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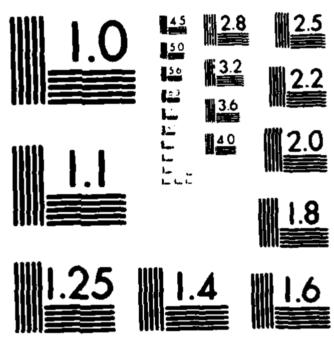
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SOME REMARKS ON THE ASYMPTOTIC BEHAVIOUR
OF THE LENGTHS OF A COLLISION RESOLUTION INTERVAL

by

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ABSTRACT

We present an operator method for obtaining upper and lower bounds for the expected length of a collision resolution interval for various protocols. The method is elementary in that it circumvents the intricate and ingenious complex variable methods of Fayolle, Flajolet and Hofri (1985). ~~It~~ is also noted that the method can be applied to computing bounds for the delay. A conjecture of Massey's and some of its implications, as well as some open questions of more than routine interest, are also discussed.

I. INTRODUCTION

Ever since the publication of the collision resolution algorithms (CRA) of Capetanakis-Tsybakov-Mikhailov (CTM) there has been a growing interest in the performance analysis of these channel access algorithms (also called protocols). The particular class of algorithms with which this paper is concerned has been clearly described in [FFHJ,1985] and thus it is not necessary to repeat that description here. We recall that the operation of the CTM-CRA leads to the following linear system of equations for $L(n) = E(\ell(n))$, where $\ell(n)$ = length of the collision resolution interval (CRI) initiated by a collision of multiplicity n.

$$(1.1) \qquad L(n) = 1 + 2 \sum_{j=0}^{\infty} L(j) p_{nj} \quad , \quad n \geq 2$$

$$L(0) = L(1) = 1$$

where

$$(1.2) \qquad p_{nj} = \sum_{i=0}^{j} \binom{n}{i} 2^{-n} \exp(-\lambda) \lambda^{j-i}/(j-i)!; \quad j \geq 0 .$$

For future reference we note that

$$(1.3) \qquad p_{n0} = 2^{-n} e^{-\lambda} \quad \text{and} \quad p_{n1} = 2^{-n} e^{-\lambda} (\lambda + n)$$

and that $p_{nj} = P(S_n + X = j)$ where S_n and X are independent random variables such that

$$P(S_n = j) = \binom{n}{j} 2^{-n} \quad , \quad 0 \leq j \leq n$$

and

$$P(X = k) = e^{-\lambda} \lambda^k / k! \quad , \quad k = 0,1,2,\ldots .$$

Equation (1.1) is the CTM-CRA with free access. The CTM-CRA with blocked access (see equation (3.12) on p. 85 of [Ma1981]) yields the following recurrence relation for the $L(n)$:

$$(1.4) \qquad \begin{cases} L(n) = 1 + 2 \sum_{i=0}^{n} L(i) \binom{n}{i} 2^{-n} , & n \geq 2 \\ L(0) = L(1) = 1 . \end{cases}$$

The sequence $L(n)$ defined by the recurrence relation (1.4) has many interesting properties of which the most unusual is the fact that

$$(1.5) \qquad \lim_{n\to\infty} L(n)/n$$

does not exist.

On the other hand, Massey conjectured (see 3.31 of [Ma1981]) the intuitively plausible inequality :

$$(1.6) \qquad L(i) + L(n-i) \leq L(n)+1 \quad , \quad 0 \leq i \leq n .$$

In part 3 of this paper we note that the validity of (1.6) implies $\lim_{n\to\infty} L(n)/n$ exists thus contradicting (1.5). Massey then used inequality (1.6) to derive an upper bound on $V(n)$ = Variance of $\ell(n)$ of the form

$$(1.7) \qquad V(n) \leq \beta n.$$

It is clear from the preceeding remarks that Massey's proof of (1.7) contains a gap, although it is not too difficult to obtain, via the methods of this paper, the bound $V(n) \leq \beta' n^2$. It is worth noting that Massey's proof of the lower bound

$$(1.8) \qquad V(n) \geq \alpha n$$

is correct. The validity of (1.7) remains an interesting open question.

The main goal of our paper lies in another direction, however, and that is to sketch (within the confines of a "correspondence") an operator method for solving linear systems of equations of the type (1.1) and (1.4) which is elementary in the sense that it circumvents the intricate and ingenious complex variable methods of [FFH1985].

We say sketch because there is a certain amount of unavoidable overlap with the previous work of Tsybakov-Vvedenskaya [TV1980] and Merakos [1983]. We note for the record, however, that as long ago as 1973 the author used similar ideas in studying the asymptotic behaviour of a sequence of expected first passage times (see [Ro1973]).

