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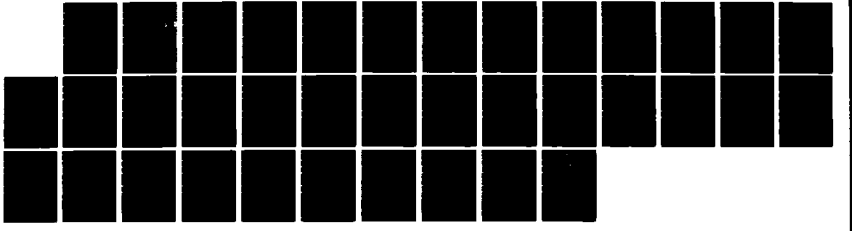

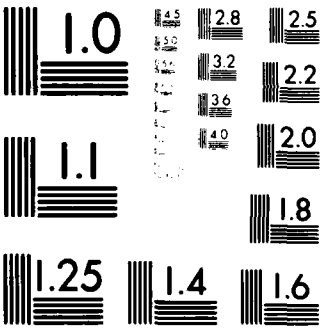

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Several notions of stochastic convexity and concavity and their properties are studied in this paper. Efficient sample path approaches are developed in order to verify the occurrence of these notions in various applications. Numerous examples are given. The use of these notions in several areas of probability and statistics is demonstrated. In queueing theory, the convexity (as a function of c) of the steady state mean waiting time in a GI/D/c queue, and (as a function of the arrival and service rates) in a GI/G/1 queue, is established. Also the convexity of the queue length in the M/M/c case (as a function of the arrival rate) is shown, thus strengthening previous results while simplifying their derivation. In reliability theory, the convexity of the payoff in the success rate of an imperfect repair is obtained and used to find an optimal repair probability. Also the convexity of the damage as a function of time in a cumulative damage shock model is shown. In branching processes, the convexity of the population size as a function of a parameter of the offspring distribution is proved. In nonparametric statistics, the

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STOCHASTIC CONVEXITY AND ITS APPLICATIONS

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Abstract

Several notions of stochastic convexity and concavity and their properties are studied in this paper. Efficient sample path approaches are developed in order to verify the occurrence of these notions in various applications. Numerous examples are given. The use of these notions in several areas of probability and statistics is demonstrated. In queueing theory, the convexity [as a function of c] of the steady state mean waiting time in a GI/D/c queue, and [as a function of the arrival and service rates] in a GI/G/1 queue, is established. Also the convexity of the queue length in the M/M/c case [as a function of the arrival rate] is shown, thus strengthening previous results while simplifying their derivation. In reliability theory, the convexity of the payoff in the success rate of an imperfect repair is obtained and used to find an optimal repair probability. Also the convexity of the damage as a function of time in a cumulative damage shock model is shown. In branching processes, the convexity of the population size as a function of a parameter of the offspring distribution is proved. In nonparametric statistics, the stochastic concavity [convexity] of the empirical distribution function is established. And, for applications in the theory of probability ineqalities, we identify several families of distributions which are convexly parametrized.

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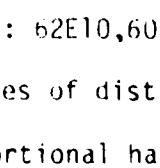

1. Introduction.

Many collections of random variables $\{X(\theta), \theta \in \Theta\}$ [e.g., some stochastic processes] have the "property" that in some sense $X(\theta)$ is stochastically convex (or concave) and perhaps also increasing (or decreasing) in θ. In this paper we discuss some notions of stochastic convexity (concavity) and develop efficient sample path approaches to verify the occurrence of these notions in various stochastic processes.

Section 2 consists of some preliminaries and definitions. Stochastic monotonicity and some notions of stochastic convexity are discussed there. Basic properties of these notions are also given in the same section. Sample path convexity and concavity notions are defined and studied in detail in Section 3. These notions have the advantage that they are sometimes easy to verify and that they yield the useful convexity and concavity concepts of Section 2. Basic preservation results, *which are* useful in the applications to follow, are also given in Section 3. A sample of collections of random variables $\{X(\theta), \theta \in \Theta\}$ which satisfy the notion of sample path convexity or concavity is given in Section 4. These collections $\{X(\theta), \theta \in \Theta\}$ are either parametric families of random variables or well studied stochastic processes. Combination of the examples in Section 4 with the properties obtained in Section 3, and with the fact that sample path convexity (concavity) implies the convexity (concavity) notions of Section 2, yields a host of applications in various areas of probability and statistics. A sample of such applications (in queueing theory, reliability theory, branching processes, nonparametric statistics, and in the theory of probability inequalities) is given in Section 5. In particular it is shown how various monotonicity and convexity (concavity) results, which have been obtained in the literature using tedious algebra, can be easily obtained (and

strengthened) using the sample path convexity (concavity) approach.

Throughout this paper 'increasing' ['decreasing'] means 'nondecreasing' ['nonincreasing']. Whenever an integral $\int \phi(x)dF(x)$ [or expected value $E\phi(X)$] is written, it is tacitly assumed that ϕ is such that the integral [or $E\phi(X)$] exists. Also, it is tacitly assumed that all the real functions mentioned in this paper are Borel measurable.

2. Preliminaries: stochastic and convex orderings.

A class $\mathbf{C}$ of functions $R \to R$ can generate a partial ordering $\prec$ on the set of distribution functions on $R \equiv (-\infty,\infty)$ by postulating that any two such distribution functions F and G satisfy $F \prec G$ if and only if

$$\int_{-\infty}^{\infty} \phi(x)dF(x) \leq \int_{-\infty}^{\infty} \phi(x)dG(x) \qquad \text{for all } \phi \in \mathbf{C}. \tag{2.1}$$

The following definitions can be found, e.g., in Stoyan (1983).

Definition 2.1. Let X and Y be random variables with distribution functions F and G respectively.

(a) Denote $X \leq_{st} Y$ and say X [or F] is stochastically less than Y [or G] if (2.1) holds for the class $\mathbf{C}$ of real increasing functions on the union of the supports of F and G.

(b) Denote $X \leq_{cx} Y$ [respectively, $X \leq_{cv} Y$] if (2.1) holds for the class $\mathbf{C}$ of real convex [respectively, concave] functions on the convex hull of the union of the supports of F and G.

(c) Denote $X \leq_{icx} Y$ [respectively, $X \leq_{icv} Y$] if (2.1) holds for the class $\mathbf{C}$ of real increasing convex [respectively, concave] functions on the convex hull of the union of the supports of F and G.

(d) Denote $X \leq_{il} Y$ if (2.1) holds for the class C of real increasing linear functions on the convex hull of the union of the supports of F and G.

For a distribution function H, let $\bar{H}$ denote 1 - H. The following results are well known (see, e.g., Stoyan (1983)).

Theorem 2.2. Let X and Y be random variables with distribution functions F and G respectively. Then

(a) $X \leq_{st} Y \iff \bar{F}(x) \leq \bar{G}(x)$ for all $x \in R$,

(b) $X \leq_{icx} Y \iff \int_x^\infty \bar{F}(y)dy \leq \int_x^\infty \bar{G}(y)dy$ for all $x \in R$,

(c) $X \leq_{icv} Y \iff \int_{-\infty}^x \bar{F}(y)dy \leq \int_{-\infty}^x \bar{G}(y)dy$ for all $x \in R$,

(d) $X \leq_{il} Y \iff EX \leq EY$.

The relation $\leq_{cx}$ is sometimes called dilation. Conditions which are equivalent to $X \leq_{cx} Y$ can be found, e.g., in Shaked (1980).

Remark 2.3. Some useful properties of the 'icx' and 'icv' conditions are (here X and Y are nonnegative random variables):

(i) $X \leq_{icx} Y \Rightarrow EX^k \leq EY^k$, $k = 1,2, \ldots,$

(ii) $X \leq_{icv} Y \Rightarrow EX^{1/k} \leq EY^{1/k}$, $k = 1,2, \ldots .$

Let '$\stackrel{st}{=}$' denote equality in law.

Theorem 2.4. If $X \leq_{st} Y$ then there exist $\hat{X}$ and $\hat{Y}$ defined on the same probability space such that $\hat{X} \stackrel{st}{=} X$, $\hat{Y} \stackrel{st}{=} Y$ and $\hat{X} \leq \hat{Y}$ a.s.

For every distribution H denote $H^{-1}(u) = \inf\{x: H(x) \geq u\}$, $u \in [0,1]$.

Remark 2.5. In Theorem 2.4 one may set $\hat{X} = F^{-1}(U)$, $\hat{Y} = G^{-1}(U)$ where U

is a uniform (0,1) random variable.

