MICROCOPY RESOLUTION TEST CHART

AD-A170 223

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION	1b. RESTRICTIVE MARKINGS
Unclassified	

2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT
	Unlimited Approved for public release;
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	distribution unlimited.

4. PERFORMING ORGANIZATION REPORT NUMBER(S)	5. MONITORING ORGANIZATION REPORT NUMBER(S)
	AFOSR-TR- 86-0355

6a. NAME OF PERFORMING ORGANIZATION	6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION
Department of Mathematics University of Arizona		

6c. ADDRESS (City, State and ZIP Code)	7b. ADDRESS (City, State and ZIP Code)
Tucson, AZ 85721	

8a. NAME OF FUNDING/SPONSORING ORGANIZATION	8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
U.S. Air Force		Grant AFOSR-84-0205

8c. ADDRESS (City, State and ZIP Code)	10. SOURCE OF FUNDING NOS			
	PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
AFOSR/NM Building 410 Bolling AFB, DC 20332-6448	61102F	2304	45	

11. TITLE (Include Security Classification)
On the first failure time of DMRS's

12. PERSONAL AUTHOR(S)
Moshe Shaked and J. George Shanthikumar

13a. TYPE OF REPORT	13b. TIME COVERED		14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT
Technical report	FROM ____ TO ____		1985, 4, 12	30

16. SUPPLEMENTARY NOTATION

17	COSATI CODES		18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB. GR	Dependent maintained reliability systems, coherent structures, repairable and nonrepairable components, performance process, NBU, IFR, stochastic ordering.

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

In this paper are considered multicomponent reliability systems where component failure and/or pair completion rates depend on the state, ages and current repair durations of the other components. This is a generalization of a model of Ross (198-). Sufficient conditions on the sets of rates which imply stochastic ordering between first failure times of two such systems are found. Sufficient conditions on the rates which imply that the first failure time of such a system is new better than used (NBU) are given. Some results of Barlow and Proschan (1976), Chiang and Niu (1980) and Ross (1976) are obtained as special cases. A counterexample to an apparently stronger result of Miller (1979) is also given. Further results and a discussion are included.

DTIC FILE COPY

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT	21. ABSTRACT SECURITY CLASSIFICATION
UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT ☐ DTIC USERS ☐	Unclassified JUL 28 1986

22a. NAME OF RESPONSIBLE INDIVIDUAL	22b. TELEPHONE NUMBER (Include Area Code)	22c. OFFICE SYMBOL
		E

DD FORM 1473, 83 APR   EDITION OF 1 JAN 73 IS OBSOLETE

AFOSR-TR- 86-0355

ON THE FIRST FAILURE TIME OF

DEPENDENT MULTICOMPONENT RELIABILITY

SYSTEMS

Disk 132

by

Moshe Shaked

Department of Mathematics

University of Arizona

and

J. George Shanthikumar

Management Science Group

School of Business Administration

University of California, Berkeley

April, 1985

X

A-1

Abstract

In this paper are considered multicomponent reliability systems where component failure and repair completion rates depend on the state, ages and current repair durations of the other components. This is a generalization of a model of Ross (1984). Sufficient conditions on the sets of rates which imply stochastic ordering between first failure times of two such systems are found. Sufficient conditions on the rates which imply that the first failure time of such a system is new better than used (NBU) are given. Some results of Barlow and Proschan (1976), Chiang and Niu (1980) and Ross (1976) are obtained as special cases. A counterexample to an apparently stronger result of Miller (1979) is also given. Further results and a discussion are included.

Key words and phrases: Dependent maintained reliability system, coherent structures, repairable and nonrepairable components, performance process, NBU, IFR, stochastic ordering, uniformization by a Poisson process, stochastic monotonicity, NBU processes.

AMS subject classification: 60K10.

1. Introduction.

The following model of <u>dependent maintained reliability system</u> (DMRS) with n components will be studied. At any time t, each component can be in one of two states: <u>up</u> (i.e., working) or <u>down</u> (i.e., failed and is in repair). The system state is also either up or down and it depends on the component states through a coherent structure function ϕ (see, e.g., Barlow and Proschan (1975)). Let $Z_i(t) = 0$ if component i is up at time t, let $Z_i(t) = 1$ if component i is down at time t and let $\underline{Z}(t) = (Z_1(t), \ldots, Z_n(t))$. Then the state of the system at time t is $\phi(\underline{Z}(t))$, that is, $\phi(\underline{Z}(t)) = 0$ if the system is up and $\phi(\underline{Z}(t)) = 1$ if the system is down.

We assume that the components are <u>repairable</u>, that is, as t varies, $Z_i(t)$ alternates between intervals in which it is up and in which it is down, $i = 1, 2, \ldots, n$. It is assumed that a repair [respectively, working] period starts immediately upon failure [respectively, repair completion] of a component.

We will study $\{\phi(\underline{Z}(t)), t \geq 0\}$ through the multivariate process $\underline{Z} = \{\underline{Z}(t), t \geq 0\}$ taking on values in $\{0, 1\}^n$. The process $\underline{Z}$ will be called the <u>performance process</u> of the components of the DMRS. We assume that, with probability one, $\underline{Z}$ has right continuous sample paths and that at any transition epoch of $\underline{Z}$, no more than one failure or one repair of a component can take place. That is, if for $\underline{\ell}, \underline{m} \in \{0, 1\}^n$, $\underline{\ell} - \underline{m} \equiv \sum_{i=1}^{n} (\ell_i - m_i)^2$, then $P\{ \underline{Z}(t) - \underline{Z}(t-) \leq 1 \text{ for all } t \} = 1$.

Note that if the n components are independently maintained and the components up and down periods are absolutely continuous then the process $\underline{Z}$ of such an independently maintained reliability system (IMRS) satisfies the above condition. In this paper, however, the system is allowed to be

dependent in the sense that the durations of the up and down periods of a component can depend on the states of the other components.

Let $W(\underline{Z},t) \subset S \equiv \{1,\ldots,n\}$ denote the set of the components which are working at time t, that is, $W(\underline{Z},t) \equiv \{i:Z_i(t) = 0\}$. Clearly, for each t, $W(\underline{Z},t)$ and $\underline{Z}(t)$ determine each other. For any set $w \subset S$, denote the complement of w by $\bar{w} \equiv S - w$. Thus $\bar{W}(\underline{Z},t)$ is the set of the components which are under repair at time t.

For $i \in W(\underline{Z},t)$ [respectively, $j \in \bar{W}(\underline{Z},t)$] let $A_i(\underline{Z},t)$ [respectively, $B_j(\underline{Z},t)$] be the age of the current up [respectively, down] period of component i [respectively, j] at time t. We will allow the instantaneous failure [respectively, repair completion] rate λ_i [respectively, μ_j] of component i [respectively, j] to depend on the set of working components and on the ages of the current up and down periods of the other components. Such a generalization is needed when the working components share an overall load (see, e.g., Schechner (1984)) or when the repair facility has limited capacity. Thus, if at some time t,
$W(\underline{Z},t) = w \subset S$, $A_i(\underline{Z},t) = a_i \geqslant 0$, $i \in w$ and $B_j(\underline{Z},t) = b_j \geqslant 0$, $j \in \bar{w}$, then for $k \in w$, we denote the instantaneous failure rate of component k by $\lambda_k(w, \underline{a}_w, \underline{b}_{\bar{w}})$, where $\underline{a}_w \equiv (a_{i_1},\ldots,a_{i_m})$ when $w = \{i_1,\ldots,j_m\}$ and
$\underline{b}_{\bar{w}} = (b_{j_1},\ldots,b_{j_{n-m}})$ when $\bar{w} = \{j_1,\ldots,j_{n-m}\}$. If w [respectively, $\bar{w}$] is empty then $\underline{a}_w$ [respectively, $\underline{b}_{\bar{w}}$] is vacuous. Formally, for $k \in w$,

(1.1)
$$\lambda_k(w, \underline{a}_w, \underline{b}_{\bar{w}}) \equiv \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} P\{Z_k(t+\Delta t)=1 \mid W(\underline{Z},t)=w, \underline{A}_w(\underline{Z},t)=\underline{a}_w, \underline{B}_{\bar{w}}(\underline{Z},t)=\underline{b}_{\bar{w}}\}.$$

Similarly, define the instantaneous repair completion rate of component $\ell \in \bar{w}$ by

$$(1.2) \qquad \mu_\ell(w, \underline{a}_{w}, \underline{b}_{\overline{w}})$$

$$\equiv \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} P\{Z_\ell(t+\Delta t)=0 \,|\, W(\underline{Z}, t)=w, A_{w}(\underline{Z}, t)=\underline{a}_{w}, B_{\overline{w}}(\underline{Z}, t)=\underline{b}_{\overline{w}}\}.$$

Note that $\lambda_k(w, \underline{a}_{w}, \underline{b}_{\overline{w}})$ and $\mu_\ell(w, \underline{a}_{w}, \underline{b}_{\overline{w}})$ depend on t only through the current set of working components and the ages of the up and down periods of the components.

Note that in general the failure or the repair completion rate of component k, say, do not remain the same during periods between transitions of $\underline{Z}$. This is because a_i, $i \in w$, and b_j, $j \in \overline{w}$, increase linearly (with slope 1) during such periods.

