not satisfy any special conditions such as $\partial f_k / \partial x_j > 0$, but our methods yield new results even in that case. Previous approaches to solving (0.1) in a constructive way seem to have relied mainly on assuming that (0.1) is a quasi-monotone system and on using the method of sub- and super-solutions. For discussions of this approach, see [10] and [11]. Since we do not assume the system is quasi-monotone, such methods do not apply, and even if the system is assumed to be quasi-monotone, it is not clear how to obtain upper and lower solutions in general.

We introduce an iterative scheme in which the iterates do not increase or decrease monotonically, but instead, they oscillate. That is, we produce a sequence (u_k) so that $u_2 \leq u_4 \leq \ldots \leq u_{2k} \leq u_{2k+1} \leq \ldots \leq u_3 \leq u_1$. One might call this a pendulum method, since the iterates swing back and forth over the solution. Much more is known about solving one equation of the form $-\Delta u + f(u) = \mu$ than is known about solving (0.1). In his survey article [8], Lions discusses the single equation $-\Delta u + f(u) = \mu$, and he points out that it is important for applications to extend the results for the single equation to systems. See also [1], [4], and [7] for the case of one equation.

For general elliptic operators, positive measures and arbitrary functions f_k, our hypotheses and results are a bit complicated to state, and we refer the reader to sections 1, 2, and 3. (In particular, Theorems (2.9) and (3.1) contain the main results.) In this introduction, we content ourselves with describing the hypotheses, methods and results in the case where we keep the Laplacian with Dirichlet boundary conditions on a connected domain E in (0.1), $f_k(0,\ldots,0) = 0$, and where

(0.2) $\qquad \mu_k(dx) = g_k(x)dx$ with $g_k > 0$ on E,

and for every k and j, there is a constant $\rho_{kj} > 0$ so that

(0.3) $\qquad \partial f_k / \partial x_j(u_1, u_2, \ldots, u_n) \geq \rho_{kj}$ for $u_1 \geq 0, \ldots, u_n \geq 0$.

We emphasize that assumptions (0.2) and (0.3) are for the purposes of discussion in this section only: in sections 2 and 3, we treat a much more general situation. Hypothesis (0.3) is a type of quasi-monotonicity assumption. But even in this

special case, one would need upper and lower solutions in order to apply the monotone iteration schemes of quasi-monotone systems. It is not at all clear how to obtain upper and lower solutions, in general, and we completely avoid this problem. With hypotheses (0.2) and (0.3) in force, we make two crucial assumptions. Here is the first one. Let $V = (-\Delta)^{-1}$ on E, and let M be Lebesgue measure on E. Assume

(0.4)   (i) $Vg_k < \infty$ a.e. (M) for every k

          (ii) For some k, $M(Vg_k > Vf_k(Vg_1,...,Vg_n)) > 0$.

Part (i) seems quite a reasonable assumption. From the potential theory point of view, part (ii) is interesting. Part (ii) is equivalent to saying that the measure $g_k dM$ is not a balayage of the measure $f_k(Vg_1,...,Vg_n)dM$. This is quite often verifiable. For example, when we pass to discussing general measures μ in section 2, the analogue of this requirement will be that "μ is not a balayage of the measure $G(V\mu)dM$" (see (2.4)). It is interesting to note that this is automatically guaranteed if the dimension of the space d is greater than or equal to 2, and if the measure μ charges a point. Condition (ii) also implies $M(Vf_k(Vg_1,...,Vg_n)<\infty)>0$, and it is well-known that we then have $Vf_k(Vg_1,...,Vg_n)<\infty$ a.e. (M).

So if this first crucial assumption (0.4) holds, do the following. Set

$$J_k(u_1,u_2,...,u_n) = f_k(u_1,u_2,...,u_n) \text{ if } u_1 \geq 0,...,u_n \geq 0,$$
$$= 0 \quad \text{otherwise.}$$

Set $u_k^1 = V[g_k - J_k(g_1,...,g_n)]$, and define inductively $u_k^{j+1} = V[g_k - J_k(u_1^j,...,u_n^j)]$.

Then it is not difficult to see that $u_k^2 \leq u_k^4 \leq \ldots \leq u_k^{2j} \leq u_k^{2j+1} \leq u_k^3 \leq u_k^1$, so u_k^{2j} increases to a function a_k and u_k^{2j+1} decreases to a function b_k. It is easy to show that $a_k = V[g_k - J_k(b_1, \ldots, b_n)]$ and $b_k = V[g_k - J_k(a_1, \ldots, a_n)]$.

In addition, for each k, either $a_k < b_k$ on all of E,

or $a_k = b_k$ on all of E. It is easy to show that $b_k \geq 0$. What is a bit more delicate is the fact that $b_k > 0$ on E. To prove this, we use $g_k > 0$ and Brownian motion (or, in general, the Markov process associated with the elliptic or integro-differential operator). If $a_k = b_k$ for every k, then

$b_k = V[g_k - J_k(b_1, \ldots, b_n)]$. Since $b_k \geq 0$, we may rewrite this as $b_k = V[g_k - f_k(b_1, \ldots, b_n)]$, and we are done (take the Laplacian of both sides -- in the sense of distributions). If $a_k < b_k$ for some k, we must restart the iteration. Now we need our second crucial hypothesis. <u>Assume</u>

(0.5)   There is a nonnegative function F_k on E and numbers δ_k and

π_k so that

(i) $\{F_k > 0\}$ is contained in a compact set K_k which is contained in $\{a_k > \delta_k\} \cap \{b_k < \pi_k\}$

(ii) $F_k \leq J_k(b_1, b_2, \ldots, b_{k-1}, b_k - VF_k, b_{k+1}, \ldots, b_n) - J_k(a_1, \ldots, a_n)$

(iii) $F_k \leq \rho_{kk}^2 VV(F_k 1_{\{a_k > VF_k\}})$.

(iv) $M(F_k > 0) > 0$.

