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STOCHASTIC BOUNDS ON DISTRIBUTIONS OF OPTIMAL
VALUE FUNCTIONS WITH APPLICATIONS TO PERT,
NETWORK FLOWS AND RELIABILITY

Gideon Weiss
Georgia Institute of Technology
and
Tel Aviv University

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REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED		1b. RESTRICTIVE MARKINGS			
2a. SECURITY CLASSIFICATION AUTHORITY		3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited.			
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE					
4. PERFORMING ORGANIZATION REPORT NUMBER(S)		5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR·TR· 86-0542			

6a. NAME OF PERFORMING ORGANIZATION Georgia Institute of Tech	6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION Air Force Office of Scientific Research
6c. ADDRESS (City, State and ZIP Code) Atlanta, GA 30332		7b. ADDRESS (City, State and ZIP Code) Directorate of Mathematical & Information Sciences, Bolling AFB DC 20332-6448

8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR	8b. OFFICE SYMBOL (If applicable) NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER AFOSR-84-0367			
8c. ADDRESS (City, State and ZIP Code) Bolling AFB DC 20332-6448		10. SOURCE OF FUNDING NOS.			
		PROGRAM ELEMENT NO. 61102F	PROJECT NO. 2304	TASK NO. A5	WORK UNIT NO.

11. TITLE (Include Security Classification)
Stochastic Bounds on Distributions of Optimal Value Functions with Applications to Pert, Network Flow and Reliability

12. PERSONAL AUTHOR(S)
Gideon Weiss

13a. TYPE OF REPORT Reprint	13b. TIME COVERED FROM _____ TO _____	14. DATE OF REPORT (Yr., Mo., Day) July 1985	15. PAGE COUNT 28

16. SUPPLEMENTARY NOTATION
Industrial and Systems Engineeirng Report Series No. J-85-10 July 1985

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number) stochastic bounds, convex majorisation, reliability lifetime
FIELD	GROUP	SUB. GR.	

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

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20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐	21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED	
22a. NAME OF RESPONSIBLE INDIVIDUAL Major Brian W. Woodruff	22b. TELEPHONE NUMBER (Include Area Code) (202) 767-5027	22c. OFFICE SYMBOL NM

DD FORM 1473, 83 APR   EDITION OF 1 JAN 73 IS OBSOLETE.   UNCLASSIFIED

STOCHASTIC BOUNDS ON DISTRIBUTIONS OF OPTIMAL VALUE FUNCTIONS

WITH APPLICATIONS TO PERT, NETWORK FLOWS AND RELIABILITY

Gideon Weiss

Georgia Institute of Technology and Tel Aviv University

Abstract

Meilijson and Nadas [1979] have obtained stochastic bounds in the convex majorisation sense to the critical path length of a project network with random activity durations. In this paper we present those results in a more general framework and, using similar techniques, obtain bounds for shortest route, maximal flow and reliability system lifetime.

Subject classification: #488   Bounds for stochastic networks

#672 Convex majorisation of project critical path length.

#725 Stochastic majorisation of reliability system lifetime.

Consider a set $I = \{1,\ldots, n\}$ of n nodes, the base set. Let I_1, $\ldots$, I_k be subsets whose union is I, and no two of which are ordered by inclusion; $\{I_j\}$ $1 < j < k$ is a clutter over I. The blocking clutter to $\{I_j\}$ is a clutter $J_1,\ldots, J_\ell$ such that I_r $J_s \neq \phi$ for all r, s, and J_j are minimal sets with this property, cf. Edmonds and Fulkerson [1970]. In a directed acyclic graph or in a two terminal network, the paths and cuts are an example of a pair of blocking clutters. We call I a system,

and $\{I_j\}$ $1 < j < k$, $\{J_j\}$ $1 < j < \ell$ the paths and cuts of the system. Let a weight X_i be associated with each node i of system I. In many combinatorial optimization problems the system has an optimal value function, a function of $X_1, \ldots, X_n$, which is defined by the clutter of paths or of cuts. To illustrate:

- Critical path of a PERT network (Elmaghraby [1977]): the nodes represent activities, the weights activity durations, the network the precedence constraints. The critical path length is the shortest time needed to complete the project, given by

$$M = \max_{1 < j < k} \sum_{i \in I_j} X_i \qquad (1)$$

over the clutter of paths.

- Maximal flow (Ford and Fulkerson [1962], Lawler [1976]): the nodes represent pipelines, the weights maximal flow capacities. The maximal flow through a network from source to sink is:

$$L = \min_{1 < j < \ell} \sum_{i \in J_j} X_i \qquad (2)$$

over the clutter of cuts.

- Shortest route (Ford and Fulkerson [1962], Lawler [1976]): the nodes represent sections of routes, the weights their lengths, the network their connections; the shortest route from source to sink is given by L, over the clutter of paths.

- Reliability system lifetime (Barlow and Proschan [1975]): the nodes represent components, the weights their lifetimes. The system lifetime can be expressed in terms of the paths $\{I_j\}$ or the cuts $\{J_j\}$ as:

$$T = \max_{1 < j < k} \min_{i \in I_j} X_i = \min_{1 < j < \ell} \max_{i \in J_j} X_i \qquad (3)$$

The formulation of M, L, T via clutters applies equally well to structures more general than networks, e.g. precedence relations among

project activities can be defined by any partial order, and a reliability system can be defined by any Boolean coherent structure function. The equality (3) holds for any pair of blocking clutters, cf. Edmonds and Fulkerson [1970].

The stochastic behaviour of the optimal value functions M, L and T is introduced as follows. Let the weights $X_1, \ldots, X_n$ be random variables, with marginal distribution functions $F_1, \ldots, F_n$ and a joint distribution P. Then M, L, T are random variables. It is extremely difficult to obtain the distributions of M, L, T – this is so even in the case where $X_1, \ldots, X_n$ are independent, since different I_j's will in general have nodes in common and not be independent. Nor is it any easier to determine single values such as $E(V)$, $P(V > y)$, $E(V - y)^{\pm}$ where V is any of M, L or T (here $z^+ = \max (z, o)$, $z^- = (-z)^+$). Let $\mathcal{P}$ denote the family of all the joint distributions of $X_1, \ldots, X_n$ with the given marginal distributions $F_1, \ldots, F_n$. The subject of this paper is the investigation of:

$$\Psi(x) = \sup_{\mathcal{P}} E(M-x)^+$$

$$\eta(x) = \sup_{\mathcal{P}} E(L-x)^- \qquad (4)$$

$$\alpha(x) = \sup_{\mathcal{P}} P(T > x)$$

$$\beta(x) = \sup_{\mathcal{P}} P(T < x)$$

We show how each of the functions Ψ, η, α, β can be calculated as the solution to an appropriate mathematical programming problem which is in general substantially easier than the calculation of $E(M-x)^+$, $E(L - x)^-$ or $P(T > x)$ for a particular $P \in \mathcal{P}$. The suprema in (4) are attained for every x, that is, for every x there exists a joint distribution P for which $\Psi(x) = E(M - x)^+$, and similar distributions attain the supremum for

n, α, β. The joint distributions which attain these suprema can be chosen to have a special form. Define the inverse of a distribution function as

$$F^{-1}(u) = \inf \{x \mid F(x) \geq u\}$$

and let U be a uniform random variable on $(0, 1)$. Then $X_1, \ldots, X_n = F_1^{-1}$ $(\phi_1(U)), \ldots, F_n^{-1}(\phi_n(U))$, where U is common to all the nodes, and the functions $\phi_1, \ldots, \phi_n$ are piecewise linear, with a finite, not exceeding max (k, ℓ), number of discontinuities. $\phi_1, \ldots, \phi_n$ are obtained explicitly from the solution of the mathematical programming problems, together with additional structural information about the system.