In general the process $\{\underline{Z}(t), \ t \geq 0\}$, or equivalently $\{W(\underline{Z}, t), \ t \geq 0\}$, is not Markovian. However

$$(W(\underline{Z}), A_{W(\underline{Z})}(\underline{Z}), B_{\overline{W}(\underline{Z})}(\underline{Z})) \equiv \{(W(\underline{Z}, t), A_{W(\underline{Z}, t)}(\underline{Z}, t), B_{\overline{W}(\underline{Z}, t)}(\underline{Z}, t)),$$

$t \geq 0\}$ is a Markov process. In the sequel we denote this process, when the initial state is $W(\underline{Z}, 0) = w$, $A_{w}(\underline{Z}, 0) = \underline{a}_{w}$, $B_{\overline{w}}(\underline{Z}, 0) = \underline{b}_{\overline{w}}$, by $[(W(\underline{Z}), A_{W(\underline{Z})}(\underline{Z}), B_{\overline{W}(\underline{Z})}(\underline{Z}) \,|\, w, \underline{a}_{w}, \underline{b}_{\overline{w}}]$. The process $\underline{Z}$ which develops from this initial state will be denoted by $[(\underline{Z}(t), \ t \geq 0) \,|\, w, \underline{a}_{w}, \underline{b}_{\overline{w}}]$.

When $\lambda_i(w, \underline{a}_{w}, \underline{b}_{\overline{w}})$ and $\mu_j(w, \underline{a}_{w}, \underline{b}_{\overline{w}})$ are independent of $(w, \underline{a}_{w}, \underline{b}_{\overline{w}})$, one obtains an IMRS with exponentially distributed up and down times.

Various aspects of the IMRS have been investigated by Barlow and Proschan (1975, 1976), Brown (1975, 1984), Chiang and Niu (1980), Keilson (1974, 1975), Miller (1979), Ross (1975, 1976) and Ross and Schechtman (1979). In particular Barlow and Proschan (1976), Theorem 2.7, and Chiang and Niu (1980), Theorem 3.5, provide an important characterization theorem for the distribution of the time to first system failure starting with all components new at time zero. One purpose of this paper is to extend this result to DMRS. Special cases of DMRS have been considered by Ross (1984) and Shanthikumar (1985).

In the reminder of this section we give some preliminaries on stochastic ordering and aging properties. In Section 2, we compare stochastically the performance processes of two different DMRS's. The results of Section 2 are used in Section 3 to obtain an aging property for some DMRS's. A counterexample for a related result (Theorem 3 of Miller, 1979) is given in Section 3.1. A discussion, further results and applications can be found in Section 4.

A random variable X is said to be stochastically smaller than [respectively, equal to] a random variable Y if $P\{X>t\} \leqslant$ [respectively, $=$] $P\{Y>t\}$ for all t. We denote this relation by $X \overset{st}{\leqslant} Y$ [respectively, $X \overset{st}{=} Y$]. A stochastic process $\underline{X} = \{(X_1(t),\ldots,X_n(t)), \ t \geqslant 0\}$ is said to be stochastically smaller than [respectively, equal to] a stochastic process $\underline{Y} = \{(Y_1(t),\ldots,Y_n(t)), \ t \geqslant 0\}$ if $Eg(\underline{X}) \leqslant Eg(\underline{Y})$ [respectively, $Eg(\underline{X}) = Eg(\underline{Y})$] for every nondecreasing functional g for which the expectations exist.

Let X be a random variable with distribution function $F(\ )$ and survival function $\overline{F}(\ ) \equiv 1 - F(\ )$. For each s, denote by $[X-s|X>s]$ the random variable whose survival function is $\overline{F}(s+\ )/\overline{F}(s)$. The nonnegative random variable X is said to be new better than used (NBU) [respectively, new worse than used (NWU)] if $X \overset{st}{\geqslant} [X-s|X>s]$ [respectively, $X \overset{st}{\leqslant} [X-s|X>s]$] for all $s \geqslant 0$. It is said to have (or to be) increasing failure rate (IFR) [respectively, decreasing failure rate (DFR)] if $[X-s|X>s] \overset{st}{\geqslant} [X-s'|X>s']$ [respectively, $[X-s|X>s] \overset{st}{\leqslant} [X-s'|X>s']$] whenever $s \leqslant s'$. The latter condition is also equivalent to $-\log \overline{F}(\ )$ being convex [respectively, concave].

2. Stochastic comparisons of two DMRS's.

Consider two DMRS's with performance processes

$\underline{X} = \{(X_1(t), \ldots, X_n(t)), \ t \geqslant 0\}$ and $\underline{Y} = \{(Y_1(t), \ldots, Y_n(t)), \ t \geqslant 0\}$. For $w \subset S \equiv \{1, \ldots, n\}$, $\underline{c}_w \geqslant \underline{0}$, $\underline{d}_{\overline{w}} \geqslant \underline{0}$, $i \in \overline{w}$, $j \in \overline{w}$, let $\alpha_i(w, \underline{c}_w, \underline{d}_{\overline{w}})$ and $\beta_j(w, \underline{c}_w, \underline{d}_{\overline{w}})$ be, respectively, the failure and repair completion rates associated with $\underline{X}$ defined analogously to (1.1) and (1.2). Similarly, let $\gamma_i(w, \underline{e}_w, \underline{f}_{\overline{w}})$ and $\delta_j(w, \underline{e}_w, \underline{f}_{\overline{w}})$ be the analogous rates associated with $\underline{Y}$.

The main result in this section shows that under some assumptions on the initial states of $\underline{X}$ and $\underline{Y}$ and on the rates α_i, β_j, γ_i, δ_j, it is possible to find processes $\underline{\hat{X}} = \{(\hat{X}_1(t), \ldots, \hat{X}_n(t)), \ t \geqslant 0\}$ and $\underline{\hat{Y}} = \{(\hat{Y}_1(t), \ldots, \hat{Y}_n(t)), \ t \geqslant 0\}$ defined on the same probability space, such that

$$(2.1) \qquad \underline{X} \stackrel{st}{=} \underline{\hat{X}} \leqslant \underline{\hat{Y}} \stackrel{st}{=} \underline{Y},$$

where $\underline{\hat{X}} \leqslant \underline{\hat{Y}}$ means $P\{(\hat{X}_1(t), \ldots, \hat{X}_n(t)) \leqslant (\hat{Y}_1(t), \ldots, \hat{Y}_n(t)), \ t \geqslant 0\} = 1$. It will follow then from (2.1) that $\underline{X} \stackrel{st}{\leqslant} \underline{Y}$.

In the statement of the next theorem and throughout the sequel, for $w = \{i_1, \ldots, i_m\}$, the notation $\underline{c}_w \leqslant \underline{e}_w$ means $(c_{i_1}, \ldots, c_{i_r}) \leqslant (e_{i_1}, \ldots, e_{i_m})$, i.e. $c_{i_\ell} \leqslant e_{i_\ell}$ for every $i_\ell \in w$. The following assumption is needed for the uniformization procedure which is used in the proof of Theorem 2.2:

Assumption 2.1. For $w \subset v \subset S$ and $\underline{c}_w$, $\underline{d}_{\overline{w}}$, $\underline{e}_v$, $\underline{f}_{\overline{v}}$ denote

$$\eta_i(w, v, \underline{c}_w, \underline{d}_{\overline{w}}, \underline{e}_v, \underline{f}_{\overline{v}}) = \alpha_i(w, \underline{c}_w, \underline{d}_{\overline{w}}) \quad \text{if} \quad i \in w,$$

$$= \delta_i(v, \underline{e}_v, \underline{f}_{\overline{v}}) \quad \text{if} \quad i \in \overline{v},$$

$$= \beta_i(w, \underline{c}_w, \underline{d}_{\overline{w}})$$

$$+ \gamma_i(v, \underline{e}_v, \underline{f}_{\overline{v}}) \quad \text{if} \quad i \in \overline{w} \cap v.$$

Assume that

$$(2.2) \qquad \tilde{\lambda} \equiv \sup\{ \sum_{i=1}^{n} \eta_i(w, v, \underline{c}_w, \underline{d}_{\overline{w}}, \underline{e}_v, \underline{f}_{\overline{v}}) : w \subset v \subset S, \underline{c}_w \leqslant \underline{e}_w, \underline{d}_{\overline{v}} \geqslant \underline{f}_{\overline{v}} \} < \infty$$

<u>Theorem 2.2.</u> Assume Assumption 2.1 and that for all choices of $w, v \subset S$ and

$\underline{c}_w$, $\underline{d}_{\overline{w}}$, $\underline{e}_v$, $\underline{f}_{\overline{v}}$ such that $w \subset v$, $\underline{c}_w \leqslant \underline{e}_w$, $\underline{d}_{\overline{v}} \geqslant \underline{f}_{\overline{v}}$, we have

$$(2.3) \qquad\qquad \alpha_i(w, \underline{c}_w, \underline{d}_{\overline{w}}) \geqslant \gamma_i(v, \underline{e}_v, \underline{f}_{\overline{v}}) , \qquad i \in w \subset v ,$$

$$(2.4) \qquad\qquad \beta_j(w, \underline{c}_w, \underline{d}_{\overline{w}}) \leqslant \delta_j(v. \underline{e}_v, \underline{f}_{\overline{v}}) , \qquad j \in \overline{v} \subset \overline{w} .$$

Then

$$\{\underline{X}(t), \ t \geqslant 0 \,|\, w^1, \ \underline{c}_w^1, \ \underline{d}_w^1\} \overset{st}{\geqslant} \{\underline{Y}(t), \ t \geqslant 0 \,|\, v^1, \ \underline{e}_v^1, \ \underline{f}_v^1\}$$

whenever

$$(2.5.i) \qquad\qquad w^1 \subset v^1 ,$$

$$(2.5.ii) \qquad\qquad \underline{c}_w^1 \leqslant \underline{e}_w^1 ,$$

$$(2.5.iii) \qquad\qquad \underline{d}_v^1 \geqslant \underline{f}_v^1 .$$

<u>Proof.</u> Let $\lambda > \tilde{\lambda}$, $\lambda < \infty$. such a λ exists by (2.2). Consider a Poisson process $N = \{N(t), \ t \geqslant 0\}$ with intensity λ . Using N, we will construct two processes $\underline{\hat{X}} = \{\underline{\hat{X}}(t), \ t \geqslant 0\}$ and $\underline{\hat{Y}} = \{\underline{\hat{Y}}(t), \ t \geqslant 0\}$ which satisfy (2.1) and the desired result then follows.