We show in section 2 that the collection of all functions F_k satisfying (i) , (ii) and (iv) is nonempty: it contains lots of functions. One need only check whether or not one of them also satisfies (iii). Part (iii) should be thought of as a geometric

condition: the larger the domain is, the more likely it is that (iii) will be satisfied. (For example, it often suffices to use an eigenfunction of $-\Delta$ for F_k. Let $\delta_k > 0$ be small and $\pi_k > 0$ be large. In "nice" situations, we expect a_k and b_k to be continuous functions, so $G_k = \{a_k > \delta_k, b_k < \pi_k\}$ will be open. Choose an open set H so that the closure of H is contained in G_k, and let ϕ be the first eigenfunction of $-\Delta$ on H (with Dirichlet boundary conditions) and eigenvalue $\lambda > 0$. Normalize ϕ so that $\sup\{\phi(x): x \in H\} = 1$. Let W be the inverse of $-\Delta$ on H -- so $W \leq V$. Assume $V\phi$ is bounded by a constant m, and set $F_k = c\phi$ for some constant c to be determined below. Then $\rho_{kk}^2 VV(c\phi 1_{(a_k > Vc\phi)}) = \rho_{kk}^2 VV(c\phi)$ provided we choose c so small that $\delta_k/c > m$. But $\rho_{kk}^2 VV(c\phi) \geq \rho_{kk}^2 c\phi/\lambda^2$. This is larger than $c\phi$ if λ is small -- that is, if G and H are large. Thus (i), (iii) and (iv) are satisfied. Condition (ii) can be achieved by choosing c small.) Hypothesis (0.5) is a simplified version of hypothesis (2.8).

If (0.5) holds, we can restart the iteration by setting $w_k^1 = b_k - VF_k$, and $w_k^{j+1} = V[g_k - J_k(w_1^j, ..., w_n^j)]$ to obtain $a_k \leq w_k^2 \leq w_k^4 \leq ... \leq w_k^3 \leq w_k^1 \leq b_k$. So w_k^{2j} increases to $a_k' \geq a_k$ and w_k^{2j+1} decreases to $b_k' \leq b_k$. Clearly it may happen again that we obtain $a_k' < b_k'$ for some k. At this point, the reader will realize that we need to apply a transfinite induction argument to show that we must restart the procedure at most countably many times. We show in the transfinite induction argument that each time we restart, we can use the functions cF_1, cF_2, ..., cF_n, where c is a suitably small positive constant. To summarize, if we assume (0.2), (0.3), (0.4), and (0.5), then there are nonnegative functions u_1, u_2, ..., u_n solving (0.1).

In order to treat (0.1) with general elliptic operators and measures, we depend heavily on using potential theory as it is formulated in the probabilistic potential theory literature. We recall most of what the reader needs to know in the text; Chapter VI of [2] is a good reference. In section 1, we introduce the potential theory framework. In particular, this is necessary to drop hypothesis (0.3). In section 2, we discuss analogues in the general situation of the hypotheses we introduced in this section, and we show how to solve (0.1). The main result is (2.9). The hypotheses look a bit forbidding when first encountered, but they are actually quite natural as we have tried to indicate, and we discuss them further in section 3. We also give another version of (2.9) in (3.1), where we drop a hypothesis (by replacing it with several more!). In section 4, we briefly indicate some extensions. The appendix is devoted to a technical result necessary to show $b_k > 0$. All Markov processes are confined to the appendix.

NOTATION In general, notation is standard and can be found, for example, in Chapter VI of [2]. We introduce most of it as needed. If E is any σ-algebra, then we use the same letter E to denote the collection of all E-measurable functions which are real-valued. If D is any collection of functions, then bD (resp. pD) is the collection of bounded (resp. positive) functions in D. Thus pbE is the collection of positive and bounded E-measurable real-valued functions. Let (A,A,μ) be a measure space, let (B,B) be a measurable space, and let $\Phi: A \rightarrow B$ be measurable.

We denote the image measure of μ under Φ by $\Phi(\mu)$; that is, $\Phi(\mu)(G) = \mu(\Phi^{-1}(G))$ for every $G \in B$. A measure putting all of its mass at the point x will be denoted by ϵ_x.

1. THE POTENTIAL THEORY FRAMEWORK

Let E be an open domain in R^d, and let E be its Borel field. For each k with $1 \le k \le n$, let P_t^k be a sub-Markov semigroup on (E,E), and let U_k^q be its resolvent. (That is, for each $x \in E$, $P_t^k(x,\bullet)$ is a sub-probability measure on (E,E). If we define $P_t^k f(x) = \int_E f(y) P_t^k(x,dy)$, then $P_t^k P_s^k f(x) = P_{s+t}^k f(x)$. The resolvent is $U_k^q f(x) = \int_0^\infty e^{-qt} P_t^k f(x) dt$.)

NOTATION. C(E) is the collection of all continuous functions on E. $C_c(E)$ (resp. $C_c^2(E)$) is the collection of all continuous functions on E with compact support in E (resp. and having two continuous derivatives). $C_0(E)$ is the collection of functions in C(E) vanishing at the boundary of E. We let bE_c be the collection of bounded E-measurable functions vanishing off a compact set contained in E.

We <u>assume</u>:

(1.0)   For each q>0, $U_k^q : bE_c \to C_0(E)$ and $\lim_{q \to \infty} q U_k^q f = f$ whenever
    $f \in C_c(E)$.

(1.1)   (reference measure) There is a Radon measure m_k on (E,E) so that
    $U_k^q(x,\bullet) << m_k$ for each x in E.

(1.2)   (duality) There is another sub-Markov semigroup R_t^k on (E,E) with
    resolvent W_k^q so that
    $$\int_E (R_t^k f) g \, dm_k = \int_E f(P_t^k g) \, dm_k$$
    for every f and g in pE. For each q>0, $W_k^q : bE_c \to C_0(E)$ and
    $\lim_{q \to \infty} q W_k^q f = f$ whenever $f \in C_c(E)$.

(1.3)   (Infinitesimal generators) There are linear operators L_k and L_k^* with domains $D_k \supset C_c^2(E)$ and $D_k^* \supset C_c^2(E)$ so that for every f in $C_c^2(E)$, $(f - P_t^k f)/t$ converges boundedly to $-L_k f$ as t tends to zero, and $(f - R_t^k f)/t$ converges boundedly to $-L_k^* f$ as t tends to zero. In addition, L_k: $C_c^2(E) \to bE_c$ and $L_k^*: C_c^2(E) \to bE_c$.

These conditions are satisfied by a huge class of strong Feller semigroups, the best known of which has $L_k = \Delta/2$ with Dirichlet boundary conditions. In this case, P_t^k is the semigroup of a Brownian motion "killed" when it reaches the boundary of E. This example also satisfies the additional hypothesis (1.5) below if E is connected. There is also a large class of semigroups satisfying these conditions for which L_k and L_k^* are non-local integro-differential operators.

DEFINITION A function $f \in pE$ is said to be excessive for a resolvent (V^α) on (E,E) if $\alpha V^\alpha f \leq f$ for every $\alpha > 0$ and if $\lim_{\alpha \to \infty} \alpha V^\alpha f = f$.

