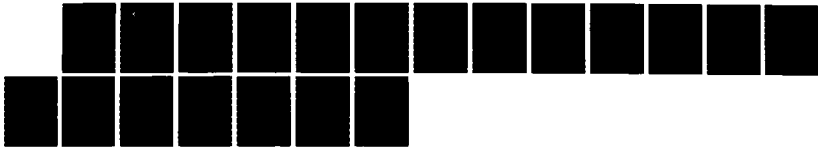
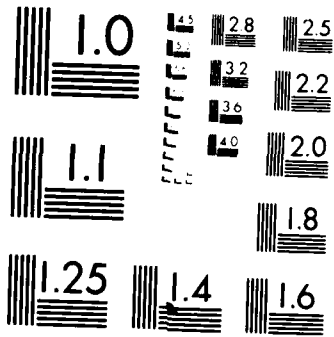


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OPTIMAL STEP TYPE DESIGNS FOR COMPARING
TEST TREATMENTS WITH A CONTROL

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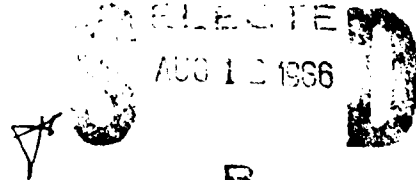
C.S. CHENG, D. MAJUMDAR, J. STUFKEN

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Optimal Step Type Designs for Comparing Test Treatments With A Control.

by

C.S. Cheng¹, D. Majumdar², J. Stufken² and T.E. Ture
University of California at Berkeley, University of Illinois at
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Abstract

The problem of obtaining A-optimal designs for comparing v test treatments with a control in b blocks of size k each is considered. A step type design is a BTIB design in which the control is replicated t times in some blocks and $t + 1$ times in the remaining blocks. A condition on the parameters (v, b, k) is identified for which optimal step type designs can be obtained. Families of such designs are given. Methods of searching for highly efficient designs are proposed, for situations where it is difficult to determine an A-optimal design.

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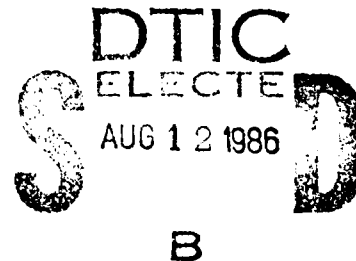
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1. INTRODUCTION.

Consider the problem of comparing v treatments called test treatments with a standard treatment called the control in b blocks of size k each. The test treatments will be labelled $1, 2, \dots, v$ and the control 0 . The model is additive and homoscedastic, that is, if Y_{ijl} is the observation on treatment i ($0 \leq i \leq v$) in block j ($1 \leq j \leq b$) and plot l ($1 \leq l \leq k$), then

$$Y_{ijl} = \mu + \tau_i + \beta_j + e_{ijl},$$

where the e_{ijl} are assumed to be uncorrelated random variables with mean 0 and a common variance σ^2 . The unknown constants μ , τ_i and β_j represent the general mean, the effect of treatment i and the effect of block j respectively. Let $D(v, b, k)$ be the set of all possible experimental designs. The primary purpose of the experiment is to draw inferences on the contrasts $(\tau_0 - \tau_i)$, $i = 1, 2, \dots, v$. If $(\hat{\tau}_{d0} - \hat{\tau}_{di})$ is the best linear unbiased estimator of $(\tau_0 - \tau_i)$ under a design $d \in D(v, b, k)$, then we want to choose a design such that the variances of $(\hat{\tau}_{d0} - \hat{\tau}_{di})$ are smallest in some sense. Formally, we want to choose an experimental design from $D(v, b, k)$ which minimizes

$$\sum_{i=1}^v \text{var}(\hat{\tau}_{d0} - \hat{\tau}_{di})$$

as d varies over all of $D(v, b, k)$. A design which attains the minimum is called an A-optimal design. Throughout this paper, we will assume that v and k satisfy

$$k \geq 3 \tag{1.1}$$

$$v \geq k. \tag{1.2}$$

Majumdar and Notz (1983) gave a method for finding A-optimal designs. Their optimal designs can basically be of two types. Using the terminology of Hedayat and Majumdar (1984), they are: rectangular (or R-)type, in which every block has the same number of replications of the control, and step (or S-)type, in which some blocks contain the control t times and the others $t + 1$ times. Optimal R-type designs were studied by Hedayat and Majumdar (1985). Families of such designs, particularly when each block has one replication of the control, were given in that paper. In this article, we intend to study optimal S-type designs. S-type designs are more complicated than R-type designs; the latter being a balanced incomplete block (BIB) design in the test treatments augmented by an equal number of controls in each block, but the former does not have such a simple characterization. Consequently, both the optimality and the construction of such designs are more involved.

