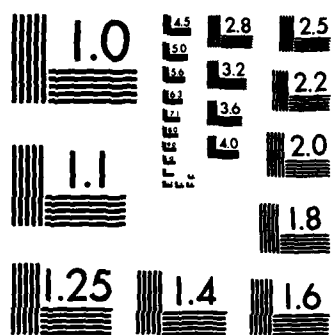MICROCOPY RESOLUTION TEST CHART
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TEARING OF CIRCUMFERENTIAL CRACKS IN PIPES
LOADED BY BENDING

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Abstract

The paper develops a theory of tearing for circumferential through-cracks in pipes. Resistance to tearing is postulated to be a consequence of invariance in shape at the tip of a growing crack.

NOTATION

E	Young's modulus	R	pipe mean radius
J	fracture parameter	h	pipe thickness
M	applied bending moment	$\overline{u}$	axial displacement
σ_F	flow stress	$\epsilon^2 = (h/R)[12(1-v^2)]^{-1/2}$	
$\sigma = M/(\pi\sigma_F hR^2)$		$u = E\overline{u}/(\sigma_F R)$	

INTRODUCTION

A method of J-estimation for a circumferentially through-cracked pipe subjected to bending loads was presented in [1]. The method was based on the Dugdale model and the semi-membrane theory of cylindrical shells. The results of that analysis can be used in conjunction with a material resistance curve to examine questions of crack growth stability by well-known procedures. The

present paper is concerned with the use of a crack growth hypothesis in place of a resistance curve to serve as an alternative theoretical foundation for stability analyses. The idea for such an hypothesis has appeared repeatedly in the literature [2] – [7] in the form of a statement that some quantity or other associated with the crack tip remains constant as the crack grows. A common feature of such crack growth or "tearing" hypotheses is that they lead to a nonlinear first-order differential equation for J as a function of crack extension. In other words the resistance curve itself is determined theoretically instead of experimentally. The determination of a resistance curve is not really necessary in applications of the idea to cracks in structures as the present paper will show. Such an approach has theoretical appeal partly because the difficulties with geometry dependance in experimentally determined resistance curves are avoided. In applications the effects of geometry dependance are an essential part of the problem. Of perhaps greater theoretical appeal is the prospect that a growth hypothesis (provided the idea proves to be correct) brings phenomenological fracture mechanics one step closer to what must be its physical foundations.

The tearing hypothesis used in the present paper is an adaptation to the pipe problem of a similar criterion set forth in [6] for the case of cracks in flat plates according to a Dugdale model. The analysis is based on results from [1] and leads to a first order nonlinear differential equation governing the effects of crack growth. The applied bending moment as a function of the crack advance angle and the increased flexibility of the pipe due to the crack follow directly from the solution to the equation.

ANALYSIS

Figure 1(a) shows a sketch of the upper half of the crack profile at the instant of crack growth initiation. The shaded region indicates plastically deformed material according to the Dugdale model. The crack tip lies at $\theta = \alpha_0$, and the Dugdale zone ends at $\theta = \beta_0$. The value of u at $\theta = \alpha_0$ is half the critical crack opening displacement $\bar{\delta}_c$ related to the critical value of the fracture parameter J_c and the flow stress by the familiar relation

$$J_c = \sigma_F \bar{\delta}_c \tag{1}$$

The picture after the crack tip has advanced to $\theta = \alpha$ is shown in Fig. 1(b). An hypothesis of the model is that the crack advances into the plastic region leaving behind a strip of "dead" material (shown unshaded). The crack face is thought of as lying along the lower edge of the dead material. If the upper curve is given by the function $\bar{u}(\theta, \alpha)$ then the width of the dead strip is by definition $\bar{u}(\theta, \theta)$. The width of the crack opening is given by

$$\Delta = 2\bar{u}(\theta, \alpha) - 2\bar{u}(\theta, \theta) , \qquad \alpha_0 < \theta < \alpha \qquad (2)$$

