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# CAMERA ROTATION INVARIANCE OF IMAGE CHARACTERISTICS

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## ABSTRACT

The image transformation due to camera rotation relative to a stationary scene is analyzed, and the associated transformation rules of "features" given by weighted averaging of the image are derived by considering infinitesimal generators on the basis of group representation theory. Three dimensional vectors and tensors are reduced to two dimensional invariants on the image plane from the viewpoint of projective geometry. Three dimensional invariants and camera rotation reconstruction are also discussed. The result is applied to the shape recognition problem when camera rotation is involved.



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## 1. INTRODUCTION

The problem we consider in this paper is as follows. Suppose the camera is rotated by a certain angle around its focus relative to a stationary scene. Then, a different projected image is seen on the image plane. However, since a point on the image plane corresponds to a *ray* in the 3D scene, occlusion is not affected by camera rotation. If the amount of camera rotation is known, the original image can be recovered. (Here, we do not consider the effect of the image boundary. We assume that the image plane is sufficiently large and that the object or scene of interest is always included in the field of view.) This means that the *information content* of the image is not affected by the 2D image transformation induced by the camera rotation.

Suppose the viewed image is characterized by a finite number of parameters or *features*. If the camera is rotated, the image is also changed so that the features change their values. If the set of features is *invariant* in the sense that these new values are completely determined by the original values and the amount of the camera rotation, we can predict the values of the features which would be obtained if the camera were rotated by a given amount. Conversely, if we are given two views of the same object obtained from different camera orientations, we can reconstruct the amount of camera rotation $R$ which would transform the values of the features to prescribed values. An important fact is that in this process *we need not know the point-to-point correspondence*. All computations are based on the observed features, which are global quantities.

These considerations are very important in many problems of computer vision and pattern recognition when the camera orientation is controlled by a computer. Even if the camera is fixed, various types of analysis of the image become easy if we apply to the image the transformation equivalent to camera rotation. This technique is used for the

shape-from-texture problem by Kanatani and Chou [7] and for the interpretation of lengths and angles by Kanatani [6]. A similar analysis is done when the object is moving and we are observing the optical flow (Kanatani [5]). In this paper, we will discuss, as a typical example, the center of gravity and principal axes of a given region to see how the invariant properties can be utilized to recognize the shape and to reconstruct the (actual or hypothetical) camera rotation.

## 2. CAMERA ROTATION AND INVARIANT FEATURES

Let $f$ be the *focal length* of the camera. The camera image is thought of as the projection onto an image plane located at distance $f$ from the *focus* $O$; a point $P$ in the scene is projected onto the intersection of the image plane with the *ray*, connecting point $P$ and the focus $O$. Let us choose an $XYZ$-coordinate system such that the focus $O$ is at the origin and the $Z$-axis coincides with the camera optical axis. Choose an $xy$-coordinate system in such a way that the $x$- and $y$-axes are parallel to the $X$- and $Y$-axes with $(0,0,f)$ as the origin. This $xy$-plane plays the role of the image plane (Fig. 1). A point $(X,Y,Z)$ in the scene is projected onto $(x,y)$ on the image plane, where

$$x=fX/Z, \qquad y=fY/Z. \tag{2.1}$$

Consider a camera rotation around its focus $O$ and the induced transformation of the image (Fig. 2). Suppose the camera is rotated by rotation matrix $\boldsymbol{R}$, which is an orthogonal matrix, i.e., $\boldsymbol{R}\boldsymbol{R}^T=\boldsymbol{I}$. Then, the point in the scene which was seen at $(x,y)$ now moves to another point $(x',y')$ given by the following theorem.

**Theorem 1**. The image transformation induced by camera rotation $\boldsymbol{R}=(r_{ij})$ is given by

$$x'=f\frac{r_{11}x+r_{21}y+r_{31}f}{r_{13}x+r_{23}y+r_{33}f}, \quad y'=f\frac{r_{12}x+r_{22}y+r_{32}f}{r_{13}x+r_{23}y+r_{33}f}. \tag{2.2}$$

*Proof.* A rotation of the camera by $R$ is equivalent to the rotation of the scene in the opposite sense. If the scene is rotated by $R^{-1}(=R^T)$, where $^T$ denotes transpose, point $(X,Y,Z)$ moves to point $(X',Y',Z')$ where

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}. \tag{2.3}$$

This point is projected to $(x',y')$ on the image plane, where $x'=fX'/Z'$ and $y'=fY'/Z'$. Combining this with eqns (2.1), we obtain eqn (2.2).

It should be emphasized that the image transformation due to camera rotation does not require any knowledge about the scene and that the transformation has an inverse, which is obtained by interchanging $R$ and $R^T$. This means that transformations of the form of eqn (2.2), which form a subgroup of the 2D *projective transformation group*, do not alter the information content of the image as long as the image boundary is ignored. (In this paper, we always regard the portion of the image near the boundary as *unimportant.*) In the following, some basic results from projective geometry are summarized in a way that is convenient in our consideration of the image plane transformation.

Suppose the image is characterized by a finite number of parameters $J_i$, $i=1,2,\ldots,N$, which we call *features* of the image (Amari [1, 2]). (They are called *properties* in Rosenfeld and Kak [9].) If the image is transformed by eqns (2.2) as a result of camera rotation $R$, these features take different values $J_i'$, $i=1,\ldots,N$. We say a set of features $J_i$, $i=1,\ldots,N$ is *invariant* if the values of $J_i'$, $i=1,\ldots,N$, are determined by the values of $J_i$, $i=1,\ldots,N$ and the amount of camera rotation $R$ alone.

This definition suggests that an invariant set of features is describing some aspects of the image that are "inherent to the scene itself" and are independent of the camera orientation (Weyl [15]).

Let $J_i$, $i=1, \ldots, N$, be an invariant set of features. We say the set is *reducible* if it splits, after an appropriate rearrangement, into two or more sets of features, each of which is itself invariant separately. If no further reduction is possible, we say the set of features is *irreducible*. This definition suggests that an irreducible invariant set of features is describing a "single" characteristic inherent to the scene while a reducible set describes two or more different characteristics at the same time (Weyl [15]).

If a quantity $c$ does not change its value under transformation (2.2), i.e.,

$$c'=c, \tag{2.4}$$

under camera rotation $R$, we call it a *scalar*. Obviously, a scalar is itself an invariant and is irreducible. Hence, it describes a characteristic inherent to the scene.

If a pair $a$, $b$ of numbers is transformed as $x$, $y$ of transformation (2.2), i.e.,

$$a'=f\frac{r_{11}a+r_{21}b+r_{31}f}{r_{13}a+r_{23}b+r_{33}f}, \quad b'=f\frac{r_{12}a+r_{22}b+r_{32}f}{r_{13}a+r_{23}b+r_{33}f}, \tag{2.5}$$

we call it a *point*. Note that any pair of numbers can be interpreted as a position *on the image plane*. However, it is interpreted as indicating a position *in the scene* if and only if it is transformed as a point. A point is also an invariant set of features and is irreducible.

A line on the image plane is expressed in the form

$$Ax+By+C=0. \tag{2.6}$$

Here, the ratio $A:B:C$ alone has a geometrical meaning; $A$, $B$, $C$ and $cA$, $cB$, $cC$ for a

non-zero scalar $c$ define one and the same line. In order to emphasize this fact, let us write $A:B:C$ to express a line. If transformation (2.2) is applied, line (2.6) is mapped into

$$A'x'+B'y'+C'=0, \tag{2.7}$$

as in the following theorem.

**Theorem 2.** A line $A:B:C$ on the image plane is transformed by camera rotation $R$ into the line

$$A':B':C'=r_{11}A+r_{21}B+r_{31}C/f:r_{12}A+r_{22}B+r_{32}C/f:f(r_{13}A+r_{23}B)+r_{33}C. \tag{2.8}$$

*Proof.* In view of eqns (2.1), eqn (2.6) is written as $A(fX/Z)+B(fY/Z)+C=0$, or

$$[A\ \ B\ \ C/f]\begin{bmatrix}X\\Y\\Z\end{bmatrix}=0. \tag{2.9}$$

From eqn (2.3), we find that $A$, $B$, $C/f$ are transformed as a vector, i.e.,

$$\begin{bmatrix}A'\\B'\\C'/f\end{bmatrix}=R^T\begin{bmatrix}A\\B\\C/f\end{bmatrix}, \tag{2.10}$$

from which eqn (2.8) is obtained.

If the ratio of three given quantities $A$, $B$, $C$ is transformed by eqn (2.8) under camera rotation, we call it a *line* and write it as $A:B:C$. It is an invariant set of features and is evidently irreducible. As in the case of a point, any triplet of numbers can be interpreted as a line on the image plane, but it is interpreted as a line in the scene if and only if it is transformed as a line.

All the invariant properties considered in this paper are invariant with respect to the "projective transformations" of the form of eqns (2.2). In traditional "projective

geometry", all equations are written in terms of "homogeneous coordinates" defined in a "projective space" (cf. Naeve and Eklundh [8]). If we regard the $xy$-image plane (with the "line at infinity" added) as a two-dimensional projective space and introduce homogeneous coordinates, eqns (2.2) are rewritten as a liner transformation. The "point" and "line" defined here are mutually "dual" and expressed exactly dually in homogeneous coordinates.

However, the purpose of this paper is to deal with applications of the ideas of projective geometry, and in dealing with real images the $xy$-Cartesian coordinate system is most convenient. Therefore, in the following, we express all the invariant properties in terms of the $xy$-"inhomogeneous" coordinates of the image plane. The aim of this paper is to translate the results known in projective geometry into "manageable" forms and to demonstrate the practical use of this type of knowledge.

## 3. IRREDUCIBLE REDUCTION OF 3D VECTORS AND TENSORS

Consider three quantities $a$, $b$, $c$ which are transformed as a *3D vector*, i.e.,

$$\begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} = R^T \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \tag{3.1}$$

for camera rotation $\boldsymbol{R}$. (Note that the rotation matrix $\boldsymbol{R}$ is tr nsposed because we adopted the convention that $\boldsymbol{R}$ is the amount of "camera rotation".) This is an invariant set of features but is not irreducible because

**Lemma 1**. If $a$, $b$, $c$ are transformed as a 3D vector, then the length $\sqrt{a^2+b^2+c^2}$ is a scalar.

There are two ways, mutually dual, to interpret a 3D vector $a,b,c$ as irreducible sets of features. One way is to regard $fa/c$, $fb/c$ as a point and the length $\sqrt{a^2+b^2+c^2}$ as its *intensity*, which is a scalar. We can easily check from Theorem 1 that

**Lemma 2**. If $a$, $b$, $c$ are transformed as a 3D vector, then $fa/c$, $fb/c$ are transformed as a point.

Hence a pair $fa/c$, $fb/c$ has an interpretation as a point invariant on the image plane in the sense described above. Here, we allow the case $c=0$, regarding it as a point located at infinity. We also make the convention that the intensity is negative if $c<0$. If we imagine that the 3D vector $(a,b,c)$ is emanating from the origin $O$ (or the camera focus) of the $XYZ$-coordinate system, the point $(fa/c,fb/c)$ is the intersection of the image plane with the ray defined by the 3D vector $(a,b,c)$.

Another way to represent a 3D vector on the image plane is to regard $a:b:fc$ as a line and the length $\sqrt{a^2+b^2+c^2}$ as its intensity. We can easily check from Theorem 2 that

**Lemma 3**. If $a$, $b$, $c$ are transformed as a 3D vector, then $a:b:fc$ is transformed as a line.

Hence, equation $ax+by+fc=0$ has an interpretation as a line invariant on the image plane in the sense described above. If we imagine that the 3D vector $(a,b,c)$ is emanating from the origin $O$ (or the camera focus) of the $XYZ$-coordinate system, the line $ax+by+fc=0$ is the intersection of the image plane with the plane passing at the origin $O$ and perpendicular to $(a,b,c)$. As before, we allow the case of $a=b=0$, regarding the line as located at infinity, and make the convention that the intensity is negative if

$c < 0$.

The above results are summarized as follows:

**Theorem 3**. A 3D vector is an invariant feature set. It can be irreducibly reduced into a point and a scalar or into a line and a scalar on the image plane.

