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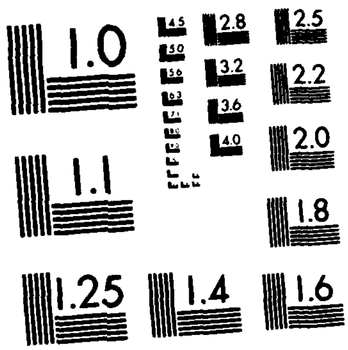

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DECONVOLUTION BY MODIFIED WIENER FILTERING: INTERPRETATION FOR AN IMPERFECTLY KNOWN WAVELET

by

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Deconvolution by modified Wiener filtering: interpretation for an imperfectly known wavelet

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ABSTRACT

Deconvolution in the presence of additive noise is a well known problem for which there exists a Wiener filter which simultaneously spectrally whitens while suppressing noise. A simple variant of this standard Wiener filter incorporates a parameter, p say, which is intended to allow further weight to be given to noise suppression. We shall call such a filter a modified Wiener filter. To design such a filter it is required to know precisely the frequency response of the spread function or wavelet, plus the spectra of the input and additive noise.

In practice some response function is taken to be appropriate, and the modified Wiener filter designed from it. If the design response function is thought of as one chosen from a set of allowable response functions — a realistic practical viewpoint — then it is shown how the selection of the design response, the chosen value of the parameter p and the noise/input power spectral ratio effectively *determine* the characteristics of this set of possible wavelet response functions. This is demonstrated for two different error criteria — (i) the minimization of the *average* mean-squared error, and (ii) the minimization of the *maximum* mean-squared error.

It is shown how to calculate deconvolution filters which solve sub-optimal versions of (i) and (ii), but which are robust to uncertainty in the wavelet's frequency response.

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Deconvolution by modified Wiener filtering: interpretation for an imperfectly known wavelet

A.T. Walden

Introduction

A most important problem in reflection seismology is that of deconvolution plus filtering. Consider fig. 1; we can write

$$y(t) = \sum_{k-\infty}^{\infty} h(t-k)r(k) + n(t) \qquad (1)$$

where $r(t)$ is the input, $h(t)$ is the channel or spread function, $n(t)$ is additive noise (uncorrelated with the signal), and $y(t)$ is the result. In this paper we shall work with the discrete-time representation (1) with unit sampling interval. In reflection seismology we would express y in discrete time as the familiar convolutional model (1) written as

$$y(t) = h(t) * r(t) + n(t)$$

where we equate $r(t)$ with the reflection sequence, $h(t)$ with the seismic wavelet, and $n(t)$ with additive (colored) noise.

We seek to design a deconvolution (or equalization) filter $g(t)$ which satisfactorily recovers, or estimates, $r(t)$; see fig. 1. There are several statistical approaches to this problem, (for a summary of other deconvolution methods see Schultz, 1985). For example, we can carry out a "blind deconvolution" where we work only with $y(t)$ and make some important assumptions: (a) reflectivity whiteness and minimum-phase wavelet for standard whitening or predictive deconvolution, or (b) reflectivity whiteness and/or sparse reflectivity for MED-type methods. Both methods attempt to suppress the noise either explicitly or implicitly. For (a) a "prewhitening" is applied to the Toeplitz matrix before inversion, while for (b) the harshness of the non-linear mapping involved (see e.g., Walden, 1985) will often provide some protection against additive noise. A second approach which is generally called signature deconvolution assumes that the wavelet is known to a very good approximation. This wavelet estimate could be model-based, a far-field measured response, or the result of other signal processing schemes. Deregowski (1971, 1978) contains excellent discussions of signature deconvolution.

A third approach involves Wiener filtering. If the power spectrum of the input, $R(\omega)$, and the noise, $N(\omega)$, plus the frequency response of the channel (wavelet) $H_T(\omega)$ are all *exactly known,* then the mean squared error $E\{(r(t)-\hat{r}(t))^2\}$ is minimized when the deconvolution filter is chosen to be the Wiener filter

$$G(\omega) = \frac{H_T^*(\omega) R(\omega)}{|H_T(\omega)|^2 R(\omega) + N(\omega)} \tag{2}$$

where $H_T^*(\omega)$ is the complex conjugate of $H_T(\omega)$. Denote the class of all possible deconvolution-filter responses by $\mathfrak{G}$. Then $G(\omega)$ is the solution of

$$\min_{\mathfrak{G}} e\,(H_T, G)$$

where

$$\begin{aligned} e(H,G) &= E\{(r(t)-\hat{r}(t))^2\} \\ &= (1/2\pi)\int_{-\pi}^{\pi} \{\,|1-H(\omega)G(\omega)|^2 R(\omega) + |G(\omega)|^2 N(\omega)\,\}\, d\omega \end{aligned}$$

Multiplying top and bottom of (2) by $H_T(\omega)$, and then dividing through by $|H_T(\omega)|^2 R(\omega)$ gives

$$G(\omega) = [\{1+\mu(\omega)\} H_T(\omega)]^{-1} \tag{3}$$

We have added a superscript "T" to $H(\omega)$ to emphasize that $H_T(\omega)$ is the true response for the realization of the system which is under investigation. Here $\mu(\omega)$ is the noise-to-signal ratio $N(\omega)/\{|H_T(\omega)|^2 R(\omega)\}$. The amplitude characteristic of the filter,

$$\frac{|H_T(\omega)|\;\; R(\omega)}{|H_T(\omega)|^2 R(\omega) + N(\omega)}$$

provides noise rejection at the frequencies where $R(\omega)$ is small relative to $N(\omega)$. Note that even a perfect all-pass wavelet demands *amplitude* as well as phase compensation.