The usual terminology isn't quite appropriate but the crack tip opening angle $\bar{\omega}$ will be defined to be

$$R\bar{\omega} = -\lim_{\theta \to \alpha} \frac{\partial \Delta}{\partial \theta}$$

$$= \frac{d\bar{\delta}}{d\alpha} - 2\left(\frac{\partial \bar{u}}{\partial \theta}\right)_{\theta = \alpha} \qquad (3)$$

where $\bar{\delta} = 2\bar{u}(\alpha, \alpha)$. With use of $J = \sigma_F \bar{\delta}$, and put in dimensionless form, Eq. (3) reads

$$\frac{E}{\sigma_F^2 R} \frac{dJ}{d\alpha} + F = \frac{E}{\sigma_F} \bar{\omega} \qquad (4)$$

where

$$F = -2\left(\frac{\partial u}{\partial \theta}\right)_{\theta = \alpha} \qquad (5)$$

Equation (4) might be called the "tearing" equation.

The crack growth hypothesis set forth in [6] is that the crack opening shape in the neighborhood of the crack tip remains invariant as the crack grows. The obvious adaptation of that hypothesis in the present case is that the angle $\bar{\omega}$ remain constant. In essence the hypotheses adopted here are the same as those in [6] but there is some difference in detail. In [6] the analysis is made according to plane stress theory for cracks in flat plates. Here the semi-membrane theory for cylindrical shells applies. Results from semi-membrane theory are certain limits of results from a more complete shell theory. What that means, for instance, is that the angle $\bar{\omega}$ given by (3) is a parameter associated with the shape of the crack opening near the tip, not an actual angle. However, it is a somewhat remarkable fact that quantities important to fracture mechanics turn out

to be given accurately by semi-membrane theory provided the crack is long enough. See the discussion after Eq. (62) in [1].

What remains to be done is to relate the angle $\bar{\omega}$ to material properties and the radius and thickness of the pipe. In order for the theory to make sense $\bar{\omega}$ cannot in any way depend upon the crack length. A material resistance curve is generally assumed to have been obtained as though small scale yielding conditions were adhered to. A crack growth hypothesis such as the one adopted here is tantamount to assuming that the whole resistance curve is determined by conditions at the initiation of crack growth. There is, however, no reason to suppose that $\bar{\omega}$ cannot depend upon the curvature and thickness of the pipe. Obviously the effect of curvature on $\bar{\omega}$ must be established on theoretical grounds because curvature is not a material property.

The required determination of $\bar{\omega}$ is accomplished here by analysis of certain hypothetical experiments. Small scale yielding conditions must exist in the limiting case $\sigma_F \rightarrow \infty$. In such a case the extent of the Dugdale zone is small and results from [1], repeated here

$$\varepsilon\delta = \sqrt{2}\,(\beta - \alpha)^2 \tag{6}$$

$$\varepsilon F = 2\sqrt{2}\,(\beta - \alpha) \tag{7}$$

imply

$$F = 2^{5/4}\sqrt{\frac{EJ}{\varepsilon\sigma_F^2 R}} \tag{8}$$

in which case the tearing equation reads

$$\frac{E}{\sigma_F^2 R}\frac{dJ}{d\alpha} + 2^{5/4}\sqrt{\frac{EJ}{\varepsilon\sigma_F^2 R}} = \frac{E}{\sigma_F}\,\bar{\omega} \tag{9}$$

The coefficients in the equation do not depend on α therefore the solution depends only on $\alpha - \alpha_0$. Initially $J = J_c$ and initially the first term in (9) is the Paris tearing modulus T_0 obtained from the initial slope of a resistance curve for small scale yielding. The angle $\bar{\omega}$ is thus determined to be given by

$$\bar{\omega} = \frac{\sigma_F}{E}\,T_0 + 2^{5/4}\sqrt{\frac{J_c}{E\varepsilon R}} \tag{10}$$

A somewhat different, but related, calculation was made based on the analysis of a semi-infinite circumferential crack in a shallow cylindrical shell. The mathematical formulation of the problem in terms of a semi-membrane theory turns out to be well-set and the problem can be solved. The details will not be reproduced here but the resulting determination of $\bar{\omega}$ is exactly the same as (10). Since the equations of the model are indifferent to the magnitude of the parameters involved the determination (10) is expected to hold in all cases. In the general case the tearing equation has the form

$$\frac{E}{\sigma_F^2 R} \frac{dJ}{d\alpha} + F(J,\alpha) = \frac{E}{\sigma_F} \bar{\omega} \qquad (11)$$

in which F is a complicated implicit function of J and α.