Let $\{P_\theta, \theta\in\Theta\}$ be a family of univariate distributions. Throughout this paper Θ is a convex set (that is, an interval) of the real line or of the set $\{0,1,2,\ldots\}$. Let $X(\theta)$ denote a random variable with distribution P_θ. We find it convenient and intuitive to replace the notation $\{P_\theta, \theta\in\Theta\}$ by $\{X(\theta), \theta\in\Theta\}$ and this notation will be used throughout this paper. Note that when we write $\{X(\theta), \theta\in\Theta\}$ we do not assume [and often we are not concerned with] any dependence (or independence) properties among the $X(\theta)$'s. We are only interested in the 'marginal distributions' $\{P_\theta, \theta\in\Theta\}$ of $\{X(\theta), \theta\in\Theta\}$ even when in some circumstances $\{X(\theta), \theta\in\Theta\}$ is a well defined stochastic process. Note also that $X(\theta)$ <u>does not mean</u> that X is a function of θ; it only indicates that the distribution of $X(\theta)$ is P_θ. Thus, for example, for $\phi:R\to R$, the notation $E\phi(X(\theta))$ stands for $\int\phi dP_\theta$ --- this is usually denoted in the literature by $E_\theta\phi(X)$. When $\{X(\theta), \theta\in\Theta\}$ is a well defined stochastic process then the notation $E\phi(X(\theta))$ is often justifiably used.

In the following definition the abbreviations SI, SCX, SCV, SICX, SIL, SD, SDCV etc. stand, respectively, for stochastically increasing, stochastically convex, stochastically concave, stochastically increasing and convex, stochastically increasing and linear, stochastically decreasing, stochastically decreasing and concave etc.

<u>Definition 2.6</u>. Let $\{X(\theta), \theta\in\Theta\}$ be a set of random variables. Denote

(a) $\{X(\theta), \theta\in\Theta\} \in$ SI [SD] if

$$\phi\in\mathbf{C} \Rightarrow \psi(\cdot) \equiv E\phi(X(\cdot)) \in \mathbf{C} \tag{2.2}$$

for $\mathbf{C}$ - the class of all increasing real functions on R, and $\mathbf{C}_c$ -

the class of all increasing [decreasing] real functions on Θ.

(b) $\{X(\theta),\theta\in\Theta\} \in$ SCX [SCV] if (2.2) holds for $\mathbf{C}$ - the class of all real convex [concave] functions on R, and $\mathbf{C}_\Theta$ - the class of all real convex [concave] functions on Θ.

(c) $\{X(\theta),\theta\in\Theta\} \in$ SICX [SICV, SIL] if $\{X(\theta),\theta\in\Theta\} \in$ SI and if (2.2) holds for $\mathbf{C}$ - the class of all increasing and convex [concave, linear] real functions on R, and $\mathbf{C}_\Theta$ - the class of all increasing and convex [concave, linear] real functions on Θ.

(d) $\{X(\theta),\theta\in\Theta\} \in$ SDCX [SDCV, SDL] if $\{X(\theta),\theta\in\Theta\}\in$SD and if (2.2) holds for $\mathbf{C}$ - the class of all <u>increasing</u> and convex [concave, linear] real functions on R, and $\mathbf{C}_\Theta$ - the class of all <u>decreasing</u> and convex [concave, linear] real functions as Θ.

Some basic properties of these notions are given next.

<u>Lemma 2.7</u>. If $\{X(\theta),\theta\in\Theta\}\in$SI [respectively, SD] then there exist random variables $\hat{X}(\theta)$, $\theta\in\Theta$, defined on a common probability space, such that $\hat{X}(\theta) \overset{st}{=} X(\theta)$, $\theta\in\Theta$, and, a.s., $\hat{X}(\theta)$ is increasing [respectively, decreasing] in $\theta\in\Theta$.

One way to construct $\{X(\theta),\theta\in\Theta\}$ in Lemma 2.7 is to set $\hat{X}(\theta) = F^{-1}(U,\theta)$ $\theta\in\Theta$, where U is a uniform (0,1) random variable and $F^{-1}(u,\theta) \equiv \inf\{x:P\{X(\theta)>x\} < u\}$.

<u>Remark 2.8</u>. It is worthwhile to note down the following 'continuous' analog of Remark 2.3.

(i) $\{X(\theta),\ \theta\in\Theta\} \in$ SICX $\Rightarrow EX^k(\theta)$ is increasing and convex in $\theta\in\Theta$ for $k = 1,2, \ldots$.

(ii) $\{X(\theta),\ \theta\in\Theta\} \in$ SICV $\Rightarrow EX^{1/k}(\theta)$ is increasing and concave in $\theta\in\Theta$

for $k = 1,2, \ldots$.

(iii) In particular, if $\{X(\theta), \theta \in \Theta\} \in$ SICX [SICV] then $EX(\theta)$ is increasing and convex [concave] in $\theta \in \Theta$.

Remark 2.9. In Definition 2.6 (a), (b) and (c) we require $\mathbf{C}$ and $\mathbf{C}_\Theta$ to be "similar" classes (e.g., for SICX, both are the classes of increasing convex functions). In general these need not be "similar". For example, of a particular interest is the class of processes $\{X(\theta), \theta\in\Theta\}$ such that $E\phi(X(\theta))$ is increasing and convex in θ for all increasing functions ϕ. Such classes will be considered elsewhere (but see Proposition 5.7).

Some of the stochastic notions of Definition 2.6 are often preserved under reparametrization (or time transformation):

Proposition 2.10. (a) If $\{X(\theta),\theta\in\Theta\} \in$ SI [SD] and $h:\Theta\to\Theta$ is increasing then $\{X(h(\theta)), \theta \in \Theta\} \in$ SI [SD].

(b) If $\{X(\theta),\theta\in\Theta\} \in$ SICX [SICV, SIL] and $h:\Theta \to \Theta$ is increasing and convex [concave, linear] then $\{X(h(\theta)), \theta\in\Theta\} \in$ SICX [SICV, SIL].

(c) If $\{X(\theta),\theta\in\Theta\} \in$ SDCX [SDCV, SDL] and $h:\Theta \to \Theta$ is increasing and concave [convex, linear] then $\{X(h(\theta)), \theta\in\Theta\} \in$ SDCX [SDCV, SDL].

The proof of Proposition 2.10 is straightforward. So is also the proof of the next result. Let '$\xrightarrow{st}$' denote convergence in law.

Proposition 2.11. Suppose $\{X_m(\theta), \theta \in \Theta\} \in \mathbf{C}$ for $m = 1,2, \ldots$. If $X_m(\theta) \xrightarrow[m\to\infty]{st} X(\theta)$ for each $\theta \in \Theta$ then $\{X(\theta),\theta\in\Theta\} \in \mathbf{C}$ where $\mathbf{C} \in$ {SI, SD, SCX, SCV, SICX, SDCX, SICV, SDCV, SIL, SDL}.

3. Sample path convexity: definition and some properties.

The interest in this paper centers around the monotone convex and concave notions SICX, SICV, SDCX and SDCV. In this section we find sufficient conditions which imply that a process $\{X(\theta),\theta\in\Theta\}$ satisfies some of these notions. Our approach is to 'put' some (more explicitly four) of the random variables $\{X(\theta),\theta\in\Theta\}$ on a common probability space and then obtain 'almost sure' results which carry back to the whole process $\{X(\theta),\theta\in\Theta\}$.

We start with a definition which formally states these conditions. For any four real numbers x_1, x_2, x_3, x_4 we abbreviate the conditions $x_1 \leq \min(x_2,x_3) \leq \max(x_2,x_3) \leq x_4$ by $x_1 \leq [x_2,x_3] \leq x_4$. Also, $x_1 \leq [x_2,x_3,x_4]$ denotes $x_1 \leq \min(x_2,x_3,x_4)$ and $[x_1,x_2,x_3] \leq x_4$ denotes $\max(x_1,x_2,x_3) \leq x_4$.

Definition 3.1. (a) The family $\{X(\theta), \theta \in \Theta\}$ is said to be stochastically increasing and convex in sample path sense if for any $\theta_i \in \Theta$, $i = 1,2,3,4$, such that $\theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4$ and $\theta_1 + \theta_4 = \theta_2 + \theta_3$, there exist four random variables $\hat{X}_i$, $i = 1,2,3,4$, defined on the same probability space such that

(st) $\quad \hat{X}_i \overset{st}{=} X(\theta_i)$, $i = 1,2,3,4$,

(cx) $\quad \hat{X}_2 + \hat{X}_3 \leq \hat{X}_1 + \hat{X}_4$ a.s. (convexity condition),

(i-cx) $\quad [\hat{X}_1,\hat{X}_2,\hat{X}_3] \leq \hat{X}_4$ a.s. (monotonicity condition).

Denote conditions (st), (cx), (i-cx) by $\{X(\theta), \theta \in \Theta\} \in$ SICX.

(b) If (st) holds and also (cv) and (i-cv), where

(cv) $\quad \hat{X}_1 + \hat{X}_4 \leq \hat{X}_2 + \hat{X}_3$ a.s.,

(i-cv) $\hat{X}_1 \preccurlyeq [\hat{X}_2, \hat{X}_3, \hat{X}_4]$ a.s.,

then denote $\{X(\theta), \theta \in \Theta\} \in$ SICV(sp).

(c) If (st) and (ℓ) hold and also (i-cx) [which is then equivalent to (i-cv)], where

(ℓ) $\hat{X}_1 + \hat{X}_4 = \hat{X}_2 + \hat{X}_3$ a.s.,

then denote $\{X(\theta), \theta \in \Theta\} \in$ SIL(sp).