Define $\hat{\underline{X}}(0) \equiv (x_1(0), \ldots, x_n(0))$ where $x_i(0) = 0$ if $i \in w^1$ and $x_i(0) = 1$ if $i \in \bar{w}^1$. Similarly define $\hat{\underline{Y}}(0) \equiv (y_1(0), \ldots, y_n(0))$ where $y_i(0) = 0$ if $i \in v^1$ and $y_i(0) = 1$ if $i \in \bar{v}^1$.

Let t_1 be the time of the first epoch of N. For $t \in [0, t_1)$ let $\hat{\underline{X}}(t) = \underline{x}(0)$ and $\hat{\underline{Y}}(t) = \underline{y}(0)$. Thus, by (2.5.i),

$$(2.6) \qquad\qquad \hat{\underline{X}}(t) \geqslant \hat{\underline{Y}}(t) , \quad t \in [0, t_1).$$

Also, for $t \in [0, t_1)$ let $W(\hat{\underline{X}}, t) = w^1$, $\underline{A}_{W(\hat{\underline{X}}, t)}(\hat{\underline{X}}, t) = \underline{c}^1_w \underline{1} + t \underline{1}$, $\underline{B}_{\overline{W}(\hat{\underline{X}}, t)}(\hat{\underline{X}}, t) = \underline{d}^1_w \underline{1} + t \underline{1}$, $W(\hat{\underline{Y}}, t) = v^1$, $\underline{A}_{W(\hat{\underline{Y}}, t)}(\hat{\underline{Y}}, t) = \underline{e}^1_v \underline{1} + t \underline{1}$ and $\underline{B}_{\overline{W}(\hat{\underline{Y}}, t)}(\hat{\underline{Y}}, t) = \underline{f}^1_v \underline{1} + t \underline{1}$, where $\underline{1}$ is a vector of ones. The dimension of $\underline{1}$ may vary from one expression to another, but it is always possible to determine it from the formula in which $\underline{1}$ appears.

Since $\underline{c}^1_w \underline{1} \leqslant \underline{e}^1_w \underline{1}$ and $\underline{d}^1_v \underline{1} \geqslant \underline{f}^1_v \underline{1}$ it follows that $\underline{A}_{w^1}(\hat{\underline{X}}, t) \leqslant \underline{A}_{w^1}(\hat{\underline{Y}}, t)$ and $\underline{B}_{\overline{v}^1}(\hat{\underline{X}}, t) \geqslant \underline{B}_{\overline{v}^1}(\hat{\underline{Y}}, t)$ for all $t \in [0, t_1)$.

For $k > 0$ let t_k be the time of the k-th epoch of N. Assume, by induction, that $\hat{\underline{X}}(t)$ and $\hat{\underline{Y}}(t)$ have already been determined over the interval $[0, t_k)$ and that, for $t \in [0, t_k)$, $\hat{\underline{X}}(t) \geqslant \hat{\underline{Y}}(t)$ [so that $W(\hat{\underline{X}}, t) \subset W(\hat{\underline{Y}}, t)$] and that $\underline{A}_{W(\hat{\underline{X}}, t)}(\hat{\underline{X}}, t) \leqslant \underline{A}_{W(\hat{\underline{X}}, t)}(\hat{\underline{Y}}, t)$ and $\underline{B}_{\overline{W}(\hat{\underline{Y}}, t)}(\hat{\underline{X}}, t) \geqslant \underline{B}_{\overline{W}(\hat{\underline{Y}}, t)}(\hat{\underline{Y}}, t)$. Denote the particular realization at time t_k by $W(\hat{\underline{X}}, t_k-) = w^k$, $\underline{A}_{w^k}(\hat{\underline{X}}, t_k-) = \underline{c}^k_{w^k}$, $\underline{B}_{w^k}(\hat{\underline{X}}, t_k-) = \underline{d}^k_{w^k}$, $W(\hat{\underline{Y}}, t_k) = v^k$, $\underline{A}_{v^k}(\hat{\underline{Y}}, t_k-) = \underline{e}^k_{v^k}$ and $\underline{B}_{\overline{v}^k}(\hat{\underline{Y}}, t_k-) = \underline{f}^k_{v^k}$, so that by the induction hypothesis $\underline{c}^k_{w^k} \leqslant \underline{e}^k_{w^k}$ and $\underline{d}^k_{v^k} \geqslant \underline{f}^k_{v^k}$.

Define $\hat{\underline{X}}(t_k)$ and $\hat{\underline{Y}}(t_k)$ as follows: At time t_k at most one component of $\hat{\underline{X}}$ and at most one component of $\hat{\underline{Y}}$ change as follows:

(a.1). With probability $1 - \lambda^{-1} \sum_{i=1}^{n} r_i(w^k, v^k, \underline{c}^k_{w^k}, \underline{d}^k_{w^k}, \underline{e}^k_{v^k}, \underline{f}^k_{v^k})$ the processes $\hat{\underline{X}}$ and $\hat{\underline{Y}}$ do not jump at time t_k. In this case

$$\hat{\underline{X}}(t_k) = \hat{\underline{X}}(t_k-), \quad W(\hat{\underline{X}}, t_k) = w^k, \quad \underline{A}_{W(\hat{\underline{X}},t_k)}(\hat{\underline{X}}, t_k) = \underline{c}^k_{\overline{w}},$$

$$\underline{B}_{\overline{W}(\hat{\underline{X}},t_k)}(\hat{\underline{X}}, t_k) = \underline{d}^k_{\overline{w}}, \quad \hat{\underline{Y}}(t_k) = \hat{\underline{Y}}(t_k-), \quad W(\hat{\underline{Y}}, t_k) = v^k,$$

$$\underline{A}_{W(\hat{\underline{Y}},t_k)}(\hat{\underline{Y}}, t_k) = \underline{e}^k_{\overline{v}} \quad \text{and} \quad \underline{B}_{\overline{W}(\hat{\underline{Y}},t_k)}(\hat{\underline{Y}}, t_k) = \underline{f}^k_{\overline{v}}.$$

(a.2). With probability $\lambda^{-1} n_r(w^k, v^k, \underline{c}^k_{\overline{w}}, \underline{d}^k_{\overline{w}}, \underline{e}^k_{\overline{v}}, \underline{f}^k_{\overline{v}})$ choose an index

$r \in \{1, \ldots, n\}$, and for $i \neq r$ set $\hat{X}_i(t_k) = \hat{X}_i(t_k-)$,

$A_i(\hat{\underline{X}}, t_k) = c^k_i$ when $i \in w^k$, $B_i(\hat{\underline{X}}, t_k) = d^k_i$ when $i \in \overline{w}^k$,

$\hat{Y}_i(t_k) = \hat{Y}_i(t_k-)$, $A_i(\hat{\underline{Y}}, t_k) = e^k_i$ when $i \in v^k$, and

$B_i(\hat{\underline{Y}}, t_k) = f^k_i$ when $i \in \overline{v}^k$.

(b.1) If $r \in w^k$, then set $\hat{X}_r(t_k) = 1$, $W(\hat{\underline{X}}, t_k) = w^k - \{r\}$,

$B_r(\hat{\underline{X}}, t_k) = 0$. Also in this case, with probability

$\gamma_r(v^k, \underline{e}^k_{\overline{v}}, \underline{f}^k_{\overline{v}})/\alpha_r(w^k, \underline{c}^k_{\overline{w}}, \underline{d}^k_{\overline{w}})$ [which is ≤ 1 by (2.3)] set

$\hat{\underline{Y}}_r(t_k) = 1$, $W(\hat{\underline{Y}}, t_k) = v^k - \{r\}$, $B_r(\hat{\underline{Y}}, t_k) = 0$, and with

probability $1 - \gamma_r(v^k, \underline{e}^k_{\overline{v}}, \underline{f}^k_{\overline{v}})/\alpha_r(w^k, \underline{c}^k_{\overline{w}}, \underline{d}^k_{\overline{w}})$ set $\hat{Y}_r(t_k) = 0$,

$W(\hat{\underline{Y}}, t_k) = v^k$ and $A_r(\hat{\underline{Y}}, t_k) = e^k_r$.