The following expression for the dimensionless axial displacement on the cracked section was derived in [1]

$$\sqrt{2}\, \varepsilon u\,(\theta,\alpha) = C \cos\theta - A + (3 \cos\theta - \tfrac{1}{2}\theta \sin\theta)\sigma \quad , \quad |\theta| < \alpha \qquad (12)$$

where A and C are functions of α, β, γ, and σ (results are reproduced in the Appendix). The load parameter σ (proportional to the applied bending moment on the pipe) is also a function of the three angles. The angle γ marks the beginning of a compressive Dugdale zone on the side of the pipe away from the crack (back yielding). In the absence of such a zone $\gamma = \pi$. From (12) and definitions the tearing equation (11) can be put in the following form

$$C' \cos\alpha - A' + (3 \cos\alpha - \tfrac{1}{2}\alpha \sin\alpha)\sigma' = \omega$$
$$(13)$$

where $\qquad\qquad \omega = (E\varepsilon/\sqrt{2}\, \sigma_F)\bar{\omega}$

and where a prime denotes total differentiation with respect to α. In terms of A, C, σ, and α the fracture parameter J is given by

$$J = (\sqrt{2}\, \sigma_F^2 R / E\varepsilon)[C \cos\alpha - A + (3 \cos\alpha - \tfrac{1}{2}\alpha \sin\alpha)\sigma] \qquad (14)$$

Since J is not a convenient dependent variable β will be used instead. For values of α less than that for which back yielding appears A, C, and σ are known explicitly in terms of α and β alone. Beyond that range γ is known implicitly as a function of α and β and a root-finding

routine must be included as part of the numerical integration procedure. In either range the differential equation for β has the form

$$\beta' = f(\beta,\alpha) \tag{15}$$

The initial value of β is determined from (14) by the condition $J = J_c$ at $\alpha = \alpha_0$, and by a root-finding process. For very tough materials back yielding can occur before crack growth initiation, in which case initial values of β and γ must be obtained by root finding.

The procedure for setting up (15) will be described in some detail in the case $\gamma < \pi$. From results in [1] there are two expressions for σ involving the three angles α, β, and γ. These have the form

$$N_1(\alpha,\beta,\gamma) - \sigma D_1(\beta,\gamma) = 0 \tag{16}$$

$$N_2(\alpha,\beta,\gamma) - \sigma D_2(\beta,\gamma) = 0 \tag{17}$$

Eliminate σ between these two equations to obtain

$$N_1 D_2 - N_2 D_1 = 0 \tag{18}$$

For α and β given this is an implicit equation for the determination of γ. It must be solved by some root-finding method. Then σ is given by N_1/D_1 or by N_2/D_2 (which are equal). The total derivative of (16) with respect to α yields

$$\left(\frac{\partial N_1}{\partial \gamma} - \sigma \frac{\partial D_1}{\partial \gamma}\right)\gamma' - D_1 \sigma' = - \frac{\partial N_1}{\partial \alpha} - \left(\frac{\partial N_1}{\partial \beta} - \sigma \frac{\partial D_1}{\partial \beta}\right)\beta' \tag{19}$$

A similar equation follows from (17) containing subscripts 2 in place of 1. These are two simultaneous equations to solve for γ' and σ' in terms of β'. Let the result be expressed in the form

$$\gamma' = \Gamma_1 + \Gamma_2 \beta' \tag{20}$$

$$\sigma' = \Sigma_1 + \Sigma_2 \beta' \tag{21}$$

Equation (13) can now be written in the form

$$g_2 \beta' + g_1 = \omega \tag{22}$$

where

$$g_1 = \frac{\partial C}{\partial \alpha} \cos \alpha - \frac{\partial A}{\partial \alpha} - \frac{\partial A}{\partial \gamma} \Gamma_1 + \left(\frac{\partial C}{\partial \sigma} \cos \alpha - \frac{\partial A}{\partial \sigma} + q\right)\Sigma_1 \qquad (23)$$

$$g_2 = \frac{\partial C}{\partial \beta} \cos \alpha - \frac{\partial A}{\partial \beta} - \frac{\partial A}{\partial \gamma} \Gamma_2 + \left(\frac{\partial C}{\partial \sigma} \cos \alpha - \frac{\partial A}{\partial \sigma} + q\right)\Sigma_2 \qquad (24)$$

and where

$$q = 3 \cos \alpha - \tfrac{1}{2} \alpha \sin \alpha \qquad (25)$$

The f in (15) can now be expressed as

$$f = (\omega - g_1)/g_2 \qquad (26)$$

The various functions required to calculate g_1 and g_2 are given in the Appendix.