(d) If (st), (cx) and (d-cx) hold, where

(d-cx) $[\hat{X}_2, \hat{X}_3, \hat{X}_4] \preccurlyeq \hat{X}_1$ a.s.,

then denote $\{X(\theta), \theta \in \Theta\} \in$ SDCX(sp).

(e) If (st), (cv) and (d-cv) hold, where

(d-cv) $\hat{X}_4 \preccurlyeq [\hat{X}_1, \hat{X}_2, \hat{X}_3]$ a.s.,

then denote $\{X(\theta), \theta \in \Theta\} \in$ SDCV(sp).

(f) If (st) and (ℓ) hold and also (d-cx) [which is then equivalent to (d-cv)] then denote $\{X(\theta), \theta \in \Theta\} \in$ SDL(sp).

In order to gain some insight into the monotonicity and the convexity conditions above consider, for example, the SICX(sp) case. First note that (i-cx) indeed implies stochastic monotonicity of $\{X(\theta), \theta \in \Theta\}$: from (i-cx) we have $X(\theta_1) \overset{st}{\leq} \hat{X}_1 \preccurlyeq \hat{X}_4 \overset{st}{\leq} X(\theta_4)$ and thus $\{X(\theta), \theta \in \Theta\} \in$ SI [see Definition 2.6(a)]. Since $\theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4$ it seems more natural to postulate the stocnastic monotonicity by replacing (i-cx) by something "simpler" such as

(i) $\hat{X}_1 \leq \hat{X}_2 \leq \hat{X}_3 \leq \hat{X}_4$, a.s.,

or

(i') $\hat{X}_1 \leq [\hat{X}_2, \hat{X}_3] \leq \hat{X}_4$, a.s..

However in some applications we found it hard [if not impossible] to verify (i) or even (i') [see, e.g., Subsection 5.3], whereas (i-cx) could be obtained. Note also that if φ is increasing and convex [concave] then the set $\{\varphi(\theta), \theta \in \Theta\}$ of degenerate random variables is SICX(sp) [SICV(sp)].

The (sp) conditions enjoy some useful preservation properties which are discussed below. The first one is

Proposition 3.2. (a) If $\{X(\theta), \theta \in \Theta\} \in$ SICX(sp) [SICV(sp), SIL(sp)] and if φ is an increasing and convex [concave, linear] function then $\{\varphi(X(\theta)), \theta \in \Theta\} \in$ SICX(sp) [SICV(sp), SIL(sp)].

(b) If $\{X(\theta), \theta \in \Theta\} \in$ SDCX(sp) [SDCV(sp), SDL(sp)] and if φ is increasing and convex [concave, linear] function then $\{\varphi(X(\theta)), \theta \in \Theta\} \in$ SDCX(sp) [SDCV(sp), SDL(sp)].

Proof. Suppose $\{X(\theta), \theta \in \Theta\} \in$ SICX(sp). Let θ_i, i = 1,2,3,4, be such that $\theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4$ and $\theta_1 + \theta_4 = \theta_2 + \theta_3$. Let $\hat{X}_i$, i = 1,2,3,4, be random variables on a common probability space such that (st), (cx) and (i-cx) of Definition 3.1 hold. In particular, a.s., $\hat{X}_3 \leq \hat{X}_1 + \hat{X}_4 - \hat{X}_2$ and $\hat{X}_2 \leq \hat{X}_4$. Now, the convexity and monotonicity of φ imply $\varphi(\hat{X}_2) - \varphi(\hat{X}_1) \leq \varphi(\hat{X}_4) - \varphi(\hat{X}_1+\hat{X}_4-\hat{X}_2) \leq \varphi(\hat{X}_4) - \varphi(\hat{X}_3)$. Also, the monotonicity of φ implies $[\varphi(\hat{X}_1), \varphi(\hat{X}_2), \varphi(\hat{X}_3)] \leq \varphi(\hat{X}_4)$ a.s.. Thus $\{\varphi(X(\theta)), \theta \in \Theta\}$ satisfies (st), (cx) and (i-cx) of Definition 3.1. The proofs of the SICV and

SIL cases and of (b) are similar. ||

The next result is a dual of Proposition 3.2. It shows that the (sp) conditions are sometimes preserved under reparametrization (time transformation). First the following lemma is proved. Roughly speaking it says that if $\hat{X}_i$, i = 1,2,3,4, exist such that (st), (cx) and (i-cx) hold whenever $\{\theta_i,\ i\in\{1,2,3,4,\}\} \in \text{SIL(sp)}$ then such $\hat{X}_i$, i=1,2,3,4, exist whenever $\{\theta_i,\ i \in \{1,2,3,4\}\} \in \text{SICX(sp)}$. Let $O^4_{cx} \equiv \{(\theta_1,\theta_2,\theta_3,\theta_4)$: $\theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4$ and $\theta_2 + \theta_3 \leq \theta_1 + \theta_4\}$ and let $O^4_{cv} = \{(\theta_1,\theta_2,\theta_3,\theta_4)$: $\theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4$ and $\theta_2 + \theta_3 \geq \theta_1 + \theta_4\}$. Below we assume $O \subset [0,\infty)$.

Lemma 3.3. Suppose that for each $\theta \in O$ the distribution of $X(\theta)$ has no atoms.

(a) If $\{X(\theta),\ \theta \in O\} \in \text{SICX(sp)}$ [SDCV(sp)] then for each $(\theta_1,\theta_2,\theta_3,\theta_4) \in O^4_{cx}$ there exist $\hat{X}_i$, i = 1,2,3,4, defined on a common probability space which satisfy (st), (cx) and (i-cx) [(cv) and (i-cv)] of Definition 3.1.

(b) If $\{X(\theta),\ \theta \in O\} \in \text{SICV(sp)}$ [SDCX(sp)] then for each $(\theta_1,\theta_2,\theta_3,\theta_4) \in O^4_{cv}$ there exists $\hat{X}_i$, i = 1,2,3,4, as in (a) which satisfy (st), (cv) and (i-cv) [(cx) and (d-cx)] of Definition 3.1.

Proof. First we prove (a). Suppose $\{X(\theta),\ \theta \in O\} \in \text{SICX(sp)}$. Let $(\theta_1,\theta_2,\theta_3,\theta_4) \in O^4_{cx}$. Set $\theta_1' = \max(0,\theta_2+\theta_3-\theta_4)$ and $\theta_4' = \min(\theta_2+\theta_3,\theta_4)$. Then $\theta_1' \leq \theta_1$, $\theta_4' \leq \theta_4$ and $\theta_1' + \theta_4' = \theta_2 + \theta_3$. Then there exist $\hat{X}_1'$, $\hat{X}_2$, $\hat{X}_3$, $\hat{X}_4'$ such that $\hat{X}_i' \overset{st}{=} X(\theta_i')$, i = 1,4, and $\hat{X}_i \overset{st}{=} X(\theta_i)$, i = 2,3, and, a.s., $[\hat{X}_1',\hat{X}_2,\hat{X}_3] \leq \hat{X}_4'$ and $\hat{X}_1' + \hat{X}_4' \geq \hat{X}_2 + \hat{X}_3$.

For each $\theta \in O$ denote $\bar{F}_\theta(x) \equiv P\{X(\theta)>x\}$ and $\bar{F}^{-1}_\theta(u) \equiv \inf\{x:\bar{F}_\theta(x)\leq u\}$, $u \in [0,1]$. Define

$$\hat{X}_1 \equiv \bar{F}^{-1}_{\theta_1}(\bar{F}_{\theta_1'}(\hat{X}_1')), \tag{3.1}$$

$$\hat{X}_4 = \bar{F}^{-1}_{\theta_4}(\bar{F}_{\theta_4'}(\hat{X}_1')). \tag{3.2}$$

Then (using the fact that the distributions of $\hat{X}_1'$ and $\hat{X}_1'$ have no atoms) $\hat{X}_1 \overset{st}{=} X(\theta_1)$, $\hat{X}_4 \overset{st}{=} X(\theta_4)$ and (by the stochastic monotonicity of $X(\theta)$ in $\theta \in \Theta$) $\hat{X}_1 \geq \hat{X}_1'$, $\hat{X}_4 \geq \hat{X}_4'$. Thus $\hat{X}_i$, $i = 1,2,3,4$, satisfy (cx) and (i-cx) of Definition 3.1. The proof for SDCV(sp) is similar.

The proof of (b) for the SICV(sp) case is similar to the above proof of the SICX(sp) case. The main difference is that that instead of 'decreasing' θ_1 and θ_4 to θ_1' and θ_4' and then 'increasing' $\hat{X}_1'$ and $\hat{X}_4'$ to $\hat{X}_1$ and $\hat{X}_4$, in the SICV(sp) case one first 'decreases' θ_2 and θ_3 to θ_2' and θ_3' and then 'increases' $\hat{X}_2'$ and $\hat{X}_3'$ to $\hat{X}_2$ and $\hat{X}_3$. The proof of the SDCX(sp) is similar. ∎

Remark 3.4. The assumption of no atoms in Lemma 3.3 is not really necessary. Even without this assumption, (a) and (b) of Lemma 3.3 hold. The proof of this statement is the same as the proof of Lemma 3.3 except that constructions (3.1) and (3.2) are to be modified so that they are proper for the general case. We omit the lengthy details.