Intuitively, under (b.1) one sees that component r has failed at time
t_k in the realization of $\hat{\underline{X}}$. As for $\hat{\underline{Y}}$, component r either fails or does
not fail. In either case it is seen that

(2.7) $$\hat{\underline{X}}(t_k) > \hat{\underline{Y}}(t_k),$$

(2.8) $$\underline{A}_{W(\hat{\underline{X}},t_k)}(\hat{\underline{X}}, t_k) > \underline{A}_{W(\hat{\underline{X}},t_k)}(\hat{\underline{Y}}, t_k),$$

(2.9) $$\underline{B}_{\overline{W}(\hat{\underline{Y}},t_k)}(\hat{\underline{X}}, t_k) > \underline{B}_{\overline{W}(\hat{\underline{Y}},t_k)}(\hat{\underline{Y}}, t_k).$$

(b.2). If $r \in \overline{v}^k$ then set $\hat{Y}_r(t_k) = 0$, $\overline{W}(\hat{\underline{Y}}, t_k) = \overline{v}^k - \{r\}$,

$A_r(\hat{\underline{Y}}, t_k) = 0$. Also in this case, with probability

$\beta_r(w^k, \underline{c}^k_{\overline{w}}, \underline{d}^k_{\overline{w}})/\delta_r(v^k, \underline{e}^k_{\overline{v}}, \underline{f}^k_{\overline{v}})$ [which is ≤ 1 by (2.4)] set

$\hat{X}_r(t_k) = 0$, $\overline{W}(\hat{\underline{X}}, t_k) = \overline{w}^k - \{r\}$, $A_r(\hat{\underline{X}}, t_k) = 0$, and with

probability $1 - \beta_r(w^k, \underline{c}^k_{\overline{w}}, \underline{d}^k_{\overline{w}})/\delta_r(v^k, \underline{e}^k_{\overline{v}}, \underline{f}^k_{\overline{v}})$ set $\hat{X}_r(t_k) = 1$,

$$\overline{W}(\hat{\underline{X}}, t_k) = \overline{w}^k \quad \text{and} \quad B_r(\hat{\underline{X}}, t_k) = d_r^k.$$

Intuitively, under (b.2) one sees that component r has completed repair at time t_k in the realization of $\hat{\underline{Y}}$. As for $\hat{\underline{X}}$, component r either completed repair or not. In either case it is seen that (2.7) - (2.9) hold.

(b.3). If $r \in \overline{w}^k \cap v^k$ then change the state of component r in either $\hat{\underline{X}}$ or $\hat{\underline{Y}}$ as follows:

(b.3.1). With probability $\beta_r(w^k, \underline{c}_{w}^{k}, \underline{d}_{\overline{w}}^{k})/n_r(w^k, v^k, \underline{c}_{w}^{k}, \underline{d}_{\overline{w}}^{k}, \underline{e}_{v}^{k}, \underline{f}_{v}^{k})$ set

$$\hat{X}_r(t_k) = 0, \quad \overline{W}(\hat{\underline{X}}, t_k) = \overline{w}^k - \{r\}, \quad A_r(\hat{\underline{X}}, t_k) = 0,$$
$$\hat{Y}_r(t_k) = \hat{Y}_r(t_k^-) = 0, \quad W(\hat{\underline{Y}}, t_k) = v^k \quad \text{and} \quad A_r(\hat{\underline{Y}}, t_k) = e_r^k.$$

(b.3.2). With probability $\gamma_r(v^k, \underline{e}_{v}^{k}, \underline{f}_{v}^{k})/n_r(w^k, v^k, \underline{c}_{w}^{k}, \underline{d}_{\overline{w}}^{k}, \underline{e}_{v}^{k}, \underline{f}_{v}^{k})$ set

$$\hat{Y}_r(t_k) = 1, \quad W(\hat{\underline{Y}}, t_k) = v^k - \{r\}, \quad B_r(\hat{\underline{Y}}, t_k) = 0,$$
$$\hat{X}_r(t_k) = \hat{X}_r(t_k^-) = 1, \quad W(\hat{\underline{X}}, t_k) = w^k \quad \text{and} \quad B_r(\hat{\underline{X}}, t_k) = d_r^k.$$

Intuitively it is seen that under (b.3.1), component r completes repair at time t_k in the realization of $\hat{\underline{X}}$. However, in the realization of $\hat{\underline{Y}}$ component r is working anyway at time t_k. Under (b.3.2) component r fails at time t_k in the realization of $\hat{\underline{Y}}$. However, in the realization of $\hat{\underline{X}}$ component r is under repair anyway at time t_k. In either case it is seen that (2.7) - (2.9) hold.

Let t_{k+1} be the time of the $(k+1)$-st epoch of . Now define $\hat{\underline{X}}$ and $\hat{\underline{Y}}$ on $[t_k, t_{k+1})$ as follows: For $t \in [t_k, t_{k+1})$ let $\hat{\underline{X}}(t) = \hat{\underline{X}}(t_k)$ and $\hat{\underline{Y}}(t) = \hat{\underline{Y}}(t_k)$. Thus

$$(2.10) \qquad\qquad \hat{\underline{X}}(t) \geq \hat{\underline{Y}}(t), \quad t \in [t_k, t_{k+1}).$$

Also for $t \in [t_k, t_{k+1})$ let $W(\hat{\underline{X}}, t) = W(\hat{\underline{X}}, t_k) = w^{k+1}$, $W(\hat{\underline{Y}}, t) = W(\hat{\underline{Y}}, t_k) = v^{k+1}$, $A_{W(\hat{\underline{X}}, t)}(\hat{\underline{X}}, t) = A_{W(\hat{\underline{X}}, t_k)}(\hat{\underline{X}}, t_k) + (t - t_k)\underline{1}$, $B_{W(\hat{\underline{X}}, t)}(\hat{\underline{X}}, t) = B_{W(\hat{\underline{X}}, t_k)}(\hat{\underline{X}}, t_k) + (t - t_k)\underline{1}$,

$$A_W(\hat{\underline{Y}}, t)^{(\hat{\underline{Y}}, t)} = A_W(\hat{\underline{Y}}, t_k)^{(\hat{\underline{Y}}, t_k)} + (t-t_k)\underline{1}, \quad B_{\overline{W}(\hat{\underline{Y}}, t)}^{(\hat{\underline{Y}}, t)} =$$

$$B_{\overline{W}(\hat{\underline{Y}}, t_k)}^{(\hat{\underline{Y}}, t_k)} + (t-t_k)\underline{1}. \text{ Thus, for } t \in [t_k, t_{k+1}),$$

$$A_W(\hat{\underline{X}}, t)^{(\hat{\underline{X}}, t)} \leqslant A_W(\hat{\underline{X}}, t)^{(\hat{\underline{Y}}, t)} \text{ and } B_{\overline{W}(\hat{\underline{Y}}, t)}^{(\hat{\underline{X}}, t)} \geqslant B_{\overline{W}(\hat{\underline{Y}}, t)}^{(\hat{\underline{Y}}, t)}. \text{ In}$$

particular, denoting the realizations at time t_{k+1}^- by

$$\underline{c}_{w}^{k+1}{}_{k+1} \equiv A_W(\hat{\underline{X}}, t_{k+1}-)^{(\hat{\underline{X}}, t_{k+1}-)},$$

$$\underline{d}_{w}^{k+1}{}_{k+1} \equiv B_{\overline{W}(\hat{\underline{X}}, t_{k+1}-)}^{(\hat{\underline{X}}, t_{k+1}-)}, \quad \underline{e}_{v}^{k+1}{}_{k+1} \equiv A_W(\hat{\underline{Y}}, t_{k+1}-)^{(\hat{\underline{Y}}, t_{k+1}-)},$$

$$\underline{f}_{w}^{k+1}{}_{k+1} \equiv B_{\overline{W}(\hat{\underline{Y}}, t_{k+1}-)}^{(\hat{\underline{Y}}, t_{k+1}-)}, \text{ it is seen that } \underline{c}_{w}^{k+1}{}_{k+1} \leqslant \underline{e}_{w}^{k+1}{}_{k+1} \text{ and}$$

$$\underline{d}_{v}^{k+1}{}_{k+1} \geqslant \underline{f}_{v}^{k+1}{}_{k+1}.$$

Thus, by induction, the procedure described above defines

$\hat{\underline{X}}(t)$ and $\hat{\underline{Y}}(t)$ for all $t \geqslant 0$.

From (2.6) and (2.10) it is seen that $\hat{\underline{X}}(t) \geqslant \hat{\underline{Y}}(t)$, $t \geqslant 0$, with

probability one. Using well known results on thinning of Poisson processes it

is not hard to verify that $\hat{\underline{X}} \stackrel{st}{=} \underline{X}$ and $\hat{\underline{Y}} \stackrel{st}{=} \underline{Y}$. The desired result then

follows from (2.1).

Remark 2.3. In some applications of Theorem 2.2 (see, for example, Section 3)

the α_i's, β_j's, γ_i's and δ_j's (see (2.3) and (2.4)) do not depend on the

ages of the working components, $\underline{c}_w$ and $\underline{e}_v$. The result of Theorem 2.2 then

is still true even if (2.5.ii) does not hold. The proof of this statement is

the same as the proof of Theorem 2.2. Similarly, if the α_i's, β_j's, γ_i's

and δ_j's do not depend on $\underline{d}_w$ and $\underline{f}_v$, then the conclusion of Theorem 2.2

is still true even (2.5.iii) does not hold.

