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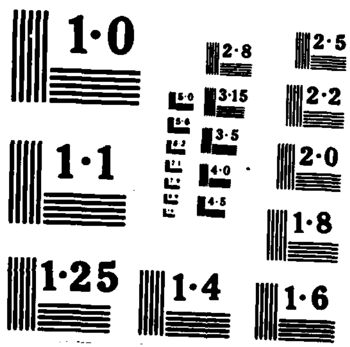
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ESTIMATION FROM BINOMIAL DATA WITH CLASSIFIERS OF KNOWN AND UNKNOWN IMPERFECTIONS

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ABSTRACT: Observations from inspection by a 'test' method and a standard method are combined to provide estimators of population proportion, and of probabilities of misclassification for the test method. Results of Hochberg and Tenenbein [3] and of Albers and Veldman [1] are extended to the case where the standard method is not perfect, but its misclassification probabilities have known values. Both moment and maximum likelihood estimators are considered and some asymptotic properties of the resulting estimators are compared.

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Key Words and Phrases: Binomial distribution; information matrix; inspection errors; maximum likelihood; method of moments; EM algorithm; statistical differentials.

079

1. INTRODUCTION

Suppose we have a large population, containing an unknown proportion, P, of individuals possessing a certain characteristic, which we will call 'nonconformance'. In a random sample, of size n, from this population, the distribution of the number, X, say, of nonconforming individuals will be binomial with parameters n,P so that

$$\Pr[X=x] = \binom{n}{x} P^x(1-P)^{n-x} \quad (x=0,1,\ldots,n).$$

We will represent this, symbolically, as

$$X \frown \text{Bin}(n,P)\ , \text{ where } \frown \text{ denotes "is distributed as".}$$

If the individuals in the sample of size n are examined by an imperfect measuring device, which detects actual nonconformance with probability p, and (incorrectly) 'detects' nonconformance, when the individual is really not nonconforming, with probability p', then the distribution of Z, the number of individuals declared to be nonconforming, as a result of this inspection, will be binomial with parameters n, $Pp+(1-P)p'$. It is clear that the only parameter that can be estimated from observations on values of Z in independent samples is $Pp+(1-P)p'$.

Various methods have been suggested for obtaining data from which estimates of P, p and p' can be derived (e.g. Albers and Veldman [1], Johnson and Kotz [5]). Tenenbein [6] suggested additional inspection of part of the sample by a perfect measuring device (for which p=1 and p'=0) and utilizing the resultant data. This method has been extended by Hochberg and Tenenbein [3] to allow for inspection of a further sample, of size n_S, say, by the perfect measuring device (S).

In this paper, we study problems arising in this latter situation if the 'established' measuring device S is not perfect, but has known values p_S, p'_S for p,p' respectively. For convenience, we will denote the (unknown) values of p,p' for the measuring device under test (T) by p_T, p'_T respectively. We will also assume (when necessary) that $p_S \gg p'_S$ and $p_T \gg p'_T$.

Problems of this kind arise when it is desired to calibrate the new device (T), by estimating p_T and p'_T. The unknown proportion (P) of NC units plays the role of a nuisance parameter in such problems.

2. ANALYSIS I (Moment Estimation)

As a consequence of the inspections we have the following sets of observations:

(i) n_S using S alone, with Z_S judged nonconforming (NC),

(ii) n_T using T alone, with Z_T judged NC,

(iii) n using both S and T, with results shown below:

S \ T	# NC	# not NC
# NC	Z_{11}	Z_{10}
# not NC	Z_{01}	Z_{00}

(# denotes 'number of'.) Evidently, $Z_{11}+Z_{10}+Z_{01}+Z_{00}=n$.

Under the assumption of random sampling from a population of effectively infinite size, we have that:

$$Z_S,\ Z_T \text{ and } \underset{\sim}{Z} = \begin{pmatrix} Z_{11} & Z_{10} \\ Z_{01} & Z_{00} \end{pmatrix} \text{ are mutually independent;} \tag{1.1}$$

$$Z_S \frown \mathrm{Bin}(n_S, \theta_S) \quad \text{with} \quad \theta_S = p_S P + p'_S(1-P)\ ; \tag{1.2}$$

$$Z_T \frown \mathrm{Bin}(n_T, \theta_T) \quad \text{with} \quad \theta_T = p_T P + p_T'(1-P) , \tag{1.3}$$

Also, assuming that the S and T classifications are independent, given the true status of the individual,

$$\underset{\sim}{Z} \frown \mathrm{Multinomial}\left(n; \begin{pmatrix} \phi & \theta_S - \phi \\ \theta_T - \phi & 1-\theta_S-\theta_T+\phi \end{pmatrix} \right) \tag{1.4}$$

with $\phi = p_S p_T P + p_S' p_T'(1-P)$, where $\frown$ denotes "is distributed as".