The functions Ψ, η, α, β can be used to define random variables $\bar{M}$, $\underline{L}$, $\bar{T}$ and $\underline{T}$ as follows:

$$
\begin{aligned}
E(\bar{M} - x)^+ &= \Psi(x) \\
E(\underline{L} - x)^- &= \mu(x) \\
P(\bar{T} > x) &= \alpha(x) \\
P(\underline{T} < x) &= \beta(x)
\end{aligned}
\qquad (5)
$$

By their definition (4, 5), $\bar{M}$ is convexly greater, $\underline{L}$ is concavely smaller, $\bar{T}$ ($\underline{T}$) is stochastically greater (smaller) than M, L, T respectively, for any distribution $P \in \mathfrak{p}$.

The definitions of X stochastically greater than Y ($X >_{ST} Y$) and of X convexly (concavely) greater than Y ($X >_c Y$, ($X <_k Y$)) are (cf Stoyan [1983]):

$$X >_{ST} Y \iff \forall x \quad P(X > x) \geq P(Y > x) \iff Eh(x) \geq Eh(Y) \quad \forall h \text{ nondecreasing}$$

$$X >_c Y \iff \forall x \quad E(X - x)^+ \geq E(Y - x)^+ \iff Eh(X) \geq Eh(Y) \quad \forall h \text{ convex nondecreasing}$$

$$X >_k Y \iff \forall x \quad E(X - x)^- \leq E(Y - x)^- \iff Eh(X) \geq Eh(Y) \quad \forall h \text{ concave nondecreasing}$$

$$\iff \forall x \quad E(x - X)^+ \leq E(x - Y)^+ \iff Eh(X) \leq Eh(Y) \quad \forall h \text{ convex nonincreasing}$$

$$\iff -X <_c -Y$$

We say that the random variables $\bar{M}$, $\underline{L}$, $\bar{T}$, $(\underline{T})$ are convex upper, concave lower and stochastic upper (lower) bounds for M, L, T. Clearly by (4, 5) they are sharp bounds, in the sense that if for example $Z \succ_c M$ for every $P \in \mathcal{P}$, then $Z \succ_c \bar{M}$. By the properties of $\succ_{ST}$, $\succ_c$, $\succ_k$, $Eh(\bar{M})$, $Eh(\underline{L})$, $Eh(\bar{T})$, $Eh(\underline{T})$ provide bounds for $Eh(M)$, $Eh(L)$, $Eh(T)$ for every $P \in \mathcal{P}$, whenever h has the appropriate monotonicity and convexity properties; these bounds are not necessarily sharp, unless $\bar{M}$, $\underline{L}$, $\bar{T}$, $\underline{T}$ are obtained within $\mathcal{P}$.

In general, $\bar{M}$, $\underline{L}$, $\bar{T}$ and $\underline{T}$ are not obtained within $\mathcal{P}$. If however the system is series parallel, then there exist joint distributions in $\mathcal{P}$ for which $\bar{M} = M$, or $\underline{L} = L$ or $T = \bar{T}$, $T = \underline{T}$; bounds for series parallel systems are discussed in section 1, together with a discussion of modular decomposition. In sections 2, 3, 4 we discuss each of the optimal value functions, M, L and T separately. We conclude in section 5 with some general remarks on the type of bounds presented in this paper, and with a comparison with other types of bounds which appear in the literature.

The present work is based on a paper of Meilijson and Nadas [1979], who derived the properties of $\Psi(x)$. Some of the results on $\alpha(x)$, $\beta(x)$ have been previously obtained by Zemel [1982]. A brief summary of the present paper appeared in Weiss [1984]. Some related results and extensions appeared in klein Haneveld [1982], and Meilijson [1984].

1. Bounds for Series Parallel Systems.

The pure series system with nodes $1, \ldots, n$ has a single path $I_1 = I = \{1, \ldots, n\}$ and n singleton cuts, $J_1 = \{1\}, \ldots, J_n = \{n\}$. The pure parallel system has paths $I_1 = \{1\}, \ldots, I_n = \{n\}$, and a single cut $J_1 = \{1, \ldots, n\}$. For the pure series system, $M = \sum_{i=1}^{n} X_i$, $L = T = \min_{1 \leq i \leq n} X_i$. For the pure parallel system $M = T = \max_{1 \leq i \leq n} X_i$, $L = \sum_{i=1}^{n} X_i$.

The following three special joint distributions of $X_1, \ldots, X_n$ are essential in this paper; they provide the bounds for the pure series and the pure parallel systems $\left(U \text{ is a uniform random variable on } (0, 1),\right.$ $\left. \bar{F}(x) = 1 - F(x) = P(X > x) \right)$:

- The "perfect tracking" distribution P*: $X_1, \ldots, X_n = F_1^{-1}(U), \ldots, F_n^{-1}(U)$.

- The "max antithetic" distribution P** defined inductively for $n = 2$ by $X_1, X_2 = F_1^{-1}(U), F_2^{-1}(1-U)$ and, given P** for $X_1, \ldots, X_{n-1}$ and $Y_{n-1} = \max_{1 \leq i \leq n-1} X_i$, X_n and Y_{n-1} are distributed by Y_{n-1}, $X_n = F_{Y_{n-1}}^{-1}(U)$, $F_n^{-1}(1-U)$.

- The "min antithetic" distribution P*** defined similarly to P** with $Z_{n-1} = \min_{1 \leq i \leq n-1} X_i$ replacing Y_{n-1}.

