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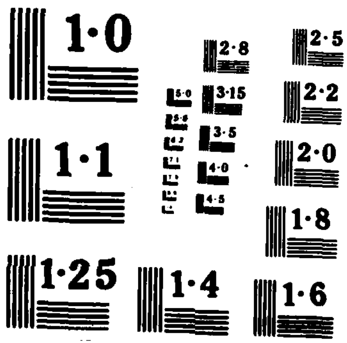
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Consider a statistical decision problem in which nature has a finite number of states. The elementary theorems of decision theory, namely the Minimax theorem, the Complete class theorem, and theorems on the structure of admissible rules, are proved in most texts under the assumptions that the risk set is closed from below and bounded from below (see Berger (1985) and Ferguson (1967)). The condition that the risk set is bounded from below is sufficient for the existence of the lower boundary points; however, that it is not necessary can be

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) seen from simple examples. The purpose of this paper is to extend the elementary theorems of decision theory to include the case in which the risk set is not bounded from below and the set of lower boundary points is nonempty. we also show that if the risk set is bounded from above then it is necessary for the risk set to be bounded from below for the set of lower boundary points to be nonempty. We present examples to illustrate our theorems.

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# On The Elementary Theorems Of Decision Theory

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Keywords and Phrases: Admissible rule, Bayes rule, Minimax principle, Complete class.

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Abstract. Considers a statistical decision problem in which nature has a finite number of states. The elementary theorems of decision theory, namely the Minimax theorem, the Complete class theorem, and theorems on the structure of admissible rules, are proved in most texts under the assumptions that the risk set is closed from below and bounded from below. The condition that the risk set is bounded from below is sufficient for the existence of the lower boundary points; however, that it is not necessary can be seen from simple examples. The purpose of this paper is to extend the elementary theorems of decision theory to include the case in which the risk set is not bounded from below and the set of lower boundary points is nonempty. The author ~~We~~ also shows that if the risk set is bounded from above then it is necessary for the risk set to be bounded from below for the set of lower boundary points to be nonempty. We present examples are presented to illustrate this ~~our~~ theorems.

1. Introduction and Definitions. A statistical decision problem consists of a triplet $(\Omega, D, R)$ where $\Omega$ is the set of all possible states of nature, also known as the parameter space. The set $D$ is the class of all randomized decision rules available to the statistician. When the statistician chooses the randomized decision rule $\delta \in D$ and the state of nature is $\theta \in \Omega$, the risk of the statistician is given by the real valued function $R(\theta, \delta)$. For a complete discussion of randomized decision rules, risk functions, the reader is referred to Ferguson (1967) and Berger (1985). Throughout this paper we shall assume that the parameter space $\Omega$ is a finite set, say $\{\theta_1, \theta_2, \ldots, \theta_k\}$, where $k \geq 1$ is fixed.

Given any decision rule $\delta \in D$, the risk vector of $\delta$ is defined as the vector $x = (R(\theta_1,\delta),\ldots\ldots,R(\theta_k,\delta))^t$. The collection

$$(1.1) \qquad S = \left\{x = (R(\theta_1,\delta),\ldots\ldots,R(\theta_k,\delta))^t : \delta \in D\right\}$$

of all risk vectors is known as the risk set. It is well known that S is a convex subset of $R^k$; that is, if $x$, $y \in S$ and $0 \le \lambda \le 1$ then $\lambda x + (1-\lambda)y$ is also in S. We adopt the following definitions from Ferguson(1967).

Definition 1.1. The subset S is said to be bounded from below if there exists a finite positive number A, such that for every $x = (x_1,\ldots\ldots,x_k)^t \in S$

$$(1.2) \qquad x_i > -A \qquad \text{for } i=1,\ldots\ldots,k.$$

We say that S is bounded from above if for every $x = (x_1,\ldots\ldots,x_k)^t \in S$,

$$(1.3) \qquad x_i < A \qquad \text{for } i=1,\ldots\ldots,k.$$

The set S is said to be bounded if it is bounded from above and below.

Definition 1.2. Let $x$ be a point of $R^k$. The lower quantant at $x$, denoted by $Q_x$, is defined as the set

$$(1.4) \qquad Q_x = \{ y = (y_1,\ldots\ldots,y_k)^t : y_j \le x_j,\ j=1,\ldots,k \}.$$

In the following definitions $\bar{S}$ denotes the closure of the set S, that is, $\bar{S}$ is the smallest closed set containing S.