As a consequence of the numbered assumptions above, there is a function $u_k(x,y) \in pE \times E$ so that:

(1.4)   (i) $x \to u_k(x,y)$ is excessive for U_k^q;

(ii) $y \to u_k(x,y)$ is excessive for W_k^q;

(iii) $U_k^0 f(x) = \int_E u_k(x,y) f(y)\, m_k(dy)$ for every $f \in pE$;

(iv) $W_k^0 f(y) = \int_E u_k(x,y) f(x)\, m_k(dx)$ for every $f \in pE$;

(v) For each $y \in E$, $x \to u_k(x,y)$ is lower semicontinuous;

(vi) For each $x \in E$, $y \to u_k(x,y)$ is lower semicontinuous.

For the construction of this function, see Chapter VI of [2]. This is the precise version of the "Green function" we need to define potentials of measures. We assume

(1.5) $u_k(x,y) > 0$ for every x, $y \in E$.

DEFINITION If μ is a positive measure on (E,E), then the potential of μ is defined to be $U_k\mu(x) = \int_E u_k(x,y)\mu(dy)$. If $\rho=\mu-\nu$ is the difference of the two positive measures μ and ν, and if $U_k(\mu+\nu) < \infty$ a.e. (m_k), then we set

$$U_k\rho(x) = U_k\mu(x)-U_k\nu(x) \text{ on } \{U_k(\mu+\nu)<\infty\}$$
$$= \infty \text{ on } \{U_k(\mu+\nu)=\infty\}.$$

The following properties are implied by (1.4 v, vi) and (1.5):

(1.6)   (i) If K is compact in E, then $\inf\{u_k(x,y):x\in K\} > 0$, and $\inf\{u_k(x,y):y\in K\} > 0$.

(ii) If ρ is a signed measure on (E,E) with $U_k|\rho|<\infty$ a.e. (m_k), then $U_k\rho$ is locally integrable. That is, $\int_K U_k\rho(x) \, m_k(dx) < \infty$ whenever K is compact in E.

See [2].

Our purpose in this section is to develop a bit of potential theory necessary to formulate conditions sufficient to insure the existence of solutions for the following system of semilinear equations:

$$(1.7) \qquad -L_1 u_1 + f_1(u_1,\ldots,u_n) = \mu_1$$

$$\vdots \qquad\quad \vdots \qquad\quad \vdots$$

$$-L_n u_n + f_n(u_1,\ldots,u_n) = \mu_n .$$

Here, each μ_k is a positive measure on $(E,\mathcal{E})$ and each $f_k: R^d \to R$. Our analysis of the problem depends on adopting the following perspective which is useful in potential theory.

Let $E_1, E_2,\ldots, E_n$ be n distinct copies of E, and let $F = \cup_{k=1}^{n} E_k$. In general, we will denote a point of E by x (without a subscript) and a point in E_k by x_k (the subscript indicating in which copy of E the point lies). Let $e_k: E \to E_k$ be the injection $e_k x = x_k$. Let d_E be a metric on E compatible with the topology of E. Define a metric d_F on F by setting

$$d_F(x_k, y_j) = d_E(x,y) \quad \text{if } k=j$$
$$= 1 \qquad \text{otherwise,}$$

whenever $x_k = e_k x$ and $y_j = e_j y$. This metric induces a topology on F which obviously extends the topology on E. Its Borel field F is characterized by the fact that its trace on E_k is $\mathcal{E}_k$, a copy of $\mathcal{E}$.

Define

$$(1.8) \qquad \mu = \sum_{k=1}^{n} e_k(\mu_k),$$

and define a new sub-Markov semigroup P_t on F by setting

$$(1.9) \qquad P_t f(x_k) = \int_{E_k} f(y) e_k(P_t^k(x,dy))$$

whenever $x_k = e_k x$ and $f \in bF$. It is easy to check that P_t is a semigroup on F, and it has infinitesimal generator

(1.10) $\quad L = \sum_{k=1}^{n} L_k {}^1 E_k$,   .

and resolvent

$$U^q = \int_0^\infty e^{-qt} P_t \, dt \ .$$

Note that $U^q(x_k, \bullet) = e_k(U_k^q(x, \bullet))$ whenever $x_k = e_k x$. If we define the measure M on (F, F) by setting

(1.11) $\quad M = \sum_{k=1}^{n} e_k(m_k)$,

then $U^q(z, \bullet) << M$ for every z in F. The potential density $u(\bullet, \bullet)$ for U^0 is given by

$$u(x_k, y_j) = u_k(x, y) \ \text{if} \ k = j$$
$$\qquad\qquad = \ 0 \quad \text{otherwise}$$

for $x_k = e_k x$ and $y_j = e_j y$.

We find this perspective handy because in solving (1.7), we need to analyze <u>not</u> the inverse of the linear system $(-L_1, -L_2, \ldots, -L_n)$, but instead the inverse of the perturbed linear system

(1.12) $\quad -L_1 u_1 - \alpha_{11} u_1 - \alpha_{12} u_2 - \cdots - \alpha_{1n} u_n$
$$\qquad\qquad \vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots$$
$$\qquad -L_n u_n - \alpha_{n1} u_1 - \alpha_{n2} u_2 - \cdots - \alpha_{nn} u_n$$

(The α_{kj} are positive functions on E to be chosen in sections 2 and 3. In general, they may be unbounded and have infinities.) We may obtain the inverse of this

system by perturbing the resolvent U^q as follows. Define a kernel $B(z,dy)$ from (F,F) to (F,F): for $1 \le k \le n$, set

$$B(x_k, \bullet) = \sum_{j=1}^{n} \alpha_{kj}(x) \epsilon_{x_j}(\bullet)$$

where $x_j = e_j x$. For each $q \ge 0$, define a new kernel on (F,F) by setting

$$V^q = \sum_{k=0}^{\infty} (U^q B)^k U^q .$$

The infinite sum makes sense since each term is positive. It is easy to check that V^q satisfies the resolvent equation: if $p \ge 0$ and $q \ge p$, then $V^p = V^q + (q-p) V^p V^q$. If the α_{kj} are too large, then $V^0(z, \bullet)$ may not be σ-finite for some z in F. We always <u>assume</u>:

(1.13)   There is a strictly positive function $\Psi \in F$ so that $V^0 \Psi(z) < \infty$
          for every z in F.

If $f \in pF$, an application of the resolvent equation shows that Vf is excessive. Since V^q is a resolvent and we are assuming (1.13), we have the following maximum principle.

(1.14) PROPOSITION ([9],IX,T68). Let $f \in pF$. If h is an excessive function for (V^q) and if $Vf \le h$ on $\{f > 0\}$, then $Vf \le h$ on F.

(1.15) NOTATION $V(x, \bullet) \equiv V^0(x, \bullet)$.

This perturbation V^q of U^q is well-understood in the classical case when the α_{kj} and U^q are well-behaved.