Recall that p_S and p_S' have known values, and P is the (unknown) proportion of NC individuals in the population.

Also $$P = (\theta_S - p_S')/(p_S - p_S') \tag{2.1}$$

$$p_T = (\phi - p_S'\theta_T)/(\theta_S - p_S') \tag{2.2}$$

$$p_T' = (p_S\theta_S - \phi)/(p_S - \theta_S) \tag{2.3}$$

Now, $$(n_S + n)\tilde{\theta}_S = Z_S + Z_{10} + Z_{11} \frown \mathrm{Bin}(n_S + n, \theta_S) \tag{3.1}$$

$$(n_T + n)\tilde{\theta}_T = Z_T + Z_{01} + Z_{11} \frown \mathrm{Bin}(n_T + n, \theta_T) \tag{3.2}$$

$$n\tilde{\phi} = Z_{11} \frown \mathrm{Bin}(n,\phi) \tag{3.3}$$

so that $\tilde{\theta}_S$ $\tilde{\theta}_T$ and $\tilde{\phi}$ (as defined in (3.1)-(3.3)) are unbiased estimators of θ_S, θ_T and ϕ respectively.

Hence $$\tilde{P} = (p_S - p_S')^{-1}(\tilde{\theta}_S - p_S') \tag{4.1}$$

is an unbiased estimator of P. Although the estimators

$$\tilde{p}_T = (\tilde{\theta}_S - p_S')^{-1}(\tilde{\phi} - p_S' \tilde{\theta}_T) \tag{4.2}$$

and $$\tilde{p}_T' = (p_S - \tilde{\theta}_S)^{-1}(p_S\tilde{\theta}_T - \tilde{\phi}) \tag{4.3}$$

are not unbiased estimators of p_T and p_T' respectively, the biases should not be large if sample sizes are adequate (see the example later in this section).

The variance-covariance matrix of the random variables in (3.1)-(3.2) is

$$\underset{\sim}{\mathrm{Var}}\left((n_S+n)\tilde{\theta}_S,(n_T+n)\tilde{\theta}_T,n\tilde{\phi}\right) = \begin{bmatrix} (n_S+n)\theta_S(1-\theta_S) & n(\phi-\theta_S\theta_T) & n\phi(1-\theta_S) \\ n(\phi-\theta_S\theta_T) & (n_T+n)\theta_T(1-\theta_T) & n\phi(1-\theta_T) \\ n\phi(1-\theta_S) & n\phi(1-\theta_T) & n\phi(1-\theta) \end{bmatrix}$$

Hence (cf. (4.1))

$$\mathrm{var}(\tilde{P}) = (n_S+n)^{-1}(p_S-p_S')^{-2}\theta_S(1-\theta_S) \tag{5.1}$$

and, using the method of statistical differentials (see, e.g. Johnson and Kotz [4, Chapter 1, Section 7.5]) we obtain, after some algebraic manipulation; the approximate formula

$$\mathrm{var}(\tilde{p}_T) \doteq p_T{}^2P^{-2}(p_S-p_S')^{-2}\Big[\Big\{n^{-1}\phi(1-\phi)-2(n_T+n)^{-1}p_S'\phi(1-\theta_T)+(n_T+n)^{-1}p_S'^2\theta_T(1-\theta_T)\Big\}p_T{}^{-2}$$
$$-2(n_S+n)^{-1}\{\phi(1-\theta_S)-n(n_T+n)^{-1}p_S'(\phi-\theta_S\theta_T)\}p_T{}^{-1}+(n_S+n)^{-1}\theta_S(1-\theta_S)\Big] \tag{5.2}$$

An approximate expression for $\mathrm{var}(\tilde{p}_T')$ is obtained from (5.2) by replacing p_T by p_T' and P by $(1-P)$, and interchanging p_S and p_S'.