Definition 1.3. A point $x$ is said to be a lower boundary point of a convex set $S \subset R^k$ if $Q_x \cap \bar{S} = \{x\}$. The set of lower boundary points of a convex set S is denoted by LB(S).

The following lemma can be found in Ferguson(1967).

Lemma 1.4. Let S be a convex subset of $R^k$. Then $\bar{S}$ is convex and the set of lower boundary points of S and $\bar{S}$ is the same, that is, $LB(S) = LB(\bar{S})$.

Definition 1.5. A convex set $S \subset R^k$ is said to be closed from below if $LB(S) \subset S$. We say that S is closed if $S = \bar{S}$.

Let $x = (x_1, \ldots, x_k)^t$ and $y = (y_1, \ldots, y_k)^t$ be two elements of $R^k$. We write $x \leq y$ if $x_i \leq y_i$ for $i=1, \ldots, k$, and $x < y$ if $x \leq y$ and $x_j < y_j$ for atleast one j. The relations $\leq$ , $<$ induce a partial ordering of the elements of $R^k$. The next few definitions are concerned with the class D of randomized decision rules available to the statistician.

Definition 1.6. Let $\delta_1$ and $\delta_2$ be two decision rules in D with risk vectors $x_1$ and $x_2$ respectively. We say that $\delta_1$ is as good as the rule $\delta_2$ if $x_1 \leq x_2$. The rule $\delta_1$ is said to be better than $\delta_2$ if $x_1 < x_2$ and $\delta_1$ is said to be equivalent to $\delta_2$ if $x_1 = x_2$.

Definition 1.7. A rule $\delta$ is said to be admissible if there exists no rule that is better than $\delta$. A rule $\delta$ is said to be inadmissible if it is not admissible.

It is clear that inadmissible rules are undesirable. Thus in looking for a best rule, the statistician usually restricts his search to the class of admissible rules, provided the class is nonempty.

Definition 1.8. A class C of decision rules, $C \subset D$, is said to be complete, if given any decision rule $\delta \in D$, not in C, there exists a rule $\delta_0 \in C$ that is better than $\delta$. The class C of decision rules is said to be minimal complete if C is complete and if no proper subclass of C is complete.

We note that a complete class C always exists for a statistical decision problem (take C=D), but a minimal complete class need not exist. However, if a minimal complete class exists then it consists exactly of the admissible rules. This result and several other results which demonstrate the relationship between minimal complete class and admissible rules can be found in Ferguson (1967) (also see Berger (1985)). We state below one important theorem that is relevant to the main theorems of this paper.

Theorem 1.9. If C is the class of admissible rules and C is a complete class then C is minimal complete.

We now proceed and develop the notion of Bayes rule and Bayes risk. Any probability distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ on the parameter space $\Omega$ is known as a prior distribution. We shall denote the class of all prior distributions by P.

Given any prior distribution $\pi \in P$ and a decision rule $\delta \in D$, the Bayes risk of $\delta$ with respect to $\pi$ is defined as

$$(1.5) \qquad BR(\pi,\delta) = E\left[R(Z,\delta)\right]$$

where $Z$ is a random variable having distribution $\pi$. The Bayes principle requires the statistician to look for a rule which minimizes the Bayes risk. This rule if it exists is known as the Bayes rule. Formally we have

Definition 1.10. A decision rule $\delta_b$ is said to be Bayes with respect to the prior distribution $\pi \in P$ if

$$(1.6) \qquad BR(\pi,\delta_b) = \inf_{\delta \in D} BR(\pi,\delta)$$

In searching for a best rule the statistician is also guided by another notion of best rule, known as the Minimax principle. A rule $\delta_1$ is preferred to a rule $\delta_2$ if

$$(1.7) \qquad \sup_{\theta \in \Omega} R(\theta,\delta_1) < \sup_{\theta \in \Omega} R(\theta,\delta_2).$$

This relation "is preferred to" also defines a linear ordering on the class of decision rules D. If the statistician decides to adopt the Minimax principle then he should look for a smallest element in this ordering. A rule that is most preferred in this ordering is called a minimax decision rule. This discussion leads us to the following

Definition 1.11. A decision rule $\delta_m$ is said to be minimax if

(1.8)
$$\sup_{\theta\in\Omega} R(\theta,\delta_m) = \inf_{\delta\in D}\sup_{\theta\in\Omega} R(\theta,\delta).$$