Proposition 3.5. (a) If $\{X(\theta), \theta \in \Theta\} \in$ SICX(sp) [SDCV(sp)] and if $\varphi:\Theta \to \Theta$ is increasing and convex then $\{X(\varphi(\theta)), \theta \in \Theta\} \in$ SICX(sp) [SDCV(sp)].

(b) If $\{X(\theta), \theta \in \Theta\} \in$ SICV(sp) [SDCX(sp)] and if $\varphi:\Theta \to \Theta$ is increasing and concave then $\{X(\varphi(\theta)), \theta \in \Theta\} \in$ SICV(sp) [SDCX(sp)].

Proof. We only prove (a) for the SICX(sp) case. The proofs of the other statements are similar. So suppose $\{X(\theta), \theta \in \Theta\} \in$ SICX(sp). Let θ_i, $i =$

1,2,3,4, be such that $\theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4$ and $\theta_1 + \theta_4 = \theta_2 + \theta_3$. Denote $\eta_i = \phi(\theta_i)$, i = 1,2,3,4. then $\eta_1 \leq \eta_2 \leq \eta_3 \leq \eta_4$ and $\eta_1 + \eta_4 \geq \eta_2 + \eta_3$, that is $(\eta_1, \eta_2, \eta_3, \eta_4) \in O^4_{cx}$. The result now follows from Lemma 3.3 and Remark 3.4. ‖

The next result shows that indeed Definition 3.1 gives sufficient conditions for stochastic monotone convexity and concavity as defined in Definition 2.6.

Theorem 3.6. If $\{X(\theta), \theta \in O\} \in$ SICX(sp) [SICV(sp), SIL(sp), SDCX(sp), SDCV(sp), SDL(sp)] then $\{X(\theta), \theta \in O\} \in$ SICX [SICV, SIL, SDCX, SDCV, SDL].

Proof. Suppose $\{X(\theta), \theta \in O\} \in$ SICX(sp). Let ϕ be an increasing convex real function. Proposition 3.2(a) shows that $\{\phi(X(\theta)), \theta \in O\} \in$ SICX(sp). That is, if θ_i, i = 1,2,3,4, are such that $\theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4$ and $\theta_1 + \theta_4 = \theta_2 + \theta_3$ then there exist four random variables $\hat{Y}_i$, i = 1,2,3,4, on a common probability space such that $\hat{Y}_i \overset{st}{=} \phi(X(\theta_i))$, i = 1,2,3,4, and

$$[\hat{Y}_1, \hat{Y}_2, \hat{Y}_3] \leq \hat{Y}_4, \quad \text{a.s.}, \tag{3.3}$$

$$\hat{Y}_1 + \hat{Y}_4 \geq \hat{Y}_2 + \hat{Y}_3, \quad \text{a.s.}. \tag{3.4}$$

The stochastic monotonicity of $\{\phi(X(\theta)), \theta \in O\}$ follows from (3.3). From (3.4) we obtain $E\phi(X(\theta_2)) + E\phi(X(\theta_3)) \leq E\phi(X(\theta_1)) + E\phi(X(\theta_4))$ which is equivalent to the convexity of $E\phi(X(\theta))$. Thus $\{X(\theta), \theta \in O\} \in$ SICX. The proof of the SICV, SIL, SDCX, SDCV, SDL cases is similar. ‖

The next result gives sufficient conditions for $\{X(\theta), \theta \in O\}$ to be SCX and SCV. It will be used in Subsection 5.7.

Proposition 3.7. If $\{X(\theta), \theta \in O\} \in$ SIL(sp) or SDL(sp) then $\{X(\theta), \theta \in O\} \in$

SCX and SCV.

Proof. We will show

$$\{X(\theta),\ \theta \in \Theta\} \in \mathrm{SIL}(sp) \Rightarrow \{X(\theta),\ \theta \in \Theta\} \in \mathrm{SCX}.$$

The other cases can be shown similarly.

Suppose $\{X(\theta),\ \theta \in \Theta\} \in \mathrm{SIL}(sp)$. Let φ be a convex function. Let θ_i, $i = 1,2,3,4$, be such that $\theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4$ and $\theta_1 + \theta_4 = \theta_2 + \theta_3$. If we show that $E\varphi(X(\theta))$ is convex in θ, that is,

$$(3.5)\quad E\varphi(X(\theta_1)) + E\varphi(X(\theta_4)) \geq E\varphi(X(\theta_2) + E\varphi(X(\theta_3)),$$

then it follows by Definition 2.6(b) that $\{X(\theta),\ \theta \in \Theta\} \in \mathrm{SCX}$.

To show (3.5) let $\hat{X}_i$, $i = 1,2,3,4$, be four random variable on a common probability space such that $\hat{X}_i \overset{st}{=} X(\theta_i)$, $i = 1,2,3,4$, and, a.s., $\hat{X}_1 + \hat{X}_4 = \hat{X}_2 + \hat{X}_3$ and $[\hat{X}_1,\hat{X}_2,\hat{X}_3] \leq \hat{X}_4$ (or, equivalently, $\hat{X}_1 \leq [\hat{X}_2,\hat{X}_3,\hat{X}_4]$). The convexity of φ implies that, a.s.,

$$(3.6)\qquad \varphi(\hat{X}_1) + \varphi(\hat{X}_4) \geq \varphi(\hat{X}_2) + \varphi(\hat{X}_3).$$

Taking expectations in (3.6) one obtains (3.5). ∎

The following lemma gives sufficient conditions for sample path convexity, concavity and linearity. Usually it is not easy to verify these conditions, but see Example 4.5.

Lemma 3.8. Let $\{X(\theta),\ \theta \in \Theta\}$ be a collection random varibles and denote $F^{-1}(u;\theta) \equiv \inf\{x: P\{X(\theta) > x\} \leq u\}$, $u \in [0,1]$, $\theta \in \Theta$. If for all

$u \in [0,1]$, $\bar{F}^{-1}(u;\theta)$ is

(i) increasing and convex [concave, linear] in $\theta \in \Theta$ then $\{X(\theta), \theta \in \Theta\} \in$ SICX(sp) [SICV(sp), SIL(sp)].

(ii) decreasing and convex [concave, linear] in $\theta \in \Theta$ then $\{X(\theta), \theta \in \Theta\}$ $\in$ SDCX(sp) [SDCV(sp), SDL(sp)].

Proof. For the SICX(sp) let θ_i, $i = 1,2,3,4$, be such that $\theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4$ and $\theta_1 + \theta_4 = \theta_2 + \theta_3$. For some uniform (0,1) random variables U let $\hat{X}_i \equiv \bar{F}^{-1}(U;\theta_i)$, $i = 1,2,3,4$. It is easily seen that $\hat{X}_i$, $i = 1,2,3,4$, satisfy (st), (cx) and (i-cx) of Definition 3.1. ∎

From the definition of sample path convexity, concavity and linearity it is immediate that one has:

Theorem 3.9. If $\{X(\theta), \theta \in \Theta\}$ and $\{Y(\theta), \theta \in \Theta\}$ belong to the class C where $C \in \{$SICX(sp), SICV(sp), SIL(sp), SDCX(sp), SDCV(sp), SDL(sp)$\}$ and if

$$Z(\theta) = X(\theta) \text{ with probability } p,$$
$$= Y(\theta) \text{ with probability } 1-p,$$

then $\{Z(\theta), \theta \in \Theta\} \in C$.

Theorem 3.10. If $\{X(\theta), \theta \in \Theta\}$ and $\{Y(\theta), \theta \in \Theta\}$ belong to the class C where $C \in \{$SICX(sp), SICV(sp), SIL(sp), SDCX(sp), SDCV(sp), SDL(sp)$\}$ and if $X(\theta)$ and $Y(\theta)$ are independent for each $\theta \in \Theta$ then $\{X(\theta) + Y(\theta), \theta \in \Theta\} \in C$.

4. Examples.

In this section we list a sample of examples using similar constructive ideas in most of them. Lemma 3.3 is also used. Some of the examples are used later in Section 5.

Example 4.1 (classes with the semigroup property). Let $\{X(\theta), \theta\in\Theta\}$ [$\Theta \subset (0,\infty)$] be a collection of random variables with the semigroup property in θ, that is, if Y_1 and Y_2 are independent and $Y_1 \stackrel{st}{=} X(\theta_1)$ and $Y_2 \stackrel{st}{=} X(\theta_2)$ then $Y_1 + Y_2 \stackrel{st}{=} X(\theta_1+\theta_2)$. In this case $\{X(\theta), \theta\in\Theta\} \in SIL(sp)$.