An approximate formula for the bias of $\tilde{p}_T$ is

$$E[\tilde{p}_T] - p_T \doteq p_T\left\{\frac{\mathrm{var}(\tilde{\theta}_S)}{(\theta_S-p_S')^2} - \frac{\mathrm{cov}(\tilde{\theta}_S,\ \tilde{\phi}-p_S'\ \tilde{\theta}_T)}{(\theta_S-p_S')(\phi-p_S'\ \theta_T)}\right\} \tag{6}$$

which, after some reduction, gives a proportional bias (i.e. $100(\mathrm{bias})/p_T\%$)

$$\frac{100\{n_Tp_S'(1-\theta_S)+n(1-p_S')\theta_S\}(\phi-\theta_S\theta_T)}{(n_S+n)(n_T+n)(\theta_S-p_S')^2(\phi-p_S'\theta_T)}\ \% \tag{7}$$

From (2.1)-(2.3)

$$\theta_S-p_S' = (p_S-p_S')P\ ;\quad \phi-p_S'\ \theta_T = p_T(\theta_S-p_S')P$$

and also $\phi-\theta_S\theta_T = P(1-P)(p_S-p_S')(p_T-p_T')$, so

the approximate proportional bias (7) is

$$\frac{100\{n_T p_S'(1-\theta_S) + n(1-p_S')\theta_S\}\ (1-P)(p_T-p_T')}{(n_S+n)(n_T+n)P^2(p_S-p_S')^2 p_T}\ \% \qquad (7)'$$

which is positive and (since $p_T' < p_T$) less than

$$\frac{100\ G(1-P)}{(n_S+n)P^2(p_S-p_S')^2}\ \% \qquad (8)$$

where

$$G = \frac{n_T}{n_T+n}\, p_S'(1-\theta_S) + \frac{n}{n_T+n}\,(1-p_S')\theta_S\,, \qquad (9)$$

which lies between $p_S'(1-\theta_S)$ and $(1-p_S')\theta_S$.

Example 1. Using as 'typical' values of the probabilities p_S, p_S' and P the values 0.9, 0.1 and 0.1 respectively we find that

$$G = (n_T+n)^{-1}(0.082n_T + 0.162n)$$

(so that G lies between 0.082 and 0.162) and the approximate proportional bias of $\tilde{p}_T$ is between 0 and 1406.25 $G(n_S+n)^{-1}$% . Note that the upper limit is less than 227.8 $(n_S+n)^{-1}$%, so if $n_S + n > 100$ the approximate proportional bias is less than 2.28%. The next section contains a numerical assessment of formula (5.2), without specifying values of p_T and p_T'.

3. SOME NUMERICAL APPROXIMATIONS

Utilizing the reasonable assumption that $p_S >> p'_S$, and neglecting terms in p'_S and p'^2_S in the numerator of (5.2) we find

$$\mathrm{var}(\tilde{p}_T) \doteqdot p_T^2 P^{-2}(p_S - p'_S)^{-2} \left\{ \frac{\phi(1-\phi)}{np_T^2} - \frac{2\phi(1-\theta_S)}{(n_S+n)p_T} + \frac{\theta_S(1-\theta_S)}{n_S+n} \right\} \qquad (10)$$

Taking $p_S = 0.9$, $p'_S = 0.1$ so that $\theta_S = 0.8P + 0.1$ and $\phi = 0.9\, p_T P + 0.1\, p'_T(1-P)$, we obtain from (10)

$$\mathrm{var}(\tilde{p}_T) \doteqdot \frac{p_T^2}{0.64P^2} \left[\frac{\{0.9p_T P + 0.1p'_T(1-P)\}\{1-0.9p_T P - 0.1p'_T(1-P)\}}{n\, p_T^2} + \frac{(0.8P+0.1)(0.9-0.8P) - \{1.8p_T P + 0.2p'_T(1-P)\}(0.9-0.8P)p_T^{-1}}{n_S + n} \right]$$