So long as the dimensionless axial stress at $\theta = \pi$ is greater than minus one back yielding does not occur. This condition can be written

$$\sigma + \tfrac{1}{2} \sqrt{2} (\beta - \alpha - \sigma \sin \beta)/\sin x < 1$$

$$(27)$$

where $\qquad\qquad x = (\pi - \beta)/\sqrt{2}$

Let α_B denote the value of α for which the condition is first violated. For $\alpha < \alpha_B$ the above formulas simplify because $\gamma = \pi$. Equation (16) determines σ, and σ' is obtained from (19) by dropping the term containing γ'. The terms with Γ_1 and Γ_2 drop out of (23) and (24) and the formulas in the Appendix hold with $\gamma = \pi$. The numerical integration of (15) proceeds in two stages, one for $\alpha < \alpha_B$ and another for $\alpha > \alpha_B$. Two difficulties arise. One is simply a matter of matching the two stages. In any step-by-step process α is not likely to equal α_B exactly at the end of a step. Some sort of interpolation scheme is required to ensure accuracy in the starting values of the second stage. A more troublesome difficulty arises from the fact that $\gamma = \pi$ is always a root, but not the desired root, of (18). the spurious root can be eliminated by division of both N_2 and D_2 by $\pi - \gamma$ and using a polynomial approximation to $\sin(z)/z$ to be found in [9]. However, finding the root when $\pi - \gamma$ is small is still difficult. Fortunately it turns out that inaccuracies in small values of $\pi - \gamma$ matter very little later on in the calculations.

For any given value of α there is a value of the load parameter σ sufficiently large to cause plastic collapse. This limit load is given by

$$\sigma_L = \frac{4}{\pi}(\cos\frac{\alpha}{2} - \frac{1}{2}\sin\alpha) \qquad (28)$$

Some example results are shown in the figures. These come from the following round-number choices for the parameters: $R/h = 10$, $\sigma_F = 100$ ksi, $E = 30\times10^6$ psi, $J_c = 1000$ lb/in., $R = 15$ in., $T_0 = 100$ or 25. Figure 2 shows load curves for initial crack angles of .3, .5, and .7 radians; the dashed curve is for $T_0 = 25$. The points where $\alpha = \alpha_B$ are indicated by tick marks. The upper curve represents the limit load according to (28). In the cases where $T_0 = 100$ the load reaches a maximum slightly below the limit load after a small amount of tearing. In the case $T_0 = 25$ the load reaches a maximum considerably below the limit load after a larger amount of tearing. Theoretically the load curves should approach the limit curve asymptotically. Numerically they appear to become tangent to the limit curve. Typically the computer indicates an overflow condition and the program may as well be stopped. Figure 3 shows computed resistance curves in three cases. In these cases (and in many computed but not shown) the initial slope corresponds very closely to T_0. There is very little dependance on α_0 in the case $T_0 = 100$.

CONCLUDING REMARKS

The results from the theory appear to be qualitatively correct but no extensive comparisons with the results of experiments have yet been made. A tearing theory such as that presented here substitutes a crack growth hypothesis for the hypothesis that there exists a "material" resistance curve. The theory calls for the experimental determination of three material constants J_c , and T instead of J_c , and a resistance curve. Such simplicity is appealing. The very need for a material resistance curve (always a rather troublesome object) is bypassed. Should the theory turn out to have validity a search for a more realistic model accounting for strain hardening would be indicated.