Proof. Let θ_i, $i = 1,2,3,4$, be such that $\theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4$ and $\theta_1 + \theta_4 = \theta_2 + \theta_3$. Let $\hat{Y}_i$, $i = 1,2,3$, be independent random variables such that $\hat{Y}_1 \stackrel{st}{=} X(\theta_1)$, $\hat{Y}_2 \stackrel{st}{=} X(\theta_2-\theta_1)$ and $\hat{Y}_3 \stackrel{st}{=} X(\theta_4-\theta_2)$ [recall $\theta_4 - \theta_2 = \theta_3 - \theta_1$]. Set $\hat{X}_1 = \hat{Y}_1$, $\hat{X}_2 = \hat{Y}_1 + \hat{Y}_2$, $\hat{X}_3 = \hat{Y}_1 + \hat{Y}_3$ and $\hat{X}_4 = \hat{Y}_1 + \hat{Y}_2 + \hat{Y}_3$. It is easily seen that $\hat{X}_i$, $i = 1,2,3,4$, satisfy (st), (ℓ) and (i-cx) of Definition 3.1. ∎

It follows that if $X(\theta)$ is a Poisson random varible with mean θ or gamma random variable with shape parameter θ or negtive binomial random variable with the proper parametrization [or binomial random variable with θ being the number of independent trials] then $\{X(\theta), \theta\in(0,\infty)\}$ [or $\{X(\theta), \theta\in\{1,2,\ldots\}\}$] belongs to SIL(sp). Also, if X_j, $j = 1,2, \ldots,$ are independent nonnegative identically distributed random variables and $S_0 \equiv 0$, $S_n \equiv \sum_{j=1}^{n} X_j$ then $\{S_i, i \in \{0,1,2,\ldots\}\}$ have the semigroup property in i. Thus $\{S_i, i \in \{0,1,2,\ldots\}\} \in SIL(sp)$.

Example 4.2. (nonhomogeneous Poisson process). Let $\{X(t), t \in [0,\infty)\}$ be a nonhomogeneous Poisson process with intensity function λ. If λ is

increasing [decreasing, constant] then $\{X(t),\ t \in (0,\infty)\} \in$ SICX(sp) [SICV(sp), SIL(sp)].

Proof. First suppose $\lambda(t) \equiv \lambda$, independent of t. Then $\{X(t),\ t\in[0,\infty)\} \in$ SIL(sp) by Example 4.1. Now suppose λ is increasing. Then $\Lambda(t) \equiv \int_0^t \lambda(u)du$ is increasing and convex. Clearly $\{X(t),\ t\in[0,\infty)\} \stackrel{st}{=} \{N(\Lambda(t)),\ t \in [0,\infty)\}$ where $\{N(t),\ t \geq 0\}$ is a homogeneous Poisson process with intensity $\lambda = 1$. But we have shown above that $\{N(t),\ t\in[0,\infty)\} \in$ SIL(sp). Thus $\{X(t),\ t \in [0,\infty)\} \in$ SICX(sp) by Proposition 3.5(a). The proof the SICV(sp) case is similar. ∎

Example 4.3. (sum of independent random variables). Let X_j, $j = 1,2, \ldots,$ be independent nonnegative random variables. Set $S_0 \equiv 0$, $S_n = \sum_{j=1}^{n} X_j$.

(a) If $\{X_j,\ j \in \{1,2,\ldots\}\} \in$ SI then $\{S_n,\ n \in \{0,1,2,\ldots\}\} \in$ SICX(sp).

(b) If $\{X_j,\ j \in \{1,2,\ldots\}\} \in$ SD then $\{S_n,\ n \in \{0,1,2,\ldots\}\} \in$ SICV(sp).

(c) If X_j, $j = 1,2,\ldots,$ are identically distributed then $\{S_n,\ n \in \{0,1,2,\ldots\}\} \in$ SIL(sp).

Proof. (a) Suppose $\{X_j,\ j \in \{1,2,\ldots\}\} \in$ SI. Let n_i, $i = 1,2,3,4$, be such that $n_1 \leq n_2 \leq n_3 \leq n_4$ and $n_1 + n_4 = n_2 + n_3$. Let $\hat{X}_j$, $j = 1, \ldots, n_3$ be random variables such that $\hat{X}_j \stackrel{st}{=} X_j$. Let $\hat{Y}_j$, $j = n_1+1, \ldots, n_2$, be such that $\hat{Y}_j \stackrel{st}{=} \hat{X}_j$ and let $\hat{Y}_j$, $j = n_3 + 1, \ldots, n_4$, be such that $\hat{Y}_j \stackrel{st}{=} X_j$ and $\hat{Y}_{n_3+j} \geq \hat{Y}_{n_1+j}$ a.s., $j = 1,2,3, \ldots, n_4-n_3$ $(=n_2-n_1)$. [Since $X_{n_3+j} \stackrel{st}{\geq} X_{n_1+j}$, it follows from Theorem 2.4 that such random variables can be constructed on a common probability space.] Furthermore, the random variables $\hat{X}_j$, $j = 1, \ldots, n_3$ and the random vectors $(\hat{Y}_{n_1+k}, \hat{Y}_{n_3+k})$, $k = 1, \ldots, n_1-n_3$ can be taken such that all are mutually independent [just generate them from independent uniform (0,1) random variables].

Now set $\hat{S}_1 = \sum_{j=1}^{n_1} \hat{X}_j$, $\hat{S}_2 = \sum_{j=1}^{n_1} \hat{X}_j + \sum_{j=n_1+1}^{n_2} \hat{X}_j$,

$\hat{S}_3 = \sum_{j=1}^{n_1} \hat{X}_j + \sum_{j=n_1+1}^{n_2} \hat{Y}_j + \sum_{j=n_2+1}^{n_3} \hat{X}_j$ and $\hat{S}_4 = \sum_{j=1}^{n_1} \hat{X}_j + \sum_{j=n_1+1}^{n_2} \hat{X}_j$ $+ \sum_{j=n_2+1}^{n_3} \hat{X}_j + \sum_{j=n_3+1}^{n_4} \hat{Y}_j$. Clearly $\hat{S}_i \overset{st}{=} S_{n_i}$, $i = 1,2,3,4$. Also, clearly, a.s., $\hat{S}_1 + \hat{S}_4 \geq \hat{S}_2 + \hat{S}_3$ and $\hat{S}_1 \leq [\hat{S}_2, \hat{S}_3] \leq \hat{S}_4$. Thus $\{S_n, n \in \{0,1,2,\ldots\}\} \in SICX(sp)$.

The proof of (b) uses a similar idea and is omitted. Statement (c) has been proven in Example 4.1. An alternative proof of (c) can be obtained from the proof of (a) by taking there $\hat{Y}_{n_3+j} \equiv \hat{Y}_{n_1+j}$, $j = 1, \ldots, n_4-n_3$. ∎

Example 4.4. (binomial random variables with parameter p). Let $\{X(p), p \in [0,1]\}$ be a collection of binomial random variables with parameters n and p (n is a fixed integer). Then $\{X(p), p \in [0,1]\} \in SIL(sp)$.

Proof. First note that $X(p) \overset{st}{=} X_1(p)+\cdots+X_n(p)$ where $X_j(p)$, $j = 1, \ldots, n$, are independent and identically distributed Bernoulli random variables with $P\{X_j(p) = 1\} = p$. We first show, for $j = 1,2, \ldots, n$, that

$$\{X_j(p), p \in [0,1]\} \in SIL(sp). \tag{4.1}$$

To see it let p_i, $i = 1,2,3,4$, be such that $p_1 \leq p_2 \leq p_3 \leq p_4$ and $p_1 + p_4 = p_2 + p_3$. Let U be a uniform (0,1) random variable. Let I_A denote the indicator function of A. Define $\hat{X}_1 = I_{\{U \leq p_1\}}$, $\hat{X}_2 = I_{\{U \leq p_2\}}$, $\hat{X}_3 = I_{\{U \leq p_1\}} + I_{\{p_2 < U \leq p_3\}}$ and $\hat{X}_4 = I_{\{U \leq p_4\}}$. Then $\hat{X}_i \overset{st}{=} X_j(p_i)$, $i = 1,2,3,4$, and $\hat{X}_i$, $i = 1,2,3,4$, satisfy (a) and (i-cx) of Definition 3.1.

From Theorem 3.10 and (4.1) it follows that $\{X(p), p \in [0,1]\} \overset{st}{=} \{\sum_{j=1}^{n} X_j(p), p \in [0,1]\} \in SIL(sp)$. ∎

Example 4.5. (location-scale parametrizations).

(a) Let Z be a nonnegative random variable and let a be a constant. Set $X(\theta) = a + \theta Z$, $\theta > 0$. Then $\{X(\theta), \theta \in [0,\infty)\} \in SIL(sp)$.

(b) Let Z be a nonnegative random variable. Set $X(\theta) = Z/\theta$, $\theta > 0$. Then $\{X(\theta), \theta \in (0,\infty)\} \in SDCX(sp)$.