Now taking $P = 0.1$, we find

$$\mathrm{var}(\tilde{p}_T) \doteqdot \frac{p_T^{\,2}}{0.0064} \left[\frac{0.09(p_T+p'_T)\{1-0.09(p_T+p'_T)\}}{n\, p^2_T} - \frac{0.1476}{n_S+n} \cdot \frac{p'_T}{p_T} \right] \qquad (11)$$

$$\doteqdot \frac{14.375}{n} (p_T+p'_T)\{1-0.09(p_T+p'_T)\} - \frac{23.06}{n_S+n} \cdot \frac{p'_T}{p_T} \qquad (12)$$

Since $0.09(p_T+p'_T)\{1-0.09(p_T+p'_T)\} < \frac{1}{4}$ (because $0.09(p_T+p'_T) < 1$) the right hand side of (12) is less than

$$(0.0256n)^{-1} < 39.1\, n^{-1}.$$

In the next section we will compare the asymptotic variances and covariances of $\tilde{\theta}_S, \tilde{\theta}_T$ and $\tilde{P}$ with those for maximum likelihood estimators $\hat{\theta}_S$, $\hat{\theta}_T$ and $\hat{P}$ of θ_S, θ_T and P respectively.

4. ANALYSIS II (Maximum Likelihood Estimators)

The likelihood function of Z_S, Z_T and Z is

$$\binom{n_S}{Z_S}\binom{n_T}{Z_T}\binom{n}{Z_{11},Z_{10},Z_{01},Z_{00}} \theta_S^{Z_S}(1-\theta_S)^{n_S-Z_S}\,\theta_T^{-Z_T}(1-\theta_T)^{n_T-Z_T}$$

$$\phi^{Z_{11}}(\theta_S-\phi)^{Z_{10}}(\theta_T-\phi)^{Z_{01}}(1-\theta_S-\theta_T+\phi)^{Z_{00}}$$

Equating derivatives of the log-likelihood to zero gives the following equations for $\hat\theta_S$, $\hat\theta_T$ and $\hat\phi$:

$$\frac{Z_S}{\hat\theta_S}-\frac{n_S-Z_S}{1-\hat\theta_S}+\frac{Z_{10}}{\hat\theta_S-\hat\phi}-\frac{Z_{00}}{1-\hat\theta_S-\hat\theta_T+\hat\phi}=0 \tag{13.1}$$

$$\frac{Z_T}{\hat\theta_T}-\frac{n_T-Z_T}{1-\hat\theta_T}+\frac{Z_{01}}{\hat\theta_T-\hat\phi}-\frac{Z_{00}}{1-\hat\theta_S-\hat\theta_T+\hat\phi}=0 \tag{13.2}$$

$$\frac{Z_{11}}{\hat\phi}-\frac{Z_{10}}{\hat\theta_S-\hat\phi}-\frac{Z_{01}}{\hat\theta_T-\hat\phi}+\frac{Z_{00}}{1-\hat\theta_S-\hat\theta_T+\hat\phi}=0 \tag{13.3}$$

subject to $0<\hat\phi<\hat\theta_S,\hat\theta_T<1$ and $\hat\phi>\hat\theta_S+\hat\theta_T-1$.

The information matrix is

$$\underset{\sim}{V}^{-1}=\begin{bmatrix} \frac{n_S}{\theta_S(1-\theta_S)}+\frac{n(1-\theta_T)}{(\theta_S-\phi)(1-\theta_S-\theta_T+\phi)}, & \frac{n}{1-\theta_S-\theta_T+\phi}, & -\frac{n(1-\theta_T)}{(\theta_S-\phi)(1-\theta_S-\theta_T+\phi)} \\ \frac{n}{1-\theta_S-\theta_T+\phi}, & \frac{n_T}{\theta_T(1-\theta_T)}+\frac{n(1-\theta_S)}{(\theta_T-\phi)(1-\theta_S-\theta_T+\phi)}, & -\frac{n(1-\theta_S)}{(\theta_T-\phi)(1-\theta_S-\theta_T+\phi)} \\ -\frac{n(1-\theta_T)}{(\theta_S-\phi)(1-\theta_S-\theta_T+\phi)}, & -\frac{n(1-\theta_S)}{(\theta_T-\phi)(1-\theta_S-\theta_T+\phi)}, & \frac{n\{\theta_S\theta_T(1-\theta_S-\theta_T)+2\theta_S\theta_T\phi-\phi^2\}}{\phi(\theta_S-\phi)(\theta_T-\phi)(1-\theta_S-\theta_T+\phi)} \end{bmatrix} \tag{14}$$

The determinant is

$$|\underset{\sim}{V}^{-1}| = \frac{n}{\phi(\theta_S-\phi)(\theta_T-\phi)(1-\theta_S-\theta_T+\phi)}\left[\frac{n_S n_T\,\gamma}{\theta_S\theta_T(1-\theta_S)(1-\theta_T)} + n(n_S+n_T+n)\right]$$

with $\gamma = \theta_S\theta_T(1-\theta_S-\theta_T+\phi) - \phi(\phi-\theta_S\theta_T)$.