Next, consider nine elements $A_{ij}$, $i,j=1,2,3$, which are transformed by camera rotation $R$ as a *3D tensor*, i.e.,

$$\begin{bmatrix} A'_{11} & A'_{12} & A'_{13} \\ A'_{21} & A'_{22} & A'_{23} \\ A'_{31} & A'_{32} & A'_{33} \end{bmatrix} = R^T \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} R. \tag{3.2}$$

By definition, this is an invariant set of features. However, it is reducible. First, it can be decomposed into a *symmetric part* and an *antisymmetric part* (or *skew part*):

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & (A_{12}+A_{21})/2 & (A_{31}+A_{13})/2 \\ (A_{12}+A_{21})/2 & A_{22} & (A_{23}+A_{32})/2 \\ (A_{31}+A_{13})/2 & (A_{23}+A_{32})/2 & A_{33} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & (A_{12}-A_{21})/2 & -(A_{31}-A_{13})/2 \\ -(A_{12}-A_{21})/2 & 0 & (A_{23}-A_{32})/2 \\ (A_{31}-A_{13})/2 & -(A_{23}-A_{32})/2 & 0 \end{bmatrix}, \tag{3.3}$$

and each part is transformed as a 3D tensor by eqn (3.2) separately. Moreover, it can be verified that the three independent elements $(A_{23}-A_{32})/2$, $(A_{31}-A_{13})/2$, $(A_{12}-A_{21})/2$ of the antisymmetric part are transformed as a 3D vector. Hence, they are, from Theorem 3, irreducibly reduced into a point and a scalar or into a line and a scalar.

Suppose $A=(A_{ij})$ is already a symmetric 3D tensor. As is well known, such a tensor is represented by three mutually perpendicular unit vectors $e_1$, $e_2$, $e_3$ indicating the principal axes and the corresponding principal values $\sigma_1$, $\sigma_2$, $\sigma_3$ in the form

$$A = \sigma_1 e_1 e_1{}^T + \sigma_2 e_2 e_2{}^T + \sigma_3 e_3 e_3{}^T . \tag{3.4}$$

Here, this representation does not change if $e_1$ (or $e_2$ or $e_3$) is replaced by $-e_1$ (or $-e_2$ or $-e_3$). (If two of $\sigma_1$, $\sigma_2$, $\sigma_3$ are identical, the corresponding principal axes are not unique and can be arbitrarily rotated rigidly around the remaining one. If all of $\sigma_1$, $\sigma_2$, $\sigma_3$ are identical, the orientations of $e_1$, $e_2$, $e_3$ are completely arbitrary as long as they are mutually orthogonal.)

The three principal values are scalars, each of which is an invariant irreducible feature. On the other hand, if we determine the orientations of two of the three principal axis orientations, say $e_1$ and $e_2$, the orientation of the remaining one is uniquely determined. ($e_3$ and $-e_3$ indicate the same orientation.) As is shown in Theorem 3, the orientations of $e_1$ and $e_2$ are represented by two points on the image plane. (If we replace $e_1$ (or $e_2$) by $-e_1$ (or $-e_2$), the corresponding points are unchanged as desired.) However, since $e_1$ and $e_2$ are perpendicular, one of the two points and the line connecting the two points are sufficient; if one point on the image plane and a line through it are given, the three orientations are determined (Appendix A). Thus, we obtain

**Theorem 4**. A 3D tensor is invariantly reduced to its symmetric part and its antisymmetric part. The antisymmetric part is irreducibly reduced into a point and a scalar or a line and a scalar. The symmetric part is irreducibly reduced to three scalars, a point and a line through it.

## 4. INFINITESIMAL GENERATORS OF THE IMAGE TRANSFORMATION

Let $F(x,y)$ represent an observed image. This many be the intensity of the gray-level or a vector-valued function corresponding to R, B and G. Here, the value of $F(x,y)$

is assumed to be inherent to the scene and independent of the viewing orientation. Color, for example, has this property. Furthermore, $F(x,y)$ is assumed to be of *finite support*, i.e., $F(x,y)$ is zero at a sufficiently large distance from the origin of the image plane.

Let us write the transformation of eqn (2.2), which is determined by the rotation matrix $\boldsymbol{R}$, symbolically as

$$(x',y')=M[R](x,y). \tag{4.1}$$

Then, we can see the (transposed) *homomorphism* in the sense that

$$M[R_2]\circ M[R_1]=M[R_1R_2]. \tag{4.2}$$

Now, define the *rotation operator* $T_R$ acting on image $F(x,y)$ by

$$T_RF(x,y)\equiv F(M[R^T](x,y)). \tag{4.3}$$

In view of our assumptions of image value constancy and finite support, the function $T_RF(x,y)$ describes the image we observe if the image plane undergoes the transformation (2.2). Operator $T_R$ induces a *representation* of the 3D rotation group $SO(3)$ in the sense that

$$T_{R_2R_1}=T_{R_2}\circ T_{R_1}. \tag{4.4}$$

As is well known, this representation is completely determined once its behavior for infinitesimal rotations (i.e., its *Lie algebra*) is known, since $SO(3)$ is a compact Lie group.

A 3D rotation is specified by the *rotation axis* $(n_1,n_2,n_3)$, which is taken to be a unit vector, and the *rotation angle* $\Omega$ (rad) screwwise around it. As is well known, the corresponding rotation matrix is given by

$$R=\begin{bmatrix} \cos\Omega+(1-\cos\Omega)n_1^2 & (1-\cos\Omega)n_1n_2-\sin\Omega n_3 & (1-\cos\Omega)n_1n_3+\sin\Omega n_2 \\ (1-\cos\Omega)n_2n_1+\sin\Omega n_3 & \cos\Omega+(1-\cos\Omega)n_2^2 & (1-\cos\Omega)n_2n_3-\sin\Omega n_1 \\ (1-\cos\Omega)n_3n_1-\sin\Omega n_2 & (1-\cos\Omega)n_3n_2+\sin\Omega n_1 & \cos\Omega+(1-\cos\Omega)n_3^2 \end{bmatrix} . \quad (4.5)$$

If the rotation is *infinitesimally small*, i.e., $\Omega$ is infinitesimally small, the rotation matrix takes the form $R=I+\delta R+o(\Omega)$, where $I$ is the unit matrix, $\delta R$ is the matrix given by

$$\delta R=\begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix} , \quad (4.6)$$

and $o(\Omega)$ denotes higher order terms of $\Omega$. (We let the context indicate whether these terms are scalars, vectors or tensors.) Here, we put $\Omega_1=\Omega n_1$, $\Omega_2=\Omega n_2$ and $\Omega_3=\Omega n_3$.

If the rotation is infinitesimal, the transformation of eqns (2.2) becomes $x'=x+\delta x+o(\Omega)$ and $y'=y+\delta y+o(\Omega)$, where

$$\delta x=-f\Omega_2+\Omega_3 y+\frac{1}{f}(-\Omega_2 x+\Omega_1 y)x, \quad \delta y=f\Omega_1-\Omega_3 x+\frac{1}{f}(-\Omega_2 x+\Omega_1 y)y. \quad (4.7)$$

Then, the image $F(x,y)$ also undergoes an infinitesimal change and becomes

$$F(x-\delta x,y-\delta y)=F(x,y)+\delta F(x,y)+o(\Omega), \quad (4.8)$$

and $\delta F(x,y)$ is given by

$$\begin{aligned} \delta F(x,y)&=-\frac{\partial F}{\partial x}\delta x-\frac{\partial F}{\partial y}\delta y \\ &=-\left[-f\Omega_2+\Omega_3 y+\frac{1}{f}(-\Omega_2 x+\Omega_1 y)x\right]\frac{\partial F}{\partial x}-\left[f\Omega_1-\Omega_3 x+\frac{1}{f}(-\Omega_2 x+\Omega_1 y)y\right]\frac{\partial F}{\partial y} \\ &=-(\Omega_1 D_1+\Omega_2 D_2+\Omega_3 D_3)F(x,y), \end{aligned} \quad (4.9)$$

where the *infinitesimal generators* are defined by

$$D_1=\frac{xy}{f}\frac{\partial}{\partial x}+(f+\frac{y^2}{f})\frac{\partial}{\partial y}, \quad D_2=-(f+\frac{x^2}{f})\frac{\partial}{\partial x}-\frac{xy}{f}\frac{\partial}{\partial y}, \quad D_3=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}. \quad (4.10)$$

Hence, operator $T_R$ becomes, for infinitesimal rotations,

$$T_R = I - (\Omega_1 D_1 + \Omega_2 D_2 + \Omega_3 D_3) + o(\Omega), \tag{4.11}$$

where $I$ is the identity operator.

It can be checked easily that these infinitesimal generators satisfy the *commutator relations*

$$[D_1, D_2] = D_3, \qquad [D_2, D_3] = D_1, \qquad [D_3, D_1] = D_2, \tag{4.12}$$

where the *commutator* is defined by $[A, B] \equiv AB - BA$. Hence, a set of functions can be found which induces a representation of the 3D rotation group $SO(3)$ [3, 4].

As is well known, a set of functions which induces an *irreducible representation* is obtained as eigenfunctions of the *Casimir operator*

$$H \equiv -(D_1^2 + D_2^2 + D_3^2). \tag{4.13}$$

The eigenvalue is $l(l+1)$ and the eigenspace is $2l+1$ dimensional, where $l$ is an integer or half-integer called the *weight* of the irreducible representation (cf. Gel'fand, et al. [3], Hammermesh [4]). In other words, the differential equation

$$HF = l(l+1)F, \quad \text{or} \quad (D_1^2 + D_2^2 + D_3^2)F + l(l+1)F = 0, \tag{4.14}$$

has $2l+1$ independent solutions, which become the basis of the irreducible representation $D_l$ of weight l (Appendix B).

## 5. ADJOINT ROTATION AND FEATURE TRANSFORMATION

Let $J$ be a feature of the image. To be precise, a feature is a *functional* mapping the image function $F(x,y)$ into a real number $J[F(x,y)]$. Consider a *linear* feature obtained by weighted averaging or *filtering*:

$$J[F(x,y)]=\int m(x,y)F(x,y)dxdy. \tag{5.1}$$

Here, $m(x,y)$ is the filter weight function and integration is performed over the entire image plane. (Recall our assumption of finite support of $F(x,y)$.) If the camera is rotated by $\boldsymbol{R}$, the image becomes $\boldsymbol{T_R}F(x,y)$ by eqn (4.2) and hence the corresponding feature becomes

$$J[\boldsymbol{T_R}F(x,y)]=\int m(x,y)\boldsymbol{T_R}F(x,y)dxdy. \tag{5.2}$$

We define the *adjoint rotation operator* $\boldsymbol{T_R^*}$ by

$$J[\boldsymbol{T_R}F(x,y)]=\int \boldsymbol{T_R^*}m(x,y)F(x,y)dxdy. \tag{5.3}$$

From this definition, we can see that operator $\boldsymbol{T_R^*}$ induces an adjoint representation of the 3D rotation group in the sense that

$$\boldsymbol{T^*_{R_2R_1}}=\boldsymbol{T^*_{R_1}}\circ\boldsymbol{T^*_{R_2}}. \tag{5.4}$$

Once we know how this adjoint rotation operator $\boldsymbol{T_R^*}$ acts, the transformation of such features is immediately computed for any given image. This is done by just considering infinitesimal transformations.