(1.16) PROPOSITION [3] Assume F is locally compact with a countable base and that P_t is a Feller semigroup with strong infinitesimal generator $(L, D(L))$ on $C_0(F)$. If B is a continuous operator on $C_0(F)$, then V^q is the resolvent of a strong Feller semigroup on $C_0(F)$ with infinitesimal generator $(L+B, D(L))$.

We will work without such strong hypotheses. In particular, P_t may not be Feller and B is usually not continuous.

(1.17) DEFINITION Set

(1.18) $v(z,y) = \sum_{k=0}^{\infty} [(UB)^k u(\bullet, y)](z) = \sum_{k=0}^{\infty} \int_F u(w,y)(UB)^k(z, dw)$.

If ρ is a positive or signed measure on F, then the potential of ρ, $V\rho$, is defined with the density v just as $U\mu$ was defined with the density u. In section 2, hypothesis (2.1c) will imply $\alpha_{kj} > 0$, and this in turn implies $v(z,y) > 0$.

(1.19) LEMMA (i) If ρ is a signed measure on $(F, \mathcal{F})$ with $V|\rho| < \infty$ a.e. (M), then $\int_K V\rho \, dM < \infty$ whenever $K \subset F$ is compact. That is, $V\rho$ is locally integrable.

(ii) Assume (2.1c). If $K \subset F$ is compact, then $\inf\{v(z,y): y \in K\} > 0$ for every z in F and $\inf\{v(z,y): z \in K\} > 0$ for every y in F.

PROOF (i) $V|\rho| = U(|\rho| + \sum_{k=0}^{\infty} B(UB)^k U|\rho|)$. Since the right hand side is the U-potential of a positive measure, it must be locally integrable by (1.6ii) and the definition of U. Since $V|\rho|$ is locally integrable, $V\rho$ is locally integrable.

(ii) This follows from (1.6i) and (2.1c) since $v(x,y) \geq u(x,y)$. Q.E.D.

(1.20) PROPOSITION Let ρ be a signed measure on $(F, \mathcal{F})$ so that $V|\rho| < \infty$ a.e.

(M). Then $(L+B)V\rho = -\rho$ in the sense that

(1.21) $\qquad \int_F (L^* f) V\rho \, dM + \int_F f(BV\rho) \, dM = -\rho(f)$

whenever $f \in C_c^2(F)$, and where

$$L^* = \sum_{k=1}^n L_k^* 1_{E_k}.$$

PROOF Assume first that ρ is a positive measure on (F, F) and let $f \in pC_c^2(E_k)$. By (1.19), $V\rho$ is locally integrable. Since the support of L_k^* is compact in E_k by hypothesis (1.3), the following integrals are well-defined and we have

$$\int_{E_k} (L_k^* f) V\rho \, dM = \int_{E_k} (L_k^* f) \sum_{j=0}^{\infty} (UB)^j U\rho \, dM.$$

Since the integrands are absolutely integrable, Fubini's theorem applies, and we may rewrite (1.22) as

(1.23) $\qquad \int_{E_k} (L_k^* f) U\rho \, dM + \sum_{j=1}^{\infty} \int_{E_k} (L_k^* f) (UB)^j U\rho \, dM.$

To analyze (1.23), we need the following result.

(1.24) LEMMA If γ is a positive measure on (E, E) with $U_k \gamma < \infty$ a.e. (m_k), then

(1.25) $\qquad \int_E (L_k^* f) U_k \gamma \, dm_k = -\gamma(f)$

whenever $f \in C_c^2(E)$.

PROOF Recall that $W_k L_k^* f = W_k[\lim_{t \to 0} (R_t^k f - f)/t] = \lim_{t \to 0} W_k (R_t^k f - f)/t$ since the convergence is bounded by hypothesis (1.3). But this last term may be rewritten as

.

$$-\lim_{t\to 0} t^{-1} \int_0^t R_s^k f \, ds = -f.$$

Since

$$\int_E |L_k^* f| U_k \gamma \, dm_k < \infty,$$

Fubini's theorem applies to the first term in (1.25) and yields

$$\int_E W_k L_k^* f \, d\gamma = -\gamma(f).$$

<div align="right">Q.E.D.</div>

Thus the first term in (1.23) may be rev ..ten as $-\rho(f)$, while the infinite sum in (1.23) may be rewritten as

$$\sum_{j=1}^{\infty} \int_{E_k} (L_k^* f) U(BU)^j \rho \, dM = -\sum_{j=1}^{\infty} \int_{E_k} f(BU)^j \rho \, dM.$$

Since $f \geq 0$ and $(BU)^j \rho \geq 0$, we may rewrite this as

$$-\int_{E_k} f \sum_{j=1}^{\infty} (BU)^j \rho \, dM = -\int_{E_k} f(BV\rho) \, dM.$$

This proves (1.21) for $\rho \geq 0$ and $f \in \rho C_c^2(E_k)$. The general case follows by linearity.

<div align="right">Q.E.D.</div>

Now how can this potential theory aid us in solving (1.7)? Formally, we may rewrite (1.7) as

(1.26)
$$-L_1 u_1 - \alpha_{11} u_1 - \ldots - \alpha_{1n} u_n = \mu_1 - G_1(x, u_1, \ldots, u_n)$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$-L_n u_n - \alpha_{n1} u_1 - \ldots - \alpha_{mn} u_n = \mu_n - G_n(x, u_1, \ldots, u_n) \quad,$$

where $G_k(x, u_1, \ldots, u_n) = \alpha_{k1}(x) u_1 + \ldots + \alpha_{kn}(x) u_n + f_k(u_1, \ldots, u_n)$. Define a function

$H: F \times R^d \to R$ by setting $H(x_k, \underline{u}) = G_k(x, u_1, \ldots, u_n)$ for $x_k = e_k x$. If $u: F \to R$, let $G(x_k, u) = H(x_k, u(x_1), u(x_2), \ldots, u(x_n))$ where $x_j = e_j x$. If we try to solve (1.26) by inverting, we find we need to solve the following equation for u:

(1.27) $u = V[\mu - G(\bullet, u(\bullet))]$,

where μ and V are defined above. Once we find such a function u, Prop. (1.20) shows that $-(L+B)u = \mu - G(\bullet, u(\bullet))$. Thus, if we set $u_1(x) = u(e_1 x)$, $u_2(x) = u(e_2 x)$, $\ldots$, $u_n(x) = u(e_n x)$, we have that the functions $u_1, u_2, \ldots, u_n$ solve (1.26), and so they solve (1.7).

2. THE MAIN ARGUMENT

In this section, we develop hypotheses which allow us to solve the equation $u=V[\mu-G(\bullet,u(\bullet))]$. We assume the notation and hypotheses discussed in section 1. (In particular, we assume (1.0) through (1.3), (1.5) and (1.13).) In addition, we assume the following conditions throughout this section.