In section 2 we give a characterization of classes $D(v,b,k)$ for which the Majumdar and Notz (1983) method gives an optimal S-type design. Hedayat and Majumdar (1985) contains a similar result for R-type designs. The results of section 2 are applied in section 3 to obtain some families of optimal S-type designs. We also suggest ways of obtaining efficient designs in classes $D(v,b,k)$ where optimal designs are not readily available.

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2. THE NATURE OF OPTIMAL STEP TYPE DESIGNS.

We start this section by giving some definitions and known results. For $d \in D(v,b,k)$, n_{dij} will denote the number of times treatment i ($0 \leq i \leq v$) occurs in block j ($1 \leq j \leq b$), and $\lambda_{dip} \equiv \sum_{j=1}^b n_{dij}n_{d pj}$ ($0 \leq i,p \leq v$). The following definition is due to Bechhofer and Tamhane (1981).

Definition 2.1. d is a Balanced Treatment Incomplete Block (BTIB) design if

$$\lambda_{d01} = \dots = \lambda_{d0v}$$

and

$$\lambda_{d12} = \lambda_{d13} = \dots = \lambda_{d,v-1,v}$$

A special type of BTIB designs will be of particular interest to us.

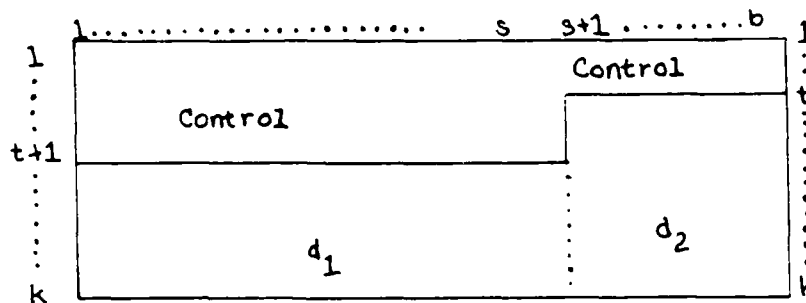
Definition 2.2. For integers $t \in \{0,1, \dots, k-1\}$, $s \in \{0,1, \dots, b-1\}$, d is a $BTIB(v,b,k ; t,s)$ if it is a BTIB design with the additional property that

$$n_{dij} \in \{0,1\}, i = 1,2, \dots, v; j = 1,2, \dots, b,$$

$$n_{d01} = \dots = n_{d0s} = t + 1,$$

$$n_{d0,s+1} = \dots = n_{d0b} = t.$$

The layout of a $BTIB(v,b,k ; t,s)$ design can be pictured as follows, with columns of the array denoting the blocks:



Here d_1 and d_2 are components of the design which involve the test treatments only. If $s = 0$ then d_1 is empty and d_2 is necessarily a BIB design. In this case the $BTIB(v,b,k ; t,0)$ is called rectangular (R-)type. Otherwise, when $s > 0$, the $BTIB(v,b,k ; t,s)$ is called step (S-)type.

Let

$$a = (v-1)^2, c = bvk(k-1), p = v(k-1) + k,$$

$$\Lambda = \{(x,z): x = 0, \dots, [k/2] - 1; z = 0,1, \dots, b, \text{ with } z > 0 \text{ when } x = 0\}.$$

Here $[\cdot]$ is the largest integer function. Let,

$$g(x,z) = a / \{c - p(bx + z) + (bx^2 + 2xz + z)\} \\ + 1 / \{k(bx + z) - (bx^2 + 2xz + z)\}.$$

Observe that $g(x,0) = g(x-1,b)$, for all $x \neq 0$. The following theorem is due to Majumdar and Notz (1983).