As previously noted the numerical implementation of the theory given here runs into difficulties when the load parameter gets close to the plastic collapse value. Examination of the

limiting case $\beta - \gamma$ small or even $\beta - \gamma = 0$ (fully yielded cracked section) has not yet been carried out analytically. Because shell theory is involved results in the limiting case should be similar to but not identical to results from theories such as those in [7] or [8] in which the cracked section is assumed to be fully yielded from the outset. The crack growth hypothesis in [7] bears an interesting similarity to the one adopted in the present paper.

ACKNOWLEDGEMENT

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APPENDIX

$Q = \sigma \sin \gamma - (\pi - \gamma)$

$P \sin x = Q \cos x - \sigma \sin \beta + \beta - \alpha$

where $\quad x = (\gamma - \beta)/\sqrt{2}$

$A = \sigma \cos \beta + \frac{1}{2} (\beta - \alpha)^2 - \sqrt{2} \, (P \cos x + Q \sin x)$

$C \sin \beta = - \frac{1}{2} (7 \sin \beta + \beta \cos \beta) \sigma + \beta - \alpha + \sin (\beta - \alpha)$

$\dfrac{\partial A}{\partial \alpha} = - (\beta - \alpha) + \sqrt{2} \, \cot x$

$\dfrac{\partial A}{\partial \beta} = - (\beta - \alpha - \sigma \sin \beta) \cot^2 x - \sqrt{2} \, (1 - \sigma \cos \beta) \cot x - [\sigma \sin \gamma - (\pi - \gamma)] \cot x / \sin x$

$\dfrac{\partial A}{\partial \gamma} = - \sqrt{2} \, (1 + \sigma \cos \gamma) / \sin x + \{ [\sigma \sin \gamma - (\pi - \gamma)] \cos x + \beta - \alpha - \sigma \sin \beta \} / \sin^2 x$

$\dfrac{\partial A}{\partial \sigma} = \cos \beta - \sqrt{2} \, \sin \gamma / \sin x + \sqrt{2} \, \sin \beta \cot x$

$\dfrac{\partial C}{\partial \alpha} = - [1 + \cos (\beta - \alpha)] / \sin \beta$

$\dfrac{\partial C}{\partial \beta} = [\frac{1}{2} (\beta - \sin \beta \cos \beta) \sigma + \sin \alpha + \sin \beta - (\beta - \alpha) \cos \beta] / \sin^2 \beta$

$\dfrac{\partial C}{\partial \gamma} = 0$

$\dfrac{\partial C}{\partial \sigma} = - \frac{1}{2} (7 \sin \beta + \beta \cos \beta) / \sin \beta$

$N_1 = [\sin \beta - \sin \alpha + (\beta - \alpha) \cos \beta] \sin x + \sqrt{2} \, (\beta - \alpha) \sin \beta \cos \alpha - \sqrt{2} \, (\pi - \gamma) \sin \beta$

$\dfrac{\partial N_1}{\partial \alpha} = - (\cos \alpha + \cos \beta) \sin x - \sqrt{2} \, \sin \beta \cos x$

$\dfrac{\partial N_1}{\partial \beta} = 2 \cos \beta \sin x + \frac{1}{2} \sqrt{2} \, [\sin \beta + \sin \alpha + (\beta - \alpha) \cos \beta] \cos x - \sqrt{2} \, (\pi - \gamma) \cos \beta$

$\dfrac{\partial N_1}{\partial \gamma} = \frac{1}{2} \sqrt{2} \, [\sin \beta - \sin \alpha + (\beta - \alpha) \cos \beta] \cos x - (\beta - \alpha) \sin \beta \sin x + \sqrt{2} \, \sin \beta$

$$D_1 = \tfrac{1}{2}(\beta + 3 \sin \beta \cos \beta)\sin x + \sqrt{2} \sin^2 \beta \cos x - \sqrt{2} \sin \gamma \sin \beta$$

$$\frac{\partial D_1}{\partial \alpha} = 0$$

$$\frac{\partial D_1}{\partial \beta} = 2 \cos^2 \beta \sin x - \tfrac{1}{4}\sqrt{2}(\beta - 5 \sin \beta \cos \beta)\cos x - \sqrt{2} \sin \gamma \cos \beta$$

$$\frac{\partial D_1}{\partial \gamma} = \tfrac{1}{4}\sqrt{2}(\beta + 3 \sin \beta \cos \beta)\cos x - \sin^2 \beta \sin x - \sqrt{2} \cos \gamma \sin \beta$$