(2.1)　　　(a) μ is a positive measure, $\mu(F) > 0$ and $V\mu < \infty$ a.e. (M).

　　　　　(b) $G(x,u)$ is continuous in u, and $G(x,0) = 0$.

　　　　　(c) The functions α_{kj} have been chosen strictly positive so that

　　　　　　$G(x,u) < G(x,v)$ whenever $0 \le u(\bullet) \le v(\bullet) \le V\mu(\bullet)$.

Recall (2.1c) implies $v(z,y) > 0$ for every z and y in F.

For most of this section, we work with a "cutoff" of G, $J(u)(z)$, defined as follows: for x in E and $x_k=e_k x$,

(2.2)　　$J(u)(x_k) = G(x_k,u) 1_{\{u(e_1 x)\ge 0,\ldots,u(e_n x)\ge 0\}}.$

(2.3) PROPOSITION Assume (2.1) and the following condition:

(2.4)　　$M(V\mu > VG(V\mu)) > 0.$

Then there are two functions u_1 and v_1 on F so that $u_1 \le v_1 \le V\mu$, $u_1 = V[\mu-J(v_1)]$, and $v_1 = V[\mu-J(u_1)]$.

NOTATION If $w \in F$, let $H(w) = \mu-J(w)$. We also adopt the convention $\infty-\infty=\infty$.

REMARK By (2.4), $M(VG(V\mu)<\infty) > 0$. This implies $VG(V\mu) < \infty$ a.e. (M). For if $M(VG(V\mu)=\infty) > 0$, then we would have $\alpha V^\alpha(VG(V\mu)) = \infty$ since $v^\alpha(z,\bullet)$ is

bounded away from zero on compacts.

PROOF Set $w_1 = V\mu$, and define $w_2 = VH(w_1)$. Note that w_2 is well-defined a.e. (M) since $V\mu < \infty$ a.e. (M) and $J \geq 0$. By (2.4), $M(w_2 > 0) > 0$. Since $J \geq 0$, $w_2 \leq w_1$. Now set $w_3 = VH(w_2)$: once again, this is well-defined a.e. (M). Since $w_2 \leq w_1$, $J(w_2) \leq J(w_1)$ by (2.1c) and (2.2). Thus $w_3 \geq w_2$. Since $J(w_2) \geq 0$, $w_3 \leq V\mu = w_1$. We have proved that $w_2 \leq w_3 \leq w_1$. This alternating or oscillating behavior continues. By induction, define $w_{k+1} = VH(w_k)$. Note that w_{k+1} is well-defined a.e. (M).

(2.5) LEMMA Assume $w_2 \leq ... \leq w_{2k} \leq w_{2k+1} \leq ... \leq w_1$. Then $w_{2k} \leq w_{2k+2} \leq w_{2k+3} \leq w_{2k+1}$.

PROOF The lemma follows immediately from these observations:

(i) $J(w_{2k+1}) \leq J(w_{2k-1})$ implies $w_{2k+2} \geq w_{2k}$.

(ii) $J(w_{2k+1}) \geq J(w_{2k})$ implies $w_{2k+2} \leq w_{2k+1}$.

(iii) $J(w_{2k+2}) \leq J(w_{2k+1})$ implies $w_{2k+3} \geq w_{2k+2}$.

(iv) $J(w_{2k+2}) \geq J(w_{2k})$ implies $w_{2k+3} \leq w_{2k+1}$.

Q.E.D.

Therefore, the sequence (w_{2k}) increases to a function u_1, while the sequence (w_{2k+1}) decreases to a function v_1 with $-\infty < u_1 \leq v_1 < V\mu$ a.e. (M). (Note that (2.4) and (1.5) imply strict inequality in $v_1 < V\mu$.) Since $w_{2k+1} = V\mu - VJ(w_{2k})$, the monotone convergence theorem yields $v_1 = V\mu - VJ(u_1)$. Since $J(w_1) \geq J(w_{2k+1})$ and $VJ(w_1) < \infty$ a.e. (M) by (2.4), we may apply the dominated convergence theorem to the equality $w_{2k+2} = V\mu - VJ(w_{2k+1})$ to obtain $u_1 = V\mu - VJ(v_1)$.

Q.E.D.

The measure μ can always be decomposed as $r \cdot dM + \mu^S$, where μ^S is singular with respect to M and $r \geq 0$. Assume

(2.6)　　$r > 0$ on F.

(2.7) LEMMA Let a and b be two functions on F so that $u_1 \leq a \leq b \leq V\mu$, $M(b > a) > 0$, and suppose that $a = V\mu - VJ(b)$ and $b = V\mu - VJ(a)$. Then $b-a > 0$ (remember $\infty - \infty = \infty$) and $b > 0$ on F.

PROOF By hypothesis, $M(b-a = V[J(b)-J(a)] > 0) > 0$, and $b \geq a$, so $M(J(b) > J(a)) > 0$ and $J(b) \geq J(a)$. Therefore, $b-a = V[J(b)-J(a)] > 0$ since $v(\bullet,\bullet) > 0$.

Since $b + VJ(a) = V\mu$, we have that $VJ(a) \leq$

$V\mu$ on $\{b > 0\}$. Thus we have $VJ(a) \leq V\mu$ on $\{a > 0\} \supset \{J(a) > 0\}$ since $a \leq b$. By (1.14), $VJ(a) \leq V\mu$ on F. Thus b must be nonnegative. Now assume $\{b=0\}$ is nonempty, and let $z \in \{b=0\}$. Then we have $V\mu(z) = VJ(a)(z) < \infty$. Since $b-a > 0$, $a(z) < 0$ and $VJ(b)(z) > V\mu(z)$. Note that $\{VJ(b) > V\mu\}$ is a finely open set (for the fine topology generated by the resolvent (V^α)). By (A.1) and (2.6), $V\mu(w) < VJ(a)(w)$ for some w in F. But this contradicts $b \geq 0$. Therefore, $\{b=0\} = \emptyset$.　　　　　　　　　　　　　　　　　　Q.E.D.

The next hypothesis will be used in a crucial way in the main theorem in this section. It will be discussed further in the remark following it and also in section 3. See also the discussion in the introduction. Assume:

(2.8)　　　(a) There are positive functions $\rho_{kj} : F \rightarrow R$ so that

(i) $J(c+\gamma)(x_k)-J(c)(x_k) \geq \sum_{j=1}^{n} \rho_{kj}(x_j)\gamma(x_j)$ a.e. (M) whenever

$x_j = e_j x$ and c and γ are functions with $u_1 \leq c \leq c+\gamma \leq v_1$;

(ii) Whenever $K \subset E_k$ is compact, $\inf\{\rho_{kk}(z): z \in K\} > 0$.