We can also define similarly, a minimax rule for nature. This will be an element of the class P. The minimax rule of the nature $\pi_f \in P$ is also known as a least favorable distribution. The probability distribution $\pi_f \in P$ satisfies the equality

(1.9)
$$\inf_{\delta\in D} BR(\pi_f,\delta) = \sup_{\pi\in P}\inf_{\delta\in D} BR(\pi,\delta).$$

2. Main Theorems. In this section we present the main results of this paper. The theorems presented in this section are analogous to the theorems in Chapter 2 of Ferguson(1967). However, Ferguson(1967) assumes that the risk set S is bounded from below. We replace this condition by the weaker condition that the set of lower boundary points of S, LB(S), is nonempty. We also present applications of our theorems which are not covered by the theorems of Ferguson(1967). The following lemma plays an important role in the proofs of our theorems.

Lemma 2.1. Let $\{y_{1n}, n \geq 1\},\ldots\ldots,\{y_{kn}, n \geq 1\}$ be k sequences which are bounded above. Assume that there exists j, $1 \leq j \leq k$, such that $y_{jn} \longrightarrow -\infty$ as $n \longrightarrow \infty$. Then there exists an r, $1 \leq r \leq k$, and a subsequence $\{m\}$ such that

(2.1)
$$\left[\, y_{im} \;/\; y_{rm} \,\right] \longrightarrow M_i$$

for all i, $1 \leq i \leq k$, where $0 \leq M_i < \infty$ and $y_{rm} \longrightarrow -\infty$ as $m \longrightarrow \infty$.

Proof. We shall prove the lemma by the method of induction. The lemma trivially holds for $k = 1$. Assume that the lemma is true for $k = (l-1)$, that is , there exists an $r$, $1 \le r \le (l-1)$, and a subsequence $\{m\}$ such that (2.1) holds for $1 \le i \le (l-1)$. Let $\{y_{lm}, m \ge 1\}$ be another sequence. Note that

$$0 \le \liminf (y_{lm} / y_{rm}) \le \infty,$$

since $\{y_{lm}\}$ is bounded above and $y_{rm} \longrightarrow -\infty$ as $m \longrightarrow \infty$. We consider the following two cases.

Case 1: $0 \le \liminf (y_{lm} / y_{rm}) < \infty$. In this case we can find a subsequence $\{m'\}$ of $\{m\}$ such that

$$\lim (y_{lm'} / y_{rm'}) = M_l, \tag{2.2}$$

where $0 \le M_l < \infty$. The subsequence $\{m'\}$ satisfies (2.1) for all $1 \le i \le l$.

Case 2: $\liminf (y_{lm} / y_{rm}) = \infty$. In this case we can find a subsequence $\{m'\}$ of $\{m\}$ such that

$$\lim (y_{rm'} / y_{lm'}) = 0. \tag{2.3}$$

Combining (2.3) with the induction hypothesis we get that for $1 \le i \le (l-1)$,

$$(2.4) \quad \lim (y_{im'}/y_{lm'}) = \lim (y_{im'}/ y_{rm'}) \lim (y_{rm'}/ y_{lm'})$$
$$= 0 = M_i \quad \text{(say)}.$$

Therefore the subsequence $\{m'\}$ and $r = l$ satisfies (2.1) for $1 \le i \le l$. This completes the proof of Lemma 2.1.

As an application of Lemma 2.1 we have the following result.

Lemma 2.2. Let S be a convex subset of $R^k$. Assume that S is bounded above. If the set of lower boundary points LB(S) is nonempty then S is bounded from below.

Proof. Suppose S is not bounded from below. Then there exists a sequence $y_n = (y_{1n},\ldots,y_{kn})^t \in S$ such that $y_{jn} \longrightarrow -\infty$ for some j, $1 \le j \le k$ as $n \longrightarrow \infty$. Since the $y_{jn}$'s are bounded above, using Lemma 2.1 we can find an $1 \le r \le k$ and a subsequence $\{m\}$ such that $y_{rm} \longrightarrow -\infty$ and

$$(2.5) \qquad (y_{im} / y_{rm}) \longrightarrow M_i \text{ as } m \longrightarrow \infty,$$

where $0 \le M_i < \infty$, for $1 \le i \le k$. Let $z=(z_1,\ldots,z_k)^t$ be an element of LB(S). Let $\epsilon > 0$ be given. Define

$$(2.6) \qquad \lambda_m = [\,(z_r-\epsilon) - y_{rm}\,] \,/\, [\,z_r-y_{rm}\,], \text{ for } m \ge 1.$$