If the image is infinitesimally changed as in eqn (4.8), feature $J$ also undergoes an infinitesimally small change $J\rightarrow J+\delta J+o(\Omega)$. Substitution of eqn (4.9) and integration by parts yield

$$\delta J=\int(\Omega_1 D_1^*+\Omega_2 D_2^*+\Omega_3 D_3^*)m(x,y)F(x,y)dxdy, \tag{5.5}$$

where $D_1^*$, $D_2^*$ and $D_3^*$ are the *adjoint infinitesimal generators* defined by

$$D_1^*=\frac{3y}{f}+\frac{xy}{f}\frac{\partial}{\partial x}+(f+\frac{y^2}{f})\frac{\partial}{\partial y},\quad D_2^*=-\frac{3x}{f}-(f+\frac{x^2}{f})\frac{\partial}{\partial x}-\frac{xy}{f}\frac{\partial}{\partial y},$$

$$D_3^* = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \tag{5.6}$$

In eqn (5.5), no boundary terms appear due to our assumption of finite support for $F(x,y)$. Hence, operator $T_R^*$ becomes, for infinitesimal rotations,

$$T_R^* = I + \Omega_1 D_1^* + \Omega_2 D_2^* + \Omega_3 D_3^* + o(\Omega). \tag{5.7}$$

It can be checked easily that these adjoint infinitesimal generators satisfy the commutator relations

$$[D_1^*, D_2^*] = D_3^*, \qquad [D_2^*, D_3^*] = D_1^*, \qquad [D_3^*, D_1^*] = D_2^*. \tag{5.8}$$

Hence, we can find a set of functions which induces a representation of the 3D rotation group SO(3). Then, operator $T_R^*$ acts as a linear transformation on them (cf. Gel'fand, et al. [3]). As before, a basis of the irreducible representation $D_l$ of weight $l$ is obtained as $2l+1$ eigenfunctions of the (adjoint) Casimir operator

$$H^* \equiv -(D_1^{*2} + D_2^{*2} + D_3^{*2}), \tag{5.9}$$

i.e., as $2l+1$ independent solutions of the differential equation

$$H^* m = l(l+1)m, \qquad \text{or} \qquad (D_1^{*2} + D_2^{*2} + D_3^{*2})m + l(l+1)m = 0. \tag{5.10}$$

From Appendix C, we find that

$$J = \int \frac{F(x,y)\,dx\,dy}{\sqrt{(x^2+y^2+f^2)^3}} \tag{5.11}$$

is an invariant (i.e., it is transformed as a scalar). This implies that

$$\rho(x,y) = \frac{1}{\sqrt{(x^2+y^2+f^2)^3}} \tag{5.12}$$

is an *invariant measure* (Appendix D).

We also see from Appendix C that

$$J_1=\int \frac{xF(x,y)dxdy}{(x^2+y^2+f^2)^2},\quad J_2=\int \frac{yF(x,y)dxdy}{(x^2+y^2+f^2)^2},\quad J_3=\int \frac{fF(x,y)dxdy}{(x^2+y^2+f^2)^2} \tag{5.13}$$

are transformed as a 3D vector. Hence, they are irreducibly reduced to a scalar $\sqrt{(J_1)^2+(J_2)^2+(J_3)^2}$ and a point $fJ_1/J_3$, $fJ_2/J_3$ (or a line $J_1{:}J_2{:}fJ_3$) on the image plane. This scalar and point (or line) are invariant in the sense that they describe characteristics inherent to the scene.

Also from Appendix C, we find that

$$\begin{aligned}
J_{11}&=\int \frac{x^2F(x,y)dxdy}{\sqrt{(x^2+y^2+f^2)^5}},\quad J_{12}=\int \frac{xyF(x,y)dxdy}{\sqrt{(x^2+y^2+f^2)^5}},\quad J_{13}=\int \frac{fxF(x,y)dxdy}{\sqrt{(x^2+y^2+f^2)^5}},\\
J_{21}&=\int \frac{xyF(x,y)dxdy}{\sqrt{(x^2+y^2+f^2)^5}},\quad J_{22}=\int \frac{y^2F(x,y)dxdy}{\sqrt{(x^2+y^2+f^2)^5}},\quad J_{23}=\int \frac{fyF(x,y)dxdy}{\sqrt{(x^2+y^2+f^2)^5}},\\
J_{31}&=\int \frac{fxF(x,y)dxdy}{\sqrt{(x^2+y^2+f^2)^5}},\quad J_{33}=\int \frac{fyF(x,y)dxdy}{\sqrt{(x^2+y^2+f^2)^5}},\quad J_{33}=\int \frac{f^2F(x,y)dxdy}{\sqrt{(x^2+y^2+f^2)^5}}
\end{aligned} \tag{5.14}$$

are transformed as a 3D (symmetric) tensor. Hence, they are irreducibly reduced to three scalars, a point and a line through it on the image plane. They are invariant and describe characteristics inherent to the scene.

## 6. INVARIANT CHARACTERIZATION OF A SHAPE

As an application of the results in the previous sections, let us consider the characterization of a shape on the image plane. Consider a region $S$ on the image plane. Its characteristic function

$$F(x,y)=\begin{cases}1 & \text{if}(x,y)\epsilon S\\ 0 & \text{otherwise}\end{cases} \tag{6.1}$$

is taken as the image function $F(x,y)$.

The simplest characteristic of the region $S$ may be its area

$$\overline{S}=\int_S dxdy(=\int F(x,y)dxdy). \tag{6.2}$$

However, this area is not invariant with respect to camera rotation. Suppose the region $S$ is located far away from the image origin. If we move it so that it comes to the center of the image plane by appropriately controlling the camera orientation, the area of eqn (6.2) changes. Consequently, eqn (6.2) is not considered to be a characteristic inherent to the scene itself. In short, eqn (6.2) is not a scalar.

On the other hand, if eqn (6.2) is replaced by

$$C=f^3\int_S \frac{dxdy}{\sqrt{(x^2+y^2+f^2)^3}}, \tag{6.3}$$

this is a scalar as was shown in the previous section. If $S$ is a small region located around the image origin, i.e., $x\approx 0$, $y\approx 0$ in $S$, then $C$ is approximately equal to its area. We call $C$ the *invariant area* of region $S$. It is interpreted as the area the region would have if the region were moved to the center of the image plane by changing the camera orientation. Geometrically speaking, this quantity is nothing but an expression of the solid angle the object makes with respect to the viewer.

Another simple but important characteristic is the center of gravity of the region $S$:

$$\overline{x}=\int_S xdxdy / \int_S dxdy, \qquad \overline{y}=\int_S ydxdy / \int_S dxdy. \tag{6.4}$$

Again these quantities do not have invariant meanings. Namely, if region $S$ is moved to another region by camera rotation and $(\overline{x}',\overline{y}')$ is its center of gravity, $(\overline{x},\overline{y})$ is not mapped into $(\overline{x}',\overline{y}')$ by the same camera rotation. In short, $\overline{x}$, $\overline{y}$ is not a point.

On the other hand, we know from the previous section that

$$a_1=\int_S \frac{x\,dx\,dy}{(x^2+y^2+f^2)^2},\quad a_2=\int_S \frac{y\,dx\,dy}{(x^2+y^2+f^2)^2},\quad a_3=f\int_S \frac{dx\,dy}{(x^2+y^2+f^2)^2} \tag{6.5}$$

are transformed as a 3D vector. Hence, $fa_1/a_3$, $fa_2/a_3$ are transformed as a point. If the region $S$ is a small region located around the image origin and $x\approx 0$, $y\approx 0$ in $S$, then $(fa_1/a_3, fa_2/a_3)$ is approximately the center of gravity of the region. We call $(fa_1/a_3, fa_2/a_3)$ the *invariant center of gravity* of region $S$. It is interpreted as the point which would be mapped into the center of gravity if the region were moved to the center of the image plane by changing the camera orientation. Geometrically, this point corresponds to the center of the solid angle the object makes with respect to the viewer.

Another useful characteristic is the moment tensor $(M_{ij})$, $i,j=1,2$, defined by

$$M_{11}=\int_S (x-\overline{x})^2 dx\,dy,\quad M_{12}=M_{21}=\int_S (x-\overline{x})(y-\overline{y})dx\,dy,\quad M_{22}=\int_S (y-\overline{y})^2 dx\,dy. \tag{6.6}$$

Its principal values indicate the amount of elongation of the region $S$ along the corresponding principal axes. However, as described above, this tensor does not have invariant properties. Namely, the principal values of $(M_{ij})$ are not scalars, and its principal axes are not lines on the image plane.

On the other hand, we know from the previous section that

$$\begin{aligned}
&B_{11}=\int_S \frac{x^2 dx\,dy}{\sqrt{(x^2+y^2+f^2)^5}},\quad B_{12}=\int_S \frac{xy\,dx\,dy}{\sqrt{(x^2+y^2+f^2)^5}},\quad B_{13}=f\int_S \frac{x\,dx\,dy}{\sqrt{(x^2+y^2+f^2)^5}},\\
&B_{21}=\int_S \frac{xy\,dx\,dy}{\sqrt{(x^2+y^2+f^2)^5}},\quad B_{22}=\int_S \frac{y^2 dx\,dy}{\sqrt{(x^2+y^2+f^2)^5}},\quad B_{23}=f\int_S \frac{y\,dx\,dy}{\sqrt{(x^2+y^2+f^2)^5}},\\
&B_{31}=f\int_S \frac{x\,dx\,dy}{\sqrt{(x^2+y^2+f^2)^5}},\quad B_{32}=f\int_S \frac{y\,dx\,dy}{\sqrt{(x^2+y^2+f^2)^5}},\quad B_{33}=f^2\int_S \frac{dx\,dy}{\sqrt{(x^2+y^2+f^2)^5}}
\end{aligned} \tag{6.7}$$

are transformed as a 3D (symmetric) tensor. Since this tensor is positive definite as long as region $S$ is not empty, it has three positive principal values $\sigma_1$, $\sigma_2$, $\sigma_3$. Let $\sigma_3$ be the

maximum principal value. Let $e_1$, $e_2$, $e_3$ be the corresponding unit eigenvectors (determined except for sign). Let $(g_1,g_2)$ be the point corresponding to vector $e_3$. Let $l_1$ be the line through $(g_1,g_2)$ and the point corresponding to vector $e_1$ (or the line representing vector $e_2$). Similarly, let $l_2$ be the line through $(g_1,g_2)$ and the point corresponding to vector $e_2$ (or the line representing vector $e_1$). By our method of construction, scalars $\sigma_1$, $\sigma_2$, point $(g_1,g_2)$ and lines $l_1$, $l_2$ are all invariant quantities. It can be checked that lines $l_1$, $l_2$ are approximately the principal axes, and $\sigma_1$, $\sigma_2$ are approximately the corresponding principal values if $S$ is a sufficiently small region around the origin. Hence, scalars $\sigma_1$ and $\sigma_2$ are the principal values the region would have if it were moved to the center of the image plane by camera rotation, and $l_1$, $l_2$ are lines which would be mapped onto the principal axes. We call point $(g_1,g_2)$ the *invariant center of inertia*, lines $l_1$, $l_2$ the *invariant principal axes*, and $\sigma_1$, $\sigma_2$ the corresponding *invariant principal values*.

## 7. INVARIANTS AND CAMERA ROTATION RECONSTRUCTION

In the previous section, scalar $C$ defined by eqn (6.3), 3D vector $\boldsymbol{a}=(a_i)$ defined by eqns (6.5) and 3D tensor $B=(B_{ij})$ defined by eqns (6.7) are interpreted as a set of two dimensional invariant quantities on the image plane. Here, let us consider their three dimensional aspects.

First, since $C$, $\boldsymbol{a}$ and $B$ are transformed as a scalar, a vector and a tensor, respectively, by camera rotation, we can extract *invariants* that do not change their values when the camera is rotated. Obviously, scalar $C$ itself is an invariant.

Second, since $\boldsymbol{a}$ is a 3D vector, it has, as was discussed in Section 3, only one invariant, namely its *length* $\|\boldsymbol{a}\|$, or equivalently $\boldsymbol{a}^T\boldsymbol{a}$.

On the other hand, $B$ is a 3D symmetric tensor, and hence it has, as was described in Section 3, three invariants, namely the three principal values $\sigma_1$, $\sigma_2$, $\sigma_3$, or equivalently any three independent algebraic expressions formed from them such as the *fundamental symmetric forms* $\sigma_1+\sigma_2+\sigma_3$, $\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_2$, $\sigma_1\sigma_2\sigma_3$. In terms of the components of the original tensor $B$, they are respectively

$$B_{11}+B_{22}+B_{33}(=\mathrm{Tr}(B)),\quad \begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix}+\begin{vmatrix} B_{22} & B_{23} \\ B_{32} & B_{33} \end{vmatrix}+\begin{vmatrix} B_{33} & B_{31} \\ B_{13} & B_{11} \end{vmatrix},$$

$$\begin{vmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{vmatrix}(=\det B). \tag{7.1}$$

Alternatively, we can use $\sigma_1+\sigma_2+\sigma_3$, ${\sigma_1}^2+{\sigma_2}^2+{\sigma_3}^2$ and ${\sigma_1}^3+{\sigma_2}^3+{\sigma_3}^3$. This set is equal to

$$\mathrm{Tr}(B),\qquad \mathrm{Tr}(B^2),\qquad \mathrm{Tr}(B^3). \tag{7.2}$$

Finally, there are invariants describing the relationship between 3D vector $\boldsymbol{a}$ and 3D tensor $B$. As was discussed in Section 3, a 3D vector is geometrically thought of as a directed axis to which its length is attached and a 3D symmetric tensor as three mutually perpendicular (undirected) axes to which their respective principal values are attached. Now that the length and the principal values have been counted, the remaining invariants are those specifying the orientation of the vector relative to the three mutually perpendicular axes. Hence, two invariants exist. We can choose, say, $\boldsymbol{a}^T B\boldsymbol{a}$ and $\boldsymbol{a}^T B^2\boldsymbol{a}$ (Smith [10], Spencer [11], Wang [12 - 14].) Of course, the choice is not unique as stated above, and other choices are also possible.