(NOTATION: Let $\Phi(\gamma)(x_k) = \sum_{j=1}^{n} \rho_{kj}(x_j)\gamma(x_j)$.

(b) There is a nonnegative bounded function f on F and numbers $\delta > 0$ and

$\pi > 0$ so that

(i) the set $\{f > 0\}$ is contained in a compact set K_f and $K_f \subset$

$\{u_1 > \delta\} \cap \{v_1 < \pi\}$; $M(f > 0) > 0$.

(ii) $f \leq J(v_1 - Vf) - J(u_1)$.

(iii) $f \leq \Phi(V[\Phi(Vf) 1_{\{u_1 > Vf\}}])$.

REMARK Condition (2.8b(iii)) is the difficult one to fulfill, consistent with

achieving (1.13). Let us show that (2.8b(i)) and (2.8b(ii)) can always be

achieved. If $u_1 < v_1$, we may choose $\delta > 0$ and $\pi > 0$ so that

$M(u_1 > \delta, v_1 < \pi) > 0$. Let g be any bounded nonnegative function so

that $\{g > 0\}$ has compact closure in F contained in $\{u_1 > \delta\} \cap \{v_1 < \pi\}$, Vg is everywhere

finite on F, $M(g > 0) > 0$ and $J(v_1) - J(u_1) \geq g$. Let $0 < p < 1$. Then $|J(v_1 - Vpg) - J(u_1)| \geq$

$\Phi(v_1 - u_1 - Vpg)$ on $\{v_1 - u_1 \geq Vpg\}$. Recall that $v_1 - u_1 = V[J(v_1) - J(u_1)]$. Since (1.5, 2.1b)

imply $\inf\{(v_1 - u_1)(z): z \in \{g > 0\}\} > 0$, we also have that $\inf\{(J(v_1) - J(u_1))(z): z \in$

$\{g > 0\}\} = c > 0$. If we choose p so small that $pg < J(v_1) - J(u_1)$ on $\{g > 0\}$, we have that

$\Phi(v_1 - u_1 - Vpg) > d > 0$ on $\{g > 0\}$ since $\inf\{\rho_{kk}(z): z \in \{g > 0\} \cap E_k\} > 0$, $1 \leq k \leq n$. Thus, by

taking an even smaller p so that $pg < d$, we obtain $pg \leq J(v_1 - Vpg) - J(u_1)$, and we set

$f = pg$.

(2.9) THEOREM Assume (2.1), (2.4), (2.6) and (2.8). There is a function u so that $0 \leq u \leq V\mu$ and $u = V[\mu - G(\bullet, u(\bullet))]$.

PROOF The heart of this proof is a transfinite induction argument. We construct two collections of functions $(u_\gamma) \subset F$ and $(v_\gamma) \subset F$: they are indexed by the ordinals $\gamma \geq 1$.

NOTATION Set $a_1 = -\infty$ and $b_1 = V\mu$. For every ordinal $\beta \geq 2$, set $a_\beta = \sup\{u_\gamma : 1 \leq \gamma < \beta\}$, and set $b_\beta = \inf\{v_\gamma : 1 \leq \gamma < \beta\}$.

For each ordinal $\beta \geq 1$, define the proposition $\phi(\beta)$ to be the statement:

(2.10)   If $M(a_\beta < b_\beta) > 0$, then there are functions u_β and v_β so that

$$a_\beta \leq u_\beta \leq v_\beta \leq b_\beta \leq V\mu, \quad M(v_\beta < b_\beta) > 0, \quad u_\beta = V[\mu - J(v_\beta)], \text{ and}$$

$$v_\beta = V[\mu - J(u_\beta)].$$

We have already verified the truth of $\phi(1)$ in Prop. (2.3). Let us assume $\phi(\gamma)$ is true for every $\gamma < \beta$, and prove $\phi(\beta)$ is true. Since $b_\gamma = V\mu - VJ(a_\gamma)$ for every $\gamma < \beta$, the monotone convergence theorem lets us conclude that $b_\beta = V\mu - VJ(a_\beta)$. Similarly, the dominated convergence theorem lets us conclude that $a_\beta = V\mu - VJ(b_\beta)$. Set $a = a_\beta$ and $b = b_\beta$, and assume $M(a < b) > 0$. By (2.7), $b > a$ and $b > 0$ on F.

Let f be the function chosen in (2.8b). We now show that for some p with $0 < p < 1$, pf satisfies the conditions in (2.8b) if we replace u_1 with a and v_1 with b. (We essentially repeat the argument given in the remark following (2.8).)

First note that $\{pf > 0\} \subset \{u_1 > \delta, v_1 < \pi\} \subset \{a > \delta, b < \pi\}$, and $M(pf > 0) > 0$. Second, since $\inf\{(b-a)(z) = V[J(b) - J(a)](z) : z \in \{pf > 0\}\} > 0$, p can be chosen small so that $J(b) - J(a) > pf$ on $\{f > 0\}$. Thus $b - a - Vpf > c > 0$ on $\{pf > 0\}$. It follows that $\Phi(b - a - Vpf) > d > 0$

on $\{pf>0\}$ since $\inf\{\rho_{kk}(z): z \in \{pf>0\}\cap E_k\} > 0$, $1 \le k \le n$.

We may take an even smaller p

so that $\Phi(b-a-Vpf) > pf$ and we obtain $pf \le J(b-Vpf)-J(a)$. Third, and finally, we

show that $pf \le \Phi(V[\Phi(Vpf)1_{\{a>Vpf\}}])$. Multiplying (2.8b(iii)) by p yields $pf \le$

$\Phi(V[\Phi(Vpf)1_{\{u_1>Vf\}}])$. Since $\{u_1>Vf\} \subset \{a>Vpf\}$, we have the result.

Set $\epsilon = Vpf$, $w_1=b-\epsilon$, $w_2=VH(w_1)$ and $w_3=VH(w_2)$.

(2.11) LEMMA $a \le w_2 \le w_3 \le w_1$.

PROOF Since $pf \le J(b-Vpf)-J(a)$, $V[J(b)-J(a)] \ge \epsilon$, or $b-\epsilon \ge a$. Thus $w_1 \ge a$. To

obtain $w_2 \ge a$, we need $VH(b-\epsilon) \ge VH(b)$, or $V[J(b)-J(b-\epsilon)] \ge 0$. This holds since J

is increasing and $b \ge b-\epsilon$. To obtain $w_2 \le w_1$, we need $VH(a)-VH(b-\epsilon) \ge \epsilon$, or

$V[J(b-\epsilon)-J(a)] \ge Vpf$. This holds since $pf \le J(b-Vpf)-J(a)$. Since $w_2 \le w_1$, we obtain

$w_3 \ge w_2$. Finally, to obtain $w_3 \le w_1$, we need $VH(w_2) \le b-\epsilon$, or $VH(VH(b-\epsilon)) \le b-\epsilon$.