And from the asymptotic variance-covariance matrix $\underset{\sim}{V}$ we obtain

$$\mathrm{var}(\hat{\theta}_S) \doteq \frac{n}{|\underset{\sim}{V}^{-1}|}\left\{\frac{n_T}{\theta_T(1-\theta_T)}\left(\frac{1}{\phi}+\frac{1}{\theta_S-\phi}+\frac{1}{\theta_T-\phi}+\frac{1}{1-\theta_S-\theta_T+\phi}\right)+n\left(\frac{1}{\theta_T-\phi}+\frac{1}{1-\theta_S-\theta_T+\phi}\right)\left(\frac{1}{\phi}+\frac{1}{\theta_S-\phi}\right)\right\}$$

$$\doteq \theta_S(1-\theta_S)\ \ (\gamma n_S{}^{-1} + \delta N^{-1})/(\gamma + \delta) \tag{15}$$

where $N = n_S+n_T+n$ (= total number of observations) and

$$\delta = \frac{nN}{n_S n_T}\,\theta_S\theta_T(1-\theta_S)(1-\theta_T)$$

The MLE of P is

$$\hat{P} = (p_S-p_S')^{-1}(\hat{\theta}_S-p_S'). \tag{16}$$

The asymptotic efficiency of $\tilde{P}$ (see (4.1)) relative to $\hat{P}$ is the same as that of $\tilde{\theta}_S$ relative to $\hat{\theta}_S$, which is

$$100(n_S+n)(\gamma\ n_S{}^{-1} + \delta N^{-1})/(\gamma+\delta)\% \tag{17}$$

Taking $p_S=p_T=0.9$, $p_S'=p_T'=0.1=P$, as in Example 1, and $n_S=n_T=n$ $(= \frac{1}{3}N)$ we find $\gamma \doteq 0.0184680$ and $\delta = 0.0653573$, so (17) becomes

$$100(n_S+n)\ \ (0.2203\ n_S{}^{-1} + 0.7797\ N^{-1})$$

$$= 2(0.2203 + 0.2599) = \underline{96.04\%}\ .$$

The asymptotic variance of the MLE $\hat{\phi}$ is

$$\mathrm{var}(\hat{\phi}) \doteq \frac{1}{n}\,\frac{\delta\phi(1-\phi) - \delta\phi^2 N^{-1}\{n_S\theta_S{}^{-1}(1-\theta_S) + n_T\theta_T{}^{-1}(1-\theta_T)\} + \phi(\theta_S-\phi)(\theta_T-\phi)(1-\theta_S-\theta_T+\phi)}{\delta+\gamma}$$

On the other hand, recalling that $\mathrm{var}(\tilde{\phi}) = n^{-1}\phi(1-\phi)$, we find for the numerical values of the parameters used above, that the asymptotic efficiency of the moment estimator $\tilde{\phi}$ is

$$100 \times \frac{0.0653573 \times 0.09\{0.91 - (2/3) \times 0.09 \times (0.18)^{-1} \times 0.82\} + 0.09 \times 0.09^2 \times 0.73}{0.09 \times 0.91(0.0653573 + 0.0184680)}$$

$$= 100 \times \frac{0.0037449 + 0.0005322}{0.0068653} = \underline{62.30\%}$$

The markedly lower asymptotic efficiency of $\tilde{\phi}$ is associated with the fact that it does not utilize the information on values of θ_S and θ_T which is available from the other (n_S+n_T) observations. Some support for this statement comes from the asymptotic efficiency of $\tilde{\phi}$ if the values of θ_S and θ_T are <u>known</u>. This is

$$100 \times \frac{(\theta_S-\phi)(\theta_T-\phi)(1-\theta_S-\theta_T+\phi)}{\gamma(1-\phi)} \tag{19}$$

With the numerical values of θ_S, θ_T and ϕ which we have been using above this would give an asymptotic efficiency of only <u>35.18</u>%.