Let $\phi : \{0, 1\}^n \to \{0, 1\}$ be any coherent structure function of n components

and let $\underline{X} = \{X_1(t), \ldots, X_n(t)), t \geqslant 0\}$ and $\underline{Y} = \{(Y_1(t), \ldots, Y_n(t)), t \geqslant 0\}$

be the performance processes of the two DMRS's. The times of the first

failure for each of the two DMRS's then are $T_{\underline{X}} \equiv \inf\{t : \phi(\underline{X}(t)) = 1\}$ and

$$T_{\underline{Y}} \equiv \inf\{t:\phi(\underline{Y}(t)) = 1\}, \quad \text{respectively.}$$

Corollary 2.4. If the two DMRS's satisfy Assumption 2.1 and (2.3) and (2.4) and the initial states and ages of the up and down periods of the two DMRS's satisfy (2.5.i) ~ (2.5.iii), then

$$(2.11) \qquad\qquad T_{\underline{X}} \overset{st}{\leqslant} T_{\underline{Y}} .$$

Proof. Let $U = \{\underline{z}\epsilon\{0,1\}^n:\phi(\underline{z}) = 1\}$ and let $\hat{\underline{X}}$ and $\hat{\underline{Y}}$ be as in the proof of Theorem 2.2. Then $T_{\underline{X}} \overset{st}{=} T_{\hat{\underline{X}}} \equiv \inf\{t:\hat{\underline{X}}(t)\epsilon U\}$, $T_{\underline{Y}} \overset{st}{=} T_{\hat{\underline{Y}}} \equiv \inf\{t:\hat{\underline{Y}}(t)\epsilon U\}$. Let $\underline{x}$ and $\underline{y}$ be members of $\{0,1\}^n$. Note that if $\underline{x} \notin U$ and $\underline{y} \leqslant \underline{x}$ (i.e., $x_i \leqslant y_i$, $i=1,\ldots,n$) then $\underline{y} \notin U$. By Theorem 2.2, $\hat{\underline{X}}(t) \geqslant \hat{\underline{Y}}(t)$ a.s. Hence $T_{\hat{\underline{X}}} \leqslant T_{\hat{\underline{Y}}}$ a.s. and (2.11) follows.

In a similar manner, using Remark 2.3, one can obtain:

Corollary 2.5. Assume that the failure and repair completion rates do not depend on the ages of the working components. If the two DMRS's satisfy Assumption 2.1 and (2.3) and (2.4) and the initial states and ages of the up and down periods of the two DMRS's satisfy (2.5.i) and (2.5.iii), then

$$T_{\underline{X}} \overset{st}{\leqslant} T_{\underline{Y}}.$$

Remark 2.6. The assumption used in Theorem 2.2 and Corollaries 2.4 and 2.5 that the failure and repair completion rates depend on the up and down ages c_i, $i \epsilon w$, d_j, $j \epsilon \overline{w}$, can clearly be modified to assuming that these rates depend on some increasing functions $g_i(c_i)$, $i \epsilon w$, $h_j(d_j)$, $j \epsilon \overline{w}$ of the ages. This modification shows that Theorem 2.2 and Corollaries 2.4 and 2.5 are useful in applications such as follows. Suppose that an "item" is a

rubber container which is being filled with some liquid at a constant rate. As the amount of liquid increases the container expands and becomes weaker and eventually "fails" by cracking. It is often the case, that the failure rate of the "item" (and of other "items") depend on the amount of liquid not linearly but through some nondecreasing function g of the amount of liquid. Theorem 2.2 and Corollaries 2.4 and 2.5 still apply in such cases.

3. NBU properties of some DMRS's.

In this section we consider only one DMRS at a time. We provide sufficient conditions on the component dynamics so that the system lifetime is NBU without any specific assumptions on the system structure ϕ. For this single DMRS, the notation of Section 1 will be used. Throughout this section we assume:

Assumption 3.1. The failure and the repair completion rates do not depend on the ages of the working components.

We still allow the rates to depend on the ages of the repair times.

Thus, in this section, (1.1) and (1.2) reduce to

$$(3.1) \qquad \lambda_k(w, \underline{b_w}) \equiv \lim_{\Delta t \to 0} \frac{1}{\Delta t} P\{Z_k(t+\Delta t) = 1 | W(\underline{Z}, t) = w, \underline{A}_w(\underline{Z}, t) = \underline{a}_w,$$
$$\underline{B}_w(\underline{Z}, t) = \underline{b}_w\}, \; k \in w,$$

$$(3.2) \qquad \mu_k(w, \underline{b_w}) \equiv \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} P\{Z_k(t+\Delta t) = 0 | W(\underline{Z}, t) = w, \underline{A}_w(\underline{Z}, t) = \underline{a}_w,$$
$$\underline{B}_w(\underline{Z}, t) = \underline{b}_w\}, \; k \in \overline{w}.$$

Since in this section the λ_k's and μ_k's do not depend on $\underline{a}_w$, it follows that, given $W(\underline{Z}, t) = w$, $\underline{A}_w(\underline{Z}, t) = \underline{a}_w$ and $\underline{B}_w(\underline{Z}, t) = \underline{b}_w$, the stochastic behaviour of $\{\underline{Z}(t), \; t \geq 0 | w, \underline{a}_w, \underline{b}_w\}$ does not depend on $\underline{a}_w$.

Thus, in this section, such a process will be denoted by $\{\underline{Z}(t),\ t \geqslant 0 | w,\ \underline{b}_{\overline{w}}\}$ and when $w = S \equiv \{1, \ldots, n\}$, we simply denote the process then by $[\underline{Z}|S]$ or by $\{\underline{Z}(t),\ t \geqslant 0 | S\}$.

We will also need a counterpart of Assumption 2.1.

<u>Assumption 3.2.</u> For $w \subset v \subset S$ and $\underline{d}_{\overline{w}}$, $\underline{f}_{\overline{w}}$, denote

$$\xi_i(w, v, \underline{d}_{\overline{w}}, \underline{f}_{\overline{v}}) = \lambda_i(w, \underline{d}_{\overline{w}}) \qquad \text{if } i \ \epsilon \ w,$$
$$= \mu_i(v, \underline{f}_{\overline{v}}) \qquad \text{if } i \ \epsilon \ \overline{v},$$
$$= \lambda_i(w, \underline{d}_{\overline{w}}) + \mu_i(v, \underline{f}_{\overline{v}}) \quad \text{if } i \ \epsilon \ \overline{w} \cap v.$$

Assume that

$$\tilde{\lambda} \equiv \sup\{\sum_{i=1}^{n} \xi_i(w, v, \underline{d}_{\overline{w}}, \underline{f}_{\overline{v}}) : w \subset v \subset S,\ \underline{d}_{\overline{v}} \geqslant \underline{f}_{\overline{v}}\} < \infty\ .$$

<u>Theorem 3.3.</u> Assume Assumptions 3.1 and 3.2 and that for all $w \subset v \subset S$ and $\underline{d}_{\overline{w}}$ and $\underline{f}_{\overline{v}}$ such that $\underline{d}_{\overline{v}} \geqslant \underline{f}_{\overline{v}}$,

$$(3.3) \qquad \lambda_i(w, \underline{d}_{\overline{w}}) \geqslant \lambda_i(v, \underline{f}_{\overline{v}})\ ,\quad i\epsilon\ w \subset v,$$

$$(3.4) \qquad \mu_i(w, \underline{d}_{\overline{w}}) \leqslant \mu_i(v, \underline{f}_{\overline{v}})\quad i\epsilon\ \overline{v} \subset \overline{w}\ .$$

Then, for all choices of w^1 and $\underline{b}_{\overline{w}^1}^1$,

$$(3.5) \qquad \{\underline{Z}(t),\ t \geqslant 0 | w^1,\ \underline{b}_{\overline{w}^1}^1\} \overset{st}{\geqslant} \{\underline{Z}(t),\ t \geqslant 0 | S\}.$$

<u>Proof.</u> We will use Theorem 2.2. Denote by $\{\underline{Y}(t),\ t \geqslant 0\}$ the right hand side of (3.5) and by $\{\underline{X}(t),\ t \geqslant 0\}$ the left hand side of (3.5). Then the

α_i's, β_j's, γ_i's and δ_j's in (2.3) and (2.4) are identified as

$$\alpha_i(w, \underline{c}_w, \underline{d}_{\overline{w}}) = \lambda_i(w, \underline{d}_{\overline{w}}), \quad i \in w,$$

$$\beta_j(w, \underline{c}_w, \underline{d}_{\overline{w}}) = \mu_j(w, \underline{d}_{\overline{w}}), \quad j \in \overline{w},$$

$$\gamma_i(v, \underline{e}_v, \underline{f}_{\overline{v}}) = \lambda_i(v, \underline{f}_{\overline{v}}), \quad i \in v,$$

$$\delta_j(v, \underline{e}_v, \underline{f}_{\overline{v}}) = \mu_j(v, \underline{f}_{\overline{v}}), \quad j \in \overline{v}.$$

Thus, from (3.3) and (3.4) it follows that (2.3) and (2.4) hold.

Assumption 3.2 ensures that (2.2) and hence Assumption 2.1 hold. Finally, since $\underline{Y}(0) = \underline{0}$ a.s., it follows that (2.5.i) and (2.5.iii) hold. By Remark 2.3, then, $\underline{X} \overset{st}{\geq} \underline{Y}$.

Consider a DMRS which starts to function with all components being up. That is $\underline{Z}(0) = \underline{0}$. Let $T = \inf\{t: \phi(\underline{Z}(t)) = 1\}$ be the time until first system failure.