Suppose now instead of merely minimizing the mean squared error $E\{(r(t)-\hat{r}(t))^2\}$ we seek to minimize

$$\lambda_1 E\{(r(t)-\hat{r}(t))^2\} + \lambda_2 E\{(n(t) * g(t))^2\} \tag{4}$$

i.e., λ_1 times the mean-squared estimation error $+\lambda_2$ times the mean-squared filtered noise. The parameters λ_1 and λ_2 are tradeoff parameters; their relative sizes determine how much effort is put into minimizing each component. Note $\lambda_1, \lambda_2 \geq 0$. (A similar tradeoff exists in the Backus-Gilbert inverse formalism, see e.g., Oldenburg, 1984, p670). Such a scheme is well-known in designing shaping filters in the presence of autocorrelated noise, e.g., Robinson, 1980, p227. As shown in the Appendix, the *modified* Wiener filter resulting from the minization of (4) is

$$G_M(\omega) = [\{1+p\,\mu(\omega)\}\,H_T(\omega)]^{-1} \tag{5}$$

where $p = (1+[\lambda_2/\lambda_1])$ is an adjustable noise control parameter. Increasing the value of p corresponds to increasing the desire for noise minimization at the expense of spectral whitening. If $\lambda_2 = 0$, $p = 1$ and we recover (3), while if both mean-square components are given equal weight, $p = 2$. For a chosen p there are two unknowns in (5), viz $\mu(\omega)$, the noise-to-signal ratio, and the frequency response of the wavelet, $H_T(\omega)$. In this paper we shall assume that $H_T(\omega)$ is not perfectly known, but merely that the estimate of the wavelet's frequency response, $H_D(\omega)$, used in the design of the deconvolution filter, has been selected from a set of possible choices. We also assume that the ratio $N(\omega)/R(\omega)$ — the ratio of noise power spectrum to input power spectrum — is known. This is not a desirable assumption, but is necessary to confine all the lack of information to the wavelet. In practice, *using only seismic data,* White (1984, p1346) has explained, and illustrated, that the noise and signal components and hence the noise-to-signal ratio $\mu(\omega) = N(\omega)/\{\,|H_T(\omega)|^2 R(\omega)\,\}$ can be well estimated using multiple coherence analysis. (However, the estimation of $H_T(\omega)$ from the estimate of the signal $|H_T(\omega)|^2 R(\omega)$, demands more assumptions of a dubious nature, e.g., white reflectivity, and is not recommended.) Let us denote our good estimate of $\mu(\omega)$ by $\hat{\mu}(\omega)$. It will be shown that if $N(\omega)/\{\,|H_D(\omega)|^2 R(\omega)\,\} = \bar{\mu}(\omega)$ say is close to $\hat{\mu}(\omega)$, then results are conveniently expressible in terms of p, $H_D(\omega)$ and $\hat{\mu}(\omega)$.

It will be demonstrated how the quantities p, $H_D(\omega)$ and $\hat{\mu}(\omega)$ effectively *determine* the characteristics of the set of possible wavelet frequency response functions. This is demonstrated for two different error criteria: (i) the minimization of the *average* mean-squared error of estimation, and (ii) the minimization of the *maximum* mean-squared error of estimation.

It is also shown that if $H_D(\omega)$ is obtained by well-documented methods, from which various error statistics can be formulated, then we can design deconvolution filters which satisfy sub-optimal versions of the two error criteria above, hence giving some robustness against uncertainty in the wavelet's frequency response.

Minimizing average mean-squared error in theory

Suppose the optimization problem is the selection of the deconvolution filter $g(t)$ which minimizes the mean-squared estimation error, averaged over all possible realizations of $h(t)$ or equivalently the set of all possible choices for the wavelet's frequency response, $H(\omega)$. Denote this class of allowable wavelet frequency responses by $\mathcal{H}$. Similarly, denote the class of all possible deconvolution-filter frequency responses by $\mathcal{G}$. Then we seek to

$$\min_{\mathcal{G}} E_{\mathcal{H}}\{ e(H,G) \} \tag{6}$$

Here $E_{\mathcal{H}}$ means "the expected value over all H in $\mathcal{H}$" and $\min\limits_{\mathcal{G}}$ means "the minimum over the class $\mathcal{G}$." The solution to this minimization problem is in fact that given by Maurer and Franks (1970), namely

$$G_1(\omega) = \frac{E_{\mathcal{H}}\{ H^*(\omega) \}}{E_{\mathcal{H}}\{ |H(\omega)|^2 \} + [N(\omega)/R(\omega)]} \tag{7}$$

Comparing (2) and (7) we see that G_1 in (7) is not just the Wiener filter for the average wavelet, since the denominator contains the term $E_{\mathcal{H}}\{ |H(\omega)|^2 \}$ rather than $|E_{\mathcal{H}}\{H(\omega)\}|^2$. Since $Var_{\mathcal{H}}\{H(\omega)\} \equiv E_{\mathcal{H}}\{|H(\omega)|^2\} - |E_{\mathcal{H}}\{H(\omega)\}|^2$:

$$G_1(\omega) = E_{\mathcal{H}}\{H^*(\omega)\} \;/\; \{ |E_{\mathcal{H}}\{H(\omega)\}|^2 + Var_{\mathcal{H}}\{H(\omega)\} + [N(\omega)/R(\omega)]\} \tag{8}$$

When the variation of H is very small, i.e., $Var_{\mathcal{H}} \approx 0$, the average wavelet is inverted. However, if $Var_{\mathcal{H}}\{H(\omega)\}\,R(\omega)$ is comparable in size with the noise term $N(\omega)$, then the characteristics of the optimum filter will be significantly influenced by the variation.