Theorem 2.3. A BTIB($v,b,k ; t,s$) is A-optimal in $D(v,b,k)$ if

$$g(t,s) = \min_{(x,z) \in A} g(x,z).$$

Some examples of optimal R- and S- type designs are available in Majumdar and Notz (1983) and in the catalogs of optimal designs given in Hedayat and Majumdar (1984). It may be noted that the designs given in Theorem 2.3 are also Maximum Variance (MV-) optimal, that is they minimize

$$\max_{1 \leq i \leq v} \text{var}(\hat{r}_{d0} - \hat{r}_{di})$$

among all $d \in D(v,b,k)$. More general results on MV-optimal designs are now available in Jacroux (1984). Hedayat and Majumdar (1985) investigated the nature of $g(x,z)$ and characterized classes $D(v,b,k)$ for which the $g(x,z)$ is minimized at $(t,0)$ for some t .

Let L be the set of integers r with $0 < r \leq [k/2]b$. Then the function g in Theorem 2.3 can also be expressed as a function G on L as follows:

$$G(r_0) \equiv g(x,z), \text{ where } r_0 = bx + z \text{ with } x = [r_0/b] \text{ and} \\ z = r_0 - b[r_0/b].$$

Thus a BTIB($v,b,k ; t,s$) is A-optimal if $G(r_0)$ is minimized at $r_0^* = bt + s$. Here r_0 is the number of observations allocated to the control. For specific values of v,b and k , the minimization of G over L can be performed on a computer without difficulty. However, if one wants to derive some general results such as the optimality of a family of designs, (for instance, Theorem 3.1 in the present paper) then it would be necessary to study some theoretical properties of the function G (or g). The following result, first conjectured in Ture (1982), provides an important technical tool for investigating the A-optimality of BTIB designs:

Theorem 2.4. If $G(r_0^*) = \min_{r_0 \in L} G(r_0)$, then G is decreasing on $\{r_0 \in L: r_0 \leq r_0^*\}$ and increasing on $\{r_0 \in L: r_0 \geq r_0^*\}$.

A Proof of Theorem 2.4 can be found at the end of this section.

Let $\Lambda_0 = \{(0,z) : z = 1, \dots, b\}$ and $\Lambda_i = \{(i,z) : z = 0, 1, \dots, b\}$ for $i = 1, 2, \dots, [k/2] - 1$. Then, by Theorem 2.4, one of the following two situations must prevail, for some $t \in \{0, 1, \dots, [k/2]\}$:

a) $g(x,z)$ is a decreasing function of z on each of $\Lambda_0, \Lambda_1, \dots, \Lambda_{t-1}$ and, if $t < [k/2]$, $g(x,z)$ is increasing on each of $\Lambda_t, \dots, \Lambda_{[k/2]-1}$. In this case, the

minimum of $g(x,z)$ over Λ is achieved at $x = t$ and $z = 0$, and an R-type design. $BTIB(v,b,k ; t,0)$ is optimal. This was the case studied by Hedayat and Majumdar (1985).

b) $g(x,z)$ has a local minimum at (t,s) on Λ_t for some $s \in \{1, \dots, b-1\}$, $g(x,z)$ is decreasing on each Λ_i , $i < t$, if any, and $g(x,z)$ is increasing on each Λ_i , $i > t$, if any. In this case, an S-type design, $BTIB(v,b,k ; t,s)$ is optimal.

An immediate consequence of Theorem 2.4 is that a local minimum of G is also a global minimum. Thus $G(r_0^*) = \min_{r_0 \in L} G(r_0)$ if and only if

$G(r_0^*) \leq \min\{G(r_0^* - 1), G(r_0^* + 1)\}$. In other words, we have

Corollary 2.5. For some $t \in \{0,1, \dots, [k/2] - 1\}$ and $s \in \{1, \dots, b-1\}$, $g(t,s) = \min_{(x,z) \in \Lambda} g(x,z)$ if and only if

$$g(t,s) \leq \min\{g(t,s-1), g(t,s+1)\}. \quad (2.1)$$

Consequently, a $BTIB(v,b,k ; t,s)$ is A-optimal in $D(v,b,k)$ whenever (2.1) holds.

Condition (2.1) substantially reduces the computation one has to do in verifying the optimality of a BTIB design. In section 3, it will be used to show the optimality of a family of S-type BTIB designs.