It is easy to check that P* (P**) stochastically minimises (maximises) $\max X_i$, and P*(P***) stochastically maximises (minimises) $\min_{1 \leq i \leq n} X_i$ over $\mathfrak{p}$ by achieving equality in:

$$\max_{1 \leq i \leq n} F_i(x) \leq P(\min_{1 \leq i \leq n} X_i \leq x) \leq \min(1, \sum_{i=1}^{n} F_i(x))$$

$$\max_{1 \leq i \leq n} \bar{F}_i(x) \leq P(\max_{1 \leq i \leq n} X_i > x) \leq \min(1, \sum_{i=1}^{n} \bar{F}_i(x)).$$

Also, P* convexly maximizes and at the same time concavely minimizes $\sum_{i=1}^{n} X_i$ as is seen by the following argument: For every x and v and every $P \in \mathfrak{p}$:

$$(\sum_{i=1}^{n} X_i - x)^+ \leq (\sum_{i=1}^{n} v_i - x)^+ + \sum_{i=1}^{n} (X_i - v_i)^+$$

$$(x - \sum_{i=1}^{n} X_i)^+ \leq (x - \sum_{i=1}^{n} v_i)^+ + \sum_{i=1}^{n} (v_i - X_i)^+$$

On the other hand, note that $\sum_{i=1}^{n} F_i^{-1}(u)$ is left continuous non-

decreasing in u, so for given x we can choose u_o such that $\sum_{i=1}^{n} F_i^{-1}(u_o) \leqslant x$

$\leqslant \sum_{i=1}^{n} F_i^{-1}(u_o+)$, and we can then choose v_i for $i = 1,\ldots,$ n for which

$F_i^{-1}(u_o) \leqslant v_i \leqslant F_i^{-1}(u_o+)$, so that $\sum_{i=1}^{n} v_i = x$. Using those v_i, for $X_i =$

$F_i^{-1}(u)$, $i = 1,\ldots,$ n as in P*, the above inequalities hold as equalities.

These properties of P*, P**, P*** ensure that the various bounds are

obtained within $\mathfrak{p}$ in the pure series and in the pure parallel case.

Theorem 1.1: For the pure series system, $\bar{M}$ and $\bar{T}$ are obtained by P^*, $\underline{L}$

and $\underline{T}$ are obtained by P***, and $\underline{L} \leqslant_{ST} L$ for all P ϵ $\mathfrak{p}$. For the pure

parallel system, $\bar{M}$ and $\bar{T}$ are obtained by P**, $\underline{L}$ and $\underline{T}$ by P*, and $\bar{M} \geqslant_{ST} M$

for all P ϵ $\mathfrak{p}$.

A useful concept in the theory of networks or clutters is that of

decomposition into modules (or autonomous sets), as discussed by Barlow

and Proschen [1975] and by Mohring and Radermacher [1984]. Consider a

set I^*, $I^* \subseteq I$, and let $I_1^*,\ldots,$ I_m^* be all the different subsets of I^* of

the form $I^* \cap I_j$, $1 \leqslant j \leqslant k$. Then, I^* is a module of the system I if:

(i) $I_1^*,\ldots,$ I_m^* form a clutter.

(ii) Whenever $I_i \cap I^* = I_j^*$ it follows that for every r, $1 \leqslant r \leqslant m$

there exists an s, $1 \leqslant s \leqslant k$, such that $(I_i - I_j^*) \cup I_r^* = I_s$.

The module I^* is called nontrivial if it has more than 1 and less than

n nodes. The quotient system I/I^* is formed by replacing all the nodes

of I^* in I by a single new node o, with a similar replacement in each

path of the clutter $\{I_j\}$. It is maybe more intuitive to think of a

system, module, and quotient system in the reverse order: Start with the

quotient system and the module (those can be any two systems), choose a

node in the quotient system (node o can be any node) and replace this

node by the base set of the module; then augment the clutter of the quotient system, by replacing each path which contains o with m new paths in which o is replaced by $I_1^*, \ldots, I_m^*$. For a nontrivial module I^*, call I a modular composition of I^*, I/I^* and call I^*, I/I^* a modular decomposition of I.

Mohring and Radermacher [1984] discuss the preservation of M, L, T under modular composition. Let V represent any of the optimal value functions M, L or T. For weights $x_1, \ldots, x_n$, let V, V^* be the optimal values for the system I and the module I^*. Then V can also be calculated in steps: Obtain V^*, assign the value V^* as the weight of node o in I/I^*, calculate the optimal value for I/I^*. For $X_1, \ldots, X_n$ random with joint distribution $P \in \mathcal{P}$, V and V^* are random variables. The distribution of V can be calculated in steps: Obtain the joint distribution of V^*, assigned to node o, joint with the weights of the other nodes of I/I^*, and obtain the distribution of the optimal value of I/I^* for that joint distribution.

In the following sections we prove that modular composition also preserves the bounds $\bar{M}$, $\underline{L}$, $\bar{T}$, $\underline{T}$. We show for each of the optimal value functions that:

Theorem 1.2: If module I^* is replaced by the single node o, with weight X_o that has as its marginal distribution the distribution of the bound for I^*, then the bounds for I/I^* and for I are identical.

A general series parallel system is defined (inductively in the number of nodes n) as a system which is either pure series or pure parallel or has a nontrivial module I^* and quotient system I/I^* both of which are series parallel. Combining theorems 1.1 and 1.2 we have:

Theorem 1.3: For a series parallel system the bounds $\bar{M}$, $\underline{L}$, $\bar{T}$, $\underline{T}$ are obtained by joint distributions within $\mathcal{P}$.

Proof: Combining theorems 1.2 and 1.1 provides a direct construction of the joint distributions for which M, L and T are extremal.

2. Convex Upper Bounds for Critical Path Length

In this section we discuss the optimal value function

$$M = \max_{1 \leqslant j \leqslant k} \sum_{i \in I_j} X_i$$

where $I = \{1, \ldots, n\}$, $I_1, \ldots, I_k$ is a clutter over I, and $X_1, \ldots, X_n$ have marginal distributions $F_1, \ldots, F_n$ and a joint distribution $P \in \mathcal{P}$ (the dependence of M on P is suppressed to simplify notation). We start by quoting the results of Meilijson and Nadas [1979].

Let $\Psi(x)$ be defined by:

$$\Psi(x) = \inf_{v} \{ (\max_{1 \leqslant j \leqslant k} \sum_{i \in I_j} v_i - x)^+ + \sum_{i \in I} E(X_i - v_i)^+ \} \qquad (6)$$

and let $x_o = \inf \{x \mid \Psi'(x) > -1\}$. It turns out that the calculation of (6) for $x > x_o$ is equivalent to the solution of the following mathematical program with a separable convex objective function and linear constraints:

$$\Psi(x) = \min_{v} \sum_{i \in I} E(X_i - v_i)^+ \qquad (7)$$

$$\text{s.t.} \quad \sum_{i \in I_j} v_i \leqslant x \quad j = 1, \ldots, k.$$

Denote by $\lambda_1, \ldots, \lambda_k$ the Lagrange multipliers (dual variables) of the constraints.

Theorem 2.1:

(i) $\quad \Psi(x) = \sup_{\mathcal{P}} E(M - x)^+$

(ii) There exists a random variable $\bar{M}$ such that for all x,

$\Psi(x) = E(\bar{M} - x)^+$, and $\bar{M} \geqslant_c M$ for all $P \in \mathcal{P}$.

(iii) For every x there exists a $P \in \mathcal{P}$ for which $E(M - x)^+ = \Psi(x)$.

(iv) A particular $P \in \mathcal{P}$ satisfying (iii) is of the form $X_1, \ldots, X_n = F_1^{-1}(\phi_1(U)), \ldots, F_n^{-1}(\phi_n(U))$, where $U \sim U(0, 1)$, and ϕ_i have at most k discontinuities and are linear inbetween.