That is, we need $V[\mu-J(VH(b-\epsilon))] \le V[\mu-J(a)]-\epsilon$, or $VJ(a)+Vpf \le VJ(VH(b-\epsilon))$. To

obtain this, we need only have $J(a)+pf \le J(VH(b-\epsilon))$, or $pf \le J(VH(b-\epsilon)) - J(a) =$

$J(a+V[J(b)-J(b-\epsilon)])-J(a)$. By (2.8a), it suffices to have $pf \le \Phi(V[J(b)-J(b-\epsilon)])$

on $\{u_1>0\} \subset \{a>0\}$. Applying (2.8a) again, we see it suffices to have $pf \le$

$\Phi(V[\Phi(Vpf)1_{\{b>Vpf\}}])$. Since $\{Vpf<a\} \subset \{Vpf<b\}$, we see that it suffices to have

(2.12) $pf \le \Phi(V[\Phi(Vpf)1_{\{Vpf<a\}}])$.

We showed before (2.11) that f has this property.   Q.E.D.

Now we can set $w_{k+1} = VH(w_k)$ as we did earlier in this section. The proof of

Prop. (2.3) shows that we obtain functions $u_\beta \le v_\beta$ so that $v_\beta = V\mu-VJ(u_\beta)$ and $u_\beta=$

$V\mu-VJ(v_\beta)$: we have verified $\phi(\beta)$. Choose a finite measure N on (F,F) which is equivalent to M so that $N(v_1-u_1)<\infty$. The sequence $\pi(\beta)=N(v_\beta-u_\beta)$ is a positive and strictly decreasing sequence. Thus $\pi(\beta)=0$ for some countable ordinal β. That is, there is a β so that $u_\beta=b_\beta=v_\beta=u$. But then $u = V\mu-VJ(u)$. By (2.7), $u \geq 0$, so we have $u = V\mu-VG(\bullet,u(\bullet))$. This concludes the proof of (2.9).

Q.E.D.

3. THE FINAL RESULT AND DISCUSSION OF HYPOTHESES

In this section, we again assume the notations and hypotheses of section 1. This next result drops the hypothesis (2.6) (but adds some others !).

(3.1) THEOREM Assume (2.1) and suppose there is a decreasing sequence of functions h_k on F so that

   (i) $h_k > 0$ on F ;

   (ii) $\lim_{k \to \infty} Vh_k = 0$;

   (iii) $M(V(\mu+h_k)) > VG(V(\mu+h_k))) > 0$.

Assume (2.8). (Note: the function f may vary with k.) Then there is a function $u \geq 0$ with $u = V[\mu - G(\bullet, u(\bullet))]$.

PROOF By (iii), $VJV(\mu+h_k) = VGV(\mu+h_k) < \infty$ on a set of positive measure, and hence a.e. (M). Let $\Phi = J(V(\mu+h_1))$, so $V\Phi < \infty$ a.e. (M). For each k, the hypotheses of Theorem (2.9) are satisfied, so there is a function u_k with $0 \leq u_k \leq V(\mu+h_k)$ and $u_k = V[\mu+h_k-J(u_k)]$. Since $u_k \leq V(\mu+h_k)$, $J(u_k) \leq \Phi$, so $c_k = J(u_k)/\Phi$ is a bounded sequence in $L^{\infty}(M)$. By Alaoglu's theorem, there is a subsequence $c_{k(\ell)}$ of c_k and a function $w \geq 0$ so that $\int c_{k(\ell)} g \, dM$ converges to $\int wg \, dM$ for every $g \in L^1(M)$. For almost every z, $v(z,\bullet)\Phi(\bullet) \in L^1(M)$, so $V\Phi c_{k(\ell)} = VJ(u_{k(\ell)})$ converges to Vw. Since $u_{k(\ell)} = V[\mu+h_{k(\ell)}-J(u_{k(\ell)})]$, $u_{k(\ell)}$ must converge (a.e.) to a function $u \geq 0$. Since $J(u_k) \leq \Phi$ and $V\Phi < \infty$ a.e. (M), we may apply the dominated convergence theorem to conclude $\lim VJ(u_{k(\ell)}) = VJ(u)$, and we have proved that $u = V[\mu - J(u)]$.   Q.E.D.

Theorem (3.1) is perhaps not quite as satisfying as the result in section 2
since u is obtained by a compactness method instead of monotone approximation.
One might hope that since the h_k decrease, the u_k should also decrease. This does
not seem to be the case, however, and one must be careful to recall that J is not a
local function of its argument. For example, suppose we try to prove the u_k
decrease. By subtracting, we obtain $u_k - u_{k+1} + V[J(u_k) - J(u_{k+1})] = V[h_k - h_{k+1}]$, so
$V[J(u_k) - J(u_{k+1})] \le V[h_k - h_{k+1}]$ on $\{u_k > u_{k+1}\}$. At this point it is tempting to say
$\{J(u_k) > J(u_{k+1})\} \subset \{u_k > u_{k+1}\}$ (which is <u>incorrect</u>), so the maximum principle yields
$V[J(u_k) - J(u_{k+1})] \le V[h_k - h_{k+1}]$ everywhere. This would lead to the (incorrect)
conclusion that $u_k > u_{k+1}$.

We now discuss the hypotheses in sections 1 and 2.

<u>Hypotheses (1.0) through (1.3), (1.5).</u> These hypotheses are quite reasonable and
cover a large class of elliptic and integro-differential operators.

<u>Hypothesis(1.13)</u> The condition $V\Psi < \infty$ is necessary to insure we have a non-
trivial potential theory: whether or not it is satisfied depends on how large the
α_{kj} are.

<u>Hypothesis (2.1)</u> Parts (a) and (b) are simple requirements. Part (c) is the tricky
one. Note that $V\mu$ depends on the α_{kj}: as the α_{kj} grow, so does $V\mu$. There are
two situations in which this hypothesis might be easily satisfied. First, if
$\partial f_k / \partial x_j > 0$ for all k and j, then the α_{kj} can be chosen to be zero. In this case,
(1.13) is instantly satisfied. Or it may be the case that $\partial f_k / \partial x_j$ is bounded below,
say by ξ_{kj}. If $\xi_{kj} < 0$, set $\alpha_{kj} = -\xi_{kj}$; otherwise, set $\alpha_{kj} = 0$. In this case, (1.13)

may or may not be satisfied. In the general case, it is best to adopt the following philosophy. Having chosen $\alpha_{kj} \geq 0$ so that (1.13) holds, hypotheses (2.1), (2.4) and (2.8) delineate a class of measures μ for which (1.7) can be solved. It would be nice to have a deeper understanding of the relationships between (2.1) and (1.13).