Theorem 2.4 is also useful for simplifying the search for the optimal $r_0^* = bt + s$, which, as will be demonstrated in the next section, is useful for constructing efficient designs when optimal designs cannot be found among BTIB designs. One simple method for finding the optimal (t,s) could be the sequence:

- (i) Determine $x_0 = \min\{x: g(x,b-1) \leq g(x,b), x = 0,1, \dots, [k/2] - 1\}$.
- (ii) Determine z_0 such that $g(x_0,z_0) = \min_{z=0,1,\dots,b} g(x_0,z)$. A method of determining z_0 could be to consider $g(x_0,z)$ as a function of a real-valued z over the interval $[0,b]$, and using techniques of Calculus to find its minimum. Suppose the minimum is at a point $p \in [0,b]$. If p is an integer, then $z_0 = p$. If p is not an integer, then $z_0 = [p]$ if $g(x_0,[p]) < g(x_0,[p] + 1)$, $z_0 = [p] + 1$ if $g(x_0,[p]) > g(x_0,[p] + 1)$, and any one of $[p]$ and $[p] + 1$ can be chosen as z_0 if $g(x_0,[p]) = g(x_0,[p] + 1)$. In the last situation a choice between $[p]$ and $[p] + 1$ should be made with an eye to the existence of an optimal BTIB design.
- (iii) If $z_0 = 0$ then $t = x_0, s = 0$. If $z_0 \in \{1, \dots, b-1\}$ then $t = x_0, s = z_0$. If $z_0 = b$ then $t = x_0 + 1, s = 0$.

Remark. It is easily seen that

$$g(x,z) = b^{-1} \tilde{g}(x, \tilde{z}),$$

where $\tilde{z} = b^{-1}z$ and \tilde{g} does not depend on b . Thus, treating \tilde{z} as a real variable on $[0,1]$, one can now minimize $\tilde{g}(x, \tilde{z})$ over $x \in \{0,1, \dots, [k/2] - 1\}$ and $\tilde{z} \in [0,1]$ to obtain optimal x_0, z_0 . The solutions x_0 and z_0 do not depend on b . If $z_0 b$ is

an integer then the optimal values are $t = x_0$ and $s = \bar{z}_0 b$. Otherwise, the minimum of $g(x, z)$ over Λ is attained at $x = x_0$ and $z = \lfloor \bar{z}_0 b \rfloor$ or $\lfloor \bar{z}_0 b \rfloor + 1$. This fact makes it possible to tabulate (x_0, \bar{z}_0) as a function of k and v . One such table is available in Ture (1985).

As an example, consider $v = 6$, $k = 5$ and $b = 18$. Here $x_0 = 1$ and the optimal value of \bar{z} is $\bar{z}_0 = .321$, but $\bar{z}_0 b = 5.778$ is not an integer. Since $g(1, 6) < g(1, 5)$, the optimal $s = 6$. A BTIB(6, 18, 5 ; 1, 6) is given in Hedayat and Majumdar (1984) (design S4 in its appendix).

We finish this section by proving Theorem 2.4. For this purpose we need two lemmas. Lemma 2.6 was first proved in Ture (1982); also see Ture (1985) and Hedayat and Majumdar (1985).

Lemma 2.6. The function

$$z \rightarrow g(x, z)$$

is, for a fixed $x \in \{0, 1, \dots, (k-2)/2\}$, a decreasing function on $[0, b]$, or an increasing function, or there is a $z_0 \in (0, b)$ such that the function is decreasing on $[0, z_0]$ and increasing on $[z_0, b]$.

Lemma 2.7. (i) If $x \in \{0, 1, \dots, \lfloor k/2 \rfloor - 2\}$ then $g(x, b) \geq g(x, b-1)$ implies $g(x+1, 1) \geq g(x+1, 0)$.

(ii) If $x \in \{1, 2, \dots, \lfloor k/2 \rfloor - 1\}$ then $g(x, 1) \leq g(x, 0)$ implies $g(x-1, b) \leq g(x-1, b-1)$.