(v)   For every x and P as in (iv), the Lagrange multipliers of (7)

satisfy:

$$\lambda_j \leq P(M \geq x, \sum_{i \in I_j} X_i = M)$$

With equality if all F_i's are non atomic (absolutely continuous).

(vi) The constant $\max_{1 \leq j \leq k} \sum_{i \in I_j} E(X_i)$ is convexly smaller than M for all

$P \in \mathfrak{p}$; in particular it is $\leq E(M)$.

<u>Outline of the proof</u>: (i) For every x and every vector v, for every

joint distribution $P \in \mathfrak{p}$ and every realisation $X_1, \ldots, X_n$ drawn from P:

$$(M - x)^+ \leq (\max_j \sum_{i \in I_j} v_i - x)^+ + \sum_{i=1}^{n} (X_i - v_i)^+ \qquad (8)$$

which shows that the right hand side of (6) is $\geq E(M - x)^+$ for every

$P \in \mathfrak{p}$. Equality to the supremum over $\mathfrak{p}$ follows from (iii).

(ii) Examination of (6) shows that $\Psi(x)$ is convex nonincreasing in x

with slopes tending to -1 and 0 as x tends to $-\infty$ and ∞. Hence $\Psi(x)$

defines a random variable $\bar{M}$ according to (5), and $\bar{M} \geq_c M$ for all $P \in \mathfrak{p}$.

For $x > x_o$, v which minimises (6) satisfies $\max_{1 \leq j \leq k} \sum_{i \in I_j} v_i = x$, and so it

minimises (7), and (6) and (7) are equivalent. The solution of (7) at

x_o, say v^o, minimises (6) for all $x \leq x_o$.

(iii, iv) Consider the Lagrangean of (7):

$$\Psi(v, \lambda, x) = \sum_{i=1}^{n} E(X_i - v_i)^+ + \sum_{j=1}^{k} \lambda_j (\sum_{i \in I_j} v_i - x) \qquad (9)$$

with $\lambda_j \geq 0$. The Kuhn Tucker saddle point conditions for it are:

$$\sum_{i \in I_j} v_i \leq x \text{ and } \sum_{i \in I_j} v_i < x \text{ implies } \lambda_j = 0$$

$$P(X_i > v_i) \leq \sum_{j | i \in I_j} \lambda_j \leq P(X_i \geq v_i) \qquad (10)$$

$$P(\bar{M} > x) \leq \sum_{j=1}^{k} \lambda_j \leq P(\bar{M} \geq x).$$

For a given $x > x_0$, let $v_1, \ldots, v_n, \lambda_1, \ldots, \lambda_k$ be an optimal solution and a set of multipliers of (7,9). Let $\lambda_{k+1} = 1 - \sum_{j=1}^{k} \lambda_j$, $I_{k+1} = \phi$, $\alpha_i = \sum_{j \mid i \in I_j} \lambda_j$, $i = 1, \ldots, n$. The joint distribution $P \in \mathcal{P}$ stated in (iv) is defined by the functions ϕ_i, $i = 1, \ldots, n$ which for $m = 1, \ldots, k + 1$ and $\sum_{j=1}^{m-1} \lambda_j < u \leq \sum_{j=1}^{m} \lambda_j$ have the value:

$$\phi_i(u) = \begin{cases} (1 - \alpha_i) + \alpha_i \, (u - \sum_{j=1}^{m-1} \lambda_j)/\lambda_m & i \in I_m \\ \\ (1 - \alpha_i) \, (u - \sum_{j=1}^{m-1} \lambda_j)/\lambda_m & i \notin I_m \end{cases} \tag{11}$$

(v) For $\lambda_m \neq 0$, with probability λ_m, $\sum_{i=1}^{m-1} \lambda_j < U < \sum_{i=1}^{m} \lambda_j$ in which case $1 - \alpha_i < \phi_i(U) < 1$, and by (10) $X_i = F_i^{-1}(\phi_i(U)) > v_i$, for all $i \in I_m$, while at the same time, $X_i \leq v_i$ for all $i \notin I_m$. By (8), we see that in this case $M - x = \sum_{i \in I_m} (X_i - v_i)$ and $\sum_{i \in I_m} v_i = x$, so $M = \sum_{i \in I_m} X_i > x$.

The required inequality follows, and equality for nonatomic distributions follows similarly. Finally, (vi) holds by Jensen's inequality.

Corollary 2.2. <u>Modular decomposition</u>: Theorem 1.2 holds for the function M.

<u>Proof</u>: Let $I^* \subseteq I$ with clutter $I_1^*, \ldots, I_m^*$ be a module of I, and let I/I^* be the quotient system, with set of nodes $I^- = (I - I^*) \cup \{o\}$ and clutter of paths $I_1^-, \ldots, I_\ell^-$. Let Ψ, Ψ_o, Ψ_1 and $\bar{M}$, $\bar{M}_o$, $\bar{M}_1$ denote the bounds for the systems I, I^*, I/I^* respectively. We look at the program (7) and the two additional programs:

$$\Psi_o(y) = \min_u \sum_{i \in I^*} E(X_i - u_i)^+$$

$$\text{s.t.} \quad \sum_{i \in I_j^*} u_i \leq y \quad j = 1, \ldots, m \tag{12}$$

and:

$$\Psi_1(x) = \min_{w} \sum_{i \epsilon I - I^*} E(X_i - w_i)^+ + \Psi_0(w_0)$$

$$\text{s.t.} \sum_{i \epsilon I_j^-} w_i \leqslant x \qquad j = 1, \ldots, \ell \qquad (13)$$

Since $\Psi_0(w_0) = E(\bar{M}_0 - w_0)^+ = E(X_0 - w_0)^+$, $\Psi_1(x)$ is the bound for the module I/I^*. We need to show that $\Psi(x) = \Psi_1(x)$ for all x.

(i) $\Psi(x) > \Psi_1(x)$: Let v be an optimal solution of (7).

Define:

$$w_0 = \max_{1 \leqslant j \leqslant m} \sum_{i \epsilon I_j^*} v_i$$

$$w_i = v_i \qquad i \epsilon I - I^*.$$

Because I^* is a module, and v is feasible for (7), w is feasible for (13).

The value of the objective function (13) for w is $\sum_{i \epsilon I - I^*} E(X_i - v_i)^+ +$

$\Psi_0(w_0)$. But $\{v_i\}$ $i \epsilon I^*$ is feasible for (12) with $y = w_0$, and so $\Psi_0(w_0)$

$\leqslant \sum_{i \epsilon I^*} E(X_i - v_i)^+$, so the value of the objective of (13) for w is $\leqslant \Psi(x)$,

and therefore $\Psi_1(x) \leqslant \Psi(x)$.

(ii) $\Psi_1(x) > \Psi(x)$: Let w be an optimal solution of (13). Let u be an

optimal solution of (12), with $y = w_0$. Let $v_i = u_i$, $i \epsilon I^*$, and

$v_i = w_i$, $i \epsilon I - I^*$. Because I^* is a module, v is feasible for (7). The

objective value of (7) for v is

$$\sum_{i \epsilon I} E(X_i - v_i)^+ = \sum_{i \epsilon I - I^*} E(X_i - w_i)^+ + \sum_{i \epsilon I^*} E(X_i - u_i)^+$$

$$= \sum_{i \epsilon I - I^*} (X_i - w_i)^+ + \Psi_0(w_0) = \Psi_1(x),$$

thus $\Psi(x) \leqslant \Psi_1(x)$.