Hypothesis (2.4) This is an interesting and suggestive one. If $V\mu \leq VG(V\mu)$ everywhere, it is known in probability and potential theory that μ is a balayage of $G(V\mu)$. It is easy to cite at least one condition forbidding this and so guaranteeing that (2.4) holds: if μ charges a polar set (i.e. a set contained in the infinities of an excessive function), then it cannot be the balayage of a function (and note that $G(V\mu)$ is a function).

Hypothesis (2.8) Part (a) is a simple requirement which can always be achieved by increasing the α_{kk} slightly if necessary. Part (b) is a nontrivial assumption. Parts (i) and (ii) can always be achieved, as is shown in the Remark following the hypothesis. Part (iii) is the hard part. Since $u_1 \geq a$, it may sometimes be easier to check (iii) with a replacing u_1. Notice that while hypotheses (2.1) and (2.4) seem pretty close to optimal for our presentation, hypothesis(2.8) is really more than is needed. We present it in this form as an attempt to render a complicated result a bit more palatable. By following the proof of (2.9), the reader will see exactly what is needed.

4.   EXTENSIONS

The methods and theorems we have discussed can be extended to parabolic

semigroups. For example, L_k could be $\partial/\partial t - \Delta$. Hypotheses (1.5) and (1.61) fail in this case, but slight modifications to some of the proofs overcome these difficulties. It is also possible to solve <u>infinite</u> systems of semilinear equations using these methods. Once again, some slight modifications to some of the hypotheses are needed, and we leave these to the interested reader. It is an elementary exercise to reformulate all of these results to solve equations in which the local nonlinearities f_k are replaced with nonlocal nonlinearities of the form $f_k(D_1 u_1(x),...,D_n u_n(x))$, where $f_k : R^n \to R$ and where $D_k = D_k(x,\bullet)$ is a positive kernel.

APPENDIX.

At one crucial point in the proof of (2.7), we need the following fact. (All hypotheses and notations of sections 1 and 2 are assumed here.)

(A.1) PROPOSITION Let $z \in \{VJ(b) > V\mu\} = \{a < 0\} \subset \{J(a)=0\}$. If $V\mu(z)=VJ(a)(z)$, then $V\mu(w) < VJ(a)(w)$ for some w in F.

This proposition depends on hypothesis (2.6), among others. There is undoubtedly a purely potential theoretic way to prove this result, but we resort to a proof using Markov processes. In order to do this, we apply some work of Bouleau [3].

Let $h = U\Psi$: this function is finite on F by (1.13) and is bounded away from zero on compacts in F by (1.61). Define two new kernels Y and C on (F,F) by setting $Y(z,\bullet)=h^{-1}(z)U(z,\bullet)$ and $C(z,\bullet)=B(z,\bullet)h(\bullet)$. Recall that Y is the zero potential Y^0 of a sub-Markovian resolvent (Y^α) on (F,F). Hypothesis (1.0) implies that (Y^α) is the resolvent of a right Markov process on (F,F). For more information about (Y^α), see [6]. For information about right processes, see [5] and [2]. Now let ∂ be a new point not contained in F, set $F' = F \cup \{\partial\}$, and $F' = \sigma(F,\partial)$. Extend the resolvent (Y^α) to F' by setting $\alpha Y^\alpha(z,\{\partial\}) = 1-\alpha Y^\alpha(z,F)$ for every z in F and $\alpha Y^\alpha(\partial,\{\partial\})=1$: (Y^α) so extended is a Markovian resolvent on (F', F'). Extend C to F' by setting $C(z,\{\partial\})=0$ for every z in F and $C(\partial,F')=0$. Note that if we define

$$Z^\alpha = \sum_{k=0}^{\infty} (Y^\alpha C)Y^\alpha,$$

then $Zf = h^{-1}Vf$ for every $f \in pF$ with $f(\partial)=0$. Let us compute the function

$$b = \sum_{k=0}^{\infty} (YC)^k 1_{F'} = 1_{F'} + h^{-1}\sum_{k=1}^{\infty} (UB)^k U\psi$$
$$\leq 1_{F'} + h^{-1}V\psi < \infty.$$

The function b is invariant for the resolvent (Z^{α}) [3]. Bouleau [3] shows that since b is finite, the resolvent (D^{α}) defined by $D^{\alpha}(z,\bullet)=b^{-1}(z)Z^{\alpha}(z,\bullet)b(\bullet)$ is the resolvent of a right Markov process $X = (\Omega, G, G_t, X_t, \theta_t, (P^X)_{x\in F'})$. We use this process to prove (A.1). Note that $Df = b^{-1}Z(bf) = b^{-1}h^{-1}V(bf)$, so $G = \{VJ(b)>V\mu\} = \{D(b^{-1}J(b))>D(b^{-1}\mu)\}$: this set is finely open for X. What does this mean? It means that if we define $T = \inf\{t > 0: X_t \in G^c\}$, then $T>0$ P^Z almost surely whenever $z \in G$. Recall that $\mu = r\bullet dM + \mu^S$ and $r>0$ on F. By the strong Markov property,

$$(A.2) \qquad E^Z[Db^{-1}r(X_T)] = E^Z\int_T^{\infty}b^{-1}r(X_s)ds < E^Z\int_0^{\infty}b^{-1}r(X_s)ds = Db^{-1}r(z),$$

since $b^{-1}r > 0$ and $T>0$ P^Z almost surely. Thus $E^Z[Db^{-1}\mu(X_T)] < Db^{-1}\mu(z) = D(b^{-1}J(a))(z)$. Now compute

$$(A.3) \qquad E^Z[D(b^{-1}J(a))(X_T)] = E^Z\int_T^{\infty}b^{-1}J(a)(X_s)ds = E^Z\int_0^{\infty}b^{-1}J(a)(X_s)\, ds$$
$$= D(b^{-1}J(a))(z),$$

since $J(a)(X_s)$ is zero provided $s<T$ (recall $J(a) = 0$ on G). Thus we see

$$E^Z[Db^{-1}\mu(X_T)] < E^Z[Db^{-1}J(a)(X_T)],$$

and so $Db^{-1}\mu(w) < Db^{-1}J(a)(w)$ for some w in F. That is, $V\mu(w)<VJ(a)(w)$.

$$\text{Q.E.D.}$$

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