Proof. (i) It follows from p.762 of Hedayat and Majumdar (1985) that $g(x, b) \geq g(x, b-1)$ if and only if

$$\begin{aligned} & b\{k - (x + 1)\}[a(x + 1)^2(p - 2x - 1) - (k - 2x - 1)\{v(k - 1) - (x + 1)\}^2] \\ & \geq v(k - 2x - 1)(p - 2x - 1)\{k - 1 + (v - 2)(x + 1)\}. \end{aligned}$$

In order that the above inequality holds, the left hand side should be positive since the right hand side is. Hence if $g(x, b) \geq g(x, b-1)$, then

$$a(x + 1)^2(p - 2x - 1) \geq (k - 2x - 1)\{v(k - 1) - (x + 1)\}^2. \quad (2.2)$$

Again, from p.761 of Hedayat and Majumdar (1985), $g(x+1, 1) \geq g(x+1, 0)$ if and only if

$$\begin{aligned} & b\{k - (x + 1)\}[(k - 2x - 3)\{v(k - 1) - (x + 1)\}^2 - a(x + 1)^2(p - 2x - 3)] \\ & \leq v(k - 2x - 3)(p - 2x - 3)\{k - 1 + (v - 2)(x + 1)\}. \end{aligned}$$

This inequality is certainly valid if the left hand side is negative, since the right hand side is positive; that is, if

$$a(x + 1)^2(p - 2x - 3) \geq (k - 2x - 3)\{v(k - 1) - (x + 1)\}^2. \quad (2.3)$$

It suffices to show that (2.2) implies (2.3). Using (2.2) we see

$$\begin{aligned} a(x+1)^2(p-2x-3) &= a(x+1)^2(p-2x-1) - 2a(x+1)^2 \\ &\geq (k-2x-1)\{v(k-1)-(x+1)\}^2 - 2a(x+1)^2 \\ &= (k-2x-3)\{v(k-1)-(x+1)\}^2 + 2\{v(k-1)-(x+1)\}^2 - 2a(x+1)^2. \end{aligned}$$

So, it suffices to show that

$$\{v(k-1)-(x+1)\}^2 \geq a(x+1)^2,$$

that is $\{v(k-1) + (v-2)(x+1)\}\{v(k-1) - v(x+1)\} \geq 0$, which is true since $x \leq (k-2) - 2$.

The proof of (ii) is completely similar and involves the following steps: $g(x,1) \leq g(x,0)$ is equivalent to

$$\begin{aligned} b(k-x)[(k-2x-1)\{v(k-1)-x\}^2 - ax^2(p-2x-1)] \\ \geq v(k-2x-1)(p-2x-1)\{k-1+(v-2)x\}. \end{aligned}$$

Hence, $(k-2x-1)\{v(k-1)-x\}^2 \geq ax^2(p-2x-1)$, so that $(k-2x+1)\{v(k-1)-x\}^2 \geq ax^2(p-2x+1)$. Hence,

$$\begin{aligned} b(k-x)[ax^2(p-2x+1) - (k-2x+1)\{v(k-1)-x\}^2] \\ \leq v(k-2x+1)(p-2x+1)\{k-1+(v-2)x\}, \end{aligned}$$

that is $g(x-1,b) \leq g(x-1,b-1)$.

Now we are ready to prove Theorem 2.4.

Proof of Theorem 2.4:

If the minimum of $g(x,z)$ over Λ is attained at some (t,z) with $z = 0$, then the result follows from Lemma 2.4 of Hedayat and Majumdar (1985). On the other hand, suppose

$$g(t,s) = \min_{(x,z) \in \Lambda} g(x,z),$$

where $s \notin \{0,b\}$. Then since

$$g(t,s) \leq g(t,s+1) \leq \dots \leq g(t,b-1) \leq g(t,b),$$

we obtain from Lemma 2.7 that

$$g(t+1,0) \leq g(t+1,1).$$

Now, using lemma 2.6,

$$g(t+1,0) \leq g(t+1,1) \leq \dots \leq g(t+1,b).$$

Continuing this chain of arguments, we see that

$$g(t,s) = \min_{\substack{x=t, \dots, \lfloor k/2 \rfloor - 1, \\ z=0, 1, \dots, b}} g(x,z)$$

Similar arguments starting from $g(t,s) \leq g(t,s-1)$ show that

$$g(t,s) = \min_{\substack{x=0,\dots,t \\ z=0,1,\dots,b \\ z>0 \text{ when } x=0}} g(x,z)$$

This completes the proof.

3. APPLICATIONS.

An important application of Corollary 2.5 is finding families of optimal step type designs. We give some families in Theorem 3.1. We shall also discuss situations where Corollary 2.5 cannot be applied, and suggest ways of finding efficient designs.