Theorem 3.4. For an n-component DMRS assume that the failure and repair completion rates satisfy the conditions of Theorem 3.3. If the system starts to function with all components being up then T is NBU.

Proof. Fix a $t > 0$ and consider some particular realization $\{\underline{z}(s): 0 \leq s \leq t\}$ of $\{\underline{Z}(s): 0 \leq s \leq t\}$ such that $\phi(\underline{z}(s)) = 0$, $s \leq t$, that is, given $H_t \equiv \{\underline{Z}(s) = \underline{z}(s), s \leq t\}$ we have $P\{T > t | H_t\} = 1$. It will be argued below that for every such history H_t,

(3.6) $$[(T-t)|H_t] \overset{st}{\leq} T.$$

It follows then, by unconditioning in (3.6) but retaining the condition $T >$

t, that $[(T-t)|T>t] \overset{st}{\leqslant} T$, that is, T is NBU.

The stochastic ordering (3.6) follows from Theorem 3.3 in the same manner that Corollary 2.4 (and Corollary 2.5) follow from Theorem 2.2.

Remark. As noted in Remark 2.6, the failure and repair completion rates need not depend directly on the down ages for the conclusion of Theorem 3.4 to be true. This conclusion is also true when the rates depend on the ages through some increasing functions.

Motivated by Barlow and Proschan (1976) and by Chiang and Niu (1980) we state a slight generalization of Theorem 3.4. We assume now that the DMRS has n (dependent) repairable components, as described in Section 1, and also m nonrepairable components with lifetimes which are independent of each other and of the states and the current ages and repair durations of the repairable components.

Theorem 3.5. For a DMRS with n + m components as described above assume:

(a) the nonrepairable components have NBU lifetimes,

(b) the failure and repair completion rates of the repairable components
     satisfy the conditions of Theorem 3.3,

(c) all components are new at time 0.

Then the time until first system failure is NBU.

The proof of Theorem 3.5 is similar to the proof of Theorems 3.3 and 3.4. The additional ingredient is that H_t now also contains the information whether and which of the nonrepairable component are alive or dead at time t. The nonrepairable components which are alive (according to the history H_t) at time t have stochastically smaller residual lives than the same new components (here we use the assumption that their lifetimes are NBU).

Defining on the same probability space the processes

$\hat{\underline{X}} = \{(\hat{X}_1(t),\dots,\hat{X}_n(t), \hat{X}_{n+1}(t),\dots,\hat{X}_{n+m}(t)), \ t \geqslant 0\}$ [the last m

coordinates of $\hat{\underline{X}}$ correspond to the nonrepairable components] and

$\hat{\underline{Y}} = \{(\hat{Y}_1(t),\dots,\hat{Y}_{n+m}(t)), \ t \geqslant 0\}$ such that $\hat{\underline{Y}} \overset{st}{=} [\underline{Z}|S]$ and

$\hat{\underline{X}} \overset{st}{=} [\underline{Z}(t+)|H_t]$, in a manner similar to the proof of Theorem 2.2, it is

seen that $\hat{\underline{X}} \geqslant \hat{\underline{Y}}$ a.s. The result then follows by partially unconditioning as

in the proof of Theorem 3.4. We omit the details.

The conclusion of Theorem 3.5 still holds even if (a) is relaxed as

follows. Assume as before that the lifetimes of the nonrepairable components

are independent of the states and the current ages and repair durations of the

repairable components. However, allow now the joint distribution of the

lifetimes of the nonrepairable components to be dependent in the sense of

Arjas (1981), that is, assume that

(a') the nonrepairable components have $MNBU\big| \mathcal{F}_t$ joint distribution with

$\mathcal{F}_t$ being the σ-field generated by the nonrepairable components.

The following result (whose proof is similar to the proof of Theorem 3.5) is

valid:

Theorem 3.6. For a DMRS with n + m components as described above assume

(a'), (b) and (c). Then the time until first system failure is NBU.

When the repairable components evolve independently of each other, that

is, $\lambda_i(w,\tfrac{d}{-w}) = \lambda_i$, $\mu_i(w,\tfrac{d}{-w}) = \mu_i(d_i)$, then Theorem 3.5 (or 3.6) reduces to

the results of Barlow and Proschan (1976), Theorem 2.7 and Chiang and Niu

(1980), Theorem 3.5. In this context a special case of Theorem 3.3 is for a

single component subject to failures and repairs. Suppose that the single

component stays up (in state 0) for an exponential time period and stays down

(in state 1) for a DFR time period. Then the component performance process

Z satisfies the stochastic monotonicity

$$(3.7) \qquad \{Z|Z(0) = 1, \; B_1(Z,0) = b\} \overset{st}{\geqslant} \{Z|Z(0) = 0\} \quad \text{for all} \quad b \geqslant 0$$

(see, e.g., Barlow and Proschan (1976), Lemma 2.5, Chiang and Niu (1980),
Lemma 3.1 and Shanthikumar (1984), Theorem 5.1). Miller (1979) has
generalized this result and showed that if

(d) the up-periods are exponential,

(e) the down-periods are NWU,

then (3.7) holds. The NWU condition is weaker than the DFR condition.
Miller's result thus tempts one to weaken (3.4) to a condition analogous to
NWU. However, in the next subsection it is shown through a counterexample
that such a modification is not possible.

3.1. A counterexample.

In this subsection we construct a class of univariate performance
processes $Z = \{Z(t), \; t \geqslant 0\}$ which satisfy conditions (d) and (e) but
violates (3.7).

For a fixed $\varepsilon > 0$, $\tau > 0$ and $\mu > 0$ let V be a random variable with
survival function

$$\overline{G}(v) = e^{-\mu(v-r\tau)} \quad \text{if} \quad v \varepsilon [(\tau+\varepsilon)r, \; (\tau+\varepsilon)r+\varepsilon], \; r = 0, 1, 2, \ldots,$$
$$= e^{-\mu(r+1)\varepsilon} \quad \text{if} \quad v \; \varepsilon [(\tau+\varepsilon)r+\varepsilon, \; (\tau+\varepsilon)(r+1)], \; r = 0, 1, 2, \ldots .$$

A graph of $-\log \overline{G}(v)$ is shown in Figure 3.1. It is easy to verify that $\overline{G}$
is NWU.

Let $Z = \{Z(t), \; t \geqslant 0\}$ be a component performance process that has
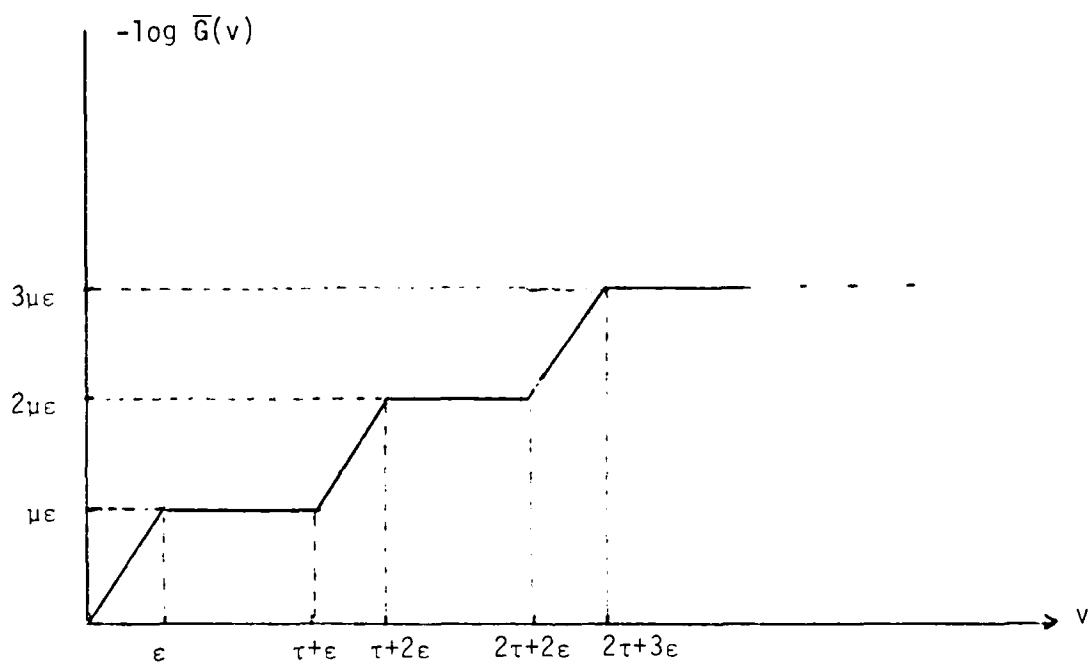
Figure 3.1

exponentially distributed up period with mean λ^{-1} (which will be a fixed constant throughout this subsection) and down periods with survival function $\overline{G}$.