The modified Wiener deconvolution filter with parameter p, design response $H_D(\omega)$ and noise to input power ratio $N(\omega)/R(\omega)$ has the form

$$\begin{aligned} G_D(\omega) &= H_D^*(\omega) \;/\; \{ |H_D(\omega)|^2 + p\,[N(\omega)/R(\omega)]\} \\ &= 1 \;/\; \{[1+p\bar{\mu}(\omega)]\, H_D(\omega)\} \end{aligned} \tag{9}$$

and so has gain

$$|G_D(\omega)| = |H_D^*(\omega)| \;/\; \{|H_D(\omega)|^2 + p[N(\omega)/R(\omega)]\} \tag{10}$$

whereas G_1 has gain

$$|G_1(\omega)| = |E_{\mathcal{H}}\{H^*(\omega)\}| / \{ |E_{\mathcal{H}}\{H(\omega)\}|^2 + Var_{\mathcal{H}}\{H(\omega)\} + [N(\omega)/R(\omega)] \} \tag{11}$$

We must now decide how to relate $H_D(\omega)$, our designed response, to the set of possible outcomes of $H(\omega)$. It seems intuitively sensible to assume $|H_D^*(\omega)| \approx |E_{\mathcal{H}}\{H^*(\omega)\}|$

because our choice of $H_D(\omega)$ almost certainly involved "averaging" using previous experience of outcomes, and possibly other parts of the seismic section. So let us take $|H_D^*(\omega)| = |E \ \{H_{\mathcal{H}}^*(\omega)\}|$. Then

$$Var_{\mathcal{H}}\{H(\omega)\} + [N(\omega)/R(\omega)] = p\,[N(\omega)/R(\omega)]$$

i.e.,

$$\begin{aligned} Var_{\mathcal{H}}\{H(\omega)\} &= (p-1)\,[N(\omega)/R(\omega)] \\ &= (p-1)\,\mu(\omega)\ |H_T(\omega)|^2 \\ &= (p-1)\,\bar{\mu}(\omega)\ |H_D(\omega)|^2 \\ &\approx (p-1)\,\hat{\mu}(\omega)\ |H_D(\omega)|^2 \qquad \text{provided } \bar{\mu} \approx \hat{\mu}. \end{aligned} \tag{12}$$

Thus we see that these three quantities determine $Var_{\mathcal{H}}\{H(\omega)\}$. Since $p = 1 + [\lambda_2/\lambda_1]$ and $\lambda_1, \lambda_2 \ge 0$ it follows that $p \ge 1$, and hence, as is obviously required, $Var_{\mathcal{H}} \ge 0$. Note that $p = 1$, i.e., $\lambda_2 = 0$, is equivalent to setting $Var_{\mathcal{H}}\{H(\omega)\} = 0$.

The phase of the modified Wiener filter is $-\theta_D(\omega)$ where $H_D^*(\omega) = |H_D(\omega)|\ e^{-i\theta_D(\omega)}$, whereas the phase of G_1 in (8) is $arg\ (E_{\mathcal{H}}\{H^*(\omega)\})$. Thus we can also identify

$$\theta_D(\omega) \equiv -arg\ (E_{\mathcal{H}}\{H^*(\omega)\})$$

The parameter p does not enter this equation; p is only greater than 1 when $\lambda_2 > 0$, and the term weighted by λ_2 in (4) is independent of phase. Thus the phase is unaffected. Note that

$$arg\ (H_D^*(\omega)) \equiv arg\ (E_{\mathcal{H}}\{H^*(\omega)\})$$

while we have chosen

$$|H_D^*(\omega)| = |E_{\mathcal{H}}\{H^*(\omega)\}|$$

To recap, we have shown that by treating the wavelet's frequency response as a random variable, and minimizing $E_{\mathcal{H}}\{e(H,G)\}$ over G, we get a deconvolution filter which is identical to the modified Wiener filter designed from $p, H_D(\omega)$ and $N(\omega)/R(\omega)$, when we set

$$|H_D^*(\omega)| = |E_{\mathcal{H}}\{H^*(\omega)\}|\ ;\ arg\ (H_D^*(\omega)) = arg\ (E_{\mathcal{H}}\{H^*(\omega)\}) \tag{13a}$$

and

$$Var_{\mathcal{H}}\{H(\omega)\} = (p-1)[N(\omega)/R(\omega)]. \tag{13b}$$

Hence $p, H_D(\omega)$ and $[N(\omega)/R(\omega)]$ can, in this sense, be interpreted as defining the characteristics of the variation of $H(\omega)$, while $H_D(\omega)$ alone defines its phase. Remember that the modified Wiener filter concerns itself explicitly with the filtered noise (when $p > 1$), through the second term in (4). The minimization of averaged mean-squared error is not concerned explicitly with filtered noise, but merely with mean-squared estimation error. Yet the deconvolution filters can be made equivalent using (13a) and (13b).