Theorem 3.1. Let $\alpha \geq 3$ be a prime or a power of a prime. Then, for any positive integer γ , there exists a $\text{BTIB}(\alpha^2 - 1, \gamma(\alpha + 2)(\alpha^2 - 1), \alpha; 0, \gamma(\alpha + 1)(\alpha^2 - 1))$ which is A-optimal in $\mathbf{D}(\alpha^2 - 1, \gamma(\alpha + 2)(\alpha^2 - 1), \alpha)$.

Proof: The A-optimality follows from Corollary 2.5 upon verifying condition (2.1).

We will establish the existence for $\gamma = 1$; for larger γ we take the union of γ copies of this design. We denote by $\text{BIB}(v, b, r, k, \lambda)$ a BIB design with the parameters v, b, r, k and λ ; the symbols enjoying their standard interpretation.

Since α is a prime or a prime power, there exists a $\text{BIB}(\alpha^2, \alpha(\alpha + 1), \alpha + 1, \alpha, 1)$, that is an Euclidean plane. There is also a $\text{BIB}(\alpha^2 - 1, (\alpha + 1)(\alpha^2 - 2), \alpha^2 - 2, \alpha - 1, \alpha - 2)$. To see this, start with an Euclidean plane based on α^2 treatments and delete one treatment from it. The remaining design has $\alpha + 1$ blocks of size $\alpha - 1$ and $\alpha^2 - 1$ blocks of size α . Take $\alpha - 2$ copies of each block of size $\alpha - 1$, and replace each block of size α by α blocks of size $\alpha - 1$, the α blocks being the α subsets of size $\alpha - 1$ of the original block of size α . These $(\alpha - 2)(\alpha + 1) + \alpha(\alpha^2 - 1) = (\alpha + 1)(\alpha^2 - 2)$ blocks give the desired BIB design.

To get a $\text{BTIB}(\alpha^2 - 1, (\alpha + 2)(\alpha^2 - 1), \alpha; 0, (\alpha + 1)(\alpha^2 - 1))$, augment one replication of the control to each block of the constructed $\text{BIB}(\alpha^2 - 1, (\alpha + 1)(\alpha^2 - 2), \alpha^2 - 2, \alpha - 1, \alpha - 2)$ and add to this the blocks of a $\text{BIB}(\alpha^2, \alpha(\alpha + 1), \alpha + 1, \alpha, 1)$, in which the control appears as one of the treatments. Hence the theorem.

Stufken (1986) has more families of optimal step type designs. There again, condition (2.1) is crucial in proving the optimality.

The problem of the existence of a $\text{BTIB}(v, b, k; t, s)$ imposes restrictions on the scope of Corollary 2.5. As in BIB designs, the parameters should satisfy some necessary conditions (Hedayat and Majumdar 1984, p.365). Even when these are satisfied, a $\text{BTIB}(v, b, k; t, s)$ may not exist. In case the parameters v, b, k cannot accommodate Corollary 2.5, one may try to find a design in $\mathbf{D}(v, b, k)$ which is highly efficient, even if it is not known to be A-optimal. It is not difficult to see from Theorem 2.3 that for any $d \in \mathbf{D}(v, b, k)$,

$$\sigma^{-2} \sum_{i=1}^v \text{var}(\hat{\tau}_{d0} - \hat{\tau}_{di}) \geq vkg(t, s) \quad (3.1)$$

The left hand side of this expression is the value of the A-criterion for a design d . We denote this by A_d . Clearly the ratio $vkg(t,s)/A_d$ gives a measure of the efficiency of d .

The lower bound in (3.1) is achieved by a BTIB($v,b,k; t,s$). Before discussing specific methods of identifying efficient designs, let us determine how close an arbitrary design in $\mathbf{D}(v,b,k)$ can come to the lower bound $vkg(t,s)$. For $d \in \mathbf{D}(v,b,k)$, let us define

$$m_{dl} = |\{j : n_{doj} = l\}|,$$

that is, m_{dl} is the number of blocks in which the control appears l times, $l = 0, 1, \dots, k-1$. Clearly

$$\sum_{l=1}^{k-1} l m_{dl} = r_{do},$$

$$\sum_{i=1}^v \lambda_{doi} = r_{do}(k-1) - \sum_{l=1}^{k-1} l(l-1)m_{dl},$$

and if $n_{dij} \in \{0,1\}$ for all $i = 1, 2, \dots, v, j = 1, 2, \dots, b$, then $\sum_{i,p=1}^v \lambda_{dip} + \sum_{i=1}^v \lambda_{doi}$

$= \frac{1}{2}bk(k-1) - \frac{1}{2} \sum_{l=1}^{k-1} l(l-1)m_{dl}$ where $r_{do} = \sum_{j=1}^b n_{doj}$, the number of replications of the control in d .