Proof. Suppose Z has the survival function $\bar{G}$. Using the notation of Lemma 3.8 we have $\bar{F}^{-1}(u;\theta) = a + \theta\bar{G}^{-1}(u)$ for (a) and $\bar{F}^{-1}(u;\theta) = \bar{G}^{-1}(u)/\theta$ for (b). The first $\bar{F}^{-1}(u;\cdot)$ is increasing and linear and the second $\bar{F}^{-1}(u;\cdot)$ is decreasing and convex. The two results then follow from Lemma 3.8. ∎

5. Applications.

5.1. GI/G/1 queue. Consider a single server queueing system at which customers arrive according to a renewal process with inter-renewal times $\{A_n(\theta), n = 1,2,\ldots\}$ for some $\theta > 0$. The service times of these customers form a sequence of independent and identically distributed random variables $\{B_n(\mu), n = 0,1,2,\ldots\}$, $\mu > 0$, independent of $\{A_n(\theta), n = 1,2,\ldots\}$. Let $W_n(\theta,\mu)$ be the waiting time of the n-th customer $[W_0(\theta,\mu) \equiv 0]$. It is well known [see, e.g., Ross (1983)] that

$$W_n(\theta,\mu) = [W_{n-1}(\theta,\mu) + B_{n-1}(\mu) - A_n(\theta)]^+, \quad n = 1,2, \ldots .$$

Theorem 5.1. (a) Fix $\mu > 0$. If $\{A_n(\theta), \theta\in(0,\infty)\} \in SICV(sp)$ for $n = 1,2, \ldots,$ then $\{W_n(\theta,\mu), \theta\in(0,\infty)\} \in SDCX(sp)$ for $n = 1,2, \ldots .$

(b) Fix $\theta > 0$. If $\{B_n(\mu), \mu\in(0,\infty)\} \in SDCX(sp)$ for $n = 0,1,2, \ldots,$ then $\{W_n(\theta,\mu), \mu\in(0,\infty)\} \in SDCX(sp)$ for $n = 1,2, \ldots .$

Proof. We prove (a); the proof of (b) is similar. Certainly $\{W_0(\theta,\mu), \theta\in(0,\infty)\} \in SDCX(sp)$. Suppose $\{W_{n-1}(\theta,\mu), \theta \in (0,\infty)\} \in SDCX(sp)$. Note that if $\{A_n(\theta), \theta \in (0,\infty)\} \in SICV(sp)$ then $\{-A_n(\theta), \theta$

$\varepsilon\ (0,\infty)\} \varepsilon$ SDCX(sp). Thus Theorem 3.10 implies that

(5.1) $\{W_{n-1}(\theta,\mu) + B_{n-1}(\mu) - A_n(\theta),\ \theta \varepsilon (0,\infty)\} \varepsilon$ SDCX(sp).

The function $[x]^+$ is increasing and convex. Thus from (5.1) and Proposition 3.2(b) it follows that $\{W_n(\theta,\mu),\ \theta \varepsilon (0,\infty)\} =$

$\{[W_{n-1}(\theta,\mu) + B_{n-1}(\mu) - A_n(\theta)]^+,\ \theta \varepsilon (0,\infty)\} \varepsilon$ SDCX(sp). ∎

Remark 5.2. Let $\{X_n,\ n = 0,1,\ldots\}$ be a sequence of independent and identically distributed nonnegative random variables and set $B_n(\mu) = X_n/\mu$. Then $\{B_n(\mu),\ \mu\varepsilon(0,\infty)\} \varepsilon$ SDCX(sp) [Example 4.5(b)]. Therefore the waiting time $\{W_n(\theta,\mu),\ \mu\varepsilon(0,\infty)\}$ in this GI/G/1 queue is SDCX(sp) and hence SDCX [Theorem 3.6]. The convexity of $E[W_n(\theta,\mu)]$ with respect to μ is proven in Weber (1983). From Proposition 2.11 it follows that the steady state mean waiting time in this queue is convex with respect to μ. This fact has been established in Tu and Kumin (1983). Observe that in Theorem 5.1(b) we do not restrict $B_n(\mu)$ to take this specific form.

5.2. GI/G/c queue with rotary assignment. Consider a single stage queueing system with c parallel serves at which customers arrive according to a renewal process with inter-arrival times $\{R_n,\ n = 1,2,\ldots\}$. The service times of these customers form a sequence $\{B_n,\ n = 1,2,\ldots\}$ of independent and identically distributed random variables independent of $\{R_n,\ n = 1,2,\ldots\}$. The customers are assigned to the servers in a rotary manner. That is, the $(cn+r)$-th customer is assigned to server r $(r=1,2,\ldots,c;n=0,1,\ldots)$. The stationary waiting time (say $W(c)$) of an arbitrary customer, when it exists [that is, when c is large enough, say $c \geq c^*$ for some c^*] has the same

distribution as that of the stationary waiting time of an arbitary customer in a GI/G/1 queue with inter-arrival times $\{A_n(c) = R_{(n-1)c+1}+\ldots+R_{nc}, n = 1,2,\ldots\}$ and service times $\{B_n, n = 1,2,\ldots\}$. In Example 4.3(c) it is shown that $\{A_n(c), c\in\{1,2,\ldots\}\}$ is SICV(sp) [in fact SIL(sp)]. Thus, from Theorem 5.1(a), Theorem 3.6 and Proposition 2.11, one has

Corollary 5.3. The stationary waiting time $W(c)$ of an arbitrary customer in a GI/G/c queue with rotary assignment satisfies $\{W(c), c \in (c^*,\infty)\} \in$ SDCX(sp).

Remark 5.4. Rolfe (1971) conjectured that $E[W(c)]$ is convex in $c > c^*$ for a GI/G/c queue with first come first served service policy and proved it for the M/D/c queue. For the GI/D/c queue the distribution function of the stationary waiting time is the same under the rotary assignment and first come first served service policy. Thus Corollary 5.3 extends the result of Rolfe (1971) to the GI/D/c queue and provides a partial answer to his conjecture.

5.3. $M^B/M(n)/1$ queue. Consider a single stage queueing system at which customers arrive according to a Poisson process with rate $\lambda > 0$. Customer n brings a random number B_n of tasks, $n = 1,2,\ldots$. The B_n's are independent and identically distributed. Denote $\bar{Q}_b = P\{B_n > b\}$, $b = 0,1,2,\ldots$. The service requirements of these tasks form a sequence of independent, identically distributed mean μ^{-1} exponential random variables independent of $\{B_n, n = 1,2,\ldots\}$. The service rate is $\gamma(n)$ when there are n tasks in the system, $n = 0,1,2,\ldots$, where $\gamma(0) = 0$ and $\gamma(n) > 0$ for $n \geq 1$. Let $Y_t(\lambda)$ denote the number of tasks in the system at time t.

Theorem 5.5. Suppose $\gamma(n)$ is increasing and concave in $n = 0,1,2,\ldots$.

(a) If $\lim_{n\to\infty} \gamma(n) < \infty$ and if $\{Y_0(\lambda), \lambda\in(0,\infty)\}\in SICX(sp)$ then

$$\{Y_t(\lambda),\ \lambda \in (0,\infty)\} \in SICX(sp) \quad \text{for all} \quad t \geq 0. \tag{5.2}$$

(b) If for some $\lambda^* > 0$ the steady state distribution exists for $\lambda \in (0,\lambda^*]$ and $Y_t(\lambda) \xrightarrow{st} Y^*(\lambda)$ as $t \to \infty$ for each $\lambda \in (0,\lambda^*]$ then $\{Y^*(\lambda),\ \lambda\in(0,\lambda^*]\} \in SICX$.

Proof. Suppose $\gamma_s \equiv \lim_{n\to\infty} \gamma(n) < \infty$. Let λ_s be an arbitrary (large) positive number and set $\eta \equiv 2(\lambda_s+\mu\gamma_s) < \infty$. For $\lambda \leq \lambda_s$ consider the Markov chain $\{X_n(\lambda),\ n = 0,1,2,\ldots\}$ with state space $\{0,1,2,\ldots\}$ and transition probabilities determined by

$$\begin{aligned}
\bar{F}_x(y;\lambda) &\equiv P\{X_n(\lambda) > y \mid X_{n-1}(\lambda) = x\} \\
&= 1 && \text{if} \quad y < x - 1, \\
&= 1 - \frac{\mu\gamma(x)}{\eta} && \text{if} \quad y = x - 1, \\
&= \frac{\lambda}{\eta}\,\bar{D}_{y-x} && \text{if} \quad y \geq x.
\end{aligned}$$

Suppose

$$X_0(\lambda) \overset{st}{\leq} Y_0(\lambda). \tag{5.3}$$

Let $\{N(t),\ t \geq 0\}$ be a Poisson process with rate η defined on the same probability space as that of $\{X_n(\lambda),\ n = 0,1,2,\ldots\}$ and independent of it. From (5.3) it follows that the uniformized process $\{X_{N(t)}(\lambda),\ t \geq 0\}$ satisfies

(5.4) $$\{Y_t(\lambda),\ t \geq 0\} \overset{st}{=} \{X_{N(t)}(\lambda),\ t \geq 0\}$$

[see, e.g., Keilson (1979)]. Since $1 - \frac{\mu\gamma(x+1)}{\eta} \geq \frac{\lambda}{\eta}$ it follows that for each y,

(5.5) $$\bar{F}_x(y;\lambda) \text{ increases in } x \in \{0,1,2,\ldots\} \text{ and in } \lambda \in (0,\lambda_s].$$

We will show that, for every $n \in \{0,1,2,\ldots\}$,

(5.6) $$\{X_n(\lambda),\ \lambda \in (0,\lambda_s]\} \in SICX(sp)\ .$$

From Theorem 3.9 and (5.4) it then follows that $\{Y_t(\lambda),\ \lambda \in (0,\lambda_s]\} \in SICX(sp)$ and since λ_s is arbitrary we obtain (5.2).