We say that two regions $S$ and $S'$ on the image plane are *equivalent* if one region can be transformed into the other by camera rotation, i.e., by changing the camera orientation. If the two regions are equivalent, the above invariants must have identical

values. If they have different values, the two regions cannot be equivalent. On the other hand, if the two regions are known to be equivalent, the camera rotation which would take one region into the other can be reconstructed by observing the invariant center of gravity and the invariant moment tensor alone. This is done as follows.

Suppose we observe $\boldsymbol{a}$ and $B$ for region $S$ and $\boldsymbol{a}'$ and $B'$ for region $S'$. Assume that $B$ (hence $B'$ as well) has three distinct eigenvalues and $\boldsymbol{a} \neq \boldsymbol{0}$. Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ be the associated eigenvectors of $B$. Since the eigenvectors are determined except for sign and magnitude, choose one set such that $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ are mutually perpendicular unit vectors forming a right-hand system in that order. Construct a matrix $R_1$ having $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ as its columns in that order. Let $\boldsymbol{e}'_1$, $\boldsymbol{e}'_2$, $\boldsymbol{e}'_3$ be the corresponding unit eigenvectors of $B'$ forming a right-hand system. Since the signs of the eigenvectors are arbitrary, there are four possibilities to make a right-hand system. For each case, construct the corresponding matrix $R_2$. Then, the rotation matrix which transforms $B$ to $B'$ is given by

$$R = R_1 R_2^T . \tag{7.3}$$

(Matrix $B$ is first transformed by $R_1^{-1}(=R_1^T)$ into a diagonal matrix, which in turn is transformed to $B'$ by $R_2$.) Finally, choose one our of those eight possible $R$s that transforms $\boldsymbol{a}$ to $\boldsymbol{a}'$.

If $B$ (hence $B'$ as well) has only two distinct eigenvalues (a single root and a pair of multiple roots), let $\boldsymbol{e}_1$ be the eigenvector associated with the single root. Suppose $\boldsymbol{a}$ is neither parallel nor perpendicular to $\boldsymbol{e}_1$. Since the sign of $\boldsymbol{e}_1$ is arbitrary, choose it so that $\boldsymbol{a}$ and $\boldsymbol{e}_1$ make an acute angle. Then, we can construct three mutually orthogonal vectors forming a right-hand system $\boldsymbol{e}_1$, $\boldsymbol{e}_2 = \boldsymbol{e}_1 \times \boldsymbol{a} / \| \boldsymbol{e}_1 \times \boldsymbol{a} \|$, $\boldsymbol{e}_3 = \boldsymbol{e}_1 \times \boldsymbol{e}_2$. We can form $R_1$ and $R_2$ as described above, and the desired rotation is given by eqn (7.3). If $\boldsymbol{a}$ is perpendicular to $\boldsymbol{e}_1$, there exist two solutions. If $\boldsymbol{a}$ is parallel to $\boldsymbol{e}_1$, or if $B$ (hence $B'$ as

well) has one eigenvalue (i.e., $B(=B')$ is a multiple of $I$), $R$ is any rotation that maps $a$ to $a'$ and we can add any rotation around $a'$. The case where $a=a'=0$ is treated similarly. These observations can be summarized as follows:

**Theorem 5.**

$$C, \quad a^T a, \quad \mathrm{Tr}(B), \quad \mathrm{Tr}(B^2), \quad \mathrm{Tr}(B^3), \quad a^T Ba, \quad a^T B^2 a \tag{7.4}$$

exhaust all the invariants constructed from $C$, $a$ and $B$. If two regions are equivalent, the amount of camera rotation which take one region into the other can be reconstructed from $a$ and $B$ alone.

An important fact is that both the equivalence test and the camera rotation reconstruction *do not require knowledge of point-to-point correspondence*, since the computation is solely based on the features (6.3), (6.5), (6.6), which are obtained by integration over the regions under consideration.

Theoretically, the camera rotation is exactly reconstructed as described above. In practice, however, the invariant center of gravity $(fa_1/a_3, fa_2/a_3)$ and the invariant center of inertia $(g_1, g_2)$ are usually located very near, and vector $a$ and vector $e_3$ are very close to each other. Therefore, the last step of choosing one out of four possible $R$s by checking $Ra$ may become difficult if much noise is involved. In this case, the final choice is done by applying the transformation (2.2) to region $S$ in four ways and choosing the one which make region $S$ sufficiently overlapping $S'$. (Since we are focusing on the principal axes, the four possibilities correspond to the four possible (skewed) "mirror image" (including identity) with respect to the principal axes.)

**Example**. Consider the three regions $S_0$, $S_1$, $S_2$ on the image plane (Fig. 2(a)). We use a scaling such that the focal length $f$ is unity. Computing the integrations of eqns (6.5) and (6.7), we find their invariant centers of gravity (Fig. 2(b)) and principal axes (Fig. 2(c)) as follows:

| $S_0$ | $S_1$ | $S_2$ |
|---|---|---|
| (-0.081,-0.202) | (0.464,0.076) | (-0.470,0.346) |
| $y=-2.814x-0.431$ | $y=1.607x-0.697$ | $y=-0.079x+0.310$ |
| $y=0.382x-0.171$ | $y=-0.476x-0.297$ | $y=-16.522x-7.424$ |

The invariants of (7.4) become as follows:

| | $S_0$ | $S_1$ | $S_2$ |
|---|---|---|---|
| $C$ | 0.1440 | 0.1440 | 0.1121 |
| $\boldsymbol{a}^T\boldsymbol{a}$ | 0.0202 | 0.0202 | 0.0123 |
| $\mathrm{Tr}(B)$ | 0.1440 | 0.1440 | 0.1121 |
| $\mathrm{Tr}(B^2)$ | 0.0197 | 0.0197 | 0.0121 |
| $\mathrm{Tr}(B^3)$ | 0.0028 | 0.0028 | 0.0013 |
| $\boldsymbol{a}^T B\boldsymbol{a}$ | 0.0028 | 0.0028 | 0.0014 |
| $\boldsymbol{a}^T B^2\boldsymbol{a}$ | 0.0004 | 0.0004 | 0.0001 |

From this result, we can conclude that regions $S_0$ and $S_1$ are equivalent but region $S_2$ is not equivalent to either. (Here, the data are exact up to rounding. If the data are affected by a large amount of error, a statistical method such as hypothesis testing becomes necessary.) By the procedure described in the previous section, the camera rotation which maps region $S_0$ onto region $S_1$ is reconstructed to be

$$R=\begin{bmatrix} 0.573 & -0.761 & -0.296 \\ 0.567 & 0.631 & -0.530 \\ 0.591 & 0.136 & 0.795 \end{bmatrix}.$$

This is the rotation around the axis of orientation (0.384, 0.512,0.768) by angle 60° screwwise.

## 8. CONCLUDING REMARKS

In this paper, we have presented invariant properties of an image with respect to camera rotation, introducing the notions of "invariance" and "irreducibility" and translating results from projective geometry in terms of the (inhomogeneous) image coordinate system. We also gave an example, computing the invariant center of gravity and the invariant principal axes and reconstructing the camera rotation. The procedure does not require the knowledge of point-to-point correspondence on the image plane. Many other applications are also possible.

Consider the problem of shape recognition. Suppose we have a reference image obtained from a certain camera orientation. If a test image is obtained from a different camera orientation, the two images cannot be compared directly due to projective distortion. However, Theorem 5 provides an easy test for their equivalence. Namely, as is also shown in the previous example, if the invariants of (7.4) have different values, the two region cannot be equivalent and the test shape is rejected.

If $C$, $\boldsymbol{a}$ and $B$ alone are sufficient to characterize the set of test shapes in question completely, the equivalence is already determined at this stage. Otherwise, we can move the test shape into the position of the reference shape in such a way that both have the same $\boldsymbol{a}$ and $B$. Then, the rest of the shape characteristics are compared to test for the equivalence. The necessary camera rotation is reconstructed as described in the previous section, and the corresponding image transformation is performed either by actually moving the camera or by numerically computing the image transformation (2.2).

We say that a region on the image plane is in the *standard position*, if the invariant center of inertia center $(g_1, g_2)$ coincides with the origin of the image plane and the invariant principal axes coincide with the $x$- and $y$-axes. Any region on the image plane

can be moved into the standard position by camera rotation $R$ such that (i) $B$ is diagonalized in the form

$$R^T B R = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix},$$

where $\sigma_3$ is the largest principal value and (ii) if

$$R^T a = \begin{bmatrix} a'_1 \\ a'_2 \\ a'_3 \end{bmatrix}$$

then $a'_3 > 0$.

Evidently, shape recognition becomes easier if the test shapes are always moved into the standard position (either by actually rotating the camera or by computation). However, this technique is not restricted to shape recognition. If a camera is tracking a moving object while the camera position is fixed, or if a camera attached to a robot or an autonomous vehicle is aiming at a fixed object in the stationary scene, the technique described above can be used so that the object in question is always seen in the standard position.

On the other hand, testing the equivalence is also viewed as detecting *active motion*. When an object image moves on the image plane, we call the motion *passive* if that motion is induced by camera rotation alone and *active* otherwise. When the camera orientation is changed, object images move on the image plane, but those objects may also have moved in the scene independently of the camera. According to the procedure described above, we can detect active motion even if the angle and orientation of camera rotation is not known. If the corresponding two object images are not equivalent, the object must have moved actively. If they are equivalent, the object has not moved in

the scene, although motion is observed on the image plane. In the previous example, if three regions $S_0$, $S_1$, $S_2$ are images of the same object, we can conclude that an active motion took place between $S_0$ (or $S_1$) and $S_2$ while no such motion took place between $S_0$ and $S_1$.

Another possible application is camera orientation registration. Even if the camera is rotated by an unknown angle around an unknown axis, the camera orientation can be determined as long as one particular region corresponding to a stationary object is identified on the image plane before and after the camera rotation. Thus, the principle we have described has a wide range of applications to many problems.

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## APPENDIX A RECIPROCITY AND CONJUGACY

Consider a line $l$ on the image plane which does not pass through the origin. Let

$$x\cos\theta+y\sin\theta=d \tag{A.1}$$

($d>0$) be its equation. We say that point

$$P(-\frac{f^2}{d}\cos\theta,-\frac{f^2}{d}\sin\theta) \tag{A.2}$$

is *reciprocal* to line $l$ with respect to the origin. Conversely, line $l$ is said to be recipro-

cal to point $P$ with respect to the origin. In other words, if we draw a line through the origin and perpendicular to line $l$, and if $d$ is the distance between the origin and line $l$, the reciprocal point $P$ is located on the other side of the perpendicular line and at distance $f^2/d$ from the origin (Fig A1). If $d=0$, point $P$ is interpreted as located at infinity (at $(\cos\theta,\sin\theta,0)$ in homogeneous coordinates), and similarly the line at infinity is regarded as the reciprocal line of the origin $O$.

Consider a line $l$ and a point $P$ on it on the image plane. Let $H$ be the foot of the perpendicular line drawn from the origin to line $l$, and let $d$ be the distance between point $P$ and point $H$. Consider a point $Q$ on the other side of line $l$ at distance $f^2/d$ from point $H$ (Fig. A2). We say that point $Q$ is *conjugate* to point $P$ on line $l$ and conversely point $P$ is conjugate to point $Q$ on line $l$. If $d=0$, $Q$ is regarded as located at infinity.

As stated in Theorem 3, a 3D vector is represented as a point or as a line on the image plane. By definition, the point and the line are easily shown to be mutually reciprocal. Hence, if one is known, the other is obtained immediately.