Note that $0 < \lambda_m < 1$ for sufficiently large m. Since $\bar{S}$ is convex, $\lambda_m z + (1-\lambda_m)y_m \in \bar{S}$. Using (2.5) and (2.6) we can easily verify that

$$(2.7) \qquad \lambda_m \longrightarrow 1$$

(2.8) $$\lambda_m z + (1-\lambda_m) y_m \longrightarrow z - \epsilon M$$

as $m \longrightarrow \infty$, where $M = (M_1, \ldots, M_k)^t$. Since $M_r=1$, the limit point $z - \epsilon M$, which belongs to $\bar{S}$, is less than $z$ and this contradicts the fact that $z \in LB(S)$. The proof of Lemma 2.2 is now complete.

The above lemma shows that if a convex subset S of $R^k$ is bounded above, then either S is bounded below or LB(S) is empty. The following example shows that we cannot relax the hypothesis that S is bounded above in Lemma 2.2.

Example 2.3. Let k=2 and $S_2 = \{(x_1,x_2)^t : x_1 \geq -x_2\}$. In this example $LB(S_2) = \{(x_1,x_2)^t : x_1 = -x_2\}$, but $S_2$ is neither bounded from above nor bounded from below.

The converse of Lemma 2.2 is well known and is stated in the Lemma 2.4. The proof of Lemma 2.4 can be found on page 69, Ferguson (1967).

Lemma 2.4. Let S be a convex subset of $R^k$. If S is bounded from below, then LB(S) is nonempty.

The next lemma is crucial to the proofs of our elementary theorems of decision theory. The proof follows easily from Lemma 2.2.

Lemma 2.5. Let S be a convex subset of $R^k$ with nonempty lower boundary points LB(S). Let $x = (x_1,\ldots,x_k)^t \in \bar{S}$ and let $S_1 = Q_x \cap \bar{S}$. Then $S_1$ is bounded from below.

Proof. Suppose $S_1$ is not bounded from below. Then we can find a sequence $y_n = (y_{1n},\ldots,y_{kn})^t \in S_1$ such that for some j, $y_{jn} \longrightarrow -\infty$ as $n \longrightarrow \infty$. Let $z = (z_1,\ldots,z_k)^t \in$ LB(S). Since $y_n$ is bounded from above, imitating the proof of Lemma 2.2 we can find $z' \in \bar{S}$ such that $z' < z$. This contradicts the fact that $z \in$ LB(S). The proof of Lemma 2.5 is complete.

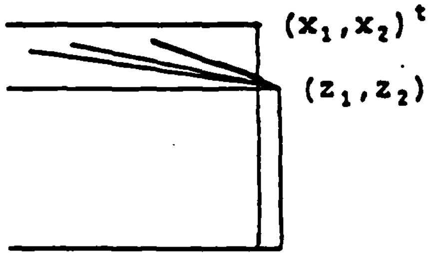


Figure 1

Lemma 2.5 is best understood when k=2 and S is a convex subset of $R^2$. Let $(x_1,x_2)^t \in \bar{S}$ and let $S_1 = Q_x \cap \bar{S}$. Let $(z_1,z_2)^t \in$ LB(S). It is easy to verify that either $x_1 > z_1$ or $x_2 > z_2$. Let us assume that $x_2 > z_2$ and $x_1 \le z_1$, the other cases can be handled similarly. If $S_1$ is not bounded from below then there exists a sequence $y_n = (y_{1n}, y_{2n})^t \in S_1$ such that $y_{1n} \longrightarrow -\infty$ and $z_2 < y_{2n} \le x_2$. Since $\bar{S}$ is convex, the line $L_n$ joining $z$ and $y_n$ is contained in $\bar{S}$. Letting n converge to $\infty$, we can check that $L_n$ coincides with the line Y= $z_2$ (see Figure 1) and therefore we can find points of $\bar{S}$ which are less than $z$. This is a contradiction because $z \in$ LB(S).