Monotonicity:

Corollary 2.3: If X_i are replaced by Z_i so that $Z_i >_c X_i$, $i = 1, \ldots, n$

then the bounds $\bar{M}_1$, $\bar{M}$ obtained for $Z_1,\ldots, Z_n$ and $X_1,\ldots, X_n$ satisfy $\bar{M}_1 >_c$ $\bar{M}$.

<u>Proof</u>: Let $\Psi_1(x)$, $v_1^{(1)},\ldots, v_n^{(1)}$ be the solution of (7) with Z_i replacing X_i. By $Z_i >_c X_i$, $\widehat{\Psi}(x) = \sum\limits_{i=1}^{n} E(X_i - v_i^{(1)})^+ \leqslant \Psi_1(x)$. Minimising (7) with X_i, we get $\Psi(x) \leqslant \widehat{\Psi}(x) \leqslant \Psi_1(x)$, so for all x, $E(\bar{M} - x)^+ \leqslant E(\bar{M}_1 - x)^+$.

<u>Computational Aspects</u>: Nadas [1979] discusses the computational aspects of solving the mathematical program (7), which with its linear constraints and separable convex objective function is relaively easy. If $E(X_i - v_i)^+$ is approximated from above by $\xi_i(v_i)$ piecewise linear and convex, the program can be solved as a linear program, and provide an upper bound for $\Psi(x)$. The approximation is equivalent to replacing each F_i by an approximating discrete distribution, and it can be chosen so that $0 \leqslant \xi_i(v) - E(X_i - v)^+ \leqslant \delta$ for any given $\delta > o$, uniformly for all v.

In the project planning application, the nodes represent activities and the clutter $I_1,\ldots, I_k$ is defined by the partial ordering of activities, and consists of all the paths from the start to the finish of the job. In that case the program (7) has the following deterministic interpretation: Find activity durations $v_1,\ldots, v_n$ so as to complete the whole project by time x at minimal cost, where doing activity i in duration v_i costs $E(X_i - v_i)^+$. This is the project cost curve problem, solved by Fulkerson [1961]. The solution is effected, parametrically for all x, by formulating the dual problem which is a minimal cost flow problem, and solving it parametrically for all flow values; this can be done by the very efficient out of kilter method, cf. Lawler [1976]. The minimal cost flow problem that arises from the dual to (7) is: For any total flow value A, find flows α_i through the nodes i, i = 1,..., n which yield total flow A, at minimal cost, that is:

$$\min_{\alpha} \sum_{i=1}^{n} h_i(\alpha_i) \tag{14}$$

$$\text{s.t.} \min_{1 \leq j \leq \ell} \sum_{i \in J_j} \alpha_i = A \qquad i = 1, \ldots, n$$

Where $J_1, \ldots, J_\ell$ is the blocking clutter of cuts, and where:

$$h_i(\alpha) = \int_o^\infty \max(\bar{F}_i(t) - \alpha, 0) \, dt = \int_o^{1-\alpha} F_i^{-1}(u) \, du. \tag{15}$$

The total flow value A, and the flows through the nodes α_i, which are obtained from the solution of (14), are related to the λ's in (9), (10) through $A = \sum_{j=1}^{k} \lambda_j$, $\alpha_i = \sum_{j \mid i \in I_j} \lambda_j$. The corresponding values of x and the v_i's in (7) can be obtained from (10).

Redesign of a PERT network: It is quite usual when designing a project with a PERT network to have a target date x for the completion of the project, and a nondecreasing convex penalty function $C(y)$ for values $M = y > x$. For such a penalty function,

$$\bar{E}(C) = C'(x) \, \Psi(x) + \int_x^\infty C''(y) \, \Psi(y) dy \tag{16}$$

where C', C'' are the 1st and 2nd derivatives of C, is an upper bound on the expected penalty.

For the target date x, the expected tardiness $E(M - x)^+$ is bounded sharply by $\Psi(x)$, and the solution of (7) provides a construction for the worst case distribution with respect to that tardiness. It also provides a host of additional information on that worst case distribution which can be used to redesign the project. Let $v = v(x) = v_1(x), \ldots, v_n(x)$ be the values of the solution of (7), $\lambda_1(x), \ldots, \lambda_k(x)$ the Lagrange multipliers, and $\alpha_i(x) = \sum_{j \mid i \in I_j} \lambda_j(x)$, $i = 1, \ldots, n$.

The values $v_i(x)$ provide target durations for the activities with respect to the general target date x. If we let $\Psi_i(x) = E(X_i - v_i(x))^+$, then $\Psi_i(x)$ is the expected contribution of node i (activity i) to the total tardiness. Similarly, for a module I^* we get by solving (13) for I/I^* and due date x, a value $w_o(x)$ which is the target duration of the module I^* with respect to the general target date x, and we can get $\Psi_{I^*}(x)$ = $E(\bar{M}_o - w_o(x))^+$ (obtained by solving (12), with $y = w_o(x)$), as the expected contribution of module I^* to the total tardiness. If $\Psi_i(x)$ or $\Psi_{I^*}(x)$ is inserted in (16) instead of Ψ, we obtain $\bar{E}_i(C)$ and $\bar{E}_{I^*}(C)$ which are the worst case bounds on the expected contribution of i or I^* to the penalty. Thus the $v_i(x)$ and $w_o(x)$ provide a way of assigning tardiness and penalties to each activity or module (on the basis of a worst case analysis).

The values $\lambda_j(x)$ provide, for the worst case distribution, the probability that tardiness beyond x occurs, and that the longest path is I_j (at least if all X_i's are continuous random variables), as stated in theorem 2.1. It is also easy to see from the proof of theorem 2.1 that $\alpha_i(x)$ is the probability that tardiness beyond x occurs and that node i is on the longest path.

Similar quantities can be calculated for a module I^*. Solution of (12) with $y = w_o(x)$ provides λ's and α's within I^*. Solution of (13) for I/I^*, provides by the value $\alpha_o(x)$ the probability that tardiness beyond x occurs and the longest path passes through I^*.

3. Concave Lower Bounds for Maximal Flow and Shortest Route

In this section we discuss the optimal value function

$$L = \min_{1 \le j \le \ell} \sum_{i \in J_j} X_i$$

where $I = \{1,\ldots, n\}$, $J_1,\ldots, J_\ell$ is a clutter over I, and $X_1,\ldots, X_n$ have marginal distributions $F_1,\ldots F_n$ and a joint distribution $P \in \mathfrak{p}$. When $J_1,\ldots, J_\ell$ are the clutter of paths in a network, L is the shortest route; when $J_1,\ldots, J_\ell$ are the clutter of cuts in a network, L is the maximal flow. The results about L exactly mirror the results about M in section 2. This is due to the duality between the various pairs of concepts occuring here: path-cuts, series-parallel, min-max, convex-concave, $P(X < x) - P(X < x)$, and $E(X - x)^+ - E(X - x)^-$.