As stated in Theorem 4, a 3D symmetric tensor is represented by three scalars, a point and a line through it. Let $e_1$, $e_2$, $e_3$ be the unit vectors of the principal axes (determined up to sign). Let $P_1$, $P_2$, $P_3$ be the points corresponding to them, and let $l_1$ be the line connecting points $P_2$ and $P_3$, $l_2$ the line connecting points $P_3$ and $P_1$, and $l_3$ the line connecting points $P_1$ and $P_2$ (Fig. A3). Then, it is easy to see that point $P_1$ and line $l_1$ are mutually reciprocal, and so are point $P_2$ and line $l_2$, and point $P_3$ and line $l_3$. It is also seen that points $P_2$ and $P_3$, points $P_3$, $P_1$, and points $P_1$ and $P_2$ are conjugate on lines $l_1$, $l_2$ and $l_3$, respectively. Hence, if point $P_3$ and line $l_1$ are given, line $l_3$ and point $P_1$ are obtained as their reciprocals. Point $P_2$ is given as the intersection between

lines $l_1$ and $l_3$, and line $l_2$ is given as the line connecting points $P_3$ and $P_1$. Thus, a point and a line passing through it are sufficient to represent on the image plane the orientations of the three principal axes of a 3D symmetric tensor.

## APPENDIX B FUNCTION BASIS OF IRREDUCIBLE REPRESENTATIONS

From eqns (4.10), the Casimir operator becomes

$$\begin{aligned} \mathit{II} = &-(f+\frac{x^2+y^2}{f})[(f+\frac{x^2}{f})\frac{\partial^2}{\partial x^2}+\frac{2xy}{f}\frac{\partial^2}{\partial x\,\partial y}+(f+\frac{y^2}{f})\frac{\partial^2}{\partial y^2}] \\ &-(f+x+\frac{2x(x^2+y^2)}{f^2})\frac{\partial}{\partial x}-(f+y+\frac{2y(x^2+y^2)}{f^2})\frac{\partial}{\partial y}, \end{aligned} \tag{B.1}$$

so that eqn (4.14) becomes

$$\begin{aligned} &(f+\frac{x^2+y^2}{f})[(f+\frac{x^2}{f})F_{xx}+\frac{2xy}{f}F_{xy}+(f+\frac{y^2}{f})F_{yy}] \\ &+(f+x+\frac{2x(x^2+y^2)}{f})F_x+(f+x+\frac{2y(x^2+y^3)}{f})F_y+l(l+1)F=0. \end{aligned} \tag{B.2}$$

Since representations of half-integer weights are not interesting because the same image must be obtained after a rotation of $2\pi$ (the sign is reversed after a rotation of $2\pi$ if the weight is a half-integer), we consider only irreducible representations of integer weights.

For $l=0$ ($l(l+1)=0$), one solution ($2l+1=1$) is easily found:

$$F_0{}^1(x,y)=1. \tag{B.3}$$

Obviously, this is invariant with respect to rotation:

$$D_1F_0{}^1=0, \qquad D_2F_0{}^1=0, \qquad D_3F_0{}^1=0, \tag{B.4}$$

and hence

$$T_R F_0{}^1 = F_0{}^1. \tag{B.5}$$

For $l=1$ ($l(l+1)=2$), the following three solutions ($2l+1=3$) are found:

$$F_1{}^1(x,y)=\frac{x}{\sqrt{x^2+y^2+f^2}},\quad F_1{}^2(x,y)=\frac{y}{\sqrt{x^2+y^2+f^2}},\quad F_1{}^3(x,y)=\frac{f}{\sqrt{x^2+y^2+f^2}}. \tag{B.6}$$

Application of the infinitesimal generators $D_1$, $D_2$, $D_3$ yields

$$D_1\begin{bmatrix} F_1{}^1 \\ F_1{}^2 \\ F_1{}^3 \end{bmatrix} = -A_1\begin{bmatrix} F_1{}^1 \\ F_1{}^2 \\ F_1{}^3 \end{bmatrix},\quad D_2\begin{bmatrix} F_1{}^1 \\ F_1{}^2 \\ F_1{}^3 \end{bmatrix} = -A_2\begin{bmatrix} F_1{}^1 \\ F_1{}^2 \\ F_1{}^3 \end{bmatrix},$$

$$D_3\begin{bmatrix} F_1{}^1 \\ F_1{}^2 \\ F_1{}^3 \end{bmatrix} = -A_3\begin{bmatrix} F_1{}^1 \\ F_1{}^2 \\ F_1{}^3 \end{bmatrix}. \tag{B.7}$$

where

$$A_1=\begin{bmatrix} & & \\ & & -1 \\ & 1 & \end{bmatrix},\quad A_2=\begin{bmatrix} & & 1 \\ & & \\ -1 & & \end{bmatrix},\quad A_3=\begin{bmatrix} & -1 & \\ 1 & & \\ & & \end{bmatrix}, \tag{B.8}$$

and the commutator relations are satisfied:

$$[A_1,A_2]=A_3,\qquad [A_2,A_3]=A_1,\qquad [A_3,A_1]=A_2. \tag{B.9}$$

Consequently, for infinitesimal rotations, we have

$$T_R\begin{bmatrix} F_1{}^1 \\ F_1{}^2 \\ F_1{}^3 \end{bmatrix} = I+(\Omega_1 A_1+\Omega_2 A_2+\Omega_3 A_3)\begin{bmatrix} F_1{}^1 \\ F_1{}^2 \\ F_1{}^3 \end{bmatrix} + o(\Omega). \tag{B.10}$$

This implies that $F_1{}^1$, $F_1{}^2$ are transformed as

$$T_R\begin{bmatrix} F_1{}^1 \\ F_1{}^2 \\ F_1{}^3 \end{bmatrix} = R\begin{bmatrix} F_1{}^1 \\ F_1{}^2 \\ F_1{}^3 \end{bmatrix}. \tag{B.11}$$

For $l=2$ ($l(l+1)=6$), the following five solutions ($2l+1=5$) are found:

$$F_2{}^1(x,y)=\frac{2x^2-y^2-f^2}{3(x^2+y^2+f^2)},\quad F_2{}^2(x,y)=\frac{-x^2+2y^2-f^2}{3(x^2+y^2+f^2)},$$
$$F_2{}^3(x,y)=\frac{xy}{x^2+y^2+f^2},\quad F_2{}^4(x,y)=\frac{fy}{x^2+y^2+f^2},\quad F_2{}^5(x,y)=\frac{fx}{x^2+y^2+f^2}. \tag{B.12}$$

Application of the infinitesimal generators $D_1$, $D_2$, $D_3$ yields

$$D_1\begin{bmatrix}F_2{}^1\\F_2{}^2\\F_2{}^3\\F_2{}^4\\F_2{}^5\end{bmatrix}=-A_1\begin{bmatrix}F_2{}^1\\F_2{}^2\\F_2{}^3\\F_2{}^4\\F_2{}^5\end{bmatrix},\quad D_2\begin{bmatrix}F_2{}^1\\F_2{}^2\\F_2{}^3\\F_2{}^4\\F_2{}^5\end{bmatrix}=-A_2\begin{bmatrix}F_2{}^1\\F_2{}^2\\F_2{}^3\\F_2{}^4\\F_2{}^5\end{bmatrix},\quad D_3\begin{bmatrix}F_2{}^1\\F_2{}^2\\F_2{}^3\\F_2{}^4\\F_2{}^5\end{bmatrix}=-A_3\begin{bmatrix}F_2{}^1\\F_2{}^2\\F_2{}^3\\F_2{}^4\\F_2{}^5\end{bmatrix}, \tag{B.13}$$

where

$$A_1=\begin{bmatrix}&&&&-2\\&&&-1&\\&&1&&\\1&2&&&\end{bmatrix},\quad A_2=\begin{bmatrix}&&&2&\\&&&&1\\-2&-1&&&\\&&-1&&\end{bmatrix},\quad A_3=\begin{bmatrix}&&-2&&\\&&2&&\\1&-1&&&\\&&&&-1\\&&&1&\end{bmatrix}, \tag{B.14}$$

and the commutator relations are satisfied:

$$[A_1,A_2]=A_3,\qquad [A_2,A_3]=A_1,\qquad [A_3,A_1]=A_2. \tag{B.15}$$

Consequently, for infinitesimal rotations, we have

$$T_R\begin{bmatrix}F_2{}^1\\F_2{}^2\\F_2{}^3\\F_2{}^4\\F_2{}^5\end{bmatrix}=I+(\Omega_1A_1+\Omega_2A_2+\Omega_3A_3)\begin{bmatrix}F_2{}^1\\F_2{}^2\\F_2{}^3\\F_2{}^4\\F_2{}^5\end{bmatrix}+o(\Omega). \tag{B.16}$$

This implies that if we put

$$F_{11}=F_2{}^1,\qquad F_{22}=F_2{}^2,\qquad F_{33}=-F_2{}^1-F_2{}^2,$$
$$F_{12}=F_{21}=F_2{}^3,\qquad F_{23}=F_{32}=F_2{}^4,\qquad F_{31}=F_{13}=F_2{}^5, \tag{B.17}$$

functions $F_2^1$, $F_2^2$, $F_2^3$, $F_2^4$, $F_2^5$ are transformed as

$$T_R \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} = R \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} R^T \tag{B.18}$$

Solutions for $l=3,4,\cdots$ are constructed similarly. In fact, functions $F_l^1$, $F_l^2$, ..., $F_l^{2l+1}$ are just the $l$-th spherical harmonics projected onto the image $xy$-plane.

## APPENDIX C FEATURE BASIS OF IRREDUCIBLE REPRESENTATIONS

From eqns (5.6), the Casimir operator becomes

$$H^* = -(f+\frac{x^2+y^2}{f})[(f+\frac{x^2}{f})\frac{\partial^2}{\partial x^2}+\frac{2xy}{f}\frac{\partial^2}{\partial x\partial y}+(f+\frac{y^2}{f})\frac{\partial^2}{\partial y^2}+8x\frac{\partial}{\partial x}+8y\frac{\partial}{\partial y}] \tag{C.1}$$

$$-6-\frac{12(x^2+y^2)}{f^2}, \tag{C.1}$$

so that eqn (5.10) becomes

$$(f+\frac{x^2+y^2}{f})[(f+\frac{x^2}{f})m_{xx}+\frac{2xy}{f}m_{xy}+(f+\frac{y^2}{f})m_{yy}+8xm_x+8ym_y]$$

$$+[l(l+1)+6+\frac{12(x^2+y^2)}{f^2}]m=0. \tag{C.2}$$

For $l=0$ ($l(l+1)=0$), the following solution ($2l+1=1$) is found:

$$m_0^1(x,y)=\frac{1}{\sqrt{(x^2+y^2+f^2)^3}}. \tag{C.3}$$

Application of the infinitesimal generators $D_1^*$, $D_2^*$, $D_3^*$ yields

$$D_1^* m_0^1=0, \qquad D_2^* m_0^1=0, \qquad D_3^* m_0^1=0, \tag{C.4}$$

and consequently $m_0^1$ is invariant for $T_R^*$:

$$T_R^* \begin{bmatrix} m_1^1 \\ m_1^2 \\ m_1^3 \end{bmatrix} = I - (\Omega_1 A_1^* + \Omega_2 A_2^* + \Omega_3 A_3^*) \begin{bmatrix} m_1^1 \\ m_1^2 \\ m_1^3 \end{bmatrix} + o(\Omega). \qquad \textbf{(C.11)}$$

This implies that $m_1^1$, $m_1^2$ are transformed as

$$T_R^* \begin{bmatrix} m_1^1 \\ m_1^2 \\ m_1^3 \end{bmatrix} = R^T \begin{bmatrix} m_1^1 \\ m_1^2 \\ m_1^3 \end{bmatrix}. \qquad \text{(C.12)}$$

Features $J_i$, $i=1,2,3$, of eqns (5.13) are obtained by

$$J_i = \int m_1^i(x,y) F(x,y)\, dx dy \qquad \text{(C.13)}$$

From eqn (C.12), they are transformed as a 3D vector.