Minimizing average mean-squared error in practice

In practice we won't know $E_{\mathcal{H}}\{H^*(\omega)\}$ or $Var_{\mathcal{H}}\{H(\omega)\}$. Suppose instead we obtain an estimate $H_D(\omega)$ of $H_T(\omega)$ using coherence methods (White, 1980; Walden and White, 1984). Then we know

$$\begin{aligned} V = Var\,\{H_D(\omega)\} &= E\,\{|H_D(\omega)|^2\} - |E\{H_D(\omega)\}|^2 \\ &\approx (1+2\sigma^2)\,|H_T(\omega)|^2 - |H_T(\omega)|^2 \\ &= 2\sigma^2 |H_T(\omega)|^2 \end{aligned}$$

where σ is the noise parameter (see e.g., Walden, 1986 for more details). We can estimate σ^2, and replace $H_T(\omega)$ by $H_D(\omega)$ to obtain an estimate of $Var\,\{H_D(\omega)\}$ as

$$\hat{V} = 2\hat{\sigma}^2\,|H_D(\omega)|^2$$

We now follow White (1984) to obtain an estimate $\hat{\mu}(\omega)$ of the noise-to-signal ratio, and assuming $\hat{\mu} \approx \bar{\mu}$ determine p from

$$\hat{V} = (p-1)\,\hat{\mu}(\omega)\,|H_D(\omega)|^2$$

(i.e., equation (12) with $Var_{\mathcal{H}}$ replaced by $\hat{V}$). Thus $G_D(\omega)$ in (9) may be computed. Instead of solving (6), $G_D(\omega)$ will now solve

$$\min_{G} E_{\mathcal{H}_1}\{e(H, G)\}$$

where $\mathcal{H}_1$ is the class of wavelet frequency responses with mean $H_D(\omega)$ and variance $\hat{V}$. While this is not as ideal as solving (6), it is a practical approach, and should prove to be robust to uncertainty in the wavelet's frequency response.

Minimizing maximum mean-squared error in theory

In this case, instead of the optimization (6), we consider

$$\min_{\mathcal{G}} \max_{\mathcal{H}} e\,(H, G) \tag{14}$$

i.e., first we maximize the mean-squared error over the class of allowable wavelet frequency responses, then we minimize the result over the class of all possible deconvolution-filter frequency responses. The solution H', G' say, is a saddlepoint to the minimax "game", satisfying

$$\min_{\mathcal{G}} e\,(H', G) \;=\; e\,(H', G') \;=\; \max_{\mathcal{H}} e\,(H, G')$$

The problem of determining the optimum G has been studied by Moustakides and Kassam (1985) who specified $\mathcal{H}$ to be the class defined by

$$A_L(\omega) \;\le\; |H(\omega)| \;\le\; A_U(\omega)$$

with $\theta(\omega) \;=\; arg\,\{H(\omega)\}$ contained in a known closed subset $\boldsymbol{\theta}$ of $(-\pi, \pi]$ (see fig. 2). Moustakides and Kassam determined that the gain, $|G_2(\omega)|$, of the optimum filter has the form

$$|G_2(\omega)| = \begin{cases} 0 & \text{if } \cos\{\alpha(\omega)\} \ge 0 \\ & \text{i.e., } \alpha(\omega) \le \pi/2 \\[2ex] \dfrac{-2\cos\{\alpha(\omega)\}}{A_L(\omega) + A_U(\omega)} & \text{if } \cos\{\alpha(\omega)\} < 0 \\ & \text{and} \\ & \dfrac{[A_L^2(\omega) R(\omega)]}{N(\omega)} \ge \dfrac{2A_L(\omega)}{[A_U(\omega) - A_L(\omega)]} \\[2ex] \dfrac{-A_L(\omega)\cos\{\alpha(\omega)\}}{A_L^2(\omega) + [N(\omega)/R(\omega)]} & \text{if } \cos\{\alpha(\omega)\} < 0 \\ & \text{and} \\ & \dfrac{[A_L^2(\omega) R(\omega)]}{N(\omega)} < \dfrac{2A_L(\omega)}{[A_U(\omega) - A_L(\omega)]} \end{cases} \tag{15}$$

Suppose $\Theta(\omega)$ is some continuous interval in $(-\pi, \pi]$, interpreted as a set of points on the unit circle in the complex plane, then $2\alpha(\omega)$ is the angle subtended at the origin by the arc on the unit circle *outside* $\Theta(\omega)$, see fig. 2. If $\Theta(\omega)$ is a single point, then $\alpha(\omega) = \pi$. The expression $(2\pi - 2\alpha(\omega))/2\pi$ denotes the proportion of the circle corresponding to the bounds of the uncertain region Θ; hence $1 - \alpha(\omega)/\pi$ is a measure of uncertainty. Note

$\Theta(\omega)$ covering more that 180° corresponds to $\cos\{\alpha(\omega)\} \geq 0$ and hence zero gain.