$$N_2 = [\sin \gamma - (\pi - \gamma)\cos \gamma]\sin x + \sqrt{2}(\pi - \gamma)\sin \gamma \cos x - \sqrt{2}(\beta - \alpha)\sin \gamma$$

$$\frac{\partial N_2}{\partial \alpha} = \sqrt{2} \sin \gamma$$

$$\frac{\partial N_2}{\partial \beta} = -\tfrac{1}{2}\sqrt{2}[\sin \gamma - (\pi - \gamma)\cos \gamma]\cos x + (\pi - \gamma)\sin \gamma \sin x - \sqrt{2} \sin \gamma$$
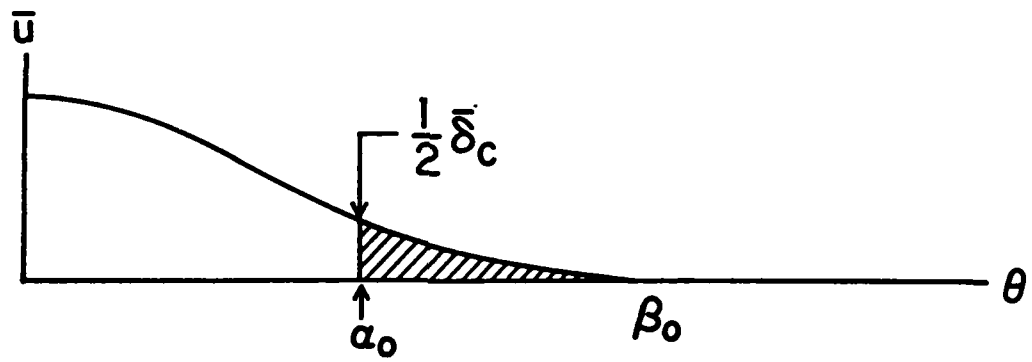
$$\frac{\partial N_2}{\partial \gamma} = 2 \cos \gamma \sin \gamma - \tfrac{1}{2}\sqrt{2}[\sin \gamma - (\pi - \gamma)\cos \gamma]\cos x - \sqrt{2}(\beta - \alpha)\cos \gamma$$

$$D_2 = \tfrac{1}{2}(\pi - \gamma - 3 \sin \gamma \cos \gamma)\sin x + \sqrt{2} \sin^2 \gamma \cos x - \sqrt{2} \sin \gamma \sin \beta$$
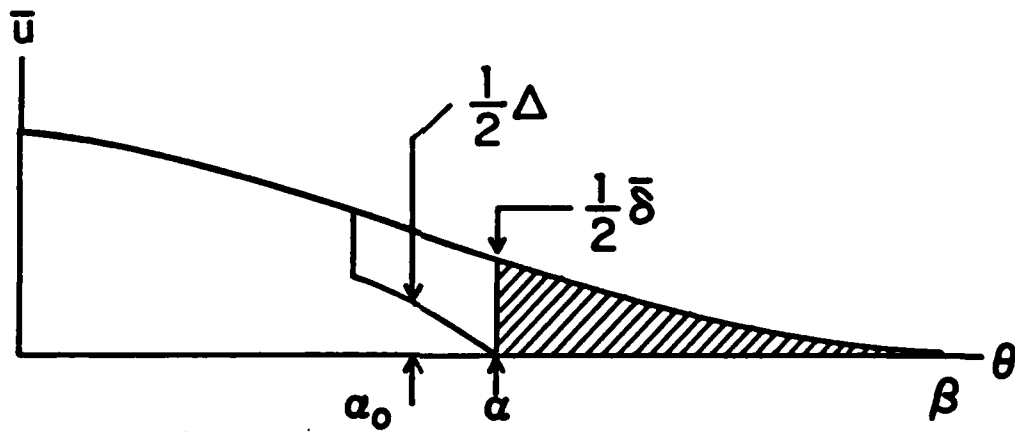
$$\frac{\partial D_2}{\partial \alpha} = 0$$