For $l=2$ ($l(l+1)=6$), we can find the following five solutions ($2l+1=5$):

$$m_2^1(x,y) = \frac{2x^2 - y^2 - f^2}{3\sqrt{(x^2+y^2+f^2)^5}}, \qquad m_2^2(x,y) = \frac{-x^2+2y^2-f^2}{3\sqrt{(x^2+y^2+f^2)^5}},$$

$$m_2^3(x,y) = \frac{xy}{\sqrt{(x^2+y^2+f^2)^5}}, \qquad m_2^4(x,y) = \frac{fy}{\sqrt{(x^2+y^2+f^2)^5}}, \qquad \text{(C.14)}$$

$$m_2^5(x,y) = \frac{fx}{\sqrt{(x^2+y^2+f^2)^5}}.$$

Application of the infinitesimal generators $D_1$, $D_2$, $D_3$ yields

$$D_1^* \begin{bmatrix} m_2^1 \\ m_2^2 \\ m_2^3 \\ m_2^4 \\ m_2^5 \end{bmatrix} = -A_1^* \begin{bmatrix} m_2^1 \\ m_2^2 \\ m_2^3 \\ m_2^4 \\ m_2^5 \end{bmatrix}, \quad D_2^* \begin{bmatrix} m_2^1 \\ m_2^2 \\ m_2^3 \\ m_2^4 \\ m_2^5 \end{bmatrix} = -A_2^* \begin{bmatrix} m_2^1 \\ m_2^2 \\ m_2^3 \\ m_2^4 \\ m_2^5 \end{bmatrix}, \quad D_3^* \begin{bmatrix} m_2^1 \\ m_2^2 \\ m_2^3 \\ m_2^4 \\ m_2^5 \end{bmatrix} = -A_3^* \begin{bmatrix} m_2^1 \\ m_2^2 \\ m_2^3 \\ m_2^4 \\ m_2^5 \end{bmatrix}, \qquad \text{(C.15)}$$

where

$$T_R^* m_0{}^1 = m_0{}^1. \tag{C.5}$$

Feature $J$ of eqn (5.11) is obtained by

$$J = \int m_0{}^1(x,y) F(x,y)\, dx dy. \tag{C.6}$$

From eqn (C.5), this is a scalar.

For $l=1$ ($l(l+1)=2$), the following three solutions ($2l+1=3$) are found:

$$m_1{}^1(x,y) = \frac{x}{(x^2+y^2+f^2)^2}, \quad m_1{}^2(x,y) = \frac{y}{(x^2+y^2+f^2)^2},$$
$$m_1{}^3(x,y) = \frac{f}{(x^2+y^2+f^2)^2}. \tag{C.7}$$

Application of the infinitesimal generators $D_1^*$, $D_2^*$, $D_3^*$ yields

$$D_1^* \begin{bmatrix} m_1{}^1 \\ m_1{}^2 \\ m_1{}^3 \end{bmatrix} = -A_1^* \begin{bmatrix} m_1{}^1 \\ m_1{}^2 \\ m_1{}^3 \end{bmatrix}, \quad D_2^* \begin{bmatrix} m_1{}^1 \\ m_1{}^2 \\ m_1{}^3 \end{bmatrix} = -A_2^* \begin{bmatrix} m_1{}^1 \\ m_1{}^2 \\ m_1{}^3 \end{bmatrix},$$
$$D_3^* \begin{bmatrix} m_1{}^1 \\ m_1{}^2 \\ m_1{}^3 \end{bmatrix} = -A_3^* \begin{bmatrix} m_1{}^1 \\ m_1{}^2 \\ m_1{}^3 \end{bmatrix}. \tag{C.8}$$

where

$$A_1^* = \begin{bmatrix} & & \\ & & -1 \\ & 1 & \end{bmatrix}, \quad A_2^* = \begin{bmatrix} & & 1 \\ & & \\ -1 & & \end{bmatrix}, \quad A_3^* = \begin{bmatrix} & -1 & \\ 1 & & \\ & & \end{bmatrix}, \tag{C.9}$$

and the commutator relations are satisfied:

$$[A_1^*, A_2^*] = A_3^*, \quad [A_2^*, A_3^*] = A_1^*, \quad [A_3^*, A_1^*] = A_2^*. \tag{C.10}$$

Consequently, for infinitesimal rotations, we have

$$A_1^*=\begin{bmatrix} & & & & -2 \\ & & & -1 & \\ & & 1 & & \\ & & & & \\ 1 & 2 & & & \end{bmatrix},\quad A_2^*=\begin{bmatrix} & & & 2 & \\ & & & & 1 \\ & & & & \\ -2 & -1 & & & \\ & & -1 & & \end{bmatrix},\quad A_3^*=\begin{bmatrix} & & -2 & & \\ & & 2 & & \\ 1 & -1 & & & \\ & & & & -1 \\ & & & 1 & \end{bmatrix}, \tag{C.16}$$

and the commutator relations are satisfied:

$$[A_1^*,A_2^*]=A_3^*,\qquad [A_2^*,A_3^*]=A_1^*,\qquad [A_3^*,A_1^*]=A_2^*. \tag{C.17}$$

Consequently, for infinitesimal rotations, we have

$$T_R^*\begin{bmatrix} m_2{}^1 \\ m_2{}^2 \\ m_2{}^3 \\ m_2{}^4 \\ m_2{}^5 \end{bmatrix}=I-(\Omega_1A_1^*+\Omega_2A_2^*+\Omega_3A_3^*)\begin{bmatrix} m_2{}^1 \\ m_2{}^2 \\ m_2{}^3 \\ m_2{}^4 \\ m_2{}^5 \end{bmatrix}+o(\Omega). \tag{C.18}$$

This implies that if we put

$$\begin{aligned} &m_{11}=m_2{}^1, \qquad m_{22}=m_2{}^2, \qquad m_{33}=-m_2{}^1-m_2{}^2, \\ &m_{12}=m_{21}=m_2{}^3, \qquad m_{23}=m_{32}=m_2{}^4, \qquad m_{31}=m_{13}=m_2{}^5, \end{aligned} \tag{C.19}$$

functions $m_2{}^1$, $m_2{}^2$, $m_2{}^3$, $m_2{}^4$, $m_2{}^5$ are transformed as

$$T_R^*\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}=R^T\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}R. \tag{C.20}$$

Features $J_{ij}$, $i,j=1,2,3$, of eqns (6.7) are obtained by

$$J_{ij}=\int\left(m_{ij}(x,y)+\frac{1}{3}m_0{}^1(x,y)\delta_{ij}\right)F(x,y)\,dxdy, \tag{C.21}$$

where $\delta_{ij}$ is the Kronecker delta. From eqns (C.5) and (C.20), they are transformed as a 3D tensor.

Solutions for $l=3,4,\cdots$ are constructed similarly. In fact, we can check, by substitution, that the solution is given by

$$m_l{}^i(x,y)=\frac{F_l{}^i(x,y)}{\sqrt{(x^2+y^2+f^2)^3}}, \qquad i=1,2,\ldots,2l+1. \tag{C.22}$$

## APPENDIX D INVARIANT MEASURE

We say that $\rho(x,y)dxdy$ is an *invariant measure* if for any image $F(x,y)$

$$\int T_R F(x,y)\rho(x,y)dxdy=\int F(x,y)\rho(x,y)dxdy. \tag{D.1}$$

In view of eqns (5.1) and (5.3), this is equivalent to

$$T_R^*\,\rho(x,y)=\rho(x,y). \tag{D.2}$$

Hence, $\rho$ is given by the solution of eqn (5.10) with $l=0$. From eqn (C.3) of Appendix C, we obtain

$$\rho(x,y)=\frac{1}{\sqrt{(x^2+y^2+f^2)^3}}. \tag{D.3}$$

This result can be interpreted intuitively in terms of fluid dynamics. Suppose the camera is rotating with rotation velocity $(\omega_1,\omega_2,\omega_3)$, namely rotating around an axis of orientation $(\omega_1,\omega_2,\omega_3)$ with angular velocity $\sqrt{\omega_1{}^2+\omega_2{}^2+\omega_3{}^2}$ (rad/sec) screwwise. (Here, $\omega_1$, $\omega_2$, $\omega_3$ are also interpreted as instantaneous angular velocities around the $x$-, the $y$-, the $z$-axis, respectively.) The *optical flow* induced on the image plane is obtained by dividing both sides of eqns (4.7) by $\delta t$:

$$u=-f\omega_2+\omega_3 y+\frac{1}{f}(-\omega_2 x+\omega_1 y)x, \qquad v=f\omega_1-\omega_3 x+\frac{1}{f}(-\omega_2 x+\omega_1 y)y. \tag{D.4}$$

If this flow is regarded as a fluid flow with density $\rho(x,y)$, the necessary and sufficient condition that the fluid is neither created or annihilated in the course of flowing is, as is

well known, given by the *equation of continuity*

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0. \tag{D.5}$$

If eqns (D.4) are substituted, eqn (D.5) becomes

$$(\omega_1 D_1^* + \omega_2 D_2^* + \omega_3 D_3^*)\rho=0. \tag{D.6}$$

This equation must be satisfied for arbitrary $\omega_1$, $\omega_2$, $\omega_3$. Hence, the invariant measure $\rho(x,y)$ is given as a solution of the differential equations

$$D_1^*\rho=0, \qquad D_2^*\rho=0, \qquad D_3^*\rho=0. \tag{D.7}$$

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## FIGURES

Fig. 1 The $XYZ$-coordinate system is fixed to the camera, the origin $O$ being the camera focus. The image plane is taken to be $Z=f$, where $f$ is the camera focal length. A point $(X,Y,Z)$ in the scene is projected onto point $(x,y)$ on the image plane.

Fig. 2 (a) Three regions $S_0$, $S_1$, $S_2$ to be tested for equivalence. (b) Computed invariant centers of gravity $G_0$, $G_1$, $G_2$ of regions $S_0$, $S_1$, $S_2$. (c) Computed invariant principal axes of regions $S_0$, $S_1$, $S_2$.

Fig. A1 Line $l$: $x\cos\theta+y\sin\theta=d$ and point $P(-(f^2/d)\cos\theta,-(f^2/d)\sin\theta)$ are mutually reciprocal with respect to the origin $O$.

Fig. A2 Points $P$, $Q$ on line $l$ are mutually conjugate with respect to the foot $H$ of the perpendicular line drawn from the origin $O$ to line $l$.

Fig. A3 Point $P_1$ and line $l_1$, point $P_2$ and line $l_2$, point $P_3$ and line $l_3$ are mutually reciprocal, and points $P_2$ and $P_3$ on line $l_1$, points $P_3$ and $P_1$ on line $l_1$, points $P_1$ and $P_2$ on line $l_1$ are mutually conjugate.