In (15) $A_L^2(\omega) R(\omega)$ is minimum signal power, so that $A_L^2 R/N(\omega)$ is a measure of the minimum possible signal-to-noise ratio at ω. Define a measure of uncertainty $\delta(\omega)$ about the channel gain characteristic at ω by $\delta(\omega) = [A_U(\omega) - A_L(\omega)]/[A_U(\omega) + A_L(\omega)]$; then the degree of certainty $= [1-\delta(\omega)]/\delta(\omega) = 2A_L(\omega)/[A_U(\omega) - A_L(\omega)]$ while the degree of uncertainty $= [A_U(\omega) - A_L(\omega)]/[2A_L(\omega)]$. Thus we see from (15) that if the minimum possible S/N ratio at ω is *greater* than the degree of certainty of channel gain

$$|G_2(\omega)| = \frac{-2\cos\{\alpha(\omega)\}}{A_L(\omega) + A_U(\omega)} \tag{16}$$

i.e., the deconvolution filter acts as the inverse of the average wavelet, apart from the attenuation due to phase uncertainty. Let us compare (16) with the modified Wiener filter gain (10):

$$|G_D(\omega)| = |H_D^*(\omega)|/\{|H_D(\omega)|^2 + p\,[N(\omega)/R(\omega)]\}$$

In practice $A_L(\omega)$ and $A_U(\omega)$ are unknown. However, if (say, 90%) symmetric confidence limits for $|H_T(\omega)|$ have been estimated, they would typically be of the form:

$$|H_D(\omega)|(1-Q(\omega)) \leq |H_T(\omega)| \leq |H_D(\omega)|\,(1+Q(\omega)$$

(more on this later). Equating $A_L(\omega)$ with $|H_D(\omega)|\,(1-Q(\omega))$ and $A_U(\omega)$ with $|H_D(\omega)|\,(1+Q(\omega))$ implies equating $|H_D(\omega)|$ with $\frac{1}{2}(A_L(\omega) + A_U(\omega))$.

For the gains to be equal, this requires:

$$\frac{-\cos\{\alpha(\omega)\}}{|H_D(\omega)|} = \frac{|H_D(\omega)|}{|H_D(\omega)|^2 + p\,[N(\omega)/R(\omega)]}$$

i.e.,

$$-\cos\{\alpha(\omega)\} = \left[1 + \frac{p\,N(\omega)}{|H_D(\omega)|^2 R(\omega)}\right]^{-1}$$

$$= [1 + p\,\bar{\mu}(\omega)]^{-1}\,. \tag{17}$$

From (15) it is a requirement that $\cos\{\alpha(\omega)\} < 0$. Since $[1+p\bar{\mu}(\omega)]^{-1} > 0$ this will always hold. Secondly, $|\cos\{\alpha(\omega)\}| \leq 1$ is a necessary condition; since $p \geq 1$ and $\bar{\mu}(\omega) \geq 0$, this will be satisfied. Suppose equal weight is given to each component in (4), i.e., $\lambda_1 = \lambda_2$, so that $p = 2$. Also assume $\hat{\mu}(\omega) \approx \hat{\mu}(\omega) = 0.2$, say. Then

$\cos\{\alpha(\omega)\} = -1/1.4 \Rightarrow \alpha(\omega) = \pi - 0.775$ giving a phase uncertainty of $1 - 2\alpha/2\pi = 1.55/2\pi$, which means that $\Theta(\omega)$ has outer limits corresponding to approximately 1.55/6.28 of 360°, i.e., about 90°. The further condition in (15), that $[A_L^2(\omega)\,R(\omega)]/N(\omega) \geq 2A_L(\omega)/[A_U(\omega) - A_L(\omega)]$ may alternatively be written

$$A_U(\omega) \;\geq\; [\{2/A_L^2(\omega)\}\{N(\omega)/R(\omega)\} + 1]\,A_L(\omega)$$

which shows the role of the noise-to-input ratio $N(\omega)/R(\omega)$. Suppose in fact that $N(\omega)/R(\omega)$ was sufficiently large that this condition was not met, but instead the minimum possible S/N ratio at ω is *less* than the degree of certainty of channel gain,

$$A_U(\omega) \;<\; [\{2/A_L^2(\omega)\}\{N(\omega)/R(\omega)\} + 1]\,A_L(\omega)$$

Then, providing $\cos\{\alpha(\omega)\} < 0$, the correct gain function is, (15),

$$|G_2(\omega)| = \frac{-A_L(\omega)\,\cos\{\alpha(\omega)\}}{A_L^2(\omega) + [N(\omega)/R(\omega)]} \tag{18}$$

Comparing (18) with the modified Wiener filter gain

$$|G_D(\omega)| = |H_D^*(\omega)|/\{|H_D(\omega)|^2 + p\,[N(\omega)/R(\omega)]\}$$

and equating $|H_D(\omega)|$ with $\frac{1}{2}(A_L(\omega) + A_U(\omega))$ as before gives this time

$$-\cos\{\alpha(\omega)\} = \frac{|H_D(\omega|\,\{A_L^2(\omega) + [N(\omega)/R(\omega)]\}}{A_L(\omega)\,\{|H_D(\omega)|^2 + p\,[N(\omega)/R(\omega)]\}}$$

$$= \left[\frac{A_L(\omega)}{|H_D(\omega)|} + \frac{\bar{\mu}(\omega)\,|H_D(\omega)|}{A_L(\omega)}\right] \cdot [1 + p\,\bar{\mu}(\omega)]^{-1} \tag{19}$$