5. <u>CALCULATION OF MAXIMUM LIKELIHOOD ESTIMATES</u>

It is not possible to obtain explicit solutions of (13.1)-(13.3) for θ_S, θ_T and ϕ, so a numerical solution must be sought.

An EM algorithm (see, e.g. Dempster et al. [2]) can be constructed in the following way. Introduce (unobserved) random variables $Z_{ij(S)}$ $(Z_{ij(T)})$ $(i,j = 0,1)$ representing the numbers of i,j decision combinations which would have been obtained if the n_S (n_T) individuals tested by S(T) had also been tested by T(S). (Clearly

$$Z_{10(S)} + Z_{11(S)} = Z_S \quad \text{and} \quad Z_{01(T)} + Z_{11(T)} = Z_T .)$$

If values of these variables had been observed the maximum likelihood estimators would have been

For θ_S: $(Z_S + Z_{10(T)} + Z_{11(T)} + Z_{10} + Z_{11})N^{-1}$;

For θ_T: $(Z_{01(S)} + Z_{11(S)} + Z_T + Z_{01} + Z_{11})N^{-1}$;

For ϕ: $(Z_{11(S)} + Z_{11(T)} + Z_{11})N^{-1}$.

Since

$$E[Z_{10(T)}|Z_T] = (n_T-Z_T)(\theta_S-\phi)(1-\theta_T)^{-1};\; E[Z_{11(T)}|Z_T] = Z_T\phi\theta_T^{-1};$$

$$E[Z_{01(S)}|Z_S] = (n_S-Z_S)(\theta_T-\phi)(1-\theta_S)^{-1};\; E[Z_{11(S)}|Z_S] = Z_S\phi\theta_S^{-1};$$

the EM algorithm leads to iteration from $\theta_S^{(\nu)}$, $\theta_T^{(\nu)}$, $\phi^{(\nu)}$ to

$$\theta_S^{(\nu+1)} = N^{-1}\left[Z_S + \frac{(n_T-Z_T)(\theta_S^{(\nu)}-\phi^{(\nu)})}{1-\theta_T^{(\nu)}} + \frac{Z_T\phi^{(\nu)}}{\theta_T^{(\nu)}} + Z_{10} + Z_{11}\right] \tag{20.1}$$

$$\theta_T^{(\nu+1)} = N^{-1}\left[\frac{(n_S-Z_S)(\theta_T^{(\nu)}-\phi^{(\nu)})}{1-\theta_S^{(\nu)}} + \frac{Z_S\phi^{(\nu)}}{\theta_S^{(\nu)}} + Z_T + Z_{01} + Z_{11}\right] \tag{20.2}$$

$$\phi^{(\nu+1)} = N^{-1}\left[\left(\frac{Z_T}{\theta_T^{(\nu)}} + \frac{Z_S}{\theta_S^{(\nu)}}\right)\phi^{(\nu)} + Z_{11}\right] \tag{20.3}$$

Example 2. Table 1 sets out results of applying the EM algorithm to three illustrative sets of values of the n's and Z's. In each case $n_S = n_T = n = 50$; $Z_T = 10$; $Z_{00} = 40$; $Z_{01} = 3$. The remaining values were

Set	Z_S	Z_{10}	Z_{11}
(I)	5	1	6
(II)	8	1	6
(III)	8	3	4

Table 1: EM algorithm solutions of equations (13.1)-(13.3)

Set	(I)			(II)			(III)		
ν	$\theta_S^{(\nu)}$	$\theta_T^{(\nu)}$	$\phi^{(\nu)}$	$\theta_S^{(\nu)}$	$\theta_T^{(\nu)}$	$\phi^{(\nu)}$	$\theta_S^{(\nu)}$	$\theta_T^{(\nu)}$	$\phi^{(\nu)}$
0 (moment estimates)	0.1200	0.1900	0.1200	0.1500	0.1900	0.1200	0.1500	0.1700	0.0800
1	0.1221	0.1839	0.1154	0.1520	0.1924	0.1248	0.1539	0.1714	0.0865
2	0.1240	0.1816	0.1131	0.1522	0.1928	0.1270	0.1553	0.1714	0.0903
3	0.1251	0.1805	0.1119	0.1522	0.1929	0.1284	0.1560	0.1712	0.0928
4	0.1256	0.1800	0.1111	0.1522	0.1930	0.1294	0.1565	0.1711	0.0945
5	0.1259	0.1798	0.1106	0.1522	0.1930	0.1300	0.1568	0.1710	0.0957
6	0.1260	0.1797	0.1103	0.1522	0.1930	0.1305	0.1570	0.1709	0.0965
FINAL	0.1261	0.1796	0.1098	0.1523	0.1931	0.1315	0.1574	0.1707	0.0984