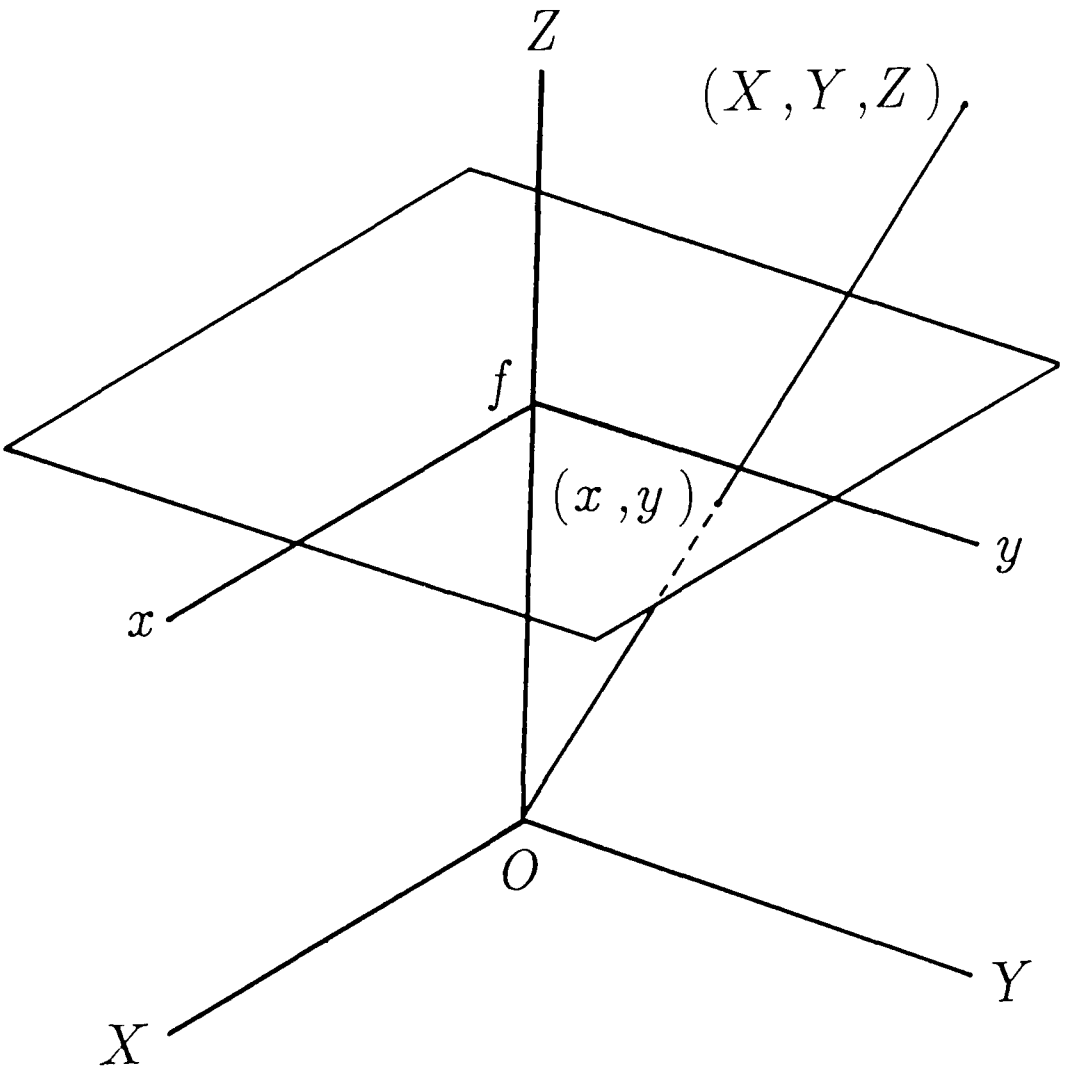
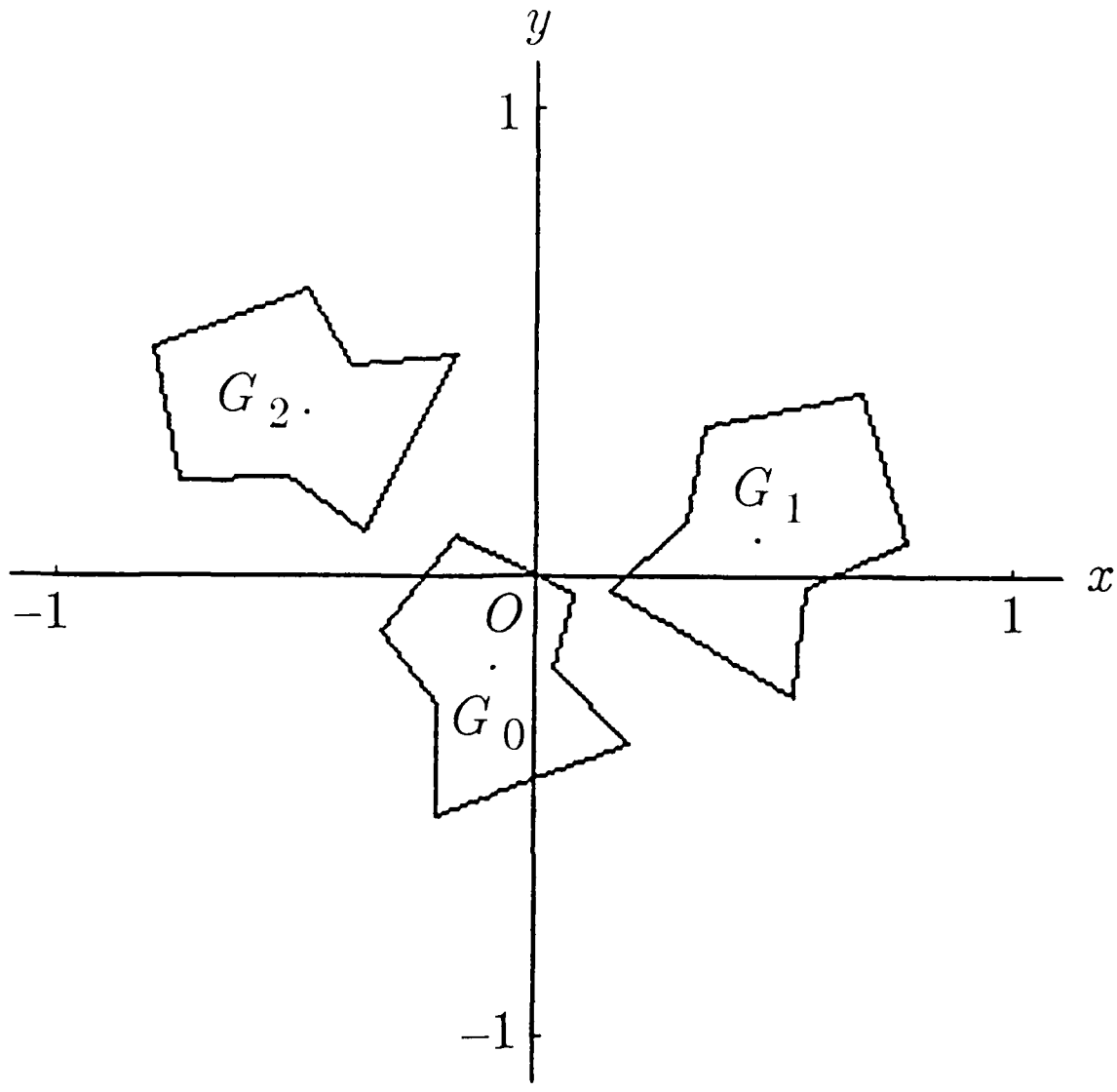
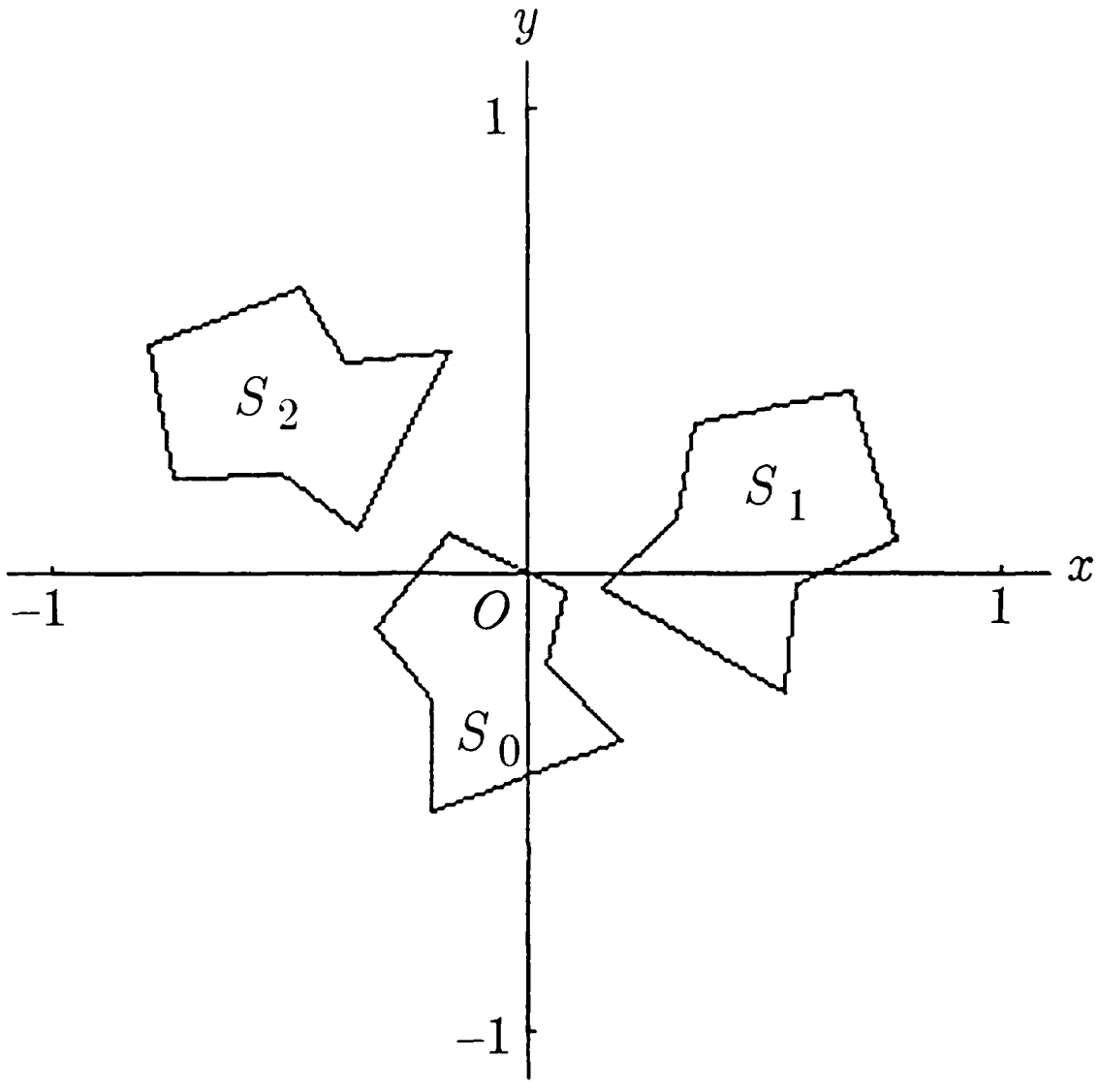



Fig. 1




(b)

Fig. 2(b)




(a)

Fig. 2(a)

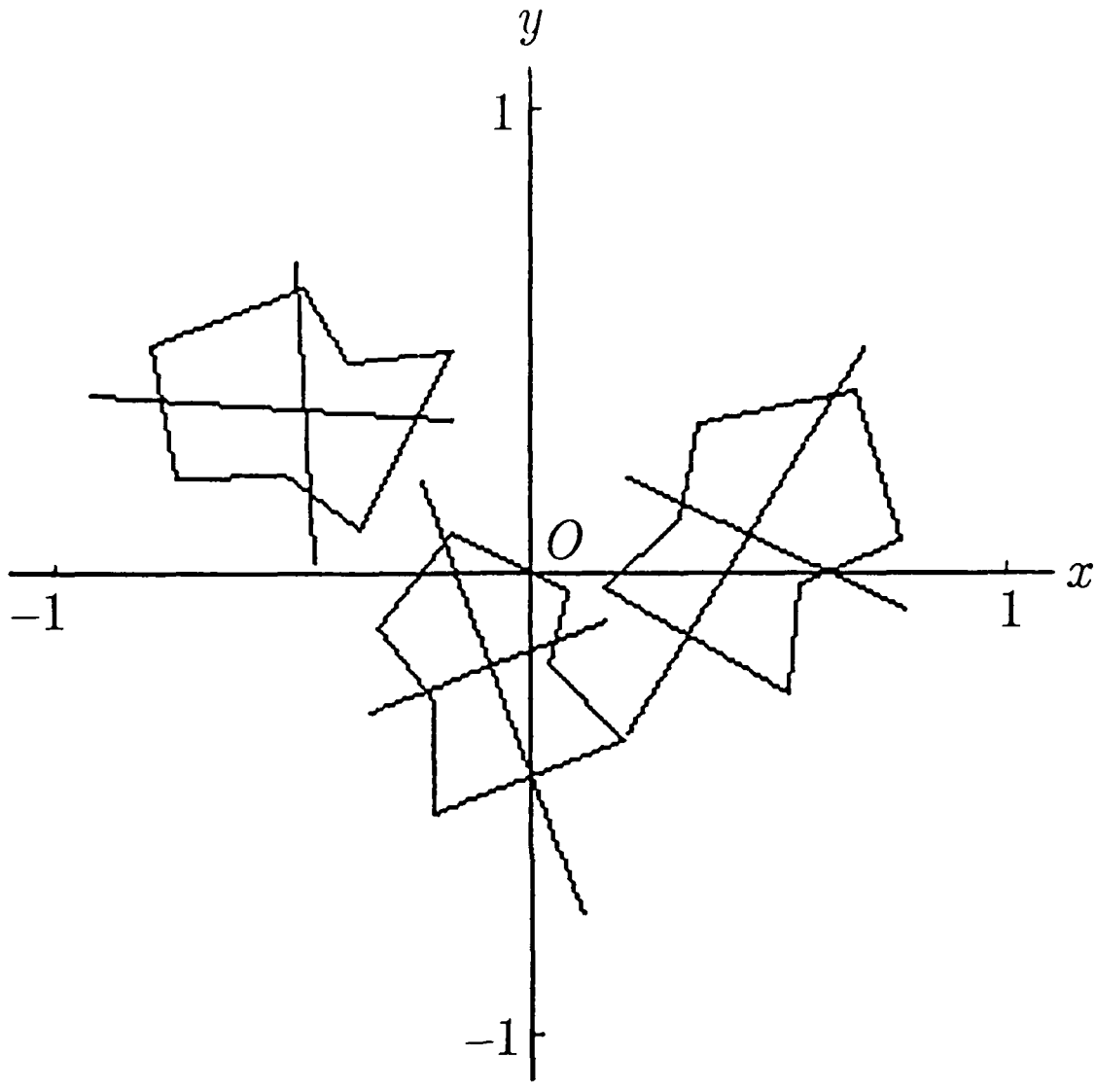



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Fig. 2(c)