The second term on the right is identical to that obtained in the case where the minimum possible S/N ratio at ω is greater than the degree of certainty of channel gain, eqn. (17). Clearly $\cos\{\alpha(\omega)\}$ is negative, since all the terms on the right in (19) are positive. What about the necessary condition $|\cos\{\alpha(\omega)\}| \leq 1$? By substituting $|H_D(\omega)| = \frac{1}{2}(A_L(\omega) + A_U(\omega))$ in (19) the condition is

$$\frac{1}{2}(A_L(\omega) + A_U(\omega))(A_L^2(\omega) + [N(\omega)/R(\omega)]) \;\leq\; A_L(\omega)\{\frac{1}{4}(A_L(\omega) + A_U(\omega))^2 + p\,[N(\omega)/R(\omega)]\}$$

which can be written as

$$A_L(\omega)(A_L^2(\omega) - [4p - 2][N(\omega)/R(\omega)]) \;\leq\; A_U(\omega)(A_L(\omega)A_U(\omega) - 2[N(\omega)/R(\omega)])$$

But $A_U(\omega) \geq A_L(\omega)$, hence it is sufficient that

$$(A_L^2(\omega) - [4p-2][N(\omega)/R(\omega)]) \leq (A_L(\omega)A_U(\omega) - 2[N(\omega)/R(\omega)])$$

Since $A_L^2(\omega) \leq A_L(\omega)A_U(\omega)$, *it is sufficient that*

$$[4p-2][N(\omega)/R(\omega)] \geq 2[N(\omega)/R(\omega)]$$

$$\text{i.e.,} \quad 4p \geq 4$$

$$p \geq 1$$

which is always true.

The equation (18) shows that $|G_2(\omega)|$ acts as a Wiener filter for the lowest gain. An additional attenuation $-\cos\{\alpha(\omega)\}$ arises because of the phase uncertainty.

The phase for the modified Wiener filter is $arg\ \{H_D^*(\omega)\} = -\theta_D(\omega)$, while the phase for the minimax filter is, in both cases discussed above,

$$\phi(\omega) = \pi - \beta(\omega) \tag{20}$$

(Moustakides and Kassam, 1985, eqn. 8). This phase is the angle between 0 and the line OL which splits into two halves the arc on the circle outside $\Theta(\omega)$. In order that these two expressions for phase be the same we require

$$-\theta_D(\omega) = \pi - \beta(\omega) \quad \text{i.e.,} \quad arg\ \{H_D^*(\omega)\} = \pi - \beta(\omega) \tag{21}$$

so that $\theta_D(\omega)$ alone defines $\beta(\omega)$ (and vice versa) and thus the position of $\Theta(\omega)$ with subtended angle $2\alpha(\omega)$. This same phase relationship applies even in the case of a large phase uncertainty, when $|G_2(\omega)| = 0$.

Minimizing maximum mean-squared error in practice

In practice we won't know the theoretical bounds $A_L(\omega)$, $A_U(\omega)$ or the subset $\Theta(\omega)$ of $(-\pi, \pi]$. However, we can readily obtain symmetric 90% confidence limits on gain and phase for the coherence methods mentioned earlier, as $\hat{A}_L(\omega) = |H_D(\omega)|\,(1-Q(\omega))$, $\hat{A}_U(\omega) = |H_D(\omega)|\,(1+Q(\omega))$ and $\hat{\theta}_L(\omega)$, $\hat{\theta}_U(\omega)$, (see e.g., Walden, 1986 for more details, especially the form of Q). From fig. 2 the values $\hat{\theta}_L(\omega)$ and $\hat{\theta}_U(\omega)$ define an estimate $\hat{\Theta}(\omega)$ of $\Theta(\omega)$. If $\hat{\Theta}(\omega)$ covers more than half the circumference of the circle, then the corresponding estimate of $\hat{\alpha}(\omega)$ satisfies $\cos\{\hat{\alpha}(\omega)\} \geq 0$ and hence from (15) the gain is zero. (The equivalent modified Winer filter is arbitrary, but with p very large.)

If $\cos\{\hat{\alpha}(\omega)\}$ is found to be less than zero, it is necessary to test whether the minimum possible S/N ratio at ω is greater than the degree of certainty of channel gain, i.e., we see if

$$\hat{A}_L^2(\omega) R(\omega)/N(\omega) \;\geq\; 2\hat{A}_L(\omega)/\{\hat{A}_U(\omega) - \hat{A}_L(\omega)\}$$

But

$$\begin{aligned}\hat{A}_L^2(\omega) R(\omega)/N(\omega) &= (1-Q(\omega))^2 \,|H_D(\omega)|^2 R(\omega)/N(\omega) \\ &= (1-Q(\omega))^2/\bar{\mu}(\omega)\end{aligned}$$

As before, assuming $\hat{\mu}(\omega)$ has been calculated, and $\hat{\mu}(\omega) \approx \bar{\mu}(\omega)$, this gives

$$\hat{A}_L^2(\omega) R(\omega)/N(\omega) \;\approx\; (1-Q(\omega))^2/\hat{\mu}(\omega)$$