2. EXISTENCE AND UNIQUENESS OF SOLUTIONS TO EQUATION (1.1)

Let f denote the vector $(f(0), f(1), \ldots, f(n), \ldots)$ and define the positive linear operator G_n via the recipe

$$G_n f \triangleq 2 \sum_{j=0}^{\infty} f(j) p_{nj} = 2E(f(S_n+X)) . \tag{2.1}$$

In this notation the infinite system of equations (1.1) can be rewritten in the more convenient operator form

$$\begin{cases} L(n) = 1 + G_n L \quad , \quad n \geq 2 \\ L(0) = L(1) = 1 . \end{cases} \tag{2.2}$$

We can get rid of the constant 1 appearing on the right hand side of (2.2) via the change of variables $f(n) = 1 + L(n)$, $n \geq 0$ and it is easy to see that $f(n)$ satisfies the system

$$\begin{cases} f(n) = G_n f \\ f(0) = f(1) = 2 . \end{cases} \tag{2.3}$$

We begin by noting that the function $h(n) = M(n-2\lambda)$, $n = 0,1,2,\ldots$ satisfies the equation (2.3) but with slightly different initial conditions : $h(0) = -2\lambda M$, $h(1) = M(1-2\lambda)$. To see this just compute

$$G_n h = 2E(h(S_n+X)) = 2ME(S_n+X) - 4M\lambda$$

$$= 2M((n/2)+\lambda) - 4M\lambda = M(n-2\lambda) = h(n) \ .$$

Note : throughout this paper we assume $0 < \lambda < 1/2$. Notice that $h(n)$ is linear in n and it is reasonable to conjecture that $f(n)$ itself is nearly linear in the sense that $f(n) \le a.n + b$ for some finite constants a and b.

(2.4) <u>Theorem</u> : (i) For any $\lambda < \lambda_c = (-5+\sqrt{41})/4 = .35078$

there exists a unique non negative solution to the system of equations

$$(2.5)\quad \begin{cases} G_n g = g(n) \\ g(0) = a_0 \ , \ g(1) = a_1 \ , \ a_i \ge 0 \ , \ i=0,1 \\ \text{satisfying the growth condition} \\ g(n) \le an + b \end{cases}$$

(ii) If $a_0 \le a_1$ then $g(n)$ is monotone increasing.

<u>Proof</u> : As noted in [TV1980] the existence of a solution is an immediate consequence of the existence of a "barrier function" $X^{(0)}(n)$. More precisely, let

$$(2.6)\quad \begin{cases} \text{(i)}\ X^{(0)}(n) \triangleq h(n) \quad , \quad n \ge 2 \quad , \quad X^{(0)}(0) = a_0 \quad , \quad X^{(0)}(1) = a_1 \\ \text{(ii)}\ M(\lambda,a_0,a_1) = \sup\limits_{n\ge 2}[(a_1\lambda n + a_0\lambda + a_1\lambda^2)/(n-(2\lambda^2+\lambda(2n+1)))] \ . \end{cases}$$

Remark : It is easily checked that $b_n(\lambda) = 2\lambda^2+\lambda(2n+1)-n$ has two roots of opposite sign. Let $\lambda(n)$ = the positive root of $b_n(\lambda) = 0$ and note that $\lambda_c = \lambda(2) \le \lambda(n)$ for $n \ge 2$. Thus for $n \ge 2$, $b_n(\lambda) < 0$ on the range $[0,\lambda_c)$, consequently the denominator in (2.6ii) is strictly positive.

(2.7) Lemma : Choose $M = \mathrm{Max}(a_0,a_1,M(\lambda,a_0,a_1))$, $\lambda < \lambda_c$. Then
(i) $X^{(0)}$ is a barrier function i.e.

$$G_n X^{(0)} \le X^0(n) \;,\; n \ge 2 \;;\; G_n X^{(0)}(i) = a_i \;,\; i=0,1 \;. \tag{2.7}$$

(ii) If $0 \le g(n)$ is monotone increasing then so is $G_n g$.

Proof : (i) (Sketch) An elementary by tedious calculation yields the expression

$$(2.8)\quad \begin{cases} \text{(i)}\;\; G_n X^0 = X^{(0)}(n) + r(n,\lambda) \;,\; n \ge 2 \text{ where} \\ \text{(ii)}\;\; r(n,\lambda) = 2^{-n}e^{-\lambda}\{Mb(n,\lambda) + a_0\lambda + a_1\lambda(\lambda+n)\} \;; \end{cases}$$

consequently a necessary condition for $r(n,\lambda) \le 0$ for all $n \ge 2$ is that $\lambda \le \lambda_c$.