The initial values $\theta_S^{(0)}$, $\theta_T^{(0)}$ and $\phi^{(0)}$ were the moment estimates. The table shows the results of the first six iterations and the final values, to four decimal places. (Speed of convergence can be improved, of course by using modified values of $\theta_S^{(\nu)}$, $\theta_T^{(\nu)}$, and $\phi^{(\nu)}$ for the $(\nu+1)$-th iteration, taking account of trends in values.)

The maximum likelihood estimates of P, p_T and p_T' are obtained by replacing θ_S, θ_T and ϕ in (2.1)-(2.3) by their maximum likelihood estimates. We obtain the following formulas (provided the values lie between 0 and 1).

Set	$\hat{p}$	$\hat{p}_T$	$\hat{p}_T'$
(I)	$(0.1261-p_S')/(p_S-p_S')$	$(0.1098-0.1796p_S')/(0.1261-p_S')$	$(0.1796p_S-0.1098)/(p_S-0.1261$
(II)	$(0.1523-p_S')/(p_S-p_S')$	$(0.1315-0.1931p_S')/(0.1523-p_S')$	$(0.1931p_S-0.1315)/(p_S-0.1523$
(III)	$(0.1574-p_S')/(p_S-p_S')$	$(0.0984-0.1707p_S')/(0.1574-p_S')$	$(0.1707p_S'-0.0984)/(p_S-0.1574$

In order to satisfy the conditions $0 \leq \hat{P}, \hat{p}_T, \hat{p}'_T \leq 1$ we need $p_S \geq \max(\hat{\theta}_S, \hat{\phi}/\hat{\theta}_T) \geq \min(\hat{\theta}_S, \hat{\phi}/\hat{\theta}_T) > p'_S$. These conditions, for sets (I)-(III), are

(I) $p_S > 0.611$; $p'_S < 0.126$

(II) $p_S > 0.681$; $p'_S < 0.152$

(III) $p_S > 0.576$; $p'_S < 0.157$.

[If the conditions are not met, then appropriate boundary values (0 if formula gives a negative value, 1 if it gives a value greater than 1) can be used.]

6. CONCLUDING REMARKS

The estimates of p_T, p'_T and P depend on the values assumed for p_S and p'_S. If these values are incorrect, biases will be introduced. The way in which the values used for p_S and p'_S affect the estimates can easily be appreciated from equations (2.1) - (2.3). For example, increase in either p_S or p'_S will tend to lead to negative bias in estimates of P (remembering that $\theta_S < p_S$).

In this paper we have been concerned with estimation of p_T, p'_T (and also P), supposing p_S, p'_S known. This has been effected via estimation of the parameters θ_S, θ_T and ϕ. The same analysis can be used in other circumstances. For example, if P (proportion of nonconforming items) and p_S are known, then p_T, p'_T and p'_S can be estimated using the relationships

$$p'_S = (\theta_S - p_S P)(1-P)^{-1}$$

$$p_T = \frac{\phi(1-P) - (\theta_S - p_S P)\theta_T}{(p_S - \theta_S)P}$$

$$p'_T = \frac{p_S \theta_S - \phi}{p_S - \theta_S}$$

Of course, if P is known, as well as p_S and p'_S, then θ_S is known and there is no need to take any observations with S alone - that is we can take $n_S = 0$.

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Observations from inspection by a "test" method and a standard method are combined to provide estimators of population proportion, and of probabilities of misclassification for the test method. Results are obtained in the case where the standard method is not perfect, but its misclassification probabilities have known values. Both moment and maximum likelihood estimators are considered and asymptotic properties of the resulting estimators are compared.

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