The proof of (5.6) is by induction on n. By assumption, (5.6) is true when $n = 0$. Suppose (5.6) is true with $n - 1$ replacing n. Let $\lambda_i \in (0,\lambda_s]$, $i = 1,2,3,4$, be such that $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$ and $\lambda_1 + \lambda_4 = \lambda_2 + \lambda_3$. Then by the induction hypothesis there exist $\hat{X}_i$, $i = 1,2,3,4$, defined on a common probability space such that $\hat{X}_i \overset{st}{=} X_{n-1}(\lambda_i)$, $i = 1,2,3,4$, and such that (cx) and (i-cx) of Definition 3.1 hold. Let x_i, $i = 1,2,3,4$, be a particular realization of $\hat{X}_i$, $i = 1,2,3,4$. Thus $[x_1,x_2,x_3] \leq x_4$ and

(5.7) $$x_1 + x_4 \geq x_2 + x_3.$$

First suppose that equality holds in (5.7), that is, $x_1 + x_4 = x_2 + x_3$. Construct $\tilde{X}_i$, $i = 1,2,3,4$, on a common probability space using two independent uniform (0,1) random variables U_1 and U_2 as follows:

(i) If $U_1 \in (0,\lambda_i/\eta)$ then $\tilde{X}_i = x_i + \bar{Q}^{-1}(U_2)$, $i = 1,2,4$, and if

$U_1 \in (0,\lambda_1/n) \cup (\lambda_2/n,\lambda_4/n)$ then $\tilde{X}_3 \equiv x_3 + \bar{Q}^{-1}(U_2)$.

(ii) If $U_1 \in (\lambda_i/n, 1-(\mu\gamma_s/n))$ then $\tilde{X}_i \equiv x_i$, $i = 1,2,4$, and if $U_1 \in (\lambda_1/n,\lambda_2/n) \cup (\lambda_4/n, 1-(\mu\gamma_s/n))$ then $\tilde{X}_3 \equiv x_3$.

(iii) If $U_1 \in (1-(\mu\gamma_2/n), 1)$ then $\tilde{X}_i \equiv x_i - \Delta_i$, $i = 1,2,3,4$, where

$\Delta_i \equiv I_{[U_2 \le \gamma(x_i)/\gamma_s]}$, $i = 1,2,4$ and

$\Delta_3 = I_{[U_2 \le \gamma(x_1)/\gamma_s]} + I_{[\gamma(x_2)/\gamma_s \le U_2 \le (\gamma(x_2)+\gamma(x_3)-\gamma(x_1))/\gamma_s]}$

It is now verified that $\tilde{X}_i \overset{st}{=} [X_n(\lambda_i) | X_{n-1}(\lambda_i) = x_i]$, $i = 1,2,3,4$. The verification of $[\tilde{X}_1,\tilde{X}_2,\tilde{X}_3] \le \tilde{X}_4$ and $\tilde{X}_1 + \tilde{X}_4 \ge \tilde{X}_2 + \tilde{X}_3$ a.s. in cases (i) and (ii) is simple. In (iii) notice that, by assumption, $x_1 + x_4 = x_2 + x_3$ and that the concavity of $\gamma(\cdot)$ implies $\gamma(x_2) + \gamma(x_3) - \gamma(x_1) \ge \gamma(x_4)$. Thus $\Delta_1 + \Delta_4 \le \Delta_2 + \Delta_3$ hence $\tilde{X}_1 + \tilde{X}_4 \ge \tilde{X}_2 + \tilde{X}_3$ a.s. To prove

$$[\tilde{X}_1,\tilde{X}_2,\tilde{X}_3] \le \tilde{X}_4 \qquad \text{a.s.,} \tag{5.8}$$

note that (5.8) can be violated only if $x_4 = x_i$ for some $i \in \{1,2,3\}$. But since we postulate $x_1 + x_4 = x_2 + x_3$ it follows that if $x_4 = x_i$ for some $i \in \{1,2,3\}$ then $\Delta_4 = \Delta_i$ for that i and thus (5.8) holds.

Summarizing the results of the preceeding paragraph we see that if equality holds in (5.7) then $[X_n(\lambda_i) | X_{n-1}(\lambda_i) = x_i]$, $i = 1,2,3,4$, "can be put" on the same probability space such that (cx) and (i-cx) of Definition 3.1 hold.

If strict inequality holds in (5.7) then using the stochastic monotonicity (5.5) it can be shown, by an argument similar to the proof of Lemma 3.3, that $[X_n(\lambda_i) | X_{n-1}(\lambda_i) = x_i]$, $i = 1,2,3,4$, "can be put" on the same probability space such that the analogs of (st), (cx) and (i-cx) hold. The proof of (5.6) is now completed using the fact that SICX(sp) is closed

under mixing [Theorem 3.9].

If $\gamma_s < \infty$ then statement (b) follows from (a), Theorem 3.6 and Proposition 2.11. By standard limiting arguments it can be shown that (b) is true also when $\gamma_s = \infty$. ||

Remark 5.6. For the M/M/c queue [that is, M/M(n)/1 with $\gamma(n) = \min(n,c)$, $n \geq 0$] Grassman (1983) and Lee and Cohen (1983) showed that $EY^*(\lambda)$ is increasing and convex in $\lambda \in (0,\lambda^*]$. The algebraic proof used there can become very cumbersome, if not impossible, if one tries to establish the convexity of $E\phi(Y^*(\lambda))$ in λ for all increasing convex functions $\phi:\{0,1,2,..\} \to R$. Thus one sees that the sample path approach is simple and very powerful.

5.4. *Proportional hazards and imperfect repair.* The main result of this subsection is

Proposition 5.7. Let $\bar{F}$ be an absolutely continuous survival function such that $\bar{F}(0) = 1$. For $p \in (0,1]$ let $X(p)$ have the survival function $\bar{F}^p$ (that is the $X(p)$'s have proportional hazards). Then $E\phi(X(p))$ is convex and decreasing in $p \in (0,1]$ whenever ϕ is increasing. In particular $\{X(P), p \in (0,1]\} \in SDCX$.

In order to motivate the proposition, and indicate its uses, consider the following imperfect repair model of Cleroux, Dubuc and Tilquin (1979) and Brown and Proschan (1983).

Model 5.8. A new item with an absolutely continuous survival function $\bar{F}$ undergoes an imperfect repair upon each time it fails before it is scrapped.

With probability p the repair is unsuccessful and the item is scrapped. With probability $1 - p$ the repair is successful and minimal, that is, after a successful repair at time t the item is as good as a working item at age t.

If $X(p)$ denote the time to scrap in Model 5.8 then the survival function of $X(p)$ is $\bar{F}^p$ (Bery and Cleroux (1982) and Brown and Proschan (1983)).

Proof of Proposition 5.7. Clearly $\bar{F}^p(t)$ is convex and decreasing in p for each $t\in[0,\infty)$. Thus $E\phi(X(p))$ is convex and decreasing in $p\in(0,1]$ whenever ϕ is a binary increasing function, and hence also whenever ϕ is an increasing step function. Standard limiting arguments complete the proof. ∎

Proposition 5.7 can be applied as follows. Suppose the cost of performing an imperfect repair with probability $1 - p$ of unsuccessful outcome is $C(p)$. It is reasonable to expect that $C(p)$ decreases in p. If the benefit associated with a lifetime $X(p)$ is an increasing function $\phi:R_+\to R$ then the total benefit $B(p)$ is

$$B(p) = E\phi(X(p)) - C(p).$$

Suppose that, due to engineering constraints, p must lie in the interval $[a,b]$ for some $0 < a < b \leq 1$, and that $C(p)$ is linear in $p\in[a,b]$. Then, by Proposition 5.7, $B(p)$ is convex in $[a,b]$. So $p = a$ or $p = b$ is an optimal solution which maximizes the benefit.

5.5. Branching processes. Consider a Galton-Watson discrete time branching process $\{X_i(\theta), i=0,1,\ldots\}$ depending on a parameter $\theta > 0$ which is the

parameter of the offspring discrete probability function $f(\cdot;\theta)$. Then the transition probabilities of this Markov chain are

$$P\{X_n(\theta) = y | X_{n-1}(\theta) = x\} = f^{(x)}(y;\theta),\ x,y\in\{0,1,2,\ldots\},$$

where $f^{(x)}(\cdot;\theta)$ denotes the x-th convolution of $f(\cdot;\theta)$.

Result 5.9. Suppose $f(\cdot;\theta)$, $\theta > 0$ has the semigroup property, that is, $f(\cdot;\theta_1)*f(\cdot;\theta_2) = f(\cdot;\theta_1+\theta_2)$ where '*' denotes convolution. If initially $X_0(\theta) \equiv x$, where x is a constant independent of θ, then $\{X_n(\theta), \theta\in(0,\infty)\}\in SICX(sp)$ for each $n = 1,2,\ldots$.