(ii) We must show that $G_{n+1}g - G_n g \ge 0$. Now $G_{n+1}g-G_n g = 2E\{g(S_{n+1}+X)-g(S_n+X)\}$, so it suffices to show that $E(g(S_{n+1}+X)-g(S_n+X)) \ge 0$. But

$$(S_{n+1}+X) \ge (S_n+X)$$

implies $g(S_{n+1}+X) \ge g(S_n+X)$, since g is monotone increasing. Consequently

$$(2.9) \qquad E(g(S_{n+1}+X) - g(S_n+X)) \geq 0.$$

The proof of Theorem (2.4) is now completed in the usual way by setting $f(n) = \lim_{k\to\infty} X^{(k)}(n)$ where $X^{(k)}(n) = G_n X^{(k-1)}$. We remark that the validity of the limit

$$(2.10) \qquad f(n) = \lim_{k\to\infty} X^{(k+1)}(n) = \lim_{k\to\infty} G_n X^{(k)} = G_n f$$

is a simple consequence of the Lebesgue derivated convergence theorem. Note also that a simple induction argument shows that $X^{(k)}(n)$ is monotone increasing for each k and consequently so is f. This is a much simpler proof than the one that appears on pp. 46-48 of [FFH1985]. It is worth noting, however, that [FFH1985] have also shown that for $\lambda > \lambda_{max} = .36017$ the solutions to (1.1) have no probabilistic meaning. Thus our method yields a maximum value for the arrival rate $\lambda_c = .35078$ which is within .01 of the true maximum!

The proof of uniqueness uses a barrier function of the form $\rho(n) = h(n)+n^2$, $n \geq 2$, $\rho(0) = \rho(1) = 0$ together with an argument familiar to aficionados of the "maximum principle" in potential theory. The readers are reperred to [Ro1984] for the details.

3. REMARKS ON THE ASYMPTOTIC BEHAVIOUR OF THE SOLUTION TO (1.4)

In the postscript to [Ma1981] reports that Vvendenskaya has shown (so far unpublished) that $\lim_{n\to\infty} L(n)/n$ does not exist although it can be shown that

$$(3.1) \qquad 2.881 \leq \liminf_{n\to\infty} L(n)/n \leq \limsup_{n\to\infty} L(n)/n \leq 2.8966 .$$

In the same paper Massey asserted, without proof, that

(3.2) $$L(i) + L(n-i) \le L(n)+1 \quad , \quad 0 \le i \le n .$$

In september of 1983 I pointed out to my colleagues Don Towsley and Jack Wolf of the Department of Electrical and Computer Engineering here at the University of Massachusetts that any inequality of the form :

(3.3) $$L(i) + L(n-i) \le L(n)+c \quad , \quad 0 \le i \le n$$

where c is an arbitrary constant, necessarily implies the existence of $\lim\limits_{n\to\infty} L(n)/n$.

In other words the nonexistence of the $\lim\limits_{n\to\infty} L(n)/n$ is inconsistent with the validity of (3.3). To see this we need to introduce the notion of a subadditive sequence.

(3.4) Definition : We say that a sequence a_n is subadditive if $a_n+a_m \ge a_{n+m}$ holds.

(3.5) Theorem : If a_n is a subadditive sequence then $\lim\limits_{n\to\infty} a_n/n$ exists.

Proof : See [P-SZ1972, p.23, problem 98] for a more complete discussion of subadditive sequences. In particular $\lim\limits_{n\to\infty} a_n/n = -\infty$ may occur although this is not the case here.

Application : Suppose (3.3) holds then the sequence $a_n = c-L(n)$ satisfies the condition $a_m+a_{n-m} \ge a_n$ which is equivalent to the condition $a_m+a_n \ge a_{m+n}$ i.e. a_n is subadditive and therefore $\lim\limits_{n\to\infty} a_n/n = -\lim\limits_{n\to\infty} L(n)/n$ exists.

Now inequality (3.2) was used by Massey to derive the upper bound (1.7). This proof must now be regarded as incomplete, although the proof for the lower bound is correct.

4. CONCLUDING REMARKS AND ACKNOWLEDMENTS

Let $c(n)$ = total sojourn time experienced by all users that became active during a CRI of multiplicity n. Then it is easy to see that

$$c(n) = 0 \ , \ n = 0$$

$$c(n) = 1 \ , \ n = 1$$

$$c(n) = n + c(S_n+X) + c(n-S_n+Y) + (n-S_n)\ell(S_n+X) \ , \ n \geq 2 \ .$$

Here X and Y are independent, identically distributed, Poisson random variables each of which is independent of S_n. This is the starting point of the delay analysis carried out in [FFHJ1985]. Letting $C(n) = E(c(n))$ leads to a system of equations similar to (1.1). Similar equations are studied in [GE-MER-PA1985] by means of the methods of part 2 and so we shall say no more about them here.

Finally it is a pleasure to acknowledge the careful reading of this paper by referees C,D,E and for drawing his attention to the extensive work of Merakos together with his colleagues Georgiadis, Papantoni-Kazakos on these problems.

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