Let $\Delta\epsilon(0, \min(\epsilon,\tau))$ and define

$$\{Z^{(1)}(t), \ t \geqslant 0\} \overset{st}{=} \{Z(t), \ t \geqslant 0 | Z(0) = 0\},$$

$$\{Z^{(2)}(t), \ t \geqslant 0\} \overset{st}{=} \{Z(t), \ t \geqslant 0 | Z(0) = 1, \ B_1(Z,0) = \epsilon+\Delta\}.$$

That is, $Z^{(2)}$ starts at state 1 at time 0 with an elapsed repair time of $\epsilon+\Delta$. Since $0<\Delta<\tau$, it is clear that no repair completion will take place for the next $\tau-\Delta$ time units (since $\overline{G}(t+\epsilon+\Delta)/\overline{G}(\epsilon+\Delta) = 1$, $0 \leqslant t \leqslant \tau-\Delta$). Then, conditioning on the repair completion instance after time $\tau-\Delta$ and no failures until time τ (recalling that $\Delta\leqslant\epsilon$) one has

$$(3.8) \qquad P\{Z^{(2)}(\tau) = 0\} \geqslant \int_{\alpha=0}^{\Delta} \mu e^{-\mu\alpha} e^{-\lambda(\Delta-\alpha)} d\alpha \equiv h_{\Delta,\lambda}(\mu),$$

where

$$h_{\Delta,\lambda}(\mu) = \frac{\mu}{\mu-\lambda} (e^{-\lambda\Delta} - e^{-\mu\Delta}) \quad \text{if} \quad \mu \neq \lambda,$$
$$= \Delta e^{-\lambda\Delta} \qquad\qquad \text{if} \quad \mu = \lambda.$$

Note that $h_{\Delta,\lambda}(\mu)$ is a continuous function of μ and that

$$h_{\Delta,\lambda}(\mu) \to e^{-\lambda\Delta} \quad \text{as} \quad \mu \to \infty,$$
$$\to 0 \qquad \text{as} \quad \mu \to 0.$$

Thus

$$
(*) \quad
\begin{cases}
\text{given } \varepsilon > 0, \ \Delta \varepsilon(0, \varepsilon) \ \text{ and } \ \varepsilon_0 < e^{-\Delta \lambda}, \\[2mm]
\text{there exists a } \ \mu = \mu_{\varepsilon_0, \Delta} \ \text{ such that,} \\[2mm]
\text{for all } \ \tau > \Delta, \ \text{ we have} \\[2mm]
P\{Z^{(2)}(\tau) = 0\} \geqslant h_{\Delta, \lambda}(\mu) > \varepsilon_0.
\end{cases}
$$

Now consider the performance process $Z^{(1)}$ and its associated age process $\{B_1(Z^{(1)}, t), \ t \geqslant 0\}$. When $Z^{(1)}(t) = 0$ then $B_1(Z^{(1)}, t)$ is vacuous. For technical reasons it is convenient to define then $B_1(Z^{(1)}, t) \equiv 0$ so that $B_1(Z^{(1)}, t)$ is well defined for all $t \geqslant 0$. Let $q(x)$ be the probability that $Z^{(1)}$ has not been in state 1 continuously for more than ε time units during $[0, x]$, i.e.,

$$
q(x) \equiv P\{\sup_{0 \leqslant t \leqslant x} B_1(Z^{(1)}, t) \leqslant \varepsilon\}, \quad x \geqslant 0.
$$

Since $\overline{G}(\tau + \varepsilon) / \overline{G}(\varepsilon) = 1$, it is clear that

$$
\{\sup_{0 \leqslant t \leqslant x} B_1(Z^{(1)}, t) > \varepsilon\} \subset \{Z^{(1)}(x) = 1\}, \quad 0 \leqslant x \leqslant \tau + \varepsilon.
$$

Therefore

$$
P\{Z^{(1)}(x) = 0\} \leqslant q(x), \quad 0 < x \leqslant \tau + \varepsilon.
$$

Note that for all $x \geqslant 0$, $q(x)$ does not depend on τ. In fact, conditioning on the first transition of $Z^{(1)}$ from state 0 to 1, one has

$$
(3.9) \quad q(x) = \int_{z=0}^{x-\varepsilon} \lambda e^{-\lambda z} \int_{u=0}^{\varepsilon} \mu e^{-\mu u} q(x - z - u) \, du \, dz + e^{-\lambda(x-\varepsilon)}, \quad x > \varepsilon,
$$

and

$$
q(x) = 1, \quad 0 \leqslant x \leqslant \varepsilon.
$$

Taking the Laplace transforms on both sides of (3.9) and solving for

$\hat{q}(s) = \int_0^\infty e^{-sx} q(x) dx,$ one can show that

(3.10)
$$\lim_{x \to \infty} q(x) = \lim_{s \to 0} s\, \hat{q}(s) = 0.$$

Thus

$(**)$ $\begin{cases} \text{given } \varepsilon > 0, \ \mu > 0, \ \varepsilon_0 > 0 \ \text{ there exists} \\ \text{a } \tau = \tau_{\varepsilon, \mu, \varepsilon_0} \ \text{ such that} \\ P\{Z^{(1)}(\tau) = 0\} \le q(\tau) < \varepsilon_0. \end{cases}$

Now, fix $\varepsilon > 0$, $\Delta \varepsilon(0, \varepsilon)$, $\varepsilon_0 \varepsilon(0, e^{-\Delta \lambda}]$. Determine $\mu = \mu_{\varepsilon_0, \Delta}$ as in (*). Determine $\tau = \tau_{\varepsilon, \mu_{\varepsilon_0, \Delta}, \varepsilon_0}$ as in $(**)$. [If this τ is not $\geqslant \Delta$ then choose a bigger $\tau \geqslant 0$ which satisfies $(**)$. Such a τ exists by (3.10).] Then, for these μ and τ,

$$P\{Z^{(2)}(\tau) = 0\} \underset{(*)}{>} \varepsilon_0 \underset{(**)}{>} P\{Z^{(1)}(\tau) = 0\}$$

in contradiction to (3.7) and to Theorem 3 of Miller (1979).

4. Discussion and further results.

It is fair to say that the set of results obtained in this paper using the method of "putting two processes on the same probability space" is only a sample of the kind of results which can be obtained by this method. Some further results and applications are indicated below.

4.1. NBU-ness of some first passage times.
One can generalize Theorem 3.3 as follows: Let $\mathcal{A} = 2^S \times [0, \infty)^n$ where 2^S is the set of all subsets of

$S = \{1, 2, \ldots, n\}$. Denote a generic element of $\mathscr{A}$ by $(w, \underline{b})$, $w \subset S$, $\underline{b} \geq \underline{0}$.
Denote by $\underset{-w}{\widetilde{b}}$ the vector $(b_1, \ldots, b_n)$ where for each $i \in w$, b_i is
replaced by 0. Consider the state space $\mathscr{B} \equiv \{(w, \underset{-w}{\widetilde{b}}), w \subset S, \underline{b} \geq 0\} \subset \mathscr{A}$.
Define a partial ordering on $\mathscr{B}$ by $(w, \underset{-w}{\widetilde{b}}) \underset{\sim}{\leq} (v, \underset{v}{\widetilde{b}'})$ if and only if
$\overline{w} \subset \overline{v}$ and $\underset{-w}{\widetilde{b}} \leq \underset{w}{\widetilde{b}'}$. Consider the right continuous process

$$(4.1) \qquad M = \{M(t), \ t \geq 0\} \equiv \{(W(\underline{Z}, t), \underset{-w(\underline{Z}, t)}{\widetilde{B}}(\underline{Z}, t)), \ t \geq 0\}$$

on $\mathscr{B}$, where $W(\underline{Z}, t)$ is as defined in Section 1 and $\underset{-w}{\widetilde{B}}(\underline{Z}, t)$ is the vector
with i-th coordinate being $B_i(\underline{Z}, t)$ [as defined in Section 1] if $i \in \overline{w}$ and
the j-th coordinate being 0 if $j \in w$. Note that if $\underline{Z}$ is a performance
process of a DMRS which satisfies Assumption 3.2, then M is a Markov process
on $\mathscr{B}$.

Using a proof similar to the proof of Theorem 3.3, it can be shown, under
Assumptions 3.1, 3.2 and conditions (3.3) and (3.4), that M is
stochastically monotone with respect to the ordering $\underset{\sim}{\leq}$ on $\mathscr{B}$, that is,

$$(4.2) \qquad \{M(t), t \geq 0 | M(0) = (w, \underset{-w}{\widetilde{b}})\} \underset{\sim}{\overset{st}{\leq}} \{M(t), t \geq 0 | M(0) = (v, \underset{v}{\widetilde{b}'})\} \text{ if } \overline{w} \subset \overline{v}, \ \underset{-w}{\widetilde{b}} \leq \underset{w}{\widetilde{b}'},$$

where the notation $\underset{\sim}{\overset{st}{\leq}}$ is self-explanatory. Note that (3.5) is a special
case of (4.2). A straightforward extension of Lemma 2.8 of Barlow and
Proschan (1976) yields the following (consult Marshall and Shaked (1985) and
Shaked and Shanthikumar (1984) for the measure-theoretic considerations). If
$M(0) = (S, \underline{0})$ a.s. then M is an NBU process on $\mathscr{B}$, that is, for every
closed upper set $U \subset \mathscr{B}$ (a set U is an upper set if $(v, \underset{v}{\widetilde{b}'}) \in U$
whenever $(v, \underset{v}{\widetilde{b}'}) \underset{\sim}{\geq} (w, \underset{-w}{\widetilde{b}})$ for some $(w, \underset{-w}{\widetilde{b}}) \in U$), the first passage time

(4.3) $T_U = \inf\{t:M(t)\in U\}$ is NBU

(see Marshall and Shaked (1985) for a discussion on NBU processes with general state space).