Also,

$$2\hat{A}_L(\omega)/\{\hat{A}_U(\omega - \hat{A}_L(\omega)\} \;=\; (1-Q(\omega))/Q(\omega)$$

so we must determine if

$$\hat{\mu}(\omega) \;\leq\; Q(\omega)(1-Q(\omega)) \tag{22}$$

Let us firstly take the case where this is true. From fig. 2 $\hat{\theta}_L(\omega)$ and $\hat{\theta}_U(\omega)$ define $\hat{\beta}(\omega)$ and $2\hat{\alpha}(\omega)$. Since $\hat{\theta}(\omega) = arg\,\{H_D(\omega)\}$, the point estimate of the phase at frequency ω, is in the middle of the interval $\hat{\theta}_L(\omega)$, $\hat{\theta}_U(\omega)$, the condition (21)

$$arg\,\{H_D^*(\omega)\} \;=\; \pi \;-\; \hat{\beta}(\omega)$$

is satisfied automatically.

By replacing $\bar{\mu}(\omega)$ in (17) by $\hat{\mu}(\omega)$ and $\alpha(\omega)$ by $\hat{\alpha}(\omega)$, p is defined. Hence the equivalent modified Wiener filter is now completely specified. Instead of solving (14), $G_D(\omega)$ will now solve

$$\min_{\mathcal{G}} \; \max_{\mathcal{H}_2} e(H, G)$$

where $\mathcal{H}_2$ is the class of wavelet frequency responses with bounds determined by the 90% confidence intervals $(\hat{A}_L(\omega), \hat{A}_U(\omega))$, $(\hat{\theta}_L(\omega), \hat{\theta}_U(\omega))$. Again, while not ideal, this approach should prove robust to uncertainty in the wavelet's frequency response.

Of course it is not *necessary* to construct the modified Wiener filter to carry out this minimization, since given $\hat{A}_L(\omega)$, $\hat{A}_U(\omega)$, $\hat{\theta}_L(\omega)$ and $\hat{\theta}_U(\omega)$ the minimax filter can be constructed from (16) and (20). The modified Wiener filter merely provides a common framework. Note that although the noise-to-signal ratio does not enter (16) or (20) it is needed to verify (22).

In the second case, when $\hat{\mu}(\omega) > Q(\omega)(1-Q(\omega))$, the only change is to find the p that solves (19) when $\hat{\alpha}(\omega)$, $\hat{A}_L(\omega)$ and $\hat{\mu}(\omega)$ replace $\alpha(\omega)$, $A_L(\omega)$ and $\bar{\mu}(\omega)$. As before, $|H_D(\omega)| = \frac{1}{2}(\hat{A}_L(\omega) + \hat{A}_U(\omega))$.

Summary

It has been demonstrated that the modified Wiener filter for the deconvolution problem can be made equivalent to (a) the minimum average mean-squared error filter, and (b) the minimum maximum mean-squared error filter, if certain quantities are equated in an intuitive way. Solving (a) and (b) requires perfect knowledge of the range of possible wavelet frequency responses. However, equivalent sub-optimal problems can be solved where estimated uncertainty characteristics can be calculated for the design response used. The deconvolution filters for the equivalent sub-optimal problems to (a) and (b) may all now be readily expressed as modified Wiener filters. Using these methods, deconvolution can be made more robust to wavelet uncertainties.

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APPENDIX

Derivation of modification parameter p

Consider the minimization of

$$\lambda_1 E\{(r(t)-\hat{r}(t))^2\} + \lambda_2 E\{(n(t) * g(t))^2\}$$

Applying the orthogonality principle:

$$\lambda_1 E\{[r(t) - \sum_{k=-\infty}^{\infty} y(t-k)g(k)] - y(j)\} + \lambda_2 E\{[\sum_{k=-\infty}^{\infty} n(t-k)g(k)]\, n(j)\} = 0$$

i.e.

$$\lambda_1\{-R_{ry}(t-j) + \sum_{k=-\infty}^{\infty} R_{yy}(t-k-j)g(k)\} + \lambda_2\{\sum_{k=-\infty}^{\infty} R_{nn}(t-k-j)g(k)\} = 0$$

where $R_{ry}(\tau) = E\{r(t)y(t-\tau)\}$; $R_{yy}(\tau) = E\{r(t)r(t-\tau)\}$ and $R_{nn}(\tau) = E\{n(t)n(t-\tau)\}$. It is assumed that $r(t), n(t)$ and hence $y(t)$ all have mean zero.

Substitute $\tau = t - j$:

$$\lambda_1\{-R_{ry}(\tau) + \sum_{k=-\infty}^{\infty} R_{yy}(\tau-k)g(k)\} + \lambda_2\{\sum_{k=-\infty}^{\infty} R_{nn}(\tau-k)g(k)\} = 0$$

Now Fourier transform throughout to obtain

$$\lambda_1\{-S_{ry}(\omega) + S_{yy}(\omega)G(\omega)\} + \lambda_2\{S_{nn}(\omega)G(\omega)\} = 0$$

where $S_{ry}(\omega)$ is the cross-power spectrum between $r(t)$ and $y(t)$, $S_{yy}(\omega)$ is the auto-power spectrum of $y(t)$, $S_{nn}(\omega)$ is the auto-power spectrum of $n(t)$ and $G(\omega)$ is the frequency response function of the deconvolution filter $g(t)$. Hence

$$G(\omega) = \lambda_1 S_{ry}(\omega) / \{\lambda_1 S_{yy}(\omega) + \lambda_2 S_{nn}(\omega)\}$$