Lemma 3.2. Let m_0, m_1, \dots, m_{k-1} be fixed numbers satisfying $m_0 + m_1 + \dots + m_{k-1} = b$. Consider the subset of $\mathbf{D}(v,b,k)$ in which $m_{dl} = m_l$, $l = 1, 2, \dots, k-1$. For these designs denote $r_{do} = r_0$ and $\lambda_{dip} = \lambda_{ip}$, $i, p = 0, 1, \dots, v$. Define the quantities

$$\bar{\lambda}_0 = \sum_{i=1}^v \lambda_{oi}/v, \quad \bar{\lambda}_1 = 2 \sum_{i,p=1}^v \lambda_{ip}/v(v-1).$$

Then, for each $d \in \mathbf{D}(v,b,k)$,

$$A_d = \sigma^{-2} \sum_{i=1}^v \text{var}(\hat{r}_{do} - \hat{r}_{di}) \geq vk(\bar{\lambda}_0 + \bar{\lambda}_1)/\{\bar{\lambda}_0(\bar{\lambda}_0 + v\bar{\lambda}_1)\}. \quad (3.2)$$

Proof. Follows from Lemma 2.2 of Majumdar and Notz (1983).

The condition (3.2) gives a sharper lower bound to A_d than (3.1). It can be used to eliminate configurations $(m_0, m_1, \dots, m_{k-1})$ for which the lower bound in (3.2) is not very close to $vkg(t,s)$.

To find an efficient design when Corollary 2.5 cannot be applied, we first determine the optimal $r_0^* = bt + s$ by the method described in section 2. If there did exist a $d \in \mathbf{D}(v,b,k)$ which was a BTIB($v,b,k; t,s$) then this d would have been A-optimal. Since there is no such design, we shall look at values r_0

close to r_0^* and various configurations of $(m_0, m_1, \dots, m_{k-1})$. For each of these we compute lower bounds given by (3.2), and consider those which are not much larger than $vkg(t,s)$, the lower bound given by (3.1). Next we consider designs having these $(m_0, m_1, \dots, m_{k-1})$ configurations which are very 'close' to BTIB designs. By this we mean $\lambda_{doi}'s (1 \leq i \leq v)$ are as close as possible and $\lambda_{dip}'s (1 \leq i, p \leq v, i \neq p)$ are as close as possible. We choose the most efficient design among all these designs. It is expected that this design would be highly efficient, in general. In particular we expect two types of designs to perform very well. One is a BTIB design with r_{do} close to r_0^* ; the other is a design 'closest' to a BTIB design, with r_{do} equal to r_0^* . We now give two illustrative examples.

Example 3.3. Let $v = 5, k = 4$ and $b = 7$. Then $t = 1$ and $s = 0, r_0^* = 7, vkg(1,0) = 2.04$.

Table 3.1

r_0	(m_0, m_1, m_2, m_3)	$(\bar{\lambda}_0, \bar{\lambda}_1)$	lower bound from (3.2)
6	(1,6,0,0)	(3.6, 2.4)	2.137
7	(1,5,1,0)	(3.8, 2.2)	2.134
7	(0,7,0,0)	(4.2, 2.1)	2.041
8	(1,4,2,0)	(4, 2)	2.143
8	(0,6,1,0)	(4.4, 1.9)	2.060
9	(0,6,0,1)	(4.2, 1.8)	2.165
9	(0,5,2,0)	(4.6, 1.7)	2.091