$$\frac{\partial D_2}{\partial \beta} = -\tfrac{1}{4}\sqrt{2}(\pi - \gamma - 3 \sin \gamma \cos \gamma)\cos x + \sin^2 \gamma \sin x - \sqrt{2} \sin \gamma \cos \beta$$

$$\frac{\partial D_2}{\partial \gamma} = -2 \cos^2 \gamma \sin x + \tfrac{1}{4}(\pi - \gamma + 5 \sin \gamma \cos \gamma)\cos x - \sqrt{2} \cos \gamma \sin \beta$$
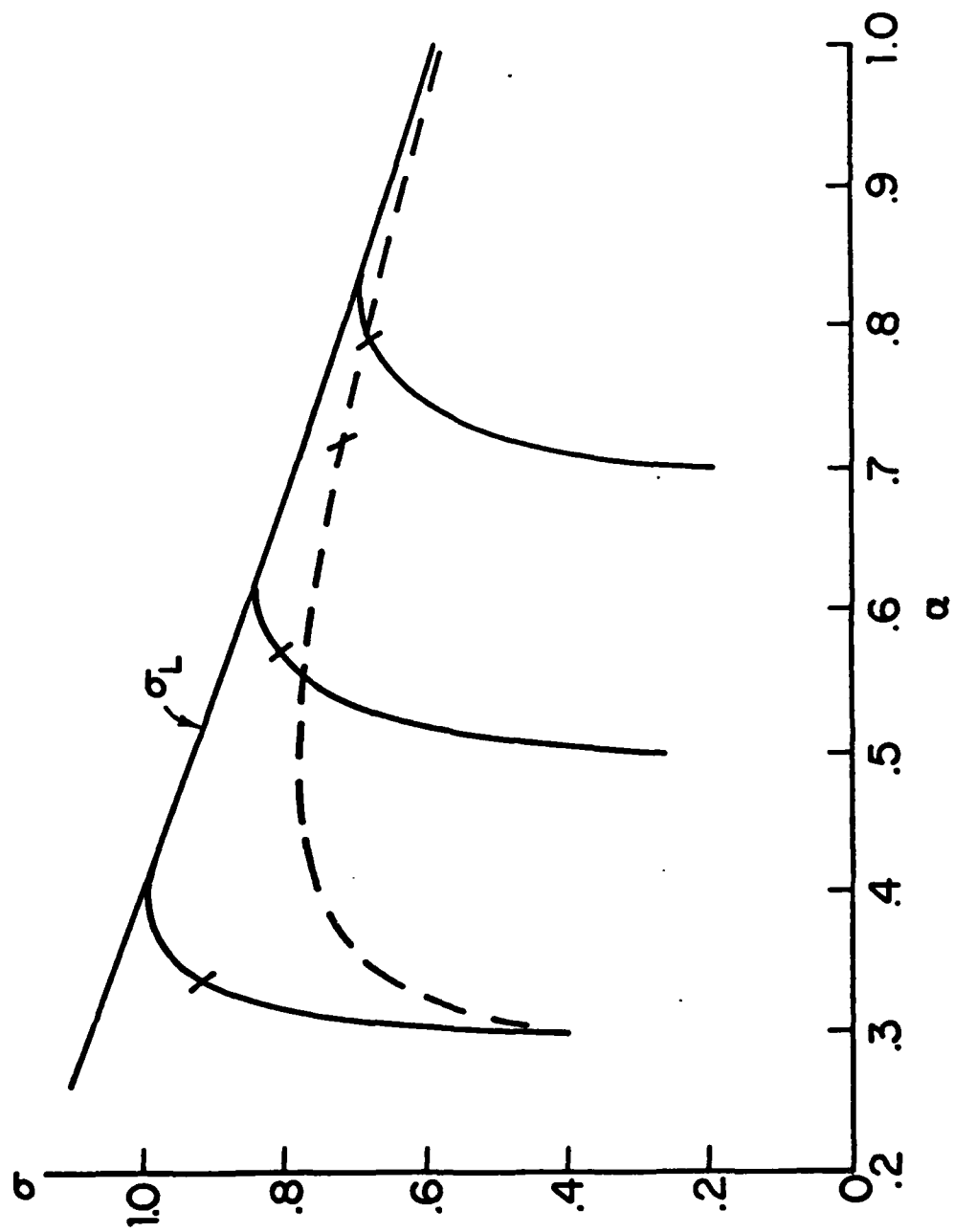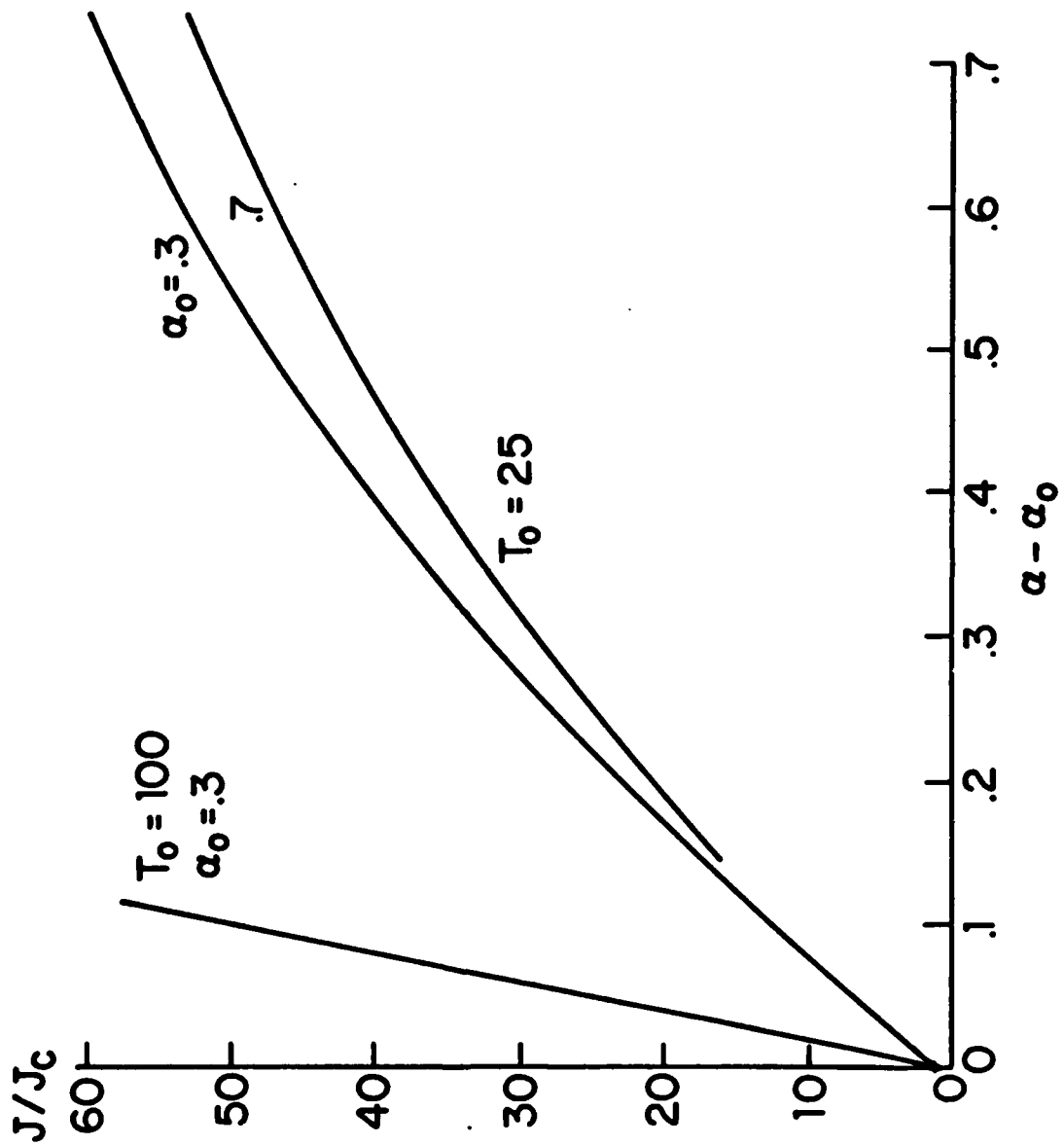
(a)

(b)

Fig. 1 CRACK PROFILES

Fig. 2 LOAD CURVES

Fig. 3 RESISTANCE CURVES

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