The function $\eta(x)$ in (4) is given by:

$$\eta(x) = \inf_{v} \{(x - \min_{1 < j < \ell} \sum_{i \in J_j} v_i)^+ + \sum_{i=1}^{n} E(v_i - X_i)^+\} \qquad (17)$$

and for $x < x_0 = \sup\{x \mid \eta'(x) < 1\}$, (17) is equivalent to

$$\eta(x) = \min_{v} \sum_{i \in I} E(v_i - X_i)^+ \qquad (18)$$

$$\text{s.t.} \sum_{i \in J_j} v_i > x \qquad j = 1,\ldots,\ell$$

with Lagrange multipliers $\lambda_1,\ldots, \lambda_\ell$.

<u>Theorem 3.1</u>:

(i) $\quad \eta(x) = \sup_{\mathfrak{p}} E(x - L)^+$

(ii) $\quad$ There exists a random variable $\underline{L}$ such that for all x, $\eta(x)$ $= E(x - \underline{L})^+$, and $\underline{L} \leq_k L$ for all $P \in \mathfrak{p}$.

(iii) $\quad$ For every x there exists a $P \in \mathfrak{p}$ for which $E(x - L)^+ = \eta(x)$.

(iv) $\quad$ A particular $P \in \mathfrak{p}$ satisfying (iii) is of the form: $X_1,\ldots, X_n$ $= F_1^{-1}(\phi_1(U)),\ldots, F_n^{-1}(\phi_n(U))$, where $U \sim U(0, 1)$, and ϕ_i have at most ℓ discontinuities and are linear in between.

(v) $\quad$ For every $x < x_0$ and P as in (iv), the Lagrange multipliers of (18) satisfy:

$$\lambda_j < P(L < x, \sum_{i \in J_j} X_i = L)$$

with equality if all F_i's are non atomic (absolutely continuous).

(vi) The constant $\min\limits_{1<j<\ell} \sum\limits_{i \in J_j} E(X_i)$ is concavely larger than L for all

P ϵ $\mathfrak{p}$; in particular it is $>$ E(L).

Proof: This is a corollary of theorem 2.1, if the problem is reformulated

in terms of $-X_i$, with $-L = \max\limits_{1<j<\ell} \sum\limits_{i \in I_j} \Sigma(-X_i)$.

The modular decomposition theorem 1.2 and monotonicity (with respect to

$>_k$) hold for L, in analogy with M.

Computational Aspects: The program (18) has a separable convex

nondecreasing objective function and linear constraints, and can be

approximated by a linear program, like (7).

For the shortest route application, the solution of (18) can be

obtained by using $Y_i = - X_i$, and solving (7).

For the maximal flow application, when J_1, ..., J_ℓ are cuts, problem

(18) has the following deterministic interpretation: Find flows (or

capacities) v_i for nodes i = 1, ..., n, so as to obtain a flow (maximal

flow) of x, at minimal cost, where the cost of flow v_i in node i is given

by $E(v_i - X_i)^+$, which is convex nondecreasing in v_i. This deterministic

problem is very similar to the dual problem for the project planning

application, given by (14). It can be solved parametrically for all flow

values x, using the out of kilter method, cf Lawler [1976].

In applications to shortest route problems one may have a design value

x and a convex decreasing reward function C(y) for values of L = y. In

applications to maximal flow problems one may have a target flow x and a

convex decreasing penalty function C(y) for value of L = y $<$ x. $\eta(x)$ and $\underline{L}$

provide upper bounds for the expected shortfall below x, $E(x - L)^+$, and of

E(C(L)). The solution of (18) provides similar information for redesign as

in the critical path applications.

4. Stochastic Upper and Lower Bounds for Reliability System Lifetime

In this section we discuss the optimal value functions

$$T' = \max_{1 \leq j \leq k} \quad \min_{i \in I_j} X_i \tag{19}$$

and

$$T'' = \min_{1 \leq j \leq \ell} \quad \max_{i \in J_j} X_i \tag{20}$$

where $I = \{1,\ldots,n\}$, $I_1,\ldots, I_k$ and $J_1,\ldots, J_\ell$ are two clutters over I, and $X_1,\ldots, X_n$ have marginal distributions $F_1,\ldots, F_n$ with joint distribution function $P \in \boldsymbol{\mathcal{p}}$. If $I_1,\ldots, I_k$ are the paths and $J_1,\ldots, J_\ell$ are the cuts of a reliability system (defined through a network or through a general Boolean coherent structure function as in Barlow and Proschan [1975]), and also for any other pair of blocking clutters as shown by Edmonds and Fulkerson [1970], $T' = T''$. For a reliability system, if nodes $1,\ldots, n$ represent components, and $X_1,\ldots, X_n$ are the component lifetimes then $T = T' = T''$ is the system lifetime.

We will show that the supremum functions $\alpha(x)$ and $\beta(x)$ of (4) are given by solution of the following linear programming problems:

$$\alpha(x) = \max_{\lambda} \sum_{j=1}^{k} \lambda_j$$

$$\text{s.t.} \sum_{j|i \in I_j} \lambda_j \leq \bar{F}_i(x) \quad i = 1,\ldots,n \tag{21}$$

$$\sum_{j=1}^{k} \lambda_j \leq 1$$

$$\lambda_j \geq 0$$

and

$$\beta(x) = \max_{\mu} \sum_{j=1}^{\ell} \mu_j$$

$$\text{s.t.} \sum_{j|i \in J_j} \mu_j \leq F_i(x) \quad i = 1,\ldots, n \tag{22}$$

$$\sum_{j=1}^{\ell} \mu_j \leq 1$$

$$\mu_j \geq 0$$

where $\bar{F}_i(x) = 1 - F_i(x) = P(X_i > x)$. The analogy with the programs (7) and (18) is seen in the dual programs to (21), (22):

$$\alpha(x) = \min_{v} \sum_{i=1}^{n} \bar{F}_i(x) v_i + w$$

$$\text{s.t.} \quad \sum_{i \in I_j} v_i + w \geqslant 1 \quad j = 1, \ldots, k \quad\quad (23)$$

$$w, v_i \geqslant 0$$

$$\beta(x) = \min_{u} \sum_{i=1}^{n} F_i(x) u_i + w$$

$$\text{s.t.} \quad \sum_{i \in J_j} u_i + w \geqslant 1 \quad j = 1, \ldots, \ell \quad\quad (24)$$

$$w, u_i \geqslant 0$$

The following theorem is implied in parts by Zemel [1982].

Theorem 4.1:

(i) $\alpha(x) = \sup_{p} P(T' > x)$, $\beta(x) = \sup_{p} P(T'' \leqslant x)$.

(ii) There exist random variables $\bar{T}$ and $\underline{T}$ such that for all x, $\alpha(x) = P(\bar{T} > x)$, $\beta(x) = P(\underline{T} \leqslant x)$, i.e. $\bar{T} \geqslant_{ST} T'$ and $\underline{T} \leqslant_{ST} T''$ for all $P \in p$.