Note that whenever $\bar{\lambda}_0$ and $\bar{\lambda}_1$ are integers, the corresponding design achieving the lower bound is a BTIB design. Consider five designs d_1, d_2, d_3, d_4, d_5 , with columns as blocks, and the corresponding values of $A_d = \sigma^{-2} \sum_{i=1}^v \text{var}(\hat{r}_{do} - \hat{r}_{di})$:

$$d_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 3 & 3 & 4 & 4 \\ 3 & 4 & 5 & 4 & 5 & 5 & 5 \end{pmatrix}, A_{d_1} = 2.156$$

$$d_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 3 & 4 & 3 & 3 \\ 4 & 5 & 4 & 5 & 5 & 4 & 5 \end{pmatrix}, A_{d_2} = 2.058$$

$$d_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 3 & 3 & 4 & 3 & 3 & 4 \\ 2 & 4 & 5 & 5 & 4 & 5 & 5 \end{pmatrix}, A_{d_3} = 2.067$$

$$d_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 & 2 & 2 \\ 1 & 3 & 3 & 4 & 3 & 4 & 3 \\ 2 & 4 & 5 & 5 & 5 & 5 & 4 \end{pmatrix}, A_{d_4} = 2.143 \text{ (BTIB design)}$$

$$d_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 2 & 3 \\ 0 & 2 & 2 & 2 & 3 & 3 & 4 \\ 5 & 5 & 3 & 4 & 4 & 4 & 5 \end{pmatrix}, A_{d_5} = 2.207$$

Clearly d_2 is the best among all these designs. In fact A_{d_2} is even smaller than the lower bounds given in table 3.1 for all other configurations of (m_0, m_1, m_2, m_3) . d_2 is not a BTIB design but $r_{d_2} = r_0^*$. It is highly efficient since

$$vkg(t,s)/A_{d_2} = .992.$$

Note that the BTIB design d_4 has a high efficiency also, since $vkg(t,s)/A_{d_4} = .952$.

Example 3.4. $v = 6, k = 5, b = 7$. Here $t = 1$ and $s = 2, r_0^* = 9, vkg(t,s) = 2.204$.

Table 3.2

r_0	$(m_0, m_1, m_2, m_3, m_4)$	$(\bar{\lambda}_0, \bar{\lambda}_1)$	lower bound from (3.2)
7	(0,7,0,0,0)	(4.67, 2.8)	2.236
8	(0,6,1,0,0)	(5, 2.6)	2.214
9	(0,5,2,0,0)	(5.33, 2.4)	2.204
10	(0,4,3,0,0)	(5.67, 2.2)	2.207
11	(0,3,4,0,0)	(6, 2)	2.222
12	(0,2,5,0,0)	(6.33, 1.8)	2.249

$$d_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 3 & 3 & 3 & 3 \\ 2 & 4 & 4 & 5 & 4 & 4 & 5 \\ 3 & 5 & 6 & 6 & 5 & 6 & 6 \end{pmatrix}, A_{d_1} = 2.243$$

$$d_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 2 & 2 & 4 & 3 \\ 3 & 5 & 3 & 3 & 4 & 5 & 4 \\ 4 & 6 & 5 & 6 & 6 & 6 & 5 \end{pmatrix}, A_{d_2} = 2.227$$

$$d_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 2 & 3 & 4 & 3 \\ 2 & 3 & 3 & 4 & 5 & 5 & 4 \\ 5 & 4 & 6 & 6 & 6 & 6 & 5 \end{pmatrix}, A_{d_3} = 2.223$$

$$d_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 3 \\ 1 & 1 & 2 & 2 & 2 & 2 & 4 \\ 3 & 4 & 3 & 4 & 3 & 5 & 5 \\ 5 & 6 & 6 & 5 & 4 & 6 & 6 \end{pmatrix}, A_{d_4} = 2.222 \text{ (BTIB design)}$$

Clearly the BTIB design d_4 is the best. Note that r_{d_4} is close, though not equal to r_0^* . This design is highly efficient, in fact,

$$vkg(t,s)/A_{d_4} = .992.$$

It may be suggested that one way of finding efficient designs is to look for an A-optimal design among all BTIB designs only. Hedayat and Majumdar (1984) shows that even though these designs perform very well, in general, there can be instances where it is quite poor. This is because the stringent combinatorial conditions of BTIB designs may enforce a very inefficient choice of r_0 . For instance, they show that in $D(10,80,2)$ it is possible to achieve at least a 24% improvement over the best BTIB design. In this case, the optimal $(x_0, \bar{z}_0) = (0, 0.49)$, so $\bar{z}_0 = 80 \times 0.49 = 39.2$. The best BTIB design has $r_0 = 80$, which is too far away from 39.2. We recommend to use the right r_0 (or approximately so) and then construct a design combinatorially close to a BTIB design.

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