The next theorem is the first of our elementary theorems of decision theory. It is well known that if the risk set S is bounded from below and closed from below then the risk vector of every admissible rule is contained in LB(S). Theorem 2.6 shows that we can extend this result to the case where the risk set S, is not bounded below but is assumed to have nonempty lower boundary points.

Theorem 2.6. Assume that the set of lower boundary points, LB(S), of the risk set S is not empty and S is closed from below. Then a rule $\delta \in D$ is admissible if and only if the risk vector $x = (x_1,\ldots,x_k)^t$ of $\delta$ is contained in LB(S).

Proof. If $x = (x_1,\ldots,x_k)^t \in LB(S)$ then it follows trivially that the decision rule $\delta$ corresponding to $x$ is admissible. We shall prove the converse by the method of contradiction. Let $\delta$ be an admissible rule and assume that the risk vector $x$ of $\delta$ is not in LB(S). Let $S_1 = Q_x \cap \bar{S}$. By Lemma 2.5, the set $S_1$ is bounded from below. It is easy to verify that $S_1$ is convex and closed. Therefore by Lemma 2.4, $LB(S_1)$ is not empty.

If $y \in LB(S_1)$ then

$$(2.9) \qquad \{y\} = Q_y \cap S_1 = Q_y \cap Q_x \cap \bar{S} = Q_y \cap \bar{S}.$$

Thus $y \in LB(S)$ and since S is closed from below, $y \in S$. Since $y < x$, $\delta$ is not admissible, which is a contradiction. This completes the proof of the theorem.

The conclusion of Theorem 2.6 need not be true if we relax the hypothesis that LB(S) is nonempty, as shown in the next example.

Example 2.7. Let $k=2$ and $S=\{(x_1,x_2)^t : x_1>0,\ x_2\le 0\} \cup \{(0,0)\}$. In this example we can easily verify that S is convex and LB(S) is empty and hence S is closed from below. The risk vector (0,0) corresponds to an admissible rule but it is not in LB(S).

The next theorem is the second of our elementary theorems of decision theory. It provides sufficient conditions on the risk set S for the existence of a minimal complete class.

Theorem 2.8. Assume that the set of lower boundary points LB(S) of the risk set S is not empty and S is closed from below. Then the class of decision rules

$$(2.10) \qquad C = \{\delta \in D : x = (R(\theta_1,\delta),\ldots\ldots,R(\theta_k,\delta))^t \in LB(S)\}$$

is a minimal complete class.

Proof. It suffices to show that C is complete, because then we can conclude that C is minimal complete from Theorem 1.9 and Theorem 2.6. Let $\delta$ be a rule not in C, $x$ be the risk vector of $\delta$, and $S_1 = Q_x \cap \bar{S}$. The set $S_1$ is bounded from below by Lemma 2.5 and hence $LB(S_1)$ is nonempty. The rest of the proof is similar to the proof of Theorem 2.6.1 of Ferguson (1967) and hence is omitted.

Example 2.9. Consider the risk set $S_3 = \{(0,0)\} \cup \{(x,y) : -x-1 \le y \le 0, x > 0\}$. The convex set $S_3$ is not closed from below since $(0,-1)$ is a lower boundary point and it is not contained in $S_3$. It is easy to verify that the minimal complete class is given by the set of all decision rules whose risk vectors belong to the subset $S_4 = \{(0,0)\} \cup \{(x,y) : y = -x-1, x > 0\}$. Thus this example shows that the condition that the risk set is closed from below in not necessary for the existence of a minimal complete class.

One should note that Theorem 2.6.1 of Ferguson (1967) follows from Lemma 2.4 and Theorem 2.8. Combining Theorem 2.5 and Theorem 2.8, we have the following corollary.

Corollary 2.10. Let C be defined by (2.10). Under the hypothesis of Theorem 2.8 the class of all admissible rules C is minimal complete.

As an application of Theorems 2.6 and 2.8 consioer the risk set $S_2$ of Example 2.3. The class of all decision rules corresponding to the points $x \in LB(S_2)$ forms an admissible, minimal complete class of decision rules.

We now take a look at the question of existence of Bayes rules in the decision problem $(\Omega,D,R)$, where $\Omega = \{\theta_1,\ldots,\theta_k\}$. The next theorem provides sufficient conditions for the existence of Bayes rules.

Theorem 2.11. Let the risk set S be bounded from above. Assume that LB(S) is nonempty and S is closed from below. Then for every prior distribution $\pi = (\pi_1,...,\pi_k)$ a Bayes rule with respect to $\pi$ exists.