(iii) For every x there exist P', $P'' \in p$ for which $P(T' > x) = \alpha(x)$, $P(T'' \leqslant x) = \beta(x)$.

(iv) In particular P', $P'' \in p$ satisfying (iii) exist which are of the form $X_1, \ldots, X_n = F_1^{-1}(\phi_1(U)), \ldots, F_n^{-1}(\phi_n(U))$, where $U \sim U(0, 1)$ and ϕ_i have at most $\max(n, k)$ ($\max(n, \ell)$) discontinuities and are linear inbetween.

(v)   For every x and P', P'' as in (iv), the solutions (21), (22) satisfy

$$\lambda_j \leqslant P(T' > x, \min_{i \in I_j} x_i = T')$$

$$\mu_j \leqslant P(T'' \leqslant x, \max_{i \in J_j} x_i = T'')$$

with equality if all F_i's are non atomic (absolutely continuous).

Proof: (ii) From the form of the objective functions of (21), (22) $\alpha(x)$ is

nonincreasing and $\beta(x)$ is nondecreasing. From (23), (24) $0 \leqslant \alpha(x) \leqslant 1$ and

$0 \leqslant \beta(x) \leqslant 1$. From (23) (24) one obtains $\beta(-\infty) = \alpha(\infty) = 0$, and from

(21), (22) $\alpha(-\infty) = \beta(\infty) = 1$. $\alpha(x)$ and $\beta(x)$ depend continuously on $F_i(x)$,

which are continuous from the right, hence $\alpha(x)$ and $\beta(x)$ are continuous

from the right. Thus $1 - \alpha(x)$ and $\beta(x)$ are distribution functions,

defining $\overline{T}$ and $\underline{T}$.

(iii)(iv), We shall describe the construction of members of P as stated in

(iv). For given x, let λ (μ) be the optimal basic solution of (24) ((22)).

Let $\lambda_{k+1} = 1 - \sum\limits_{j=1}^{k} \lambda_j$, $\mu_{\ell+1} = 1 - \sum\limits_{j=1}^{\ell} \mu_j$, $I_{k+1} = J_{\ell+1} = \phi$,

and let:

$$\alpha_i = \sum\limits_{j \mid i \in I_j} \lambda_j \leqslant \overline{F}_i(x)$$

$$\beta_i = \sum\limits_{j \mid i \in J_j} \mu_j \leqslant F_i(x),$$

and define for each $i = 1, \ldots, n$ the following functions for $0 \leqslant u \leqslant 1$:

If $\sum\limits_{j=1}^{m-1} \lambda_j \leqslant u \leqslant \sum\limits_{j=1}^{m} \lambda_j$, and $\lambda_m \neq 0$, $(1 \leqslant m \leqslant k + 1)$:

$$\phi'(u) = \begin{cases} (1 - \alpha_i) + \alpha_i (u - \sum\limits_{j=1}^{m-1} \lambda_j)/\lambda_m & i \in I_m \\[3mm] (1 - \alpha_i)(u - \sum\limits_{j=1}^{m-1} \lambda_j)'\lambda_m & i \notin I_m \end{cases}$$

and if $\sum\limits_{j=1}^{m-1} \mu_j \leqslant u \leqslant \sum\limits_{j=1}^{m} \mu_j$, and $\mu_m \neq 0$, $(1 \leqslant m \leqslant \ell + 1)$:

$$\phi_i''(u) = \begin{cases} \beta_i (u - \sum\limits_{j=1}^{m-1} \mu_j)/\mu_m & i \in J_m \\[3mm] \beta_i + (1 - \beta_i)(u - \sum\limits_{j=1}^{m-1} \mu_j)/\mu_m & i \notin J_m \end{cases}$$

It is clear from the construction that $F_i^{-1}(\phi_i'(U))$ and $F_i^{-1}(\phi_i''(U))$ are both distributed as X_i if $U \sim U(0,1)$ and we have P', $P'' \in \mathfrak{p}$. With probability λ_m (μ_m), $\sum_{j=1}^{m-1} \lambda_j < U \leqslant \sum_{j=1}^{m} \lambda_j$ ($\sum_{j=1}^{m-1} \mu_j < U \leqslant \sum_{j=1}^{m} \mu_j$), and in that case, $\phi_i'(U) > 1 - \alpha_i \geqslant F_i(x)$ ($\phi_i''(U) \leqslant \beta_i \leqslant F_i(x)$) so $X_i > x$ ($X_i \leqslant x$) for all $i \in I_m$ ($i \in J_m$). But then $\min_{i \in I_m} X_i > x$ ($\max_{i \in I_m} X_i \leqslant x$) and so $T' > x$ ($T'' \leqslant x$). Thus $\alpha(x) \leqslant P(T' > x)$ and $\beta(x) \leqslant P(T'' \leqslant x)$ for P' and P'' respectively, with equality actually occuring because of (i).

(i) We prove (i) for $\alpha(x)$, the proof for $\beta(x)$ is analogous. Define for each $j = 1, \ldots, k$, $A_j = \bigcap_{i \in I_j} \{X_i > x\}$, A_j^c the complement event of A_j, and define for each $\phi \neq K \subseteq \{1, \ldots, k\}$ $B_K = \bigcap_{j \in K} A_j \bigcap_{j \notin K} A_j^c$. Let $\lambda_K = P(B_K)$ for an arbitrary $P \in \mathfrak{p}$. Then $P(T > x) = P(\bigcup_j A_j) = P(\bigcup_K B_K) = \sum_K \lambda_K$. Let i be some node in I. Then if $i \in I_j$ and $j \in K$, $B_K \subseteq \{X_i > x\}$, hence

$$\sum \lambda_K \leqslant \bar{F}_i(x) \qquad i = 1, \ldots, n \qquad (25)$$

where the summation is over all K's such that $i \in I_j$, $j \in K$ for some j. If we set up the program

$$\max_K \sum \lambda_K \qquad (26)$$

$$\text{s.t. (25) and } \sum_K \lambda_K \leqslant 1, \ \lambda_K \leqslant 0 \qquad (26)$$

then the solution is $\geqslant P(T' > x)$ for all $P \in \mathfrak{p}$. The linear program (26) has a variable λ_K for each subset $K \subseteq \{1, \ldots, k\}$. In particular this includes $\lambda_1, \ldots, \lambda_k$ which correspond to the singletons.

Consider any K which is not a singleton, say $K \supseteq \{i, j\}$. Then the variable λ_K appears in every constraint in which λ_i or λ_j appear. It is

then clear (by examining the dual problem) that λ_k need not be basic in an optimal solution, and so it is a redundant variable. Thus (21) has the same solution as (26) and so $\alpha(x) \geqslant P(T' > x)$. This completes the proof.

<u>Modular decomposition and Monotonicity</u>: hold for $\alpha(x)$, $\beta(x)$ as for Ψ, η.