4.2. <u>Multistate coherent structures.</u> Let $\phi(z_1,\ldots,z_n)$ be a multistate coherent structure function, i.e., $\phi(\underline{z})$ is a nondecreasing function of $\underline{z}\in\{0,1\}^n$ (with a real range which may be more general than $\{0,1\}$). If Assumptions 3.1 and 3.2 hold and if conditions (3.3) and (3.4) are satisfied and if $\underline{Z}(0) = \underline{0}$ a.s., then a special case of (4.3) is

$$T_u \equiv \inf\{t:\phi(\underline{Z}(t)) \geqslant u\} \quad \text{is NBU,} \quad u \geqslant 0.$$

4.3. <u>Total ages of items under repair.</u> From Theorem 3.8 of Marshall and Shaked (1985) it follows that for every nondecreasing continuous functon $\psi:R_+^n \to R$, such that $\psi(\underline{0}) = 0$, the random time

(4.4) $S_u = \inf\{t:\psi(\widetilde{\underline{R}}_{\underline{W}(\underline{Z},t)}(\underline{Z},t)) \geqslant u\}$ is NBU, $u \geqslant 0$,

where here $\widetilde{\underline{R}}_{(\underline{Z},t)}$ is as defined in (4.1). For example, from (4.4) it follows that if a DMRS satisfies Assumptions 3.1 and 3.2 and conditions (3.3) and (3.4) and has all components new at time 0, then the first time at which the total ages of items under repair exceeds u, is NBU.

4.4. <u>A DMRS with a backup unit.</u> Consider a DMRS as described in Section 1 and suppose that a backup unit with a random life L is available. The backup unit works only during the down periods of the system (provided it has

not already failed). During the up periods the backup unit is automatically switched off. Denote by $D(t) \equiv \int_0^t \phi(\underline{Z}(u))du$ the cumulative down time of the DMRS where here ϕ is a binary nondecreasing function. Backed by the backup unit, the DMRS experiences its first actual failure at the random time

$$T_L \equiv \inf\{t \geqslant 0 : D(t) > L\}.$$

We will show now that if Assumptions 3.1 and 3.2 and conditions (3.3) and (3.4) hold and if L is NBU and independent of $\{M(t), t \geqslant 0\}$ of (4.1), then T_L is NBU.

Since D is an increasing functional of M, it follows that for every $w \subset S$, $\underline{b_w}$ and $s \geqslant 0$, we have

$$(4.5) \qquad \{D(t), t \geqslant 0 | S, 0\} \overset{st}{\leqslant} \{D(s+t), t \geqslant 0 | \underline{Z}(u) = \underline{z}(u), 0 < u \leqslant s, W(\underline{Z}, s) = w, B_w(\underline{Z}, s) = \underline{b_w}\}.$$

Since L is NBU, we have

$$(4.6) \qquad L \overset{st}{\geqslant} [L-s | L>s], \quad s \geqslant 0.$$

Conditioning on the history $\{\underline{Z}(u) = \underline{z}(u), 0 < u < s, W(\underline{Z}, s) = w, B_w(\underline{Z}, s) = \underline{b_w}\}$ of M up to time s and on $\{L > \int_0^s \phi(\underline{z}(u))du\}$, we obtain from (4.5), (4.6) and the independence of M and L, that for $s > 0$,

$$\{D(t) - L, \ t \geqslant 0 | S, 0\} \overset{st}{\leqslant} \{D(s+t) - L - d(s), \ t \geqslant 0$$
$$| \underline{Z}(u) = \underline{z}(u), \ 0 < u \leqslant s, \ W(\underline{Z}, s) = w, \ B_w(\underline{Z}, s) = \underline{b_w}, \ L > \int_0^s \phi(\underline{z}(u)du\}.$$

where $d(s) \equiv \int_0^s \phi(\underline{z}(u)du$. Hence for $s > 0$,

$$(4.7) \qquad T_L \overset{st}{\geqslant} [\inf\{t \geqslant 0 : D(s+t) - L - d(s) \geqslant 0 \,|\, \underline{Z}(u) = \underline{z}(u),\ 0 \leqslant u \leqslant s,$$

$$W(\underline{Z}, s) = w, \quad \underline{B}_{\overline{w}}(\underline{Z}, s) = \underline{b}_{\overline{w}},\ L > d(s)].$$

Unconditioning in (4.7), retaining the condition $T_L > s$, we obtain
$T_L \overset{st}{\geqslant} [T_L - s \,|\, T_L > s]$, $s \geqslant 0$, that is, T_L is NBU.

4.5. Stochastic comparisons of up periods.

We present a result that compares the first up period (denoted by V_1) of a DMRS with its subsequent up periods $V_2, V_3, \ldots$. Denote by $U_1, U_2, \ldots$ the sequence of down periods. Fix a positive integer m. Conditioning on the state of the process M at the random time $\sum_{i=1}^{m-1} (V_i + U_i) + u$, $u \geqslant 0$, under the condition $\{V_m > u\}$, one easily obtains

Theorem 4.1. Under the conditions of Theorem 3.3,

$$(4.8) \qquad V_1 \overset{st}{\geqslant} [V_m - u \,|\, V_m > u], \quad u \geqslant 0, \quad m = 1, 2, \ldots .$$

The above theorem is a generalization of Theorem 3.6 of Chiang and Niu (1980). Such a generalization is needed in Shanthikumar (1985) to establish the NBU property of the first failure time of a dependent parallel system with safety periods.

Condition (4.8) is also sufficient for the validity of some results of Shanthikumar (1984). For example, Theorem 4.4 of Shanthikumar (1984) shows that the first passage time to overflow, in a dam with level dependent release rule and compound renewal input, is NBU whenever the inter-renewal times of the input satisfy (4.8).

4.6. DMRS's with safety periods. Consider a DMRS as described in Section 1. Suppose that every time the system is down, a random safety period is provided. Denote the i-th safety period (at the i-th time the system switches down) by Y_i, i=1, 2, ... If the system is repaired before the safety period is over, then the system continues to operate normally. Otherwise, the operation stops. Let τ be the first time in which the operation stops.

Result 4.2. Under the conditions of Theorem 3.3, if Y_1, Y_2, ... are NBU, independent and identically distributed then τ is NBU.

This result extends Corollary 5 of Shanthikumar (1985). The proof uses Theorem 4.1 and its idea is similar to the proof of the result in subsection 4.4. We omit the details.

REFERENCES

1. Arjas, E. (1981), A stochastic process aproach to multivariate
        reliability: notions based on conditional stochastic order, Math of
        Opns. Res., 6, 263-276.

2. Barlow, R. E. and Proschan, F. (1975), Statistical Theory of Reliability
        and Life Testing, Holt, Rinehart and Winston, New York.

3. Barlow, R. E. and Proschan, F. (1976), Theory of maintained systems:
        distribution of time to first system failure, Math. of Opns. Res., 1,
        32-42.

4. Brown, M. (1975), The first passage time distribution for a parallel
        exponential system with repair, in Reliability and Fault Tree
        Analysis, R. E. Barlow, J. B. Fussel and N. D. Singpurwalla, editors,
        SIAM, Philadelphia, PA, 365-396.

5. Brown, M. (1984), On the reliability of repairable systems, Opns. Res.,
        32, 607-615.

6. Chiang, D. T. and Niu, S. C. (1980), On the distribution of time to first
        system failure, J. Appl. Prob., 17, 481-489.

7. Keilson, J. (1974), Monotonicity and convexity in system survival
        functions and metabolic disappearence curve, in Reliability and
        Biometry, F. Proschan and R. Serfling, editors, SIAM, Philadelphia,
        PA, 81-98.

8. Keilson, J. (1975), Systems of independent Markov components and their
        transient behavior, in Reliability and Fault Tree Analysis, R. E.
        Barlow, J. B. Fussel and N. D. Singpurwalla, editors, SIAM,
        Philadephia, PA, 351-364.

9. Marshall, A. W. and Shaked, M. (1985), NBU processes with general state
        space, Math. of Opns. Res., to appear.

10. Miller, D. R. (1979), Almost sure comparison of renewal processes and
        Poisson processes, with application to reliability theory, Math. of
        Opns. Res., 4, 406-413.

11. Ross, S. M. (1975), On the calculation of asymptotic system reliability
        characteristic, in Reliability and Fault Tree Analysis, R. E. Barlow,
        J. B. Fussel and N. D. Singpurwalla, editors, SIAM, Philadelphia, PA,
        331-350.

12. Ross, S. M. (1976), On the time to first failure in multicomponent
        exponential reliability systems, Stoch. Proc. Appl., 4, 167-173.

13. Ross, S. M. (1984), A model in which component failure rates depend on
        working set, Nav. Res. Log. Quart., 31, 297-300.

14. Ross, S. M. and Schechtman, J. (1979), On the first time a seperately
        maintained parallel system has been down for a fixed time, Nav. Res.

Log. Quart., 26, 285-290.

15. Schechner, Z. (1984), A load-sharing model: the linear breakdown rule, Nav. Res. Log. Quart., 31, 137-144.

16. Shaked, M. and Shanthikumar, J. G. (1984), Multivariate IFRA properties of some Markov jump processes with general state space, Technical report, Department of Mathematics, University of Arizona.

17. Shanthikumar, J. G. (1984), Processes with new better than used first-passage times, Adv. Appl. Probl., 16, 667-686.

18. Shanthikumar, J. G. (1985), First failure time of dependent parallel systems with safety periods, Technical report, School of Business Administration, University of California, Berkeley.

END

DTIC

8—86