Proof. Under the hypothesis of the theorem, Lemma 2.2 shows that S is bounded. Theorem 2.11 now follows from the remark on page 69 of Ferguson (1967).

The next example shows that we cannot relax the condition that S is bounded from above in Theorem 2.11.

Example 2.12. Consider the risk set $S_2$ of Example 2.3. Let the prior distribution be (1/4, 3/4). It is easy to verify that the Bayes risk is $-\infty$ and therefore a Bayes rule does not exist with respect to the prior distribution (1/4, 3/4).

Our next concern is to find sufficient conditions which guarantee the existence of minimax strategies for both nature and the statistician. The fundamental theorem of game theory, namely the Minimax theorem, states that if the risk set is closed and bounded from below, then the decision problem viewed as a two-person game, has a value and both the players have minimax strategies. We show below that the Minimax theorem holds under the weaker conditions that the risk set has nonempty lower boundary points and is closed from below.

Theorem 2.13. Minimax Theorem. Consider the decision problem $(\Omega, D, R)$, where $\Omega = \{\theta_1, \ldots, \theta_k\}$. Assume that the lower boundary points, $LB(S)$, of the risk set $S$ constitute a nonempty set. Then

$$(2.11) \qquad \inf_{\delta \in D} \sup_{\pi \in P} BR(\pi, \delta) = \sup_{\pi \in P} \inf_{\delta \in D} BR(\pi, \delta) = \eta$$

and there exists a least favorable distribution $\pi_f$. Moreover, if $S$ is closed from below, then there exists an admissible minimax decision rule $\delta_0$ and $\delta_0$ is Bayes with respect to $\pi_f$.

Proof. Let $\eta = \sup\{\alpha : Q_\alpha \cap S = \Phi\}$, where $\alpha = (\alpha, \ldots, \alpha)^t$. Since $LB(S)$ is nonempty, $\eta$ is finite. For each $n$ we can find a rule $\delta_n \in D$ such that

$$(2.12) \qquad y_{jn} = R(\theta_j, \delta_n) \le \eta + 1/n \qquad \text{for all } j = 1, \ldots, k.$$

Let $y_n = (y_{1n}, \ldots, y_{kn})^t$. Clearly $y_n$ is bounded above. Since $LB(S)$ is nonempty, imitating the proof of Lemma 2.2 we can show that $y_n$ is bounded from below. Let $y$ be the limit point of $y_n$. Clearly $y \in \bar{S}$ and Lemma 2.5 shows that $Q_y \cap \bar{S}$ is bounded from below. The rest of the proof is similar to the proof of Theorem 2.9.1 of Ferguson (1967) and hence is omitted.

Example 2.14. Consider the risk set $S_2 = \{(x_1, x_2)^t : x_1 \ge -x_2\}$ of Example 2.3. Since the risk set $S_2$ is not bounded from below, Theorem 2.9.1 of Ferguson (1967) is not applicable. However Theorem 2.13 is applicable since $S_2$ actually has lower boundary points. In this example we can easily verify that the value $\eta = 0$ and the least favorable distribution $\pi_f = (1/2, 1/2)$.

There is an interesting relationship between admissible rules and Bayes rules when the parameter space $\Omega$ is finite. The theorem below does not assume any conditions on the risk set S. It is an easy consequence of the separating hyperplane theorem (see page 86, Ferguson(1967)).

<u>Theorem 2.15.</u> Let the parameter space $\Omega$ be finite. If $\delta$ is an admissible rule, then there exists a prior distribution $\pi$ such that $\delta$ is Bayes with respect to $\pi$.

We are now in a position to state and prove the fundamental theorem of decision theory under the weaker hypothesis that the risk set has nonemty lower boundary points.

<u>Theorem 2.16. Complete Class Theorem.</u> Consider the decision problem $(\Omega,D,R)$ where $\Omega$ is a finite set. Assume that the risk set S is closed from below and LB(S) is nonempty. Then the class of Bayes rules is complete and the admissible Bayes rules form a minimal complete class.

<u>Proof.</u> The theorem is immediate from Corollary 2.10 and Theorem 2.15.

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2. Berger, J.O.,(1985). Statistical Decision Theory and Bayesian Analysis. Second Edition, Springer-Verlag, New York.

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