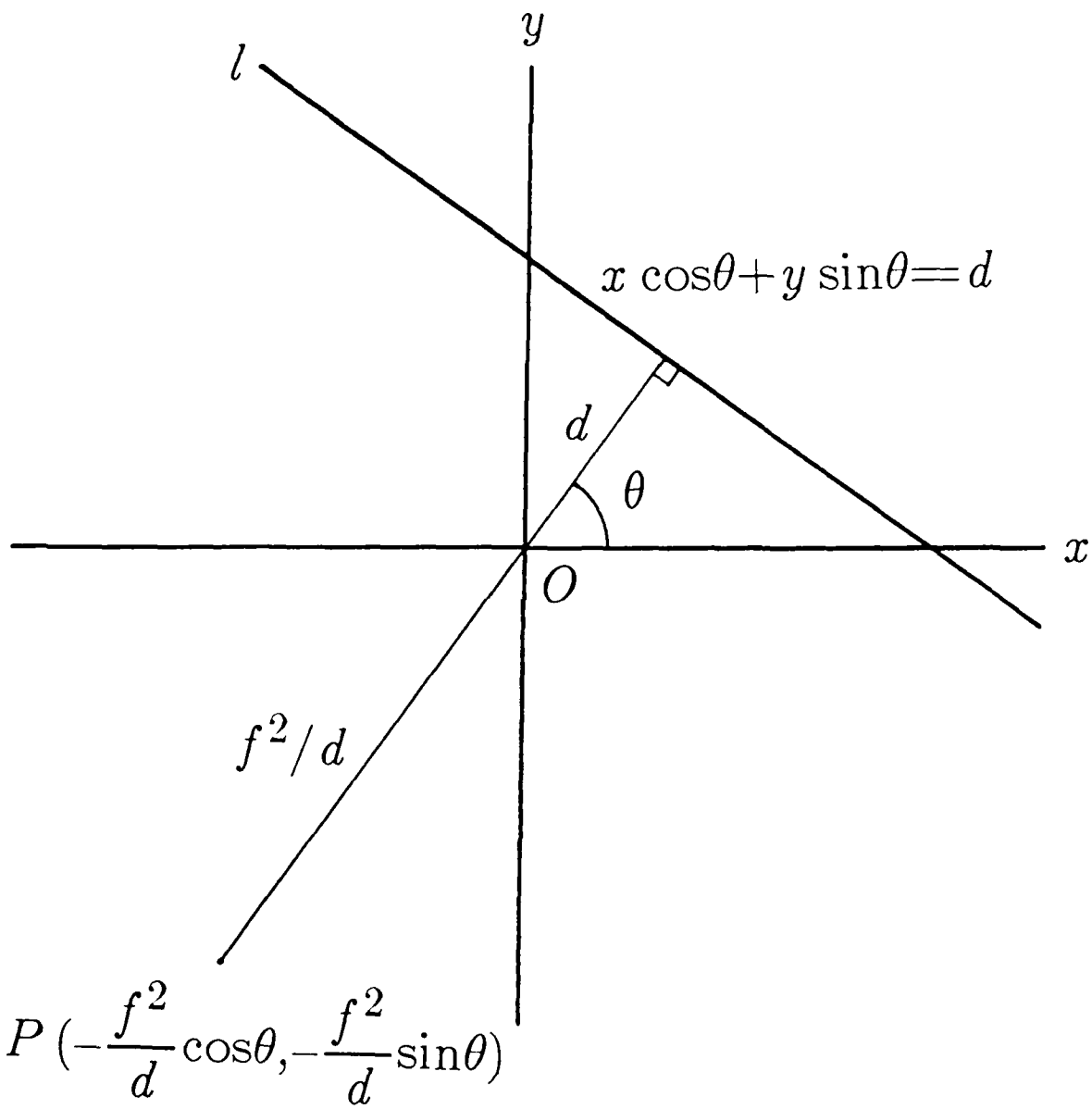
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Fig. A1

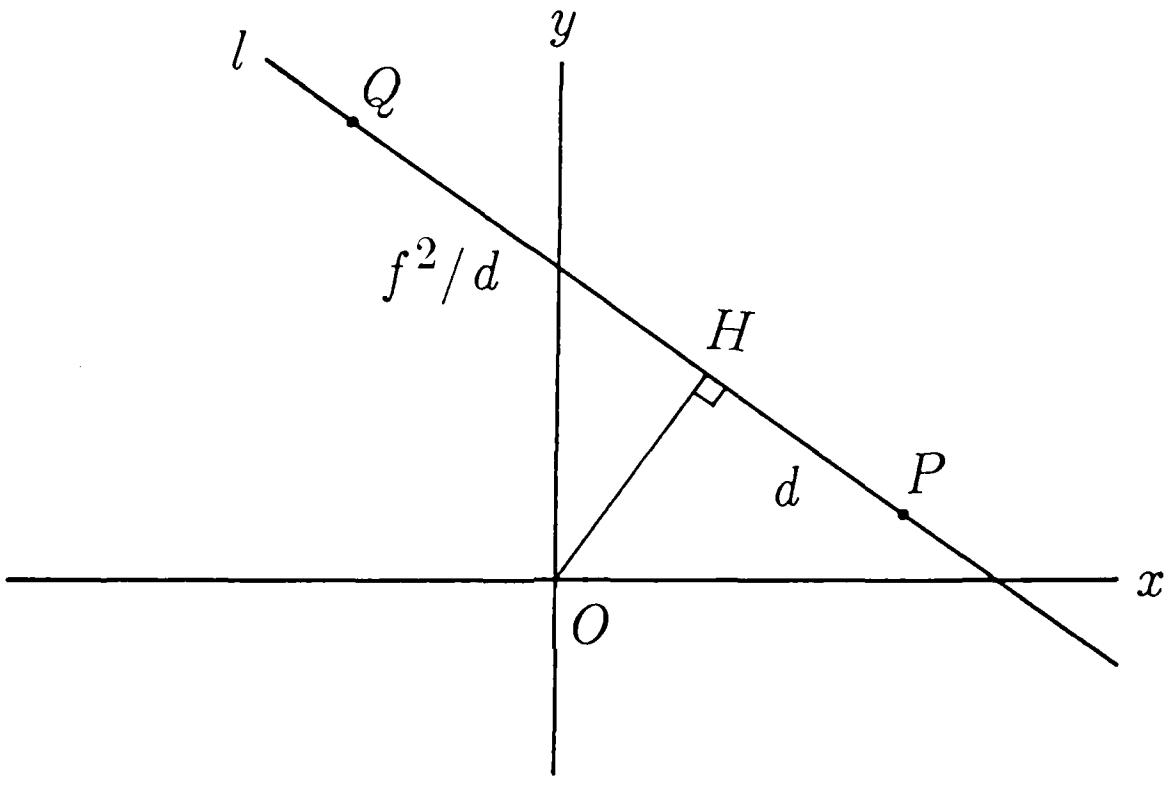


Fig. A2

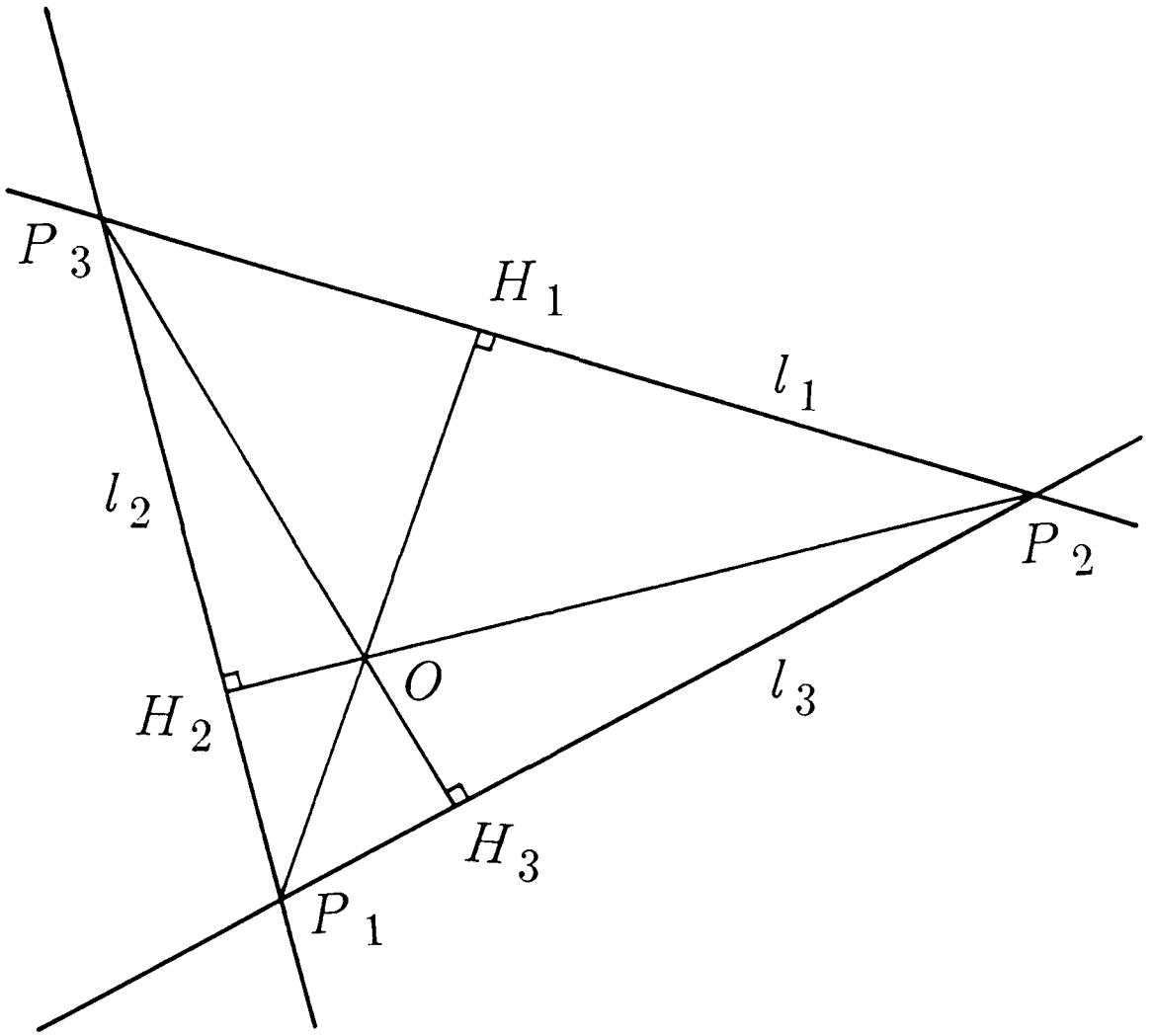


Fig. A3

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The image transformation due to camera rotation relative to a stationary scene is analyzed, and the associated transformation rules of features given by weighted averaging of the image are derived by considering infinitesimal generators on the basis of group representation theory. Three dimensional vectors and tensors are reduced to two dimensional invariants on the image plane from the viewpoint of projective geometry. Three dimensional invariants and camera rotation reconstruction are also discussed. The result is applied to the shape recognition problem when camera rotation is involved

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