Now $y(t) = r(t) * h(t) + n(t)$, and the noise is uncorrelated with the input. Hence

$$S_{ry}(\omega) = S_{rr}(\omega) H^*(\omega) = R(\omega) H^*(\omega)$$

$$S_{yy}(\omega) = S_{rr}(\omega)\, |H(\omega)|^2 + S_{nn}(\omega) = R(\omega)\, |H(\omega)|^2 + N(\omega)$$

$$S_{nn}(\omega) = N(\omega)$$

Hence we obtain

$$\begin{aligned} G(\omega) &= \lambda_1 H^*(\omega)\, R(\omega) / \{\lambda_1\, |H(\omega)|^2 R(\omega) + \lambda_1 N(\omega) + \lambda_2 N(\omega)\} \\ &= H^*(\omega)\, R(\omega) / \{|H(\omega)|^2 R(\omega) + [1 + (\lambda_2/\lambda_1)]\, N(\omega)\} \\ &= H^*(\omega)\, R(\omega) / \{|H(\omega)|^2 R(\omega) + p\, N(\omega)\} \end{aligned}$$

where $p = 1 + (\lambda_2/\lambda_1)$

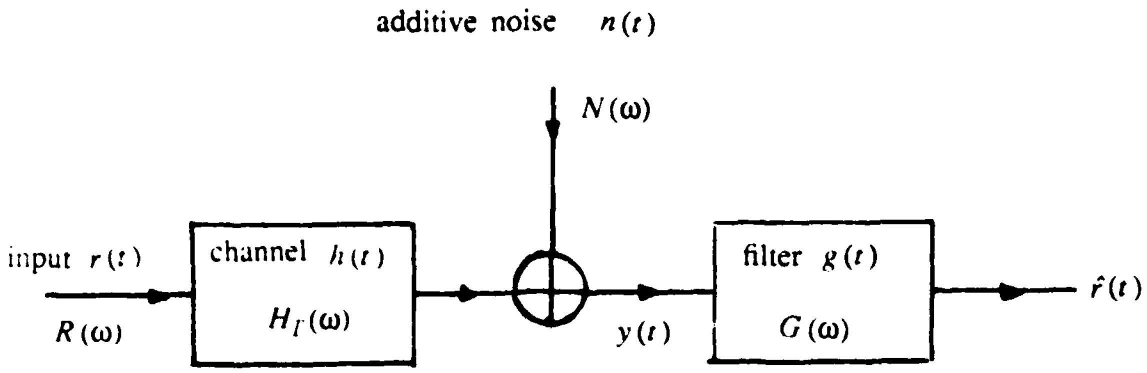
additive noise $n(t)$
$N(\omega)$
input $r(t)$
$R(\omega)$
channel $h(t)$
$H_I(\omega)$
$y(t)$
filter $g(t)$
$G(\omega)$
$\hat{r}(t)$

fig. 1.

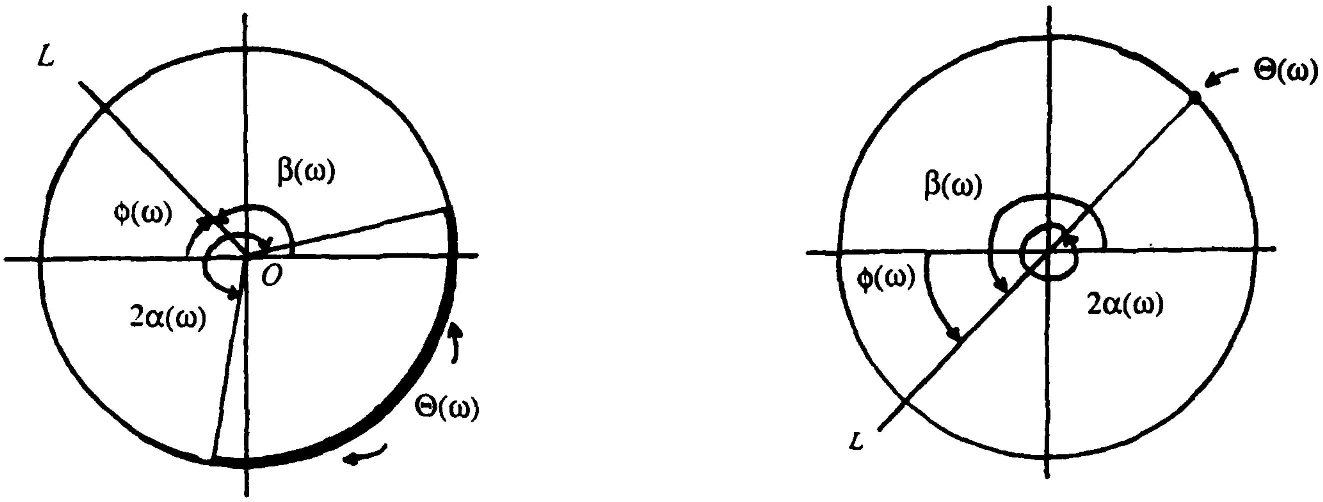

fig. 2.

END

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