<u>Computational aspects</u>: The main difficulty in solving the LP's (21), (22), is the number of variables, one for each path, which may grow exponentially with the number of nodes. Consider solution by the simplex method. Then one can avoid the necessity of handling such a large number of variables, by using column generation, as follows: Let λ be a basic feasible solution to (21) and let v be its simplex multipliers; assume $\Sigma \lambda_j < 1$ and so $w = 0$. This solution is optimal if v is dual feasible, that is

$$\sum_{I_j} v_i \geqslant 1 \qquad j = 1, \ldots, k$$

to check that, one can solve the shortest route problem with the weights v_i. If the solution is < 1, it gives a new path I_j for which λ_j should enter the basis.

The solution for all values of x can again be done using parametric LP.

Zemel [1982] has shown that if the shortest route problem, $\min_{1 \leqslant j \leqslant k}$ $\sum_{I_j} v_i$, can be solved in polynomial time then so can the LP (21), by employing an ellipsoid type algorithm. In particular, for a two terminal network the shortest route problem is indeed polynomial. This is in

sharp contrast to the fact that the calculation of the reliability of a
two terminal network with independent components is NP hard, as shown by
Rosenthal [1975], Ball [1980] and Valiant [1979]. This great difference
in difficulty of calculation between the independent case and the bounds
of the present paper is the most remarkable feature of these bounds.

Redesign of a reliability system: The solution of (21, 22, 23, 24)
provides similar information for redesigning the reliability system as we
had for M and L. Typically one may have a target system lifetime x and be
interested in $P(T > x)$. Else one may have a monotone decreasing penalty
function $C_1 (y)$ for $T = y < x$, or a monotone increasing reward function
$C_2(y)$ for $T = y > x$, with (C'_1, C'_2 are the derivatives of C_1, C_2):

$$\bar{E}(C_1) = C_1(x)\beta(x) + {\textstyle\int_{-\infty}^x} - C_1'(y)\beta(y)dy$$
$$\bar{E}(C_2) = C_2(x)\alpha(x) + {\textstyle\int_x^\infty} C_2'(y)\alpha(y)dy$$

(27)

providing upper bounds for penalty or reward.

If $\beta(x)$, $\alpha(x)$ are replaced by $u_i(x) F_i(x)$ and by $v_i(x) F_i(x)$ in (27)
where $u(x)$, $v(x)$ are the solutions to (24), (23), one gets an assignment
of penalties or rewards to the various nodes for the extreme cases.
Similar assignments are obtained for modules. The probabilistic
interpretations of λ_j, μ_j, α_i, β_i can also be used in redesign, as in
section 2.

5. Discussion

In this paper we have developed various bounds of various types for
the optimal value functions M, L, T. It is fortunate that each of those
bounds is exactly of the type most useful in application. The bounds for
T are stochastic, so that given a target date x we have upper and lower

bounds on the system reliability $P(T > x)$ at the date x. The bound for M is an upper bound in the convex majorisation sense, which in the critical path length application gives an upper bound to the expected tardiness beyond a target day x, $E(M - x)^+$. The bound on L is a lower bound in the concave majorisation sense, which in the maximal flow application gives an upper bound on the expected shortfall of the flow below a target level x, $E(x - L)^+$.

For aesthetic reasons, or for some unforeseen applications one would nevertheless desire the bounds not obtained here, i.e. stochastic bounds on M, L, convex lower and concave upper bounds on M and L respectively, and convex and concave bounds for T. In the following discussion we explain why these are unlikely to possess the same nice properties as the bounds already obtained.

In section 1 we introduced three special joint distributions, the perfect tracking distribution, P^*, and the max and min antithetic distributions, P^{**}, P^{***}, which achieve the various bounds for the pure series and the pure parallel system. We state the following simple lemma:

<u>Lemma 5.1</u>: Let $\mathfrak{M}$ be a family of distributions. If $M^* \in \mathfrak{M}$ satisfies $M^* >_c M$ for all $M \in \mathfrak{M}$, then there is no member $\bar{M} \in \mathfrak{M}$ for which $\bar{M} >_{ST} M$ for all $M \in \mathfrak{M}$ unless $\bar{M} = M^*$.

<u>Proof</u>: Assume existence of $\bar{M}$. Then $\bar{M} >_{ST} M^* >_c \bar{M}$, so $\bar{M} >_c M^* >_c \bar{M}$, so

$$E_{\bar{M}}(X - x)^+ = E_{M^*}(X - x)^+ \text{ for all } x, \text{ and } \bar{M} = M^*.$$

Since P^* does not in general stochastically majorize or stochastically minorize ΣX_i (the only exclusions one can think of are when n-1 of the variables are deterministic and $\mathfrak{P}$ has only one member), stochastic bounds on M (L) in the purely series (purely parallel) case will not be obtained within $\mathfrak{P}$.

We do not know how to construct convex lower bounds or concave upper bounds for $\sum_{i=1}^{n} X_i$. Such bounds if they exist within will minimize the variance of $\sum_{i=1}^{n} X_i$ and will provide the optimum distributions for variance reduction in Monte Carlo simulation, as discussed by Hammerseley and Mauldon [1950], Handscomb [1958] and Whitt [1976].

Next we note that M is nondecreasing convex, L is nondecreasing concave and T', T'' are nondecreasing but neither convex nor concave. The generalization of the bounds from the pure series or pure parallel case to general series parallel systems and the property of modular decomposition follow from the fact that $\geq_c$, $\geq_k$, $\geq_{ST}$ are preserved respectively by nondecreasing convex, nondecreasing concave and nondecreasing functions. We cannot however expect the same to hold for concave bounds on M, convex bounds on L or convex and concave bounds on T, T'. The stochastic bounds on T if obtained within $\mathfrak{p}$ are of course also sharp convex and concave bounds. If they are obtained outside $\mathfrak{p}$ we do not know how to construct sharp convex or concave bounds.

The bounds discussed in the present paper provide one way of circumventing the impossible problem of calculating the exact distributions of M, L or T. We conclude by mentioning other approaches that appear in the literature. These are based on special models for which the distributions of M, L and T are tractable, namely:

- Markovian systems: If the weights have independent exponential distributions then M, L, T have phase type distributions in the sense of Neuts [1981], and their distributions can be evaluated, though at considerable computational effort, e.g. Kulkarni and Adlakha [1984].

- Series Parallel systems with independent weights: The distributions of M, L, T are obtained by convolutions products and complementations.

- "Perfect tracking" (Nadas [1979]): Each weight X_i is of the form $X_i = a_i + b_i Z$, $i = 1, \ldots, n$, where Z is a single random variable, common to all the X_i's. If $p = P(Z < z)$, then the p percentiles of M, L, T are obtained from the values $x_i = a_i + b_i z$.

By approximating F_1, ..., F_n with phase type distributions one can presumably get bounds on M, L or T using the first approach.

Series parallel systems form the basis for calculations of bounds on values of $P(V > y)$ where V is M, L or T - see Robillard & Trahan [1977], Shogan [1977], Devroye [1979] for PERT and shortest route applications, and Barlow and Proschan [1975], Esary & Proschan [1970], Shogan [1978], Natvig [1980] for reliability applications.

The approach of the present paper utilises the third approach of perfect tracking.

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