Proof. We will show that for each $n = 0,1,2, \ldots$, and for each θ_i, $i = 1,2,3,4$, such that $\theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4$ and $\theta_1 + \theta_4 = \theta_2 + \theta_3$, there exist four random variables $\hat{X}_i$, $i = 1,2,3,4$, defined on the same probability space such that

$$\hat{X}_i \overset{st}{=} X_n(\theta_i), \qquad i = 1,2,3,4, \tag{5.9.i}$$

$$\hat{X}_1 + \hat{X}_4 \geq \hat{X}_2 + \hat{X}_3, \quad \text{a.s.}, \tag{5.9.ii}$$

$$\hat{X}_1 \leq [\hat{X}_2,\hat{X}_3] \leq \hat{X}_4, \quad \text{a.s.} \tag{5.9.iii}$$

It follows then that $\{X(\theta), \theta \in (0,\infty)\} \in SICX(sp)$.

The proof of (5.9) is by induction on n. Clearly (5.9) holds for $n = 0$. Suppose (5.9) holds with $n - 1$ replacing n. Then there exist $\hat{Y}_i$, $i = 1,2,3,4$, defined on some common probability space $\mathcal{A}$ such that $\hat{Y}_i \overset{st}{=} X_{n-1}(\theta_i)$, $i = 1,2,3,4$, and, a.s., $\hat{Y}_1 \leq [\hat{Y}_2,\hat{Y}_3] \leq \hat{Y}_4$ and $\hat{Y}_1 + \hat{Y}_4 \geq \hat{Y}_2 + \hat{Y}_3$.

Define now the following mutually independent random varibles which are

assumed to be independent of $\hat{Y}_i$, $i = 1,2,3,4$ (we can take Ω to be rich enough to support these):

$Z_j^{\theta_1}$ having probability function $f(\cdot,\theta_1)$, $j = 1,2, \ldots,$

$Z_j^{\theta_2-\theta_1}$ having probability function $f(\cdot,\theta_2-\theta_1)$, $j = 1,2, \ldots,$

$Z_j^{\theta_4-\theta_2}$ having probability function $f(\cdot,\theta_4-\theta_2)$, $j = 1,2, \ldots,$

and set

$$\hat{X}_1 \equiv \sum_{j=1}^{\hat{Y}_1} Z_j^{\theta_1},$$

$$\hat{X}_2 \equiv \sum_{j=1}^{\hat{Y}_2} [Z_j^{\theta_1} + Z_j^{\theta_2-\theta_1}],$$

$$\hat{X}_3 \equiv \sum_{j=1}^{\hat{Y}_1} [Z_j^{\theta_1} + Z_j^{\theta_4-\theta_2}] + \sum_{j=\hat{Y}_2+1}^{\hat{Y}_2+\hat{Y}_3-\hat{Y}_1} [Z_j^{\theta_1} + Z_j^{\theta_4-\theta_2}],$$

$$\hat{X}_4 \equiv \sum_{j=1}^{\hat{Y}_4} [Z_j^{\theta_1} + Z_j^{\theta_2-\theta_1} + Z_j^{\theta_4-\theta_2}].$$

Then, from the semigroup property it follows that $\hat{X}_i \stackrel{st}{=} X_n(\theta_i)$, $i = 1,2,3,4$. Clearly $\hat{X}_1 \leq [\hat{X}_2,\hat{X}_3] \leq \hat{X}_4$ a.s., and since $\hat{Y}_4 \geq \hat{Y}_2 + \hat{Y}_3 - \hat{Y}_1$, also $\hat{X}_1 + \hat{X}_4 \geq \hat{X}_2 + \hat{X}_3$, a.s. ∎

Similar results hold also for continuous time branching processes. These will be discussed elsewhere.

5.6. Empirical distribution functions. Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables with a common distribution function F. Suppose the support $\mathcal{O}$ of F is an interval of real numbers or integers. Denote by $F_n(t)$, $t \in \mathcal{O}$, the empirical distribution function constructed from the Y_i's.

Result 5.10. If F is convex [concave, linear] on O then $\{F_n(t), t \in O\} \in SICX(sp)$ [$SICV(sp)$, $SIL(sp)$].

To prove Result 5.10 note that $F_n(t)$ has a binomial distribution with parameters n and $F(t)$. Thus the result follows from Example 4.4 and Proposition 3.5.

5.7. Convex parametrization. According to Schweder (1982), a family $\{P_\theta, \theta \in O\}$ [O is an interval of real numbers or of integers] of distributions is called convexly parametrized if $E\phi(X(\theta))$ is convex for every convex function ϕ where $X(\theta)$ is distributed according to P_θ. That is, $\{X(\theta), \theta \in O\} \in SCX$ [see Definition 2.6(b)]. By Proposition 3.7, $\{X(\theta), \theta \in O\} \in SCX$ whenever $\{X(\theta), \theta \in O\} \in SIL(sp)$ or $SDL(sp)$. Thus Examples 4.1, 4.4 and 4.5 yield a host of convexly parametrized families of distributions (some of which have already been noticed by Schweder (1982)). For these families of distributions, Schweder (1982) and Shaked (1980) obtained various inequalities useful in biology and statistics.

5.8. Majorization and Schur-convexity. Let $X(\theta_j)$, $j = 1,2,\ldots,n$, be independent random variables having distributions $F(\cdot;\theta_j)$, $j = 1,2, \ldots, n$, respectively, where $\{F(\cdot;\theta), \theta \in O\}$ is a family of distributions. Marshall and Olkin (1979, p. 102) showed that if $\{F(\cdot;\theta), \theta \in O\}$ satisfies the semigroup property [see, e.g., Example 4.1] and if $F(x;\theta)$ has some total positivity property then $E\psi(X(\theta_1), \ldots, X(\theta_n))$ is Schur-convex in $(\theta_1,\ldots,\theta_n)$ whenever $\psi(x_1,\ldots,x_n)$ is Schur-convex. In particular, if $\phi:R\to R$ is convex then, under the conditions above, $E\sum_{j=1}^{n}\phi(X(\theta_j))$ is Schur-convex in $(\theta_1,\ldots,\theta_n)$.

From Example 4.1 and Proposition 3.7 we see that for each $j = 1, \ldots,$

n, $E\varphi(X(\theta_j))$ is convex in θ_j whenever $\{F(\cdot;\theta), \theta \in \Theta\}$ has the semigroup property. Thus, if φ is convex, then $E\sum_{j=1}^{n}\varphi(X(\theta_j))$ is symmetric and convex in $(\theta_1,\ldots,\theta_n)$ and hence Schur-convex in $(\theta_1,\ldots,\theta_n)$.

The latter conclusion (which is weaker than the conclusion of Marshall and Olkin (1979,p.102)) is obtained without the total positivity assumption of Marshall and Olkin (1979, p. 102).

5.9. Cumulative damage shock models. Esary, Marshall and Proschan (1973) considered the folowing model for wear processes.

Model 5.11. An item is subjected to shocks occurring randomly in time according to a Poisson process $\{N(t),t\geq 0\}$ with rate λ. The i-th shock causes a nonnegative random damage X_i. The damages are independent and accumulate additively.

Denote $S_0 \equiv 0$, $S_n = S_{n-1} + X_n$, $n = 1,2,\ldots$. Thus at time $t > 0$ the accumulated damage is $S_{N(t)}$.

Result 5.12. (a) If $\{X_i, i\in\{1,2,\ldots\}\} \in SI$ then

$$\{S_{N(t)}, t\in[0,\infty)\} \in SICX(sp).$$

(b) If $\{X_i, i\in\{1,2,\ldots\}\} \in SD$ then $\{S_{N(t)}, t\in[0,\infty)\}\in SICV(sp)$.

(c) If the X_i's are identically distributed then

$$\{S_{N(t)}, t\in[0,\infty)\}\in SIL(sp).$$

The proof of this result consists of two steps. For t_i, $i = 1,2,3,4$, such that $t_1 \leq t_2 \leq t_3 \leq t_4$ and $t_1 + t_4 = t_2 + t_3$, first construct $\hat{N}_i$, $i = 1,2,3,4$, such that $\hat{N}_i \overset{st}{=} N(t_i)$, $i = 1,2,3,4$, and a.s. $\hat{N}_1 \leq [\hat{N}_2,\hat{N}_3] \leq \hat{N}_4$ and $\hat{N}_1 + \hat{N}_4 = \hat{N}_2 + \hat{N}_3$, as in Example 4.1. Then construct $\hat{S}_i$, $i = 1,2,3,4$, such that $\hat{S}_i \overset{st}{=} S_{\hat{N}_i}$, $i = 1,2,3,4$, and, a.s.,

$\hat{S}_1 + \hat{S}_4 \geq [\leq,=]\hat{S}_2 + \hat{S}_3$ and $\hat{S}_1 \leq [\hat{S}_2,\hat{S}_3] \leq \hat{S}_4$.

Other convexity results can be obtained for Model 5.11. For example, under the assumptions of Result 5.12(a) [(b), (c)] the wear process $S_{N(t)}$ at time t is SICX(sp) [SICV(sp), SIL(sp)] in $\lambda \in (0,\infty)$. If the distribution of the X_i's of Model 5.11 depends on θ then for each $t > 0$, stochastic convexity [concavity, linearity] of $S_{N(t)}$ in θ follows from stochastic convexity [concavity, linearity] of each X_i in θ provided the X_i's are identically distributed. We omit